Completely Regular Semirings

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Introduction

As the name suggests completely regular semigroups are a specialization of regular semigroups. The class of completely regular semigroups $CoR$ is one of the best studied classes of semigroups since they allow various structural descriptions.

The first paper on these semigroups has been published by Clifford in 1941 ([Cli41]). Already in this paper the first decompositions into groups and completely simple semigroups were mentioned which justified their alternative name “union of groups”. The term “completely regular” we will use here was coined by Lyapin in the monograph “Polygruppy” ([Lya60]). During the time thereafter, numerous publications dealt with different subclasses and described their structure. For example, this includes the lattice of subvarieties of all bands ([Fen70]) or of completely simple semigroups ([Jon81]). Bands of groups, which will be of special interest for us, were observed by Petrich in [Pet77] and Rasin in [Ras81]. Another fundamental class of completely regular semigroups, also for this thesis, namely the one of orthogroups, has been topic of [Pet03]. These are completely regular semigroups where the idempotents form a subsemigroup. A summary about this large material is given in [PR99].

As it can be expected because of the title of this thesis, we will transfer some of this knowledge onto semirings. Semirings were first considered explicitly in an article of Vandiver from 1934 ([Van34]). They extent the concept of rings in the way that the additive reduct only has to be a semigroup instead of an Abelian group. Meanwhile, various papers concerning different aspects of this algebraic structure appeared. Good references for these topics are the monographs [HW98] and [Gol99].

The first steps towards completely regular semirings were done in [Zel81] where a good overview about additively regular semirings has been given. In the last decade, there have been different approaches to introduce a concept of completely regular semirings so that these
inherit the structural regularity from their additive reducts. Pastijn and Guo ([PG02]) analyzed
semirings which are composed of disjoint rings. A more generalized approach was chosen by
Sen, Maity and Shum in [SMS06] when they claimed that for each element $a$ of a completely
regular semiring there has to exist an element $x$ such that $a = a + x + a, a + x = x + a$ and
$a(a + x) = a + x$. This leads to semirings which are composed of skew-rings, i.e. semirings
whose additive reduct is a group but not necessarily a commutative one. So, this is a slightly
more general way to adopt the ideas of completely regular semigroups in semirings theory
as the approach of Pastijn and Guo. On the base of this definition several subclasses like
completely simple semirings ([SMW05]), Clifford semirings ([SMS05]) and generalized rect-
angular Clifford semirings ([CCY10]) have been in the center of further papers. But still there
are only rare publications explicitly on orthodox semirings. Zeleznekow ([Zel80]) and Sen and
Maity ([SM04]) investigated semirings whose multiplicative reduct are orthodox. Since we are
interested in completely regular semirings, we will concentrate on semirings with an orthodox
additive reduct as it is done in a more specific case in [PG02].

The starting point of this dissertation is the mentioned work of Pastijn and Guo. Using the
definition of Sen, Maity and Shum, several results of [PG02] will be extended. The needed
knowledge from the field of universal algebra along with the necessary fundamentals in semi-
group and semiring theory will be provided in chapter 1. Naturally, the store of knowledge
about completely regular semigroups will be very helpful for our purpose. To avoid the prelimi-
aries getting to large several results in these topics which are not necessary for the general
understanding are collected in the appendix. Chapter 2 will be concerned with the basic
properties of the semirings under investigation. This especially involves generalizations of the
decompositions known from the semigroup case. Furthermore, we will show that the class of
all completely regular semirings $\mathcal{CR}$ is in fact a variety. In the third and fourth chapter, certain
subvarieties of $\mathcal{CR}$ will be considered in more detail. So, chapter 3 contains various results
about completely simple semirings including a Rees like representation of them. The con-
cluding chapter deals with orthodox semirings, some characterizations of them and several
subclasses. In this context, the core $C(S)^+$ of a semiring will be proven to be very useful.
1. Preliminaries

1.1. Universal algebra

In this section a short introduction on the theory of universal algebras will be given. This will include the main ideas on these topics used in this thesis. In [BS81] a more comprehensive study on universal algebras can be found.

For our purpose, we will interpret a universal algebra as a pair \((S, \Phi)\) consisting of a nonempty set \(S\) and a family \(\Phi\) of operations on \(S\), where the number and arity of these operations determine the type of \(S\). Often, we will not mention the family \(\Phi\) explicitly and will just refer to the universal algebra \(S\) if the operations are clear without ambiguity. Now, terms as subalgebras, homomorphisms and congruences could be defined similarly as it is done for example in semigroup theory.

An arbitrary nonempty class \(\mathcal{V}\) of universal algebras of a given type is called variety if there exists a family of identities \([p_i = q_i]_{i \in I}\) such that \(\mathcal{V}\) consists exactly of all universal algebras (of the given type) which satisfy these identities for all their elements.

Example 1.1. The class of all semigroups, denoted by \(\mathcal{SG}\), is a variety within the class of all universal algebras of type \((2)\), i.e. with one binary operation. Using \(+\) as the symbol of this operation, the characterizing identity is

\[
a + (b + c) = (a + b) + c.
\]  

(1.1)

By \(\mathcal{G}\) we denote the class of all groups. This can be seen as a variety of algebras of type \((2, 1)\) where the unary operation maps an element \(a\) to its inverse \(-a\), so that, beside the identity
(1.1) which ensures again the associativity, the identities

\[-a + a = -b + b, \quad -a + a + a = a\]

have to hold.

Another characterization of varieties is given by the following famous theorem, which is known as Birkhoff’s theorem ([Bir35]):

**Theorem 1.2** (Birkhoff). A nonempty class \( \mathcal{V} \) of universal algebras of a given type is a variety if and only if it is closed under subalgebras, homomorphic images and direct products.

One aim of this thesis is the structural analysis of completely regular semirings. Therefore, we will observe subvarieties of a given variety \( \mathcal{V} \), e.g. of the variety \( \mathcal{CR} \) of completely regular semirings (see Corollary 2.7), which form the lattice of subvarieties of \( \mathcal{V} \) denoted by \( \mathcal{L}(\mathcal{V}) \).

In this lattice, the meet of two varieties is given by their intersection and the join by the least variety containing both.

Let \( S \) be a universal algebra of some fixed type and \( \mathcal{V'} \) be a class of universal algebras of the same type. Moreover, a congruence \( \rho \) on \( S \) is called a \( \mathcal{V}' \)-congruence if \( S/\rho \in \mathcal{V'} \), and \( \rho \) is over \( \mathcal{V}' \) if all \( \rho \)-classes which are subalgebras of \( S \) belong to \( \mathcal{V}' \). If the least \( \mathcal{V}' \)-congruence exists, we will denote it by \( \rho_{\mathcal{V'}} \). Let \( \mathcal{V'} \) be a variety and \( \mathcal{U}_1, \mathcal{U}_2 \subseteq \mathcal{V'} \). Using the foregoing terminology the Malcev product of \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) within \( \mathcal{V'} \) (see [Mal67]) is defined by

\[
\mathcal{U}_1 \circ_{\mathcal{V'}} \mathcal{U}_2 = \{ S \in \mathcal{V'} \mid \text{there exists a } \mathcal{U}_2\text{-congruence over } \mathcal{U}_1 \text{ on } S \}.
\]

Subsequently, we will use the Malcev product mainly within \( \mathcal{SG} \) or within the variety \( \mathcal{SR} \) of all semirings. That is why we will omit mostly the additional specification of the variety \( \mathcal{V}' \) at the symbol of the Malcev product and will just use \( \circ \) alternatively.

Since for any universal algebra \( S \) of an arbitrary variety \( \mathcal{V} \) the equality congruence \( id = \{(x,x) \in S \times S\} \) is a \( \mathcal{V}' \)-congruence and the universal congruence \( \omega = S \times S \) is a congruence over \( \mathcal{V}' \), the Malcev product \( \mathcal{U}_1 \circ \mathcal{U}_2 \) contains \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) if both classes are in fact subvarieties of \( \mathcal{V}' \). If this is the case, we come to the following result.
Lemma 1.3. If $\mathcal{V}$ is a variety and $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{L}(\mathcal{V})$, the Malcev product can be determined by

$$\mathcal{U}_1 \circ \mathcal{U}_2 = \{ S \in \mathcal{V} \mid \text{the least } \mathcal{U}_2\text{-congruence } \varrho_{\mathcal{U}_2} \text{ on } S \text{ is over } \mathcal{U}_1 \}.$$ 

Proof. $\supseteq$ This is trivial.

$\subseteq$ Let $S \in \mathcal{U}_1 \circ \mathcal{U}_2$ and $\varrho_{\mathcal{U}_2}$ be the least $\mathcal{U}_2$-congruence on $S$. The existence of $\varrho_{\mathcal{U}_2}$ is known by Lemma A.1. Furthermore, there exists a $\mathcal{U}_2$-congruence $\varrho$ over $\mathcal{U}_1$ on $S$. Now, assume that $e \in S$ is such that the congruence class $e \varrho_{\mathcal{U}_2}$ is a subalgebra of $S$. Let $\mu$ be an arbitrary $n$-ary operation on $S$. $e \varrho_{\mathcal{U}_2}$ has to be closed under this operation, i.e. for all elements $a_1, \ldots, a_n \in e \varrho_{\mathcal{U}_2}$ there exists an element $a \in S$, $a = \mu(a_1, \ldots, a_n)$ and $a \varrho_{\mathcal{U}_2} e$. Since $e \varrho_{\mathcal{U}_2} \subseteq e \varrho$, $a, a_1, \ldots, a_n \in e \varrho$ as well, and since $\varrho$ is a congruence, for arbitrary elements $b_1, \ldots, b_n$ of $e \varrho$, $\mu(b_1, \ldots, b_n) \varrho a \varrho e$. So, $e \varrho$ is a subalgebra, too and $e \varrho \in \mathcal{U}_1$. As a variety, $\mathcal{U}_1$ is closed with respect to subalgebras, and this results in $e \varrho_{\mathcal{U}_2} \in \mathcal{U}_1$. ☐

A slightly more specialized form of this lemma and its proof can be found in Lemma IX.1.2 of [PR99].

Please note that in general the Malcev product, even of two varieties, need not to be a variety. But it is always a quasivariety and hence at least closed under subalgebras and direct products.

1.2. Completely regular semigroups

Completely regular semigroups are well-studied, and naturally, the results are a base for the investigation of completely regular semifields. So in this section, we will recall the most important definitions and theorems we will need for our considerations. A collection of further results in these topics can be found in [PR99]. Since the statements mentioned here will be used by us mainly in the additive reduct of semifields, we will present them already here in additive notation.
Definition 1.4. Let \((S, +)\) be a semigroup. An element \(a \in S\) is called regular if there exists some element \(x \in S\) such that
\[
a = a + x + a.
\]
(1.2)
If in addition holds that
\[
a + x = x + a,
\]
(1.3)
a is called completely regular. If all elements of \(S\) are (completely) regular, \(S\) itself is called (completely) regular. The class of all completely regular semigroups will be denoted by \(\text{CoR}\).

Notation 1.5. If the binary operation of a semigroup is clear from the context, we just abbreviate \((S, +)\) to \(S\).

Using the same notations as in the foregoing definition, it is easy to see that for the regular element \(a\) holds that \(x + a + x \in V(a)\) and that \(a + x\) and \(x + a\) are idempotents. The latter is a first hint that we will deal frequently with idempotents, which is why we introduce the following notation.

Definition 1.6. Let \((S, +)\) be a semigroup and \(E(S)^+ = \{e \in S \mid e + e = e\}\) the set of all idempotents of \(S\). Then \(S\) is called a band if \(S = E(S)^+\), and the class of all bands is denoted by \(\mathcal{B}\). If \(S\) is regular and \(E(S)^+\) a subsemigroup of \(S\), \(S\) is called an orthodox semigroup. It is an orthogroup if it is even completely regular.

Corollary 1.7. The class \(\mathcal{B}\) is in fact a subvariety of \(\mathcal{SG}\) which is defined by the additional identity
\[
a + a = a.
\]

Example 1.8. Every band \((S, +)\) is a completely regular semigroup, i.e. \(\mathcal{B} \subseteq \text{CoR}\), and for an arbitrary \(a \in S\), \(a\) is itself an inverse of \(a\). Furthermore, every group is completely regular, where \(-a \in V(a)\) for all \(a \in S\). So we have as well \(\mathcal{G} \subseteq \text{CoR}\).

Throughout the thesis, we will deal with several classes of bands whose notations will be collected below.
Notation 1.9. The following classes of bands form subvarieties of the variety of all bands \( B \). They are characterized by the given additional identities and are denoted by the given symbols.

- **SL** semilattices \( a + b = b + a \)
- **LZ** left-zero bands \( a + b = a \)
- **RZ** right-zero bands \( a + b = b \)
- **Re** rectangular bands \( a = a + b + a \)

Please note that even if we used for example the term “left-zero”, the left element of a sum in such a band does not act like a zero in the sense that it is neutral to the right one. Instead, it even absorbs the right one. But several statements used later are known for these notations, which is why I decided to use them nevertheless.

Before we now can come to Green’s relations, the well known notion of an ideal has to be stated.

Definition 1.10. Let \((S, +)\) be a semigroup and \( I \) be a nonempty subset of \( S \). \( I \) is a left (resp. right) ideal of \( S \) if

\[
\forall a \in I \ s \in S : s + a \in I \text{ (resp. } a + s \in I). \quad (1.4)
\]

If \( I \) is a left and a right ideal, then it is called a (two-sided) ideal.

Because the intersection of left ideals is a left ideal again, for an element \( a \in S \) we can consider the principal left ideal generated by \( a \):

\[
(a)_l = \bigcap \{ I \subseteq S \mid a \in I \text{ and } I \text{ is a left ideal} \}. \quad (1.5)
\]

Similarly, one can define the principal right ideal \( (a)_r \), and the principal (two-sided) ideal \( (a) \).

Now, the definition of Green’s relations is possible.
Definition 1.11. Let \((S, +)\) be a semigroup and \(a, b \in S\). Then we define Green’s relations on \(S\) by:

\[
\begin{align*}
\overset{L}{\downarrow} a & \iff \exists u, v \in S_0 : u + a = b \quad \text{and} \quad v + b = a, \\
\overset{R}{\downarrow} a & \iff \exists u, v \in S_0 : a + u = b \quad \text{and} \quad b + v = a, \\
\overset{J}{\downarrow} a & \iff \exists u_1, u_2, v_1, v_2 \in S_0 : u_1 + a + u_2 = b \quad \text{and} \quad v_1 + b + v_2 = a.
\end{align*}
\]

In the foregoing definition, \(\overset{LR}{\downarrow}\) means the product of the relations \(\overset{L}{\downarrow}\) and \(\overset{R}{\downarrow}\), i.e. \(a \overset{LR}{\downarrow} b\) if there exists some \(z \in S\) such that \(a \overset{L}{\downarrow} z\) and \(z \overset{R}{\downarrow} b\).

Definition 1.12. Let \((S, +)\) be a semigroup and \(0 \not\in S\). Then we can extend the addition on \(S\) to \(S \cup \{0\}\) by defining \(s + 0 = 0 + s = s\) for all \(s \in S \cup \{0\}\). In the following, \((S_0, +)\) will denote this semigroup with the adjoined zero 0 if \((S, +)\) is not a monoid. If this is the case, \((S_0, +)\) is just the monoid itself.

The following lemma contains characterizations of some of Green’s relations which we will use without reference subsequently.

Lemma 1.13. Let \((S, +)\) be a semigroup and \(a, b \in S\). Then we have:

\[
\begin{align*}
\overset{L}{\downarrow} a & \iff \exists u, v \in S_0 : u + a = b \quad \text{and} \quad v + b = a, \\
\overset{R}{\downarrow} a & \iff \exists u, v \in S_0 : a + u = b \quad \text{and} \quad b + v = a, \\
\overset{J}{\downarrow} a & \iff \exists u_1, u_2, v_1, v_2 \in S_0 : u_1 + a + u_2 = b \quad \text{and} \quad v_1 + b + v_2 = a.
\end{align*}
\]

It is easy to see that in the observation of completely regular semigroups we can replace \(S_0\) in the above characterization by \(S\). Evidently, \(\overset{H}{\downarrow} \subseteq \overset{L}{\downarrow}, \overset{R}{\downarrow} \subseteq \overset{J}{\downarrow}\) holds. Except for \(\overset{D}{\downarrow}\), it is obvious that Green’s relations are in fact equivalence relations. But also for \(\overset{D}{\downarrow}\) this can be proven easily (see for instance Lemma I.7.2 of [PR99]). The corresponding equivalence classes containing a certain element \(a\) will be denoted by \(\overset{L}{\downarrow} a, \overset{R}{\downarrow} a, \overset{J}{\downarrow} a, \overset{H}{\downarrow} a\) and \(\overset{D}{\downarrow} a\). In several cases, the relations are even congruences.

The following characterization for completely regular elements in semigroups is well-known and can be found for example in Proposition II.1.3 of [PR99].
Theorem 1.14. Let $a$ be an element in a semigroup $(S,+)$. Then the following statements are equivalent:

1) $a$ is completely regular.

2) There exists an element $y \in V(a)$ for which $a + y = y + a$.

3) There exists a unique element $y \in V(a)$ for which $a + y = y + a$.

4) $H_a$ is a subgroup of $S$.

Remark 1.15. If $a \in S$ is completely regular and $x \in S$ is such that $a + x + a = a$ and $a + x = x + a$, it is well-known that the uniquely determined element $y$ in 3) is in fact $y = x + a + x$. Moreover, $y$ is the inverse of $a$ in the subgroup $H_a$ and $a + y$ the neutral element of this subgroup (see [PR99]).

Notation 1.16. For a completely regular element $a$ of a semigroup $(S, +)$ we will denote by $-a$ the uniquely determined element $y$ in 3) and the neutral element $a + (-a)$ of $H_a$ by 0. Furthermore, we will simplify $b + (-a)$ to $b - a$ for arbitrary elements $b \in S$.

Example 1.17. If $(S, +)$ is a group, $-a$ is obviously the inverse in the group theoretical meaning, and if an element $a$ of an arbitrary semigroup is idempotent, $-a = a = 0_a$ holds.

As seen in 4) of Theorem 1.14, the $H$-relation will have a special significance in our context. A handy characterization of $H$-related elements is given by the following fact which easily can be observed.

Lemma 1.18. Let $(S, +)$ be a completely regular semigroup and $a, b \in S$. Then

$$a H b \iff 0_a = 0_b.$$ 

Using this lemma one easily verifies

Corollary 1.19. Let $S \in \text{CoR}$ and $a \in S$. Then $0_{0_a} = 0_a$.

Because of Remark 1.15, the following result is obvious.

Lemma 1.20. Let $S \in \text{CoR}$ and $a \in S$. Then $-(-a) = a$. 
Proof. This trivially holds, since \( a \) is the unique inverse of \(-a\) in the group \((H_a, +)\).

Now we will show that the idempotents \( 0_a \) are in some sense general. Similarly this was proved for completely regular semirings in [SMS06].

**Lemma 1.21.** Let \( S \in \text{CoR} \). Then \( E(S)^+ = \{0_a | a \in S\} \).

**Proof.** Let \( f \in E(S)^+ \). As already seen in Example 1.17, this leads to \( f = f + f = f - f = 0 \) for \( f \in \{0_a | a \in S\} \).

On the other hand, \( 0_a + 0_a = a - a + a - a = a - a = 0 \) and so \( \{0_a | a \in S\} \subseteq E(S)^+ \).

Altogether, we obtain the statement.

The operation "−" assigns to every completely regular element a certain inverse. So we can interpret a member of \( \text{CoR} \) as a universal algebra of type \((2, 1)\). They can be considered as a certain kind of unary semigroups which satisfy the following identities and turn \( \text{CoR} \) into a variety (see [PR99, section II.2]).

**Lemma 1.22.** The class \( \text{CoR} \) is a variety of type \((2, 1)\). The following family of identities is determining for the variety \( \text{CoR} \):

\[
\begin{align*}
a + (b + c) &= (a + b) + c \\
a + (-a) + a &= a \\
-(-a) &= a \\
a + (-a) &= (-a) + a.
\end{align*}
\]

An important subclass is defined as follows.

**Definition 1.23.** A semigroup \((S, +)\) is **simple** if \( S \) is the only ideal in \( S \).

If \((S, +)\) is in addition completely regular, it is called **completely simple**. The class of all completely simple semigroups is denoted by \( \text{CSi} \).

**Example 1.24.** Groups are classical examples of completely simple semigroups.
Remark 1.25. Since \((a)\) is an ideal for an arbitrary element \(a \in S\), the question whether the semigroup \(S\) is simple is obviously equivalent to the question whether \((a) = S\) holds for all \(a \in S\) or in other words if all elements of \(S\) are \(^+_J\)-related.

Remark 1.26. In [PR99], it is shown, that the class of all completely simple semigroups \(\mathcal{CSi}\) is a subvariety of \(\mathcal{CSR}\) satisfying the additional identity \(a = 0_{a+x} + a\).

The following theorem due to Clifford ([CP61]) is fundamental for the investigations of the global structure of completely regular semigroups.

Theorem 1.27 (Clifford). Let \((S, +)\) be a semigroup. Then the following conditions are equivalent:

1) \(S\) is completely regular.

2) Every \(^+_H\)-class of \(S\) is a (maximal) subgroup.

3) \(S\) is a union of (disjoint) subgroups.

4) \(S \in \mathcal{CSi} \circ SL\) within \(SG\).

Because of 3), completely regular semigroups are often called union of groups and condition 4) justifies the notation of semilattices of completely simple semigroups. Even if we introduced \(\mathcal{CSi}\) as the notation for a class of \((2, 1)\) algebras, as in the theorem, we also will use the same symbol for the class of type \((2)\) algebras if we disregard the unary operation. In the same manner, we will proceed later on in semirings.

In the last point, the required \(SL\)-congruence on \(S\) over \(\mathcal{CSi}\) is in fact the \(^+_J\)-relation.

1.3. Semirings

As the title of the thesis suggests, our main focus will be on semirings. We use here a rather general definition where the binary operations are only claimed to be associative.
Definition 1.28. Let \((S,+)\) and \((S,\cdot)\) be arbitrary semigroups. If the following distributive laws hold for all elements \(a, b, c \in S\) the triple \((S,+,\cdot)\) is called a semiring. The semigroups \((S,+)\) and \((S,\cdot)\) are called the additive and multiplicative reduct of the semiring.

A nonempty subset \(U \subseteq S\) is a subsemiring of \(S\) if \(U\) is a subsemigroup in both reducts, and an ideal if it is in addition a semigroup ideal in the multiplicative one.

Notation 1.29. Evidently, the class of all semirings, denoted by \(\mathcal{SR}\), is a variety of type \((2,2)\). As in semigroups, we will abbreviate the semiring \((S,+,\cdot)\) to \(S\) if the binary operations are obvious.

Now, we will define several classes of semirings which will be important in this thesis. The first one, the class of skew-rings, was introduced by Grillet in [Gri75]. Also Weinert in [Wei75a] and [Wei75b] deals with them.

Definition 1.30. A semiring \((S,+,\cdot)\) is a skew-ring if its additive reduct is a group. If 0 is the neutral element in the additive semigroup, it is called the zero of \(S\), and if for all \(a,b \in S\) hold that \(ab = 0\), the skew-ring is a null skew-ring.

The class of all skew-rings is denoted by \(\mathcal{SR}\), and its subclass of all null skew-rings by \(\mathcal{Nu}\).

So in contrast to a ring, the additive reduct of a skew-ring does not need to be commutative.

Example 1.31. If a non-commutative group \((S,+)\) is given, where 0 denotes the neutral element, and a multiplication on \(S\) is defined by \(a \cdot b = 0\) for all \(a,b \in S\), \((S,+,\cdot)\) is a null skew-ring which is not a ring.

Beside these trivial examples, there are much more non-ring skew-rings, so it is clear that the class \(\mathcal{Ri}\) of rings is a proper subclass of \(\mathcal{SR}\).

Lemma 1.32. As in rings, the additive neutral element of a skew-ring is multiplicatively absorbing.
Remark 1.33. In skew-rings, ideals can be defined so that they have to be subgroups in the additive reduct, but these are not in the scope of our interest. So, if we will speak about ideals of skew-rings, we will always refer to the ones defined in Definition 1.28.

Definition 1.34. A semiring \((S, +, \cdot)\) is called idempotent if both reducts are bands. The class of all idempotent semirings is denoted by \(I\).

Example 1.35. Trivial examples for idempotent semirings are distributive lattices.

Lemma 1.36. The classes \(SR\) and \(I\) are varieties of algebras of type \((2, 2)\), and \(SkR\) can be seen as a variety of algebras of type \((2, 2, 1)\) where the unary operation \(\sim\) maps an element to its additive inverse.

Remark 1.37. As already suggested in the section about semigroups, we will not distinguish strictly between the notations for classes of \((2, 2, 1)\) algebras and the ones for classes of the \((2, 2)\) algebras arising from these by disregarding their unary operations. So, we will also use \(SkR\) as the symbol for the class of skew-rings if we use them as a subclass of \(SR\), for example in Malcev products within \(SR\). In the same way, we will see later that the class \(I\) also can be interpreted as a variety of type \((2, 2, 1)\) algebras. But please note that some of these classes, for example \(SkR\), are varieties as \((2, 2, 1)\) algebras but not as \((2, 2)\) algebras.

For our purpose, several classes of idempotent semirings will be important. Their notations could be derived from band varieties in the following manner.

Notation 1.38. Let \(V \in L(B)\). By \(\overset{+}{V}\), we denote the subvariety of \(I\) containing all idempotent semirings whose additive reduct is in \(V\).

In semirings, Green’s relations could be defined using semiring ideals analogously to the semigroup case. But for our aim, even more meaningful will be the equivalences on the additive reduct of the semiring for which we can directly adopt the notations \(\overset{+}{H}, \overset{+}{L}, \overset{+}{R}, \overset{+}{J}\) and \(\overset{+}{D}\). As easily seen using Lemma 1.13 we obtain the following result.

Lemma 1.39. Let \(S\) be a semiring. In the multiplicative reduct of \(S\), Green’s relations \(\overset{+}{H}, \overset{+}{L}, \overset{+}{R}, \overset{+}{J}\) and \(\overset{+}{D}\) are in fact congruences.
Hence, if Green’s relations turn out to be even congruences on the additive reduct, they are congruences as well on the semiring. This fact is already stated in [SMS06]. Like Green’s relations, also the notation $E(S)^+$ will be taken for the set of all idempotents in the additive reduct of a semiring $(S, +, \cdot)$ and correspondingly, the idempotents in the multiplicative reduct will be notated by $E(S)^\cdot$.

**Lemma 1.40.** Let $(S, +, \cdot)$ be a semiring. Then $E(S)^+$ is empty or an ideal in $(S, \cdot)$.

**Proof.** Assume that $E(S)^+$ is not empty, so let $e \in E(S)^+$ and $s \in S$. Then $es = (e + e)s = es + es$ and hence $es \in E(S)^+$. Similarly, $se \in E(S)^+$ can be shown. \qed

In this section, only a short introduction to some aspects of semirings have been given. We can advise the reading of [HW98] and [Gol99] to those readers who are interested in a more comprehensive introduction to semirings.
2. Basic properties of completely regular semirings

Pastijn and Guo observed in [PG02] semirings which are the disjoint union of their maximal subrings. This class herein after will be referred as \( \mathcal{UR} \). In this chapter, we will use the following terminology introduced by Sen, Maity and Shum in [SMS06], and will see that this generalizes the mentioned approach.

**Definition 2.1.** Let \((S, +, \cdot)\) be a semiring and \(a \in S\). Then \(a\) is called completely regular if there exists an element \(x\) satisfying

\[
a = a + x + a \quad (2.1)
\]

\[
a + x = x + a \quad (2.2)
\]

\[
a(a + x) = (a + x) . \quad (2.3)
\]

Furthermore, \(S\) is called completely regular if every element of \(S\) is completely regular.

The class of all semirings whose additive reduct is completely regular, i.e. where (2.1) and (2.2) hold, is denoted by \(\mathcal{ACR}\) and the class of all completely regular semirings by \(\mathcal{CR}\).

**Remark 2.2.** From the definition it is obvious that any element which is completely regular in the semiring is in particular completely regular in the additive reduct. This means that \(\mathcal{CR}\) is a subclass of \(\mathcal{ACR}\).

Furthermore, it does not make any difference whether we define completely regular semirings using the equations (2.1), (2.2) and (2.3) or replacing (2.3) by \((a + x)a = a + x\) (see [SMS06]).

Of course, trivial subclasses of \(\mathcal{CR}\) are \(I\) and \(\mathcal{SKR}\). But there are as well more complex examples as the following example shows.
Example 2.3. On \( S = \{a, b, c, d, e, f, g\} \) the binary operations are given by

\[
\begin{array}{cccccccc}
+ & a & b & c & d & e & f & g \\
\hline
a & a & b & c & d & e & f & g \\
b & b & b & c & d & e & f & g \\
c & c & c & b & e & d & g & f \\
d & d & d & f & b & g & c & e \\
e & e & e & g & c & f & b & d \\
f & f & f & d & g & b & e & c \\
g & g & g & e & f & c & d & b \\
\end{array}
\quad
\begin{array}{cccccccc}
\cdot & a & b & c & d & e & f & g \\
\hline
a & a & a & a & a & a & a & a \\
b & b & b & b & b & b & b & b \\
c & b & c & c & b & b & c & b \\
d & c & c & b & c & b & b & c \\
e & e & b & b & b & b & b & b \\
f & a & b & b & b & b & b & b \\
g & a & c & c & b & b & c & c \\
\end{array}
\]

This completely regular semiring was calculated using the program Mace4 developed by McCune (see [McC10]). As computed using this powerful tool, it is an example with the least order so that it is neither a member of \( \mathcal{UR} \), nor of the trivial subclasses \( \mathcal{I} \) or \( \mathcal{SR} \).

Notation 2.4. For any semiring \( S \in \mathcal{ACR} \) and element \( a \in S \), the notations \( -a \) and \( 0_a \), appointed for semigroups in Notation 1.16, will be used in \( S \) as well and refer to the additive reduct. In the same way, we proceed with other notations introduced for completely regular elements in semigroups like \( +V(a) \).

Analogously to Theorem 1.14, the following theorem contains several characterizations for being a completely regular element in a semiring. Some of them can already be found in [SMS06] while 3) is derived from a lemma in [PG02] concerning \( \mathcal{UR} \). The proof of the former paper could be shortened using the knowledge about \( \mathcal{CoR} \).

Theorem 2.5. The following conditions for an element \( a \) of a semiring \( S \) are equivalent:

1) \( a \) is completely regular.

2) There exists a unique element \( y \in V(a) \) such that \( a(a + y) = a + y \) and \( a + y = y + a \).

3) \( a \) is additively completely regular and \( a H a^2 \).

4) \( \overset{+}{H}_a \) is a skew-ring.

Proof. 1) \( \Rightarrow \) 2) Let \( a \in S \) be completely regular and \( x \in S \) such that the equations (2.1), (2.2) and (2.3) hold. Using Theorem 1.14 we obtain for the additive reduct that \( y = -a \) is the only
element in \( V^+(a) \) satisfying \( a + y = y + a \). As mentioned in Remark 1.15, \(-a = x + a + x\) and so \( a(a + y) = a(a + x + a + x) = a(a + x) = a + x = a + x + a + x = a + y \).

2) \( \Rightarrow \) 3) Suppose \( y \in V^+(a) \) satisfies 2). By Theorem 1.14 we already know that \( a \) is additively completely regular. It remains to show that \( a + Ha \). But \( y \in V^+(a) \) implies \( a = a + y + a \).

Furthermore, \( a = a + (y + a) = a + (a + y) = a + a(a + y) = a + a(y + a) = (a + ay) + a^2 \) and \( a^2 = a(a + y + a) = a^2 + a(y + a) = a^2 + a(a + y) = a^2 + a + y = (a^2 + y) + a \) yield \( a + Ha \). In a similar, even more straightforward way, \( a + Ra \) and hence \( a + Ha \) is proved.

3) \( \Rightarrow \) 4) Again using semigroup theory, namely Theorem 1.14, it can be stated that \( a + Ha \) is a subgroup in the additive reduct. It suffices to show that this \( H \)-class is multiplicatively closed as well. Since \( H \) is a congruence on \((S, \cdot)\) as already mentioned in Lemma 1.39, for arbitrary \( b, c \in H_a \) holds that \( bc + Ha = a^2 + Ha \) and so it follows that \( H_a \) is a skew-ring.

4) \( \Rightarrow \) 1) Using Lemma 1.32, it is easy to see that the additive inverse \( x = -a \) of \( a \) in the skew-ring \( H_a \) satisfies the conditions (2.1), (2.2) and (2.3), and so \( a \) is completely regular.

Remark 2.6. Sen, Maity and Shum denoted the uniquely determined element \( y \) in condition 2) of the foregoing theorem by \( a' \). Because of Remark 2.2 and the uniqueness of inverses satisfying (2.2) of completely regular elements in a semigroup, for every semiring \( S \) and completely regular element \( a, a' \) coincides with \(-a\). So, if \( a \) is completely regular in the semiring, \(-a \) can be characterized as the additive inverse element of \( a \) in the skew-ring \( H_a \) whose zero is \( 0_a \), and it satisfies not only (2.1) and (2.2) but even (2.3).

Furthermore, we perceive that the rather unnatural condition (2.3) guarantees that the \( H \)-classes in a completely regular semiring are closed under multiplication.

We have already mentioned that skew-rings can be considered as algebras of type \((2, 2, 1)\), where \(-\) is the unary operation. Using the foregoing theorem, completely regular semirings can be considered as algebras of type \((2, 2, 1)\), too, since they are unions of \( H \)-classes and thereby of skew-rings.

**Corollary 2.7.** The classes \( ACR \) and \( CR \) are in fact varieties of type \((2, 2, 1)\). The following
2. Basic properties of completely regular semirings

Family of identities is determining for the variety $\mathcal{ACR}$:

\[
\begin{align*}
    a(bc) &= (ab)c \\
    a + (b + c) &= (a + b) + c \\
    a(b + c) &= ab + ac \\
    (a + b)c &= ac + bc \\
    a + (-a) + a &= a \\
    -a + a + (-a) &= -a \\
    a + (-a) &= (-a) + a.
\end{align*}
\]

To get the subvariety $\mathcal{CR}$, the identity

\[
a(a + (-a)) = a + (-a)
\]

has to be added.

Please note that comparing this corollary with Lemma 1.22, we have exchanged (1.8) by (2.9) so that on the basis of Theorem 2.5 it is easier to see that the given identities are the determining ones for $\mathcal{ACR}$ respectively $\mathcal{CR}$. Because of Lemma 1.20 and (2.8) the mentioned equations imply each other.

If we use in the following the symbol $0_a$ in the formulation of identities of varieties, we do not want to see this as a unary operation itself. Instead, it should be just an abbreviation for $(-a) + a$.

Furthermore, as already explained in Remark 1.37, subsequently, we will use $\mathcal{CR}$ as well as the symbols of its subclasses also for the notation of the corresponding type $(2, 2)$ algebra classes with disregarded unary operation. If it is discussed whether such a subclass of $\mathcal{CR}$ is a variety or not, we will always tacitly assume that the class of algebras of type $(2, 2, 1)$ is studied.

The following example, taken as well from [SMS06], shows that $\mathcal{CR}$ is in fact a proper subvariety of $\mathcal{ACR}$.

**Example 2.8.** Let $S$ be a semiring whose additive reduct is a band, but whose multiplicative one is not idempotent. Then $S \in \mathcal{ACR}$, but there is an element $a \in S$ for which $a^2 \neq a$. 
Assume that \( a \) is completely regular in the semiring. Since \((S,+)\) is a band, as seen in Example 1.17, \(-a = a\) and \(-a\) also satisfies (2.3). So \( a^2 = a(a + a) = a + a = a \) and this is a contradiction to our choice of \( a \). Thus, \( a \) can not be completely regular and \( S \notin CR \).

In this thesis, we will discuss frequently several congruences and homomorphisms on completely regular semirings. Because of the following results, already stated in [SMS06], we do not need to distinguish between homomorphisms between completely regular semirings and homomorphisms between the type \((2,2,1)\) algebras mentioned in Remark 2.6. The same simplification holds for their congruences.

**Lemma 2.9** ([SMS06]). Let \( S \in CR \) and \( T \) be an arbitrary semiring. Furthermore, let \( \varphi : S \rightarrow T \) be a homomorphism with respect to the semiring operations. Then:

1) For any \( a \in S \), \( \varphi(-a) = -\varphi(a) \) and \( \varphi(0_a) = 0_{\varphi(a)} \).

2) \( \varphi(S) \in CR \).

**Proof.** 1) This is directly implied by the corresponding result for the additive reduct of \( \varphi(S) \) (see Lemma A.2).

2) Because of 1) we have only to check whether \(-\varphi(a)\) satisfies condition (2.3). For any \( a \in S \), this is easily observed by

\[
\varphi(a)(\varphi(a) - \varphi(a)) = \varphi(a(a - a)) = \varphi(a - a) = \varphi(a) - \varphi(a)
\]

(2.12)

if we use 1). \( \square \)

**Corollary 2.10** ([SMS06]). Let \( S \in CR \) and \( \varrho \) a semiring congruence on \( S \). Then for any \( a, b \in S \), \( a \varrho b \) implies \(-a \varrho -b \) and \( 0_a \varrho 0_b \).

**Proof.** This is due to Lemma A.3. \( \square \)

The consequence is that in order to prove that an equivalent relation or a mapping is a congruence or a homomorphism, we only have to check its compatibility with respect to + and ·, even if we interpret the semirings as universal algebras of type \((2,2,1)\).
2. Basic properties of completely regular semirings

For the additive inverse, the following calculation rules exist, which can be found for $UR_\mathbb{R}$ in [PG02] as well as for $CR_\mathbb{R}$ in [SMS06] whose proof again can be shortened.

**Lemma 2.11.** Let $S \in ACR_\mathbb{R}$ and $a, b \in S$. Then

1) $a(-b) = -ab = (-a)b$

2) $(-a)(-b) = ab$

3) $a0_b = 0_ab = 0_a0_b$

**Proof.** 1) Let $a, b \in S$. We have $a(-b) + ab = a(-b + b) = a(b + (-b)) = ab + a(-b)$ and similarly $a(-b) \in V(ab)$ can be shown. Since $ab$ has a unique inverse satisfying (2.2), this is sufficient to prove $a(-b) = -ab$. Analogously the proof of $(-a)b = -ab$ can be done.

2) For arbitrary $a, b \in S$ we clearly have $(-a)(-b) = -(a(-b)) = -(-ab) = ab$ because of 1) and Lemma 1.20.

3) Let $a, b \in S$. Now, $0_b$ is an abbreviation for $b - b$ and using 1) this leads to $a0_b = a(b - b) = ab - ab = 0_ab$. $0_a0_b = 0_ab$ can be proven analogously. Moreover, because of Corollary 1.19, we obtain $0_a0_b = 0_a0_b = 0_a0_b = 0_ab$. 

In [SMS06] we can find the following lemma.

**Lemma 2.12 ([SMS06]).** If $S \in CR_\mathbb{R}$ it holds $E(S)^+ \subseteq E(S)^*$.

**Proof.** Let $S \in CR$ and $a \in E(S)^+$, then $a = 0_a$ is the zero in the skew-ring $H_a$. But applying Lemma 1.32, we already know that this is multiplicatively absorbing and in particular multiplicatively idempotent.

In contrast, Example 2.8 shows that for arbitrary semirings of $ACR_\mathbb{R}$, the additive idempotents do not need to be multiplicatively idempotent.

Beside other possible characterizations, we will see in the following theorem that for a semiring $S \in ACR_\mathbb{R}$ one even readily verifies that the condition $E(S)^+ \subseteq E(S)^*$ is not only necessary, it is even sufficient in order to prove that $S \in CR_\mathbb{R}$. These characterizations, and parts of
the proof, are given in a similar manner as it is done in [PG02] for \( \mathcal{L} \). Some of them are also contained in [SMS06].

**Theorem 2.13.** Let \( S \) be a semiring. Then the following conditions are equivalent:

1) \( S \in \mathcal{CR} \).

2) \( S \in \mathcal{ACR} \) and \( E(S)^+ \subseteq E(S)^* \).

3) \( S \in \mathcal{ACR} \) and \( E(S)^+ \) is a subband of \((S, \cdot)\).

4) \( S \) is the disjoint union of its maximal sub-skew-rings.

5) \( S \) is a union of sub-skew-rings.

**Proof.**

1) \( \Rightarrow \) 2) Because of Remark 2.2 and Lemma 2.12, this is already done.

2) \( \Rightarrow \) 3) Since \( E(S)^+ \) is not empty, this is implied by Lemma 1.40.

3) \( \Rightarrow \) 4) Since \( S \in \mathcal{ACR} \), Theorem 1.27 implies that the \( \hat{H} \)-classes are maximal subgroups in the additive reduct. Let \( 0_a \) be an additive idempotent of such a subgroup. By the assumption and Lemma 2.11, we can conclude \( 0_a = 0^2_a = 0_a^2 \). Now, Theorem 2.5 implies that \( \hat{H}_a \) must be in fact a skew-ring which is maximal by Theorem 1.27. The \( \hat{H} \)-classes are trivially disjoint and their union is \( S \).

4) \( \Rightarrow \) 5) This is obvious.

5) \( \Rightarrow \) 1) Let \( a \in S \) and \( K \) be a skew-ring containing \( a \). Then there is a zero element \( 0 \) of \( K \) and an additive inverse element \( \bar{a} \in K \) such that \( a + \bar{a} + a = a \) and \( a + \bar{a} = 0 = \bar{a} + a \). But since \( 0 \) is multiplicatively absorbing, also \( a(a + \bar{a}) = a0 = 0 = a + \bar{a} \) holds. So, \( a \) is completely regular (in the semiring).

Note that Pastijn and Guo in [PG02] have to distinguish between semirings which are an arbitrary union of subrings and semirings which are a union of disjoint subrings. Mainly, they studied the latter ones. In our case there is no need to distinguish between disjoint or arbitrary unions of sub-skew-rings.

As seen in the foregoing proof, we can state explicitly:
Corollary 2.14. Let $S$ be a semiring. Then $S \in \mathcal{CR}$ if and only if the $\mathcal{H}$-classes are the maximal sub-skew-rings of $S$.

Of course, as already mentioned in [SMS06], the range of possible examples begins with the class of idempotent semirings $I$, where every $\mathcal{H}$-class contains only one single element, and ends with the class of skew-rings $\mathcal{SKR}$, where the $\mathcal{H}$-relation is in fact the relation $\omega$.

Very important for the analysis of the structure of completely regular semirings is the following lemma whose proof can be done as seen for semirings of $\mathcal{UR}$ in [PG02].

Lemma 2.15. Let $S \in \mathcal{CR}$. Then $\mathcal{H}$ is the least $\mathcal{I}$-congruence on $S$.

Proof. At first we will show that $\mathcal{H}$ is a congruence on $(S, +)$. Therefore, let $a, c \in S$ be arbitrary and let us define the two idempotents $f_1 = 0_{a+c}$ and $f_2 = 0_{a+c}$. We have

$$f_1 = (0_{a+c}) - (0_{a+c}) = (0_a + 0_c) + c - (0_a + c) = 0_a + (0_a + c) - (0_a + c)$$

$$= 0_a + f_1$$  \hspace{1cm} (2.13)

and

$$f_2 = 0_{a+c} = (a + c) - (a + c) = (0_a + a) + c - (a + c)$$

$$= 0_a + f_2.$$  \hspace{1cm} (2.14)

Combining these two equations and using Lemma 2.11 and Lemma 2.12 we obtain

$$f_1 \overset{2.12}{=} f_1 \overset{2.11}{=} f_1(0_{a+c}) = f_1((-a) + a + c) = f_1(-a) + f_1(a + c)$$

$$\overset{2.11}{=} f_10_{-a} + f_1f_2 = f_1(0_a + f_2) \overset{(2.14)}{=} f_1f_2 \overset{(2.13)}{=} (0_a + f_1)f_2 = 0_a f_2 + f_1f_2$$

$$\overset{2.11}{=} af_2 + (0_a + c)f_2 = (a + 0_a + c)f_2 = (a + c)f_2 \overset{2.11}{=} 0_{a+c} f_2 = f_2$$

$$\overset{2.12}{=} f_2.$$  \hspace{1cm} (2.12)

This implies $0_a + c \mathcal{H} a + c$.

Now, let $b \in \mathcal{H} a$. This means that $0_a = 0_b$ and so

$$a + c \mathcal{H} 0_a + c = 0_b + c \mathcal{H} b + c.$$  \hspace{1cm} (2.12)

Using the left-right-dual of this result we can summarize that $\mathcal{H}$ is an additive congruence. By Lemma 1.39 and Corollary 2.10, we can conclude that $\mathcal{H}$ is a congruence on the type $(2, 2, 1)$.
algebra $S$. Since every $H$-class $H_a$ contains the additive and multiplicative idempotent $0_a$, it is an $I$-congruence.

It remains to show that it is in fact the least one. So, let $\varrho$ be an arbitrary $I$-congruence on $S$ and remember that $a H b$. Then it holds $a \varrho a + a$. This leads to $0_a = a - a \varrho a + a - a = a$ and $a \varrho 0_a = 0_b \varrho b$. This verifies $H \subseteq \varrho$.

This is a significant difference to completely regular semigroups. There, the $H$-relation need not to be a congruence. Such special semigroups, where $H$ is a congruence, are called cryptic. If they are in addition completely regular, they are called cryptogroup or band of groups, which form a proper subvariety of all completely regular semigroups (see [PR99, Section II.8]). As seen, they can be regained in the additive reduct of completely regular semirings.

The following lemma holds for $H$ as well as for other $I$-congruences.

**Lemma 2.16.** Let $\varrho$ be an arbitrary $I$-congruence on a semiring $S$. Then each $\varrho$-class is a semiring. If $S \in \mathcal{CR}$, all $\varrho$-classes are even subalgebras of type $(2, 2, 1)$.

**Proof.** Let $a \in S$ and $a \varrho$ be the $\varrho$-class containing $a$. Furthermore, $b, c \in a \varrho$. Then $(b + c) \varrho = b \varrho + c \varrho = a \varrho + a \varrho = a \varrho$, i.e. $b + c \in a \varrho$. Similarly, we obtain $bc \in a \varrho$.

Now, for the last part, let $S \in \mathcal{CR}$. To show that $a \varrho$ is a subalgebra of the universal algebra $(S, +, \cdot, -)$ it suffices to prove that $-b \in a \varrho$. Corollary 2.10 implies that $\varrho$ is as well a congruence with respect to the unary operation and because $(-a) \varrho = -(a \varrho) = a \varrho$ holds in the idempotent semiring $S/\varrho$, we obtain $-b \varrho = a \varrho a$.

**Remark 2.17.** In particular, the last statement implies that each congruence class of an $I$-congruence on a completely regular semiring is itself in $\mathcal{CR}$.

The subvarieties of $\mathcal{CR}$, which are varieties of universal algebras of type $(2, 2, 1)$, are the main topic of this thesis. On the other hand, we often deal with Malcev products determined in $\mathcal{SR}$, which is a variety of universal algebras of type $(2, 2)$. That we can omit these details in several cases is caused by the following corollary.
Corollary 2.18. Let $\mathcal{U}_1 \in \mathcal{L}(\mathcal{CR})$ and $\mathcal{U}_2 \in \mathcal{L}(I)$. Then

$$\mathcal{U}_1 \circ_{SR} \mathcal{U}_2 = \mathcal{U}_1 \circ_{CR} \mathcal{U}_2.$$ 

Proof. [⊆] Let $S \in \mathcal{U}_1 \circ_{SR} \mathcal{U}_2$. By the definition of the Malcev product, there exists a semiring congruence $\varrho$ over $\mathcal{U}_1$ such that $S/\varrho \in \mathcal{U}_2 \subseteq I$. Lemma 2.16 implies that each $\varrho$-class is a semiring and hence a completely regular one. Following Theorem 2.13, each $\varrho$-class is a union of skew-rings and thus, $S$ itself is composed by skew-rings, i.e. $S$ is completely regular. But from Corollary 2.10 it follows that $\varrho$ is as well a congruence for the universal algebra of type $(2, 2, 1)$, and so, $\varrho$ is actually the required $\mathcal{U}_2$-congruence over $\mathcal{U}_1$ to construct the Malcev product within $\mathcal{CR}$.

[⊇] Now, assume that $S \in \mathcal{U}_1 \circ_{CR} \mathcal{U}_2$ and let $\varrho$ be again the appropriate $\mathcal{U}_2$-congruence over $\mathcal{U}_1$. Hence, $\varrho$ is an $I$-congruence and Lemma 2.16 can be used to state that each $\varrho$-class has to be a member of $\mathcal{U}_1$. This leads to $S \in \mathcal{U}_1 \circ_{SR} \mathcal{U}_2$. 

Remar 2.19. In particular, this corollary says that every semiring in $\mathcal{U}_1 \circ_{SR} \mathcal{U}_2$ is completely regular. Furthermore, we need not to distinguish the Malcev product with the constraints as in the foregoing corollary and determined in $\mathcal{SR}$ from the Malcev product determined in $\mathcal{CR}$. This allows us to formulate theorems like the following one in $\mathcal{SR}$, but also let us use their results in the lattice of subvarieties of $\mathcal{CR}$.

Theorem 2.20. The variety $\mathcal{CR}$ can be expressed as the Malcev product of $\mathcal{SkR}$ and $I$ within $\mathcal{SR}$:

$$\mathcal{CR} = \mathcal{SkR} \circ I.$$ 

Proof. [⊆] Because of Corollary 2.14 and Lemma 2.15, we see that $\hat{H}$ is such an $I$-congruence whose classes are in $\mathcal{SkR}$.

[⊇] As just mentioned in the previous remark, this is known by Corollary 2.18. 

The following notion was introduced for semigroups in [BT84].

Definition 2.21. A semiring $(S, +, \cdot)$ is additively coregular if $a + a + a = a$ for all $a \in S$. The variety of all additively coregular semirings is denoted by $\mathcal{ACoR}$. 

As clarified in the cited article, the origin of the name is the equivalence of such semigroups with ones for which for any element \( a \in S \) there exists an element \( b \in S \) such that \( a + b + a = a = b + a + b \).

Using this notation, we conclude the following lemma which is partially contained and proved in [SMS06].

**Lemma 2.22.** Let \( (S, +, \cdot) \) be a semiring whose multiplicative reduct is a band. Then the following conditions are equivalent:

1) \( S \in \mathbb{ACR} \)

2) \( S \in \mathbb{CR} \)

3) \( S \in \mathbb{ACoR} \)

Then we have \(-a = a\) for all \( a \in S\).

**Proof.** 1) \( \Rightarrow \) 2) This is a direct consequence of Theorem 2.13 2).

2) \( \Rightarrow \) 3) For \( a \in S \) and by Lemma 2.11 2) we obtain \( a = a^2 = (-a)(-a) = -a \) and hence \( a = a - a + a = a + a + a \).

3) \( \Rightarrow \) 1) Now, let \( a = a + a + a \) for arbitrary \( a \in S \). Obviously, \( -a = a \) fulfills (2.1) and (2.2).

Sen and Bhuniya introduced in [SB10] the notion of almost idempotent semirings.

**Definition 2.23.** A semiring \( S \) whose additive reduct is a semilattice is almost idempotent if \( a + a^2 = a^2 \) is satisfied for all \( a \in S \).

Completely regular almost idempotent semirings are characterized in the following lemma.

**Lemma 2.24.** Let \( S \) be a completely regular semiring. Then \( S \) is almost idempotent if and only if it is idempotent.

**Proof.** \( \Rightarrow \) Let \( S \) be almost idempotent and \( a \in S \). It follows

\[
a = a + 0_a = a + 0_{a^2} = a + a^2 - a^2 = a^2 - a^2 = 0_{a^2} = 0_a,
\]
i.e. all elements are additively as well as multiplicatively idempotent.

\[ \Rightarrow \] That each idempotent semiring is almost idempotent holds since \( a + a^2 = a^2 + a^2 = a^2 \)
for all \( a \in S \).

As seen in the cited article, there exist semirings which are almost idempotent but not idempotent, so there exist as well almost idempotent semirings which are not completely regular.

Beside the property that completely regular semigroups are union of groups another fundamental description was given by Clifford ([Cli41]) by the fact that they are as well semilattices of completely simple semigroups. The following results can be regarded as the semiring theoretical equivalent.

As completely simple semigroups are components for completely regular semigroups, we can define completely simple semirings in the following way so that they are basic components for completely regular semirings. This definition was introduced in [SMS06].

**Definition 2.25.** A completely regular semiring \( S \) is called completely simple if \( S \) is the only \( + \)-class in \( S \). The class of all completely simple semirings will be denoted by \( \mathcal{CS} \).

Because of Theorem A.4, this is equivalent thereto that a completely regular semiring only contains a single \( D \)-class.

Immediately, we come to the following corollary.

**Corollary 2.26.** Let \( S \in \mathcal{CR} \). Then \( S \in \mathcal{CS} \) if and only if \((S, +) \in \mathcal{CSI}\).

Pastijn and Guo ([PG02]) used in the same way semirings of \( \mathcal{UR} \) which have a completely simple additive reduct. Of course, in the literature, there are various different definitions for simple semirings. For example in [Gol99] it is defined as a semiring with a multiplicative identity \( 1 \) for which \( a + 1 = 1 \) for all elements \( a \) of the semiring. Other authors claimed simple semirings to have only trivial ideals (see [BZ57]). To obtain semirings whose nontrivial homomorphic images are not “smaller”, congruence-simple semirings were introduced, which have only the trivial congruences \( id \) and \( \omega \) (see [MF88], [Mon04]). In contrast to these concepts, we use the Definition 2.25, since already Sen, Maity and Weinert found out that they form the components of completely regular semirings (see [SMW05]).
Using the last corollary, the knowledge about $\mathcal{CSi}$ can be transferred to semirings.

**Lemma 2.27.** The class $\mathcal{CS}$ of completely simple semirings is in fact a subvariety of $\mathcal{CR}$ with the additional identity $a = 0_{a+x} + a$.

**Proof.** As already mentioned in Remark 1.26, it can be seen in [PR99] that $\mathcal{CSi}$ forms a subvariety of the variety of completely regular semigroups $\mathcal{CR}$. Therefore $\mathcal{CS}$ is the subvariety of $\mathcal{CR}$ whose semirings satisfy $a = 0_{a+x} + a$. □

Other authors characterized $\mathcal{CSi}$ by other identities, e.g. Howie in [How95] by $0_x = 0_{x+y} + x$. Of course, we could use this identity as well to identify the completely simple semirings within $\mathcal{CR}$.

The following three results can be found in [SMS06]. The proofs are adopted from there as well, but they could be simplified since $^+\mathcal{J}$ is known to be the least $SL$-congruence on the additive reduct. In this article the notation of $b$-lattices was introduced for the elements of $SL$.

**Lemma 2.28.** Let $S \in \mathcal{CR}$. Then $^+\mathcal{J}$ is the least $SL$-congruence.

**Proof.** As proven in Lemma A.5, $^+\mathcal{J}$ is the least $SL$-congruence on the additive reduct of $S$. Furthermore, we have already stated as well that $^+\mathcal{J}$ is a congruence on $(S, \cdot)$, which contains $^+H$. This implies that $^+\mathcal{J}$ is an $I$-congruence and actually the least $^+SL$-congruence. □

**Lemma 2.29.** Let $S \in \mathcal{CR}$. Then the $^+\mathcal{J}$-classes of $S$ are completely simple semirings.

**Proof.** Because of $S \in \mathcal{CR} \subseteq A\mathcal{CR}$ and Theorem 1.27 it is already known that the $^+\mathcal{J}$-classes form completely simple semigroups in the additive reduct. Moreover, Lemma 2.28 and Lemma 2.16 show that the $^+\mathcal{J}$-classes are closed under multiplication as well. So, if we are able to prove that each $^+\mathcal{J}$-class is completely regular in the semiring sense, we have finished. However, this was already concluded in Remark 2.17. □
Theorem 2.30. The variety $\mathcal{CR}$ can be expressed as the Malcev product of $\mathcal{CS}$ and $\mathcal{SL}$ within $\mathcal{SR}$:

$$\mathcal{CR} = \mathcal{CS} \circ ^{+} \mathcal{SL}.$$ 

Proof. $\subseteq$ Immediate from Lemma 2.28 and Lemma 2.29.

$\supseteq$ As already mentioned in Remark 2.19, each semiring of $\mathcal{CS} \circ ^{+} \mathcal{SL}$ is completely regular. □
3. Completely simple semirings

In the preceding chapter, we identified the crucial role played by the class of completely simple semirings \( CS \). This is why, we dedicate this chapter to them to gain more insight into the variety \( CS \). This includes especially an extension of the well-known Rees matrix representation of completely regular semigroups to semirings.

Lemma 3.1. Let \( S \in CS \) and \( a \in S \). Then \( \hat{L} \) is a congruence on \( S \), and \( \hat{L}_a \) is a semiring.

Proof. The first part can be concluded because \( \hat{L} \) is a congruence on the completely simple additive reduct by Lemma A.6 and on the multiplicative one by Lemma 1.39.

Now, Lemma 2.16 implies that each class of the \( I \)-congruence \( \hat{L} \) is a semiring. \( \square \)

So, the notation \( E(\hat{L}_a)^+ \) in the following lemma, stated for \( URE \)-semirings in [PG02], is justified.

Lemma 3.2. Let \( S \in CS \) and \( a \in S \). Then:

1) For each \( b \in S \), there exists a unique represent of \( \hat{R}_b \) in \( E(\hat{L}_a)^+ \), namely \( 0_{b+a} \).

2) \( E(\hat{L}_a)^+ \in \hat{LZ} \).

3) \( \hat{R} \) is the least \( \hat{LZ} \)-congruence on \( S \) and \( \varphi : S/\hat{R} \mapsto E(\hat{L}_a)^+, \varphi(\hat{R}_b) = 0_{b+a} \) is an isomorphism.

4) \( \hat{H} \) is the least \( \hat{RF} \)-congruence on \( S \) and \( \psi : S/\hat{H} \mapsto E(\hat{L}_a)^+ \times E(\hat{R}_a)^+, \psi(\hat{H}_b) = (0_{b+a}, 0_{a+b}) \) is an isomorphism.
Proof. 1) Let $b \in S$. Since $(S, +)$ is completely simple, Theorem A.7 implies $b R b + a$ and $a L b + a$. Thus, $R_b \cap L_a$ contains $b + a$, and this intersection is an $L$-class, namely $H_{b+a}$. The unique additive idempotent of this skew-ring is $0_{b+a}$.  \\
2) Let $0_b, 0_c \in E(L_a)^+$. Due to Lemma 3.1 the elements $0_b + 0_c, 0_b 0_c \in L_a$. Now, again by Theorem A.7, we obtain $0_b + 0_c \not\in R_b 0_b$. Since $L_a = L_b$, this leads to $0_b + 0_c \in R_b \cap L_b = H_b$ (\textstar{}).

But in this skew-ring, $0_b$ is the zero and it follows:

$$0_b + 0_c = 0_b + (0_c + 0_c) = (0_b + 0_c) + 0_c \equiv ((0_b + 0_c) + 0_b) + 0_c = (0_b + 0_c) + (0_b + 0_c),$$

i.e. $0_b + 0_c \in E(L_a)^+$. Because of (\textstar{}), this means $0_b + 0_c = 0_b$ and thus $(E(L_a)^+, +) \in LZ$.

Lemma 2.12 yields that the elements of $E(L_a)^+$ are multiplicatively idempotent as well, which proves in combination with Lemma 1.40 $E(L_a)^+, +, \cdot) \in B$ and $(E(L_a)^+, +, \cdot) \in LZ$.

3) That $R$ is a semiring congruence is the left-right dual of Lemma 3.1. Using 1) and 2), we deduce that $S/R \cong E(L_a)^+ \in LZ$, where $\varphi$ is exactly the isomorphism between these two semirings. It is obvious that $\varphi$ is well-defined.

Assume that $\varphi$ is an $LZ$-congruence and let $x R y$. This implies $x + S = y + S$, and there must exist an element $\bar{x} \in S$ such that $x + \bar{x} = y$. Consequently $x \varphi = x \varphi + \bar{x} \varphi = y \varphi$, since $\varphi$ is an $LZ$-congruence. This clearly shows $x \varphi y$ and hence $R \subseteq \varphi$. As claimed, $R$ is the least $LZ$-congruence.

4) Because $LZ$ is a variety and following Birkhoff’s Theorem, $(S/H, +)$ has to be a completely simple band since $H$ is known to be an $L$-congruence. Lemma A.8 proves that $(S/H, +)$ is in fact a rectangular band and this implies that the semiring $S/H$ is in $R^L$. That $H$ is the least $R^L$-congruence can be perceived as follows. Let $\varrho$ be an $R^L$-congruence on $S$ and $x H y$. Then $x + S = y + S$ and $S + x = S + y$. There have to exist $z_1, z_2 \in S$ that satisfy $x + z_1 = y$ and $z_2 + x = y$. We obtain:

$$x \varrho x + z_1 + z_2 + x = y + z_1 + y \varrho y.$$

This shows that $H$ is the least $R^L$-congruence.

The remaining statement is clear by 3) and its left-right dual since each $H$-class can be described uniquely as an intersection of the $R$ and $L$-class it is contained in. \qed
If we interchange the role of left and right we come to the left-right dual statement that the idempotents of an $\hat{+}R$-class form an $\hat{+}R\mathbb{Z}$-semiring.

Since $\hat{+}H$ is a congruence, the foregoing lemma gives us some information in which $\hat{+}H$-class the sum of arbitrary elements of a completely simple semiring can be found. In 1) we see once again that any two elements of a completely simple semiring are $\hat{+}D$-related. As in semigroups, the so called egg-box picture uses this property to illustrate the addition in completely simple semirings.

\[
\begin{array}{ccc}
\hat{+}R_a & \hat{+}L_a & \hat{+}L_b \\
\hat{+}R_a & a & \hat{+}H_{a+b} & \cdots \\
\hat{+}R_b & b+a & b & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}
\]

As seen, the rows represent the $\hat{+}R$-classes, the columns the $\hat{+}L$-classes, and each cell pictures a certain $\hat{+}H$-class, namely the intersection of the corresponding $\hat{+}R$ and $\hat{+}L$-class.

Next, we state explicitly the following consequence of the second item of the previous lemma for further reference.

**Corollary 3.3.** Let $S \in CS$ and $b, c \in \hat{+}L_a$. Then

\[0_{b+c} = 0_b + 0_c = 0_b\]  \hspace{1cm} (3.1)

**Proof.** $0_{b+c}$ is $\hat{+}H$-related with $0_b + 0_c = 0_b$ and of course an additive idempotent. \qed

Using Corollary 2.14 and 4) of the foregoing lemma, we obtain the next theorem which is given analogue to a theorem in [PG02] about $UR$:

**Theorem 3.4.** The variety $CS$ can be expressed as the Malcev product of $SKR$ and $\hat{+}Re$ within $SR$:

\[CS = SKR \circ \hat{+}Re.\]
3. Completely simple semirings

Proof. [⊆] Let \( S \) be a completely simple semiring. Hence, it is completely regular, and according to Corollary 2.14 the \( H \)-classes are skew-rings. Following Lemma 3.2 4), \( H \) is in this case not only a general \( I \)-congruence. Actually, \( H \) is even an \( R_e \)-congruence. This proves the first inclusion.

[⊇] Now, let \( S \in SKR \circ R_e \). There exists an \( R_e \)-congruence \( \varrho \) over \( SKR \). Because of Lemma 2.16, every \( \varrho \)-class is a semiring and, since \( \varrho \) is over \( SKR \), even a skew-ring. Theorem 2.13 implies that \( S \) is completely regular. If we are able to show that for an arbitrary element \( x \in S \) holds that \( S + +x + S = S \), we are ready because of Remark 1.25. That the left side is included in \( S \) is obvious. So, let \( y \in S \). Since \( S/\varrho \) is rectangular, we have \( y \varrho = y \varrho + x \varrho + y \varrho \) or rather \( y + x + y = z \varrho y \) for some \( z \in S \) and using that \( y \varrho \) has to be a skew-ring, there must exist an element \( \bar{z} \in y \varrho \) such that \( y = z + \bar{z} \). Then \( y = z + \bar{z} = y + x + y + \bar{z} \in S + x + S \).

Combining this theorem and Theorem 2.30 we obtain:

Theorem 3.5. The variety \( CR \) can be expressed as the Malcev product of \( SKR \circ R_e \) and \( SL \) within \( SR \):

\[ CR = (SKR \circ R_e) \circ SL. \]

With the help of Lemma 3.2 we can determine that Green’s equivalence classes of a \( D \)-class of a completely regular semiring are isomorphic to each other. An analogue result for semirings in \( UR \) can be found in [PG02].

Lemma 3.6. Let \( S \in CS \) and \( a, b \in S \). Then we have:

1) The mapping \( \varphi : \overset{+}{L}_a \mapsto E(\overset{+}{L}_a)^+ \times \overset{+}{H}_a \), \( \varphi(c) = (0_c, 0_a + c) \) is an isomorphism.

2) The mapping \( \psi : \overset{+}{H}_a \mapsto \overset{+}{H}_b \), \( \psi(e) = 0_b + c + 0_{a+b} \) is an isomorphism.

3) The mapping \( \chi : \overset{+}{L}_a \mapsto \overset{+}{L}_b \), \( \chi(c) = 0_{c+b} + 0_a + c + 0_{a+b} \) is an isomorphism.

Proof. 1) Let \( c \in \overset{+}{L}_a \) and let us first check whether \( \varphi \) really maps into \( E(\overset{+}{L}_a)^+ \times \overset{+}{H}_a \). \( 0_c \) is known to be the additive idempotent element of \( \overset{+}{H}_c \subseteq \overset{+}{L}_c = \overset{+}{L}_a \), i.e. \( 0_c \in E(\overset{+}{L}_a)^+ \). Since \( H \) is a congruence \( 0_a + c \overset{+}{H} 0_a + 0_c = 0_a \in H_a \) because \( 0_a \) and \( 0_c \) are in \( \overset{+}{L}_a \) and 2) of Lemma 3.2. Now, let \( d \in \overset{+}{L}_a \) such that \( \varphi(c) = \varphi(d) \). Indeed, it follows \( 0_c = 0_d \) and \( 0_a + c = 0_a + d \). Adding
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On the left side, the latter implies \(0_c + 0_a + c = 0_c + 0_a + d\) and again by 2) of Lemma 3.2 \(0_c + c = 0_c + d\). We conclude \(c = 0_c + c = 0_c + d = d\) which shows that \(\varphi\) is injective.

For proving the surjectivity of \(\varphi\), let \((x, y)\) be an arbitrary element of \(E^+(L_a)^+ \times H_a\). Define \(c = x + y\). Obviously, this is an element of \(L_a\). Since \(x\) is additive idempotent, i.e. \(0_x = x\), and Corollary 3.3, we obtain

\[
\varphi(c) = (0_c, 0_a + c) = (0_{x+y}, 0_a + x + y) = (0_x, 0_a + 0_x + y) = (0_x, 0_a + y) = (x, y)
\]

since \(y \in H_a\).

The homomorphism property with respect to the addition can be proven for any \(c, d \in H_a\) as follows

\[
\varphi(c + d) = (0_{c+d}, 0_a + c + d)
\]

\[
\overset{(*)}{=} (0_c + 0_d, (0_a + c) + 0_a + d) = (0_c, 0_a + c) + (0_d, 0_a + d)
\]

\[
= \varphi(c) + \varphi(d)
\]

where \((*)\) is justified since \(0_a + c \in H_a\) and Corollary 3.3.

It remains to show that \(\varphi\) is as well a homomorphism with respect to the multiplication. Using Lemma 2.11, we come to

\[
\varphi(c \cdot d) = (0_{cd}, 0_a + cd)
\]

\[
\overset{(**)}{=} (0_c \cdot 0_d, 0_a \cdot (0_a \cdot 0_d + 0_{ca} + cd)) = (0_c, 0_a + c) \cdot (0_d, 0_a + d)
\]

\[
= \varphi(c) \cdot \varphi(d)
\]

where \((**\)) holds because \(0_a = 0_{a^2}\) and \(0_{ad} + 0_{ca} \in E^+(L_a)^+\) and so 2) of Lemma 3.2 is applicable.

2) We start as above and prove at first that \(\psi\) is well-defined. Therefore, let \(c \in H_a\). Because \(H_c = H_a\) we have

\[
\psi(c) = 0_b + c + 0_a + b \in \overset{+}{R_c} \cap \overset{+}{L_a} = \overset{+}{H_b}
\]

as required.

To show the injectivity of \(\psi\), we take two elements \(c, d \in H_a\) for which \(\psi(c) = \psi(d)\). So

\[
0_b + c + 0_a + b = 0_b + d + 0_a + b.
\]
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By adding $0_{a+b}$ on the left side, we can deduce

$$0_{a+b} + b + 0_{a+b} = 0_{a+b} + 0_{a+b} + 0_{a+b}$$

which yields

$$c + 0_{a+b} = d + 0_{a+b}.$$

Using $0_{a+b} \not{\mathcal R} 0_a$ and hence $0_{a+b} + 0_a = 0_a$ this results in

$$c = c + 0_a = c + 0_{a+b} + 0_a = d + 0_{a+b} + 0_a = d + 0_a = d.$$

To see that $\psi$ is a mapping onto $\hat{+} H_b$, we let $y \in \hat{+} H_b$ and choose $x = 0_{a+b} + y + 0_a \in \hat{+} H_a$. Then we can determine

$$\psi(x) = 0_b + 0_{a+b} + y + 0_a + 0_{a+b} = 0_b + y + 0_{a+b} = y + 0_b + 0_{a+b} = y + 0_b = y.$$

Now, let $c, d \in \hat{+} H_a$ be arbitrary. We have

$$\psi(c + d) = 0_b + c + d + 0_{a+b} = 0_b + c + 0_a + d + 0_{a+b} = 0_b + c + (0_{a+b} + 0_a) + d + 0_{a+b}$$

$$= 0_b + c + (0_{a+b} + 0_b) + (0_a + d) + 0_{a+b} = (0_b + c + 0_{a+b}) + (0_b + d + 0_{a+b})$$

and

$$\psi(c)\psi(d) = (0_b + c + 0_{a+b})(0_b + d + 0_{a+b})$$

$$= 0_b^2 + 0_b(d + 0_{a+b}) + (c + 0_{a+b})0_b + (c + 0_{a+b})(d + 0_{a+b})$$

$$= 0_b + 0_b(d + 0_{a+b}) + (c + 0_{a+b})0_b + cd + c0_{a+b} + 0_{a+b}d + 0_{a+b}$$

$$\in \hat{+} H_a$$

$$\in \hat{+} H_a$$

$$\in \hat{+} E(L_b)$$

$$\in \hat{+} E(L_b)$$

$$= 0_b + cd + 0_{a+b} = \psi(cd)$$

This completes this part.

3) $\chi$ can be composed by the isomorphisms just constructed. Therefore, let us denote the
isomorphism of 1) by \( \varphi_a \) and the corresponding one for \( L_b \) by \( \varphi_b \). Furthermore, the inverse of the isomorphism given in Lemma 3.2 3), i.e. the natural homomorphism for the \( R \)-congruence restricted to the set of additive idempotents of \( L_a \), is denoted by \( \nu_a \). Let us define by \( \omega : E(L_a)^+ \times H_a \mapsto E(L_b)^+ \times H_b \), \( \omega(c,d) = (\nu_b^{-1} \circ \nu_a)(c), \psi(d) \) a mapping which is an isomorphism due to its isomorphic components as well. So, the situation is as illustrated in the following diagram

\[
\begin{array}{c}
E(L_a)^+ \\
\varphi_a \\
\nu_b^{-1} \circ \nu_a \\
H_a \\
\end{array} \xrightarrow{\chi} \begin{array}{c}
E(L_b)^+ \\
\varphi_b^{-1} \\
H_b \\
\end{array}
\]

As seen in the proof of 1), \( \varphi_b^{-1} = x + y \) for \( (x,y) \in E(L_b)^+ \times H_b \). Then we can determine

\[
(\varphi_b^{-1} \circ \omega \circ \varphi_a)(c) = (\varphi_b^{-1} \circ \omega)(0_c,0_a + c) = \varphi_b^{-1}(\nu_b^{-1} \circ \nu_a)(0_c), \psi(0_a + c)) = \varphi_b^{-1}(0_{0_c+b},0_b + 0_a + c + 0_{a+b}) = 0_{0_c+b} + 0_b + 0_a + c + 0_{a+b} = 0_{c+b} + 0_a + c + 0_{a+b} = \chi(c)
\]

which is, as a composition of isomorphisms, itself an isomorphism. \( \square \)

For further information on completely simple semirings in our terminology, the reader is referred to [SMW05]. In particular, this article contains in Theorem 3.1 a representation similar to the famous one given for completely simple semigroups by Rees. We want to cite here a slightly revised version for the subsequent observations. In a similar way this was done in [PG02] for the kind of completely simple semirings in their usage.
Theorem 3.7. Let $R$ be a skew-ring with zero $0$, $I \in \mathbb{LZ}$ and $\Lambda \in \mathbb{RZ}$ and $\{0\} = I \cap \Lambda$. Furthermore, let $P = (p_{\lambda,i})_{\lambda \in \Lambda, i \in I}$ be a mapping of $\Lambda \times I$ into $R$ satisfying the following conditions

\begin{align*}
 p_{\lambda,0} &= p_{0,j} = 0 \quad (3.2) \\
 p_{\lambda \mu,kj} &= p_{\lambda \mu,ij} - p_{\nu \mu,ij} + p_{\nu \mu,kj} \quad (3.3) \\
 p_{\mu \lambda,jk} &= p_{\mu \lambda,ji} - p_{\mu \nu,ji} + p_{\mu \nu,jk} \quad (3.4) \\
 a p_{\lambda,i} &= p_{\lambda,i} a = 0 \quad (3.5) \\
 ab + p_{0\mu,i0} &= p_{0\mu,i0} + ab \quad (3.6) \\
 ab + p_{\lambda0,0j} &= p_{\lambda0,0j} + ab \quad (3.7)
\end{align*}

for all $i, j, k \in I$, $\lambda, \mu, \nu \in \Lambda$ and $a, b \in R$.

On $M = I \times R \times \Lambda$, define the binary operations

\begin{align*}
 (i, a, \lambda) + (j, b, \mu) &= (i + p_{\lambda,j} + b, \mu) \quad (3.8) \\
 (i, a, \lambda) \cdot (j, b, \mu) &= (ij, -p_{\lambda \mu,ij} + ab, \lambda \mu). \quad (3.9)
\end{align*}

Then $(M, +, \cdot)$ is a completely simple semiring, i.e. $M \in CS$.

Conversely, let $(S, +, \cdot) \in CS$ and $0 \in E(S)^+$. Moreover, let a semiring $(M, +, \cdot)$ be constructed as above by defining $I = E(L_0)^+$, $\Lambda = E(R_0)^+$ and $R = H_0$ and the mapping $P$ by

\begin{equation}
P : (\lambda, i) \mapsto p_{\lambda,i} = \lambda + i \quad (3.10)
\end{equation}

for all $(\lambda, i) \in \Lambda \times I$.

Then $P$ satisfies the conditions (3.2) - (3.7) and the mapping $\varphi$ given by

\begin{equation}
\varphi : a \mapsto (0_{a+0}, 0 + a + 0, 0_{0+a}) \quad (3.11)
\end{equation}

for all $a \in S$ is an isomorphism of $S$ onto $(M, +, \cdot)$.

Proof. The direct part was already proven in [SMW05].

For the converse, we can follow the same proof and its references to see that the sets $I, \Lambda$ and $R$ are in fact the ones used to construct the underlying Rees matrix semigroup in the additive
reduct. From these, we know that the sets can be defined as in the theorem (see the cited Lemma in A.9) and as seen in Lemma 3.2 and its left-right dual, $I$ and $\Lambda$ are in $L^+Z$ and $R^+Z$, respectively. 

**Definition 3.8.** The completely simple semiring $(M, +, \cdot)$ constructed above is called the Rees matrix semiring with the index semirings $I$ and $\Lambda$ and the sandwich matrix $P$ normalized at 0. It will be denoted by $(\mathcal{M}(I, R, \Lambda; P), +, \cdot)$ where it is tacitly assumed to be normalized at an element $0 \in I \cap \Lambda$. The isomorphism $\varphi$ given in (3.11) is the Rees representation of $S$.

If in the subsequent discussions it will be spoken about a Rees matrix semiring of a completely simple semiring, we will always mean the construction and notations above without further reference.

Using this theorem, we can determine the idempotent element in an arbitrary $H^+$-class of a completely simple semiring in the following way.

**Corollary 3.9.** Let $S \in \mathcal{CS}$ and $\mathcal{M}(I, R, \Lambda; P)$ an isomorphic Rees matrix semiring. Then

$$0_{i+\lambda} = i - p_{\lambda,i} + \lambda$$

(3.12)

for all $(\lambda, i) \in \Lambda \times I$.

**Proof.** That $i - p_{\lambda,i} + \lambda \in H_{i+\lambda}^+$ is known by Lemma 3.2. If we are able to prove the idempotency of this element, we are finished. Therefore, we have

$$(i - p_{\lambda,i} + \lambda) + (i - p_{\lambda,i} + \lambda) = i - p_{\lambda,i} + (\lambda + i) - (\lambda + i) + \lambda = i - p_{\lambda,i} + 0 + \lambda$$

$$= i - p_{\lambda,i} + \lambda.$$

Using the notation of Theorem 3.7, another useful corollary describes the image of $E(L_0)^+$ under the Rees representation $\varphi$.

**Corollary 3.10.** Let $S \in \mathcal{CS}$ and $\mathcal{M}(I, R, \Lambda; P)$ be the corresponding Rees matrix semiring normalized at $0 \in E(S)^+$. Then

$$\varphi(E(L_0)^+) = I \times \{0\} \times \{0\}$$
where \( \varphi \) is the isomorphism given in (3.11).

**Proof.** \([\subseteq]\) Let \( e \in E(L_0)^+ \). Then

\[
\varphi(e) = (0_{e+0}, 0 + e, 0_{0+e}) = (0_e, 0, 0) = (e, 0, 0) \in I \times \{0\} \times \{0\}
\]

since \( 0, e \in E(L_0)^+ \in \mathbb{LZ} \).

\([\supseteq]\) Let \( (i, 0, 0) \in I \times \{0\} \times \{0\} \). Then \( i \in I = E(L_0)^+ \) and with the same calculations as in the first part, \( (i, 0, 0) = \varphi(i) \in \varphi(E(L_0)^+) \) can be proved. \( \square \)
4. Orthodox semirings

As already seen in the previous chapters, the set of all additive idempotents $E(S)^+$ is of particular importance in the study of completely regular semirings $S$. In this chapter, we will be concerned with semirings where $E(S)^+$ is itself a semiring.

**Definition 4.1.** A semiring $S \in CR$ is called orthodox if $E(S)^+$ forms a subsemiring of $S$. $O$ is the class of all such semirings.

**Example 4.2.** Again, $SkR$ and $I$ are trivial examples of orthodox semirings.

**Corollary 4.3.** In fact, $O$ is a subvariety of $CR$ with the additional identity

$$0_a + 0_b = 0_a + 0_b + 0_a + 0_b.$$  

**Proof.** Since we only observe completely regular semirings $S$, $E(S)^+$ is not empty. Because of Lemma 1.21, for every idempotent $e \in E(S)^+$ holds $0_e = e$. So, the additional identity is trivial, since $E(S)^+$ is known to be multiplicatively closed by Lemma 1.40.

**Remark 4.4.** We directly derive that the semiring $(S, +, \cdot) \in CR$ is orthodox if and only if its additive reduct is an orthogroup.

Please note that other authors (see [SM04]) used the term of an orthodox semiring also for a semiring with an orthodox multiplicative reduct and an inverse additive one.

4.1. The core

A very useful tool in the research of orthodox semirings is given in the definition below. In [PR99], Petrich and Reilly used it in the same context for semigroups.
Definition 4.5. Let $S$ be a semiring with a regular additive reduct. The least additive sub-semigroup containing $E(S)^+$ will be called core of $S$ and will be denoted by $C(S)^+$.

Remark 4.6. As it can be seen easily, for a semiring $S \in \mathcal{ACR}$, Lemma 1.21 implies that $C(S)^+ = \{ x \in S \mid \exists n \in \mathbb{N} \exists a_1, \ldots, a_n \in S : x = a_1 + \ldots + a_n \}$.

The following lemma including its proof are contained for $\mathcal{UR}$ in [PG02].

Lemma 4.7. Let $S \in \mathcal{CR}$, $a \in C(S)^+$ and $b \in S$. Then $ab = 0_{ab}$ and $ba = 0_{ba}$.

Proof. Let $a \in C(S)^+$. Using Remark 4.6, we know that there must exist $a_1, \ldots, a_n \in S$ for which $a = a_1 + \ldots + a_n$. Because of Lemma 2.11 it follows

$$ab = (0_{a_1} + \ldots + 0_{a_n})b = 0_{a_1}b + \ldots + 0_{a_n}b = 0_{a_1}0b + \ldots + 0_{a_n}0b = a0b = 0_{ab}$$

and similarly $ba = 0_{ba}$ can be obtained.

The following two results generalize results of [PG02] concerning the core of semirings of $\mathcal{UR}$ to semirings of $\mathcal{CR}$.

Lemma 4.8. Let $S \in \mathcal{CR}$. Then:

1) The core $C(S)^+$ is an ideal of $S$.

2) $C(S)^+ \in \mathcal{CR}$.

3) The maximal sub-skew-rings of $C(S)^+$ are in $\mathcal{Nu}$.

Proof. 1) Because the core is not empty, additively closed by definition and Lemma 4.7, this is obvious.

2) Already 1) shows that $C(S)^+$ is a semiring which is because of Lemma A.10 in $\mathcal{ACR}$. Now, Theorem 2.13 implies that $E(S)^+ \subseteq E(S)^\ast$. But since $E(S)^+ = E(C(S)^+)^+$, i.e. the additive idempotents of $C(S)^+$ are multiplicative idempotents as well, $C(S)^+$ is completely regular as a semiring.

3) Let $T$ be a maximal sub-skew-ring of the completely regular semiring $C(S)^+$ and $a, b \in T$. Lemma 4.7 yields $ab = 0_{ab}$ which is the zero element of $T$. Hence, this semiring is a null skew-ring.
Remark 4.9. We can derive that a completely regular semiring $S$ is orthodox if and only if $E(S)^+ = C(S)^+$.

Corollary 4.10. For every maximal sub-skew-ring $R$ of a semiring $S \in \mathcal{CR}$, $C(S)^+ \cap R$ is an ideal of $R$. Furthermore, it is a maximal sub-skew-ring in $C(S)^+$. Conversely, every maximal sub-skew-ring of $C(S)^+$ can be obtained in such a way.

Proof. Let $T = C(S)^+ \cap R$. As an intersection of two subsemirings of $S$, $T$ is a subsemiring of $S$ or empty, but since $R$ contains an additive idempotent, $T$ is in fact nonempty. Let $a \in T$. Moreover, to prove the first assumption, let $r \in R$. Due to Lemma 4.8 1), $ar$ is in $C(S)^+$ and, because $R$ is a semiring, as well in $R$, hence $ar \in T$ and similarly $ra \in T$.

Because both, $C(S)^+$ and $R$, are completely regular semirings, there exist inverses of $a$ satisfying (2.2) in $C(S)^+$ and in $R$, which coincide due to the uniqueness in $S$ with $-a$. It follows $-a \in T$, and $T$ is a sub-skew-ring of $C(S)^+$. To show that $T$ is maximal, assume there exists a maximal sub-skew-ring $T' \supset T$ of $C(S)^+$ containing an element $b \notin T$. So, $a$ and $b$ are $\overset{+}{H}_{C(S)^+}$-related, where $\overset{+}{H}_{C(S)^+}$ is the $\overset{+}{H}$-relation of the semiring $C(S)^+$. Now, it follows that $a \overset{+}{H}_S b$, where $\overset{+}{H}_S$ represents the $\overset{+}{H}$-relation in $S$. Since $R$ is the $\overset{+}{H}_S$-class containing $a$, we obtain that $b \in R$. We can conclude that $b \in R \cap C(S)^+ = T$, a contradiction to our assumption. So, $T$ is in fact a maximal sub-skew-ring in $C(S)^+$.

Let $M$ be a maximal sub-skew-ring in $C(S)^+$. Then it is as well a sub-skew-ring in $S$ and must be contained in some maximal sub-skew-ring of $S$, say $N$. But as seen before, $N \cap C(S)^+$ is a sub-skew-ring of $C(S)^+$, which obviously contains $M$. The maximality of $M$ leads to $M = N \cap C(S)^+$. \[ \square \]

As a direct consequence, we can formulate the first of the following two corollaries which both are contained for $\mathcal{UR}$ in [PG02].

Corollary 4.11. Let $S \in \mathcal{CR}$. If $S$ does not contain any maximal sub-skew-ring $\overset{+}{H}_a$ which has an ideal of $\mathcal{N}(\mu$ different to $\{0_a\}$, $S$ is orthodox.

Proof. Assume there is an element $b \in C(S)^+ \setminus E(S)^+$. Then there must exist a maximal sub-skew-ring of $S$ containing $b$, say $\overset{+}{H}_{b}$. Since $b$ is no idempotent, $b \neq 0_b$ but both elements
are in $H_b \cap C(S)^+$ which is due to Corollary 4.10 an ideal of $H_b$. Furthermore, since the intersection is a maximal sub-skew-ring of $C(S)^+$, it is in fact a null sub-skew-ring. But this existence contradicts our condition that no maximal sub-skew-ring of $S$ contains a nontrivial ideal which is an element of $Nu$. So, our assumption is rejected and $C(S)^+ = E(S)^+$ which implies according to Remark 4.9 that $S$ is orthodox.

Lemma 4.8 shows that if the additive reduct of a completely regular semiring is generated by its idempotents, this semiring is contained in $Nu \circ I$. We can conclude for these semirings:

**Corollary 4.12.** Let $S \in CR$ be a semiring for which $S = C(S)^+$. Then $E(S)^+ = E(S)^\bullet$.

**Proof.** That $E(S)^+ \subseteq E(S)^\bullet$ is clear because of Lemma 2.12. So, let $e \in E(S)^\bullet$. We have $e \in S = C(S)^+$ which is a union of null skew-rings as seen in Lemma 4.8. We obtain $e = ee = 0_e \in E(S)^+$, i.e. $e$ is as well additively idempotent.

4.2. The lattice $L(O)$

In this section we will concentrate our work on the subvarieties of the variety of orthodox semirings.

For completely simple semirings, the following characterization for being orthodox can be given which is analogue to Corollary 1.5 of [PG02] whose proof is adopted partially.
Lemma 4.13. Let $S \in \mathcal{CS}$. Then the following statements are equivalent:

1) $S$ is orthodox.

2) $S$ is isomorphic to $\mathcal{M}(I, R, \Lambda; P)$ where $p_{\lambda,i} = 0$ for all $\lambda \in \Lambda$ and $i \in I$.

3) For $0 \in E(S)^+$, the mapping $\varphi$ defined by

$$\varphi : a \mapsto (0_{a+0}, 0 + a, 0_0+0)$$

for all $a \in S$ is an isomorphism of $S$ onto $E(L_0)^+ \times H_0 \times E(R_0)^+$ where the latter semiring is equipped with coordinatewise operations.

Proof. 1) $\Rightarrow$ 2) Let $\mathcal{M}(I, R, \Lambda; P)$ be the Rees matrix semiring as constructed in Theorem 3.7. For arbitrary $(\lambda, i) \in \Lambda \times I$, $p_{\lambda,i}$ has to be an element of the skew-ring $H_0$. Since $S$ is assumed to be orthodox and due to equation (3.10), $p_{\lambda,i} = \lambda + i$ is an idempotent and hence $p_{\lambda,i} = 0$.

2) $\Rightarrow$ 3) By Theorem 3.7, it is already known that $\varphi$ is an isomorphism onto the Rees matrix semiring equipped with the operations (3.8) and (3.9). If we keep in mind that $I$ and $\Lambda$ are in $\mathcal{L}\mathbb{Z}$ respectively $\mathcal{R}\mathbb{Z}$ and that $p_{\lambda,i} = 0$ for all $\lambda \in \Lambda$ and $i \in I$, this directly shows that these operations coincide with the coordinatewise ones.

3) $\Rightarrow$ 1) Let $e, f \in E(S)^+$. Since $\varphi$ is an isomorphism, $\varphi(e)$ and $\varphi(f)$ must be additively idempotent, too. In particular, $0 + e + 0$ and $0 + f + 0$ must be an idempotent in $\mathcal{H}_0$, which means $0 + e + 0 = 0 = 0 + f + 0$. So, we are finished if we can show that $\varphi(e + f)$ is idempotent in $E(L_0)^+ \times H_0 \times E(R_0)^+$. Since $E(L_0)^+$ and $E(R_0)^+$ are idempotent semirings and $0 + 0 = 0$, $\varphi(e + f)$ and hence $e + f$ are in fact idempotent.

Remark 4.14. So, if we use the notations of Theorem 3.7, we see that an orthodox completely simple semiring is isomorphic to $I \times R \times \Lambda$.

Because of the following lemma, we also can combine the direct product of $I$ and $\Lambda$ to a semiring of $\mathcal{R}\mathbb{Z}$. This and the following theorem, which is based thereon, were already stated in [PG02].

Lemma 4.15. Let $S$ be a semiring. Then $S \in \mathcal{R}\mathbb{Z}$ if and only if there exist some $L \in \mathcal{L}\mathbb{Z}$ and $R \in \mathcal{R}\mathbb{Z}$ such that $S \cong L \times R$. 
Proof. [$\Rightarrow$] By definition, it is clear that the additive reduct of $S$ is in $Re$, and by Lemma A.11 $a + b + c = a + c$ (***) holds for all $a, b, c \in S$. In addition, the proof of the mentioned lemma shows, that $(S, +) \cong (L, +) \times (R, +)$ for $L = S + c \in LZ$ and $R = c + S \in RZ$ for some arbitrarily but fixed $c \in S$. The isomorphism $\varphi$ which is used there is defined by $\varphi : S \mapsto L \times R, \varphi(x) = (x + c, c + x)$. It is sufficient to prove that $\varphi$ is as well a homomorphism with respect to the multiplication. So, let $x, y \in S$ and we obtain the following:

$$\varphi(x) \varphi(y) = (x + c, c + x)(y + c, c + y) = (xy + xc + cy + c^2, c^2 + xc + cy + xy) \cong (xy + c, c + xy) = \varphi(xy)$$

since $c = c^2$. Because $S \in I$, $L$ and $R$ are idempotent semirings. This concludes this part of the proof.

[$\Leftarrow$] Since $L$ and $R$ are in $I$, a variety, $L \times R$ is in $I$ as well. Moreover, by Lemma A.11, we obtain that the additive reduct of $L \times R$ is a semigroup in $Re$. Hence, $S \in Re$. □

Consequently, for $L(O)$ as a sublattice of $L(OR)$ we obtain:

**Theorem 4.16.**

$$Re = \text{LZ} \lor \text{RZ}$$

Proof. [$\subseteq$] The variety $\text{LZ} \lor \text{RZ}$ contains all direct products of semirings $L \in \text{LZ}$ and $R \in \text{RZ}$. So, applying Lemma 4.15 results in $Re \subseteq \text{LZ} \lor \text{RZ}$.

[$\supseteq$] $Re$ is a subvariety of $OR$ containing $\text{LZ}$ and $\text{RZ}$. But $\text{LZ} \lor \text{RZ}$ is the least one satisfying these conditions. Thus, this part is proven as well. □

Now, we will have a second look on the class of orthodox completely simple semirings. Therefore, we introduce the following notation.

**Definition 4.17.** A semiring which is isomorphic to a direct product of an element of $Re$ and a skew-ring will be called rectangular skew-ring. The class of all rectangular skew-rings will be denoted by $ReSkR$.

**Remark 4.18.** Obviously, $ReSkR$ is a subclass of $O$ since, as a variety, $O$ has to be closed with respect to arbitrary direct products.
As seen in Lemma A.12, the semigroups which are direct products of a group and a rectangular band, are exactly the orthodox completely simple semigroups. As the analogon for semirings we obtain the following lemma. Parts of it can also be found in a similar result for $\mathcal{UR}$ in [PG02], other ones are inspired by corresponding results for semigroups in [PR99].

The penultimate step of the cycle of proof is similar the one in the semigroup case.

**Lemma 4.19.** Let $S \in \mathcal{CR}$. Then the following conditions are equivalent:

1) $S \in \mathcal{ReSkR}$.

2) $E(S)^+ \in \mathcal{R}_{\mathcal{R}}$.

3) $S$ satisfies the identity $0_a = 0_a + 0_b + 0_a$.

4) $S$ is an orthodox completely simple semiring.

5) The least $\mathcal{SkR}$-congruence on $S$ is over $\mathcal{R}_{\mathcal{R}}$.

**Proof.**

1) $\Rightarrow$ 2) $S \in \mathcal{ReSkR}$ implies that the additive reduct is a direct product of a group and a rectangular band, Lemma A.12 and Theorem 2.13 yield $E(S)^+ \in \mathcal{R}_{\mathcal{R}}$.

2) $\Rightarrow$ 3) Since $(E(S)^+, +) \in \mathcal{R}_{\mathcal{R}}$ this is already known (see again Lemma A.12).

3) $\Rightarrow$ 4) By Lemma A.12, we directly obtain that the additive reduct is an orthogroup and completely simple. But the Corollary 2.26 and Remark 4.4 enable us to transfer this to semirings.

4) $\Rightarrow$ 5) Using Remark 4.14 and the notations of Theorem 3.7, we know that $S \cong I \times R \times \Lambda$. For clarity reasons, we even assume $S = I \times R \times \Lambda$. Now, let us define a relation $\varrho$ on $S$ by

$$(i, g, \lambda) \varrho (j, h, \mu) \iff g = h.$$ 

Obviously, $\varrho$ is a congruence relation and $S/\varrho \cong R \in \mathcal{SkR}$, i.e. it is an $\mathcal{SkR}$-congruence. Let $e \in E(S)^+$, then $e\varrho$ must be the zero element of the skew-ring $S/\varrho$, which is of course unique, i.e. $E(S)^+ \subseteq e\varrho$. Since 0 is the only additive idempotent of $R$, there must exist $i \in I$ and $\lambda \in \Lambda$ such that $e = (i, 0, \lambda)$. Furthermore, let $a \in e\varrho$ and $a = (j, h, \mu)$. This means that $h = 0$ and therefore $a \in E(S)^+$. We proved that $E(S)^+ = e\varrho$ (*).

To show that $\varrho$ is the least $\mathcal{SkR}$-congruence, assume the existence of an $\mathcal{SkR}$-congruence
There exists a pair \((a, b) \in \rho \setminus \theta\). Since \(S/\theta\) is a skew-ring, there must exist an element \(x \in S\) such that \(b\theta + x\theta = a\theta + x\theta = E(S)^+\). We obtain \(a + x = f \in E(S)^+\) and \(b + x = f' \in E(S)^+\). Since \(f\theta\) and \(f'\theta\) have to be the unique additive idempotent in \(S/\theta\), we can conclude \(a\theta + x\theta = f\theta = f'\theta = b\theta + x\theta\). Because \(a\theta \neq b\theta\), this is a contradiction to the uniqueness of the additive inverse of \(x\theta\).

Because the additive reduct is a completely simple orthogroup, we can conclude, using once again Lemma A.12 and Theorem 2.13, that \(e\) is in \(R_{s}^+\). Assume, there is another \(\rho\)-class which is a subalgebra of \(S\), say \(a\rho\). Then \((a + a)\rho a\) and \(a\rho + a\rho = a\rho\). But the skew-ring \(S/\rho\) contains only one additive idempotent, namely \(e\rho\), i.e. \(a\rho e\). So, \(\rho\) is a congruence over \(R_{s}^+\).

5) \(\Rightarrow\) 1) Let \(\rho_{SKR}\) be the least \(SKR\)-congruence on \(S\). This is assumed to be over \(R_{s}^+\). The unique additive idempotent of the skew-ring \(S/\rho_{SKR}\), say \(e\rho\) for some \(e \in S\), contains \(E(S)^+\) and because \(e\rho\) is a subalgebra of \(S\), it must be in \(R_{s}^+\). We have \(E(S)^+ = e\rho_{SKR} \in R_{s}^+\).

Lemma A.12 and Remark 4.14 imply that \(S \cong I \times R \times \Lambda\) where \(R \in SKR\) and \(I \times \Lambda \in R_{s}^+\) due to Lemma 4.15. Thus, \(S\) is isomorphic to a direct product of a skew-ring and a semiring of \(R_{s}^+\) as required.

As a consequence, we deduce the following result for \(L(O)\) respectively \(L(CR)\). An analogue conclusion can be achieved in \(UR\) as seen in [PG02].

**Theorem 4.20.**

\[
R_{s}^+ \lor SKR = R_{s}SKR = O \cap (SKR \circ_{SR} R_{s}^+) = R_{s}^+ \circ_{CR} SKR
\]

and hence, \(R_{s}SKR\) is a variety of universal algebras of type \((2, 2, 1)\).

**Proof.** If we remember Remark 4.18, Theorem 3.4 and Lemma 1.3, the equality of the last three given classes is directly obtained by Lemma 4.19.

Because of the characterization contained in 3) of this lemma, \(R_{s}SKR\) is a subvariety of \(CR\) which contains \(R_{s}^+\) and \(SKR\) by definition. So, \(R_{s}^+ \lor SKR \subseteq R_{s}SKR\). But since each semiring of \(R_{s}SKR\) is a direct product of a semiring of \(R_{s}^+\) and of a skew-ring, and since, as a variety, \(R_{s}^+ \lor SKR\) has to be closed with respect to arbitrary direct products, the claim follows. \(\square\)
Please note that in this case we can not determine the Malcev product of $R_{\text{g}}$ and $SkR$ within $SR$. A counterexample is given by the following.

**Example 4.21.** On $S = \{a, b, c, d, e\}$ the binary operations are given by

\[
\begin{array}{c|ccccc}
+ & a & b & c & d & e \\
\hline
a & a & a & c & c & c \\
b & c & c & a & a & a \\
c & c & c & a & a & a \\
d & e & e & b & b & b \\
e & e & e & b & b & b
\end{array}
\quad
\begin{array}{c|ccccc}
\cdot & a & b & c & d & e \\
\hline
a & a & a & a & a & a \\
b & b & b & e & e & e \\
c & c & c & a & a & a \\
d & e & e & b & b & b \\
e & e & e & b & b & b
\end{array}
\]

That $(S, +, \cdot)$ is in fact a semiring can be proven easily. Moreover $E(S)^+ = \{a, b\}$ is an idempotent semiring whose additive reduct is rectangular. The equivalence relation determined by the equivalence classes $\{a, b\}$ and $\{c, d, e\}$ is in fact an $SkR$-congruence over $R_{\text{g}}$, i.e. $S \in R_{\text{g}} \circ SkR$, but $S$ is not completely regular since there is no idempotent $f \in E(S)^+$ such that $d + f = d$.

The additive reduct of this example was determined using Gap 4 ([GAP12]) and the Smallsemi package ([DM12]) of this tool, which offers a library of all semigroups with certain properties. A suitable multiplicative reduct was computed by a program written by myself in C++. If we observe a completely simple semiring such that $(E(L_a)^+, \cdot)$ has a neutral or an absorbing element we can state that it is orthodox as already mentioned for $UR$ in [PG02]. There, they only proved the case with the neutral element which is quite similar to the one with an absorbing element we proved here explicitly.

**Lemma 4.22.** Let $S \in CS$ and $a \in S$. If the multiplicative reduct of $S$ contains an absorbing element $0$ or an identity $1$ of $E(L_a)^+$, then $S$ is orthodox.

**Proof.** Assume that $0$ is an absorbing zero in $(E(L_a)^+, \cdot)$ and $M(I, R, \Lambda; P)$ be the corresponding Rees matrix semiring of $S$ normalized at $0$ as constructed in Theorem 3.7. Let $\varphi$ be the isomorphism as defined there. Hence, $\varphi(0) = (0_{0+0}, 0, 0_{0+0}) = (0, 0, 0)$ has to be absorbing in the multiplicative reduct of $\varphi(E(L_a)^+)$ which is equal to $I \times \{0\} \times \{0\}$ according to Corollary 3.10. In particular, this means that $i0 = 0 = 0i$ for all $i \in I$. 
It is known that (3.2) and (3.3) are satisfied for arbitrary \( i, j, k \in I \) and \( \lambda, \mu, \nu \in \Lambda \). So, fix \( k = j \) and \( i = 0 \), and we obtain

\[
p_{\lambda \mu, j} = p_{\lambda \mu, jj} \quad (3.3) = p_{\lambda \mu, 0} + p_{\nu \mu, jj}
\]

(4.1)

and similarly \( p_{\mu \lambda, j} = p_{\mu \nu, j} \) using (3.4). Now, \( \mu = \lambda \) is chosen implying \( p_{\lambda j} = p_{\lambda \lambda, j} = p_{\nu \lambda, j} \) and \( p_{\lambda, j} = p_{\lambda \lambda, j} = p_{\lambda \nu, j} \). We gather \( p_{\nu \lambda, j} = p_{\lambda, j} = p_{\lambda \nu, j} \). Because of this, we can interchange \( \lambda \) and \( \nu \), which yields \( p_{\lambda, j} = p_{\nu, j} = p_{0, j} \) if \( \nu = 0 \) is fixed. Due to Lemma 4.13, this is a sufficient condition for being orthodox.

The same considerations will be successful in order to proof the second case, i.e. if there is an identity 1 in \((E(L_a)^+, \cdot)\). The only difference is that we have to choose \( i = j = 1 \) in (4.1) to obtain the same result.

Please note that because of Lemma 3.6 3) the conditions of the foregoing lemma are equivalent to the existence of such an element in the set of idempotents of an arbitrary \( L \)-class.

Now, we will be concerned with the determination of the least \( O \)-congruence \( \varrho_O \). The following definition is necessary in order to use the knowledge about semigroups.

**Definition 4.23.** A nonempty subset \( K \) of a regular semigroup \((S, +)\) is self-conjugate if \( x + K + x' \subseteq K \) for all \( x \in S \) and \( x' \in \tilde{V}(x) \) and full if \( E(S)^+ \subseteq K \).

We may assume in our considerations that there is always a least self-conjugate subsemigroup of the additive reduct which is full.

**Lemma 4.24.** Let \( S \in \mathcal{CR} \). In \((S, +)\), there exists the least self-conjugate full subsemigroup.

**Proof.** Define \( K \) by

\[
K = \cap \{ T \subseteq S \mid T \text{ is a self-conjugate full subsemigroup of } (S, +) \}.
\]

Obviously, \( S \) is full and self-conjugate so that at least \( S \) is one of such semigroups about which the intersection is determined. Furthermore, each of these semigroups contains \( E(S)^+ \) so that the intersection is nonempty. As easily seen, \( K \) is itself a self-conjugate subsemigroup.
This set allows us to give the following description of $\varrho_O$.

**Lemma 4.25.** Let $S \in CR_c$, $K$ be the least full self-conjugate subsemigroup in the additive reduct of $S$ and the relation $\theta$ be defined by

\[ \theta = \{(a, b) \in S \times S \mid a + H b \text{ and } a - b, -a + b \in K\}. \]

Then the least $O$-congruence $\varrho_O$ on $S$ can be determined as the least semiring congruence containing $\theta$.

**Proof.** As seen in the foregoing lemma, the least full self-conjugate subsemigroup exists which we denote here by $K$. Combining Theorem 2 and 4 of [Tro83], we obtain that the least orthodox congruence $\theta'$ on $(S, +)$ satisfies

\[ \theta' = \{(a, b) \in S \times S \mid 0_a \triangleright \theta' 0_b \text{ and } a - b, -a + b \in K\}, \]

where $\triangleright \theta' = \theta' \cap (E(S)^+ \times E(S)^+)$ is the trace of $\theta'$.

Now, we will show $\theta = \theta'$.

[$\subseteq$] Therefore, we choose $(a, b) \in \theta$. Lemma 1.18 implies that $0_a = 0_b$ so that $(a, b) \in \theta'$ because of the reflexivity of $\triangleright \theta'$.

[$\supseteq$] Vice versa, we have to prove that $(a, b) \in \theta'$ results in $(a, b) \in \theta$. By Lemma 2.15, it is known that $S/\hat{H} \in I$ that is $\hat{H}$ is an $O$-congruence and due to Remark, 4.4 the additive reduct of this factor semiring has to be orthodox. Since $\theta'$ is the least orthodox semigroup congruence on $(S, +)$, $\hat{H}$ contains $\theta'$ and so $\theta = \theta'$.

In general, $\theta$ is no congruence on the multiplicative reduct. But we can determine a congruence $\vartheta$ on $S$ as the least semiring congruence containing $\theta$. Since this contains $\theta$ and orthogroups form a variety as well, $\vartheta$ must be an orthodox congruence on the additive reduct and by Remark 4.4 an $O$-congruence on $S$. On the other hand, any $O$-congruence $\varrho$ must involve $\theta$ so that $\vartheta$ is in fact the least $O$-congruence, i.e. $\vartheta = \varrho_O$.  

On semirings of $UR_c$, Pastijn and Guo used in [PG02] a congruence which is more concrete. They benefit from a result in [PT91] that the least self-conjugate full subsemigroup in a band of commutative groups is exactly the core. This allows as well an analysis of the corresponding
congruence classes so that the variety of $\mathcal{UR}_\circ$ can be described as a Malcev product of null rings and orthodox semirings, where the latter was used of course in their notation. Unfortunately, we are not in such a comfortable situation so that for a good description of the $\varrho_0$-classes more research will be needed.
5. Summary

In the present thesis we discussed completely regular semirings. Starting with a short introduction on relevant topics from the theory of universal algebras, semigroup and semiring theory, we proceed with the definition of these semirings. It was derived that several well-known characterizations can be adopted from the semigroup case. This includes the decompositions of a completely regular semiring into an idempotent semiring of skew-rings or into an $\mathcal{SL}$-semiring of completely simple semirings. But we obtained that the $\mathcal{H}$-relation is always a congruence, which is a significant difference to semigroups. Because of their crucial role, semirings of $\mathcal{CS}$ were considered in chapter 3 in more detail. We showed that the congruence classes of $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ are isomorphic to each other and that the corresponding factor semirings are in $\mathcal{RZ}$, $\mathcal{LZ}$ and $\mathcal{Re}$ respectively. This helped us to concretize an analogon of the Rees matrix representation for completely simple semirings. Chapter 4 was addicted to orthodox semirings where we concentrated us especially on completely simple ones. Using an associated Rees matrix semiring, it was pointed out that these are exactly the ones with a null sandwich matrix for an arbitrary idempotent 0 or the completely regular semirings which are isomorphic to a direct product of a semiring of $\mathcal{Re}$ and one of $\mathcal{SKR}$. Difficulties occurred at the attempt to determine the least orthodox congruence $\varrho_0$ so that we have been able to give only a rough description of this congruence.

Of course, there are still a lot of gaps in the knowledge about the lattice of subvarieties of $\mathcal{CR}$ which have to be filled. In particular, for the orthodox case, Pastijn and Guo already stated several results for $\mathcal{UR}$ which might are able to be generalized to $\mathcal{CR}$. A series of papers dealing with other subclasses of $\mathcal{CR}$ have been already mentioned in the introduction.
A. A more comprehensive introduction to completely regular semigroups

In this appendix, I will collect several statements so that the thesis becomes more self-contained and my proofs can be understood without the study of additional books.

Lemma A.1. [PR99, Corollary I.8.13] Let $V$ be a variety and $U \in \mathcal{L}(V)$. Then for any $S \in V$ the least $U$-congruence $\varrho_U$ on $S$ exists.

Lemma A.2. [PR99, Lemma II.2.4] Let $S \in \text{CoR}$ and $T \in \text{SG}$. Furthermore $a \in S$ and $\varphi : S \mapsto T$ is a semigroup homomorphism. Then:

- $\varphi(S) \in \text{CoR}$.
- $-\varphi(a) = \varphi(-a)$, $0_{\varphi(a)} = \varphi(0_a)$.

Lemma A.3. [PR99, Corollary II.2.5] Let $\varrho$ be a congruence on $S \in \text{CoR}$. Then $a \varrho b$ implies $-a \varrho -b$ and $0_a \varrho 0_b$.


Lemma A.5. [PR99, Corollary II.1.5] If $S \in \text{CoR}$, $+J$ is the least $SL$-congruence on $S$.

Lemma A.6. [PR99, Corollary III.1.8] For $S \in \text{CSI}$, Green’s relations $+H, +L, +R$ and $+D$ are congruences.

Theorem A.7. [PR99, Theorem III.1.3] Let $S \in \text{CSI}$ and $a, b \in S$. Then $b +R b + a$ and $a +L b + a$.

Lemma A.8. [PR99, Lemma II.1.6.] Let $S \in \text{SG}$. Then $S \in \text{Re}$ if and only if $S \in \text{CSI} \cap \text{B}$.
Lemma A.9. [PR99, Lemma III.2.3] Let $S \in CSi$ and fix $0 \in E(S)^+$. Define $I = E(\bar{L}_0)^+$, $\Lambda = E(\bar{R}_0)^+$, and $R = \bar{H}_0$ and the mapping $P : \Lambda \times I \mapsto R$ by

$$P : (\lambda, i) \mapsto p_{\lambda,i} = \lambda + i$$

(A.1)

for all $(\lambda, i) \in \Lambda \times I$. Then the mapping $\varphi$ given by

$$\varphi : a \mapsto (0_{a+0}, 0 + a, 0_{0+a})$$

(A.2)

for all $a \in S$ is an isomorphism of $S$ onto the Rees matrix semigroup $M = I \times R \times \Lambda$ equipped with the addition

$$(i, a, \lambda) + (j, b, \mu) = (i, a + p_{\lambda,j} + b, \mu)$$

for all $(i, a, \lambda), (j, b, \mu) \in M$.

Lemma A.10. [PR99, Lemma II.6.1.] Let $S \in CoR$. Then $C(S)^+ \in CoR$, too.

Lemma A.11. [How95, Theorem 1.1.3] Let $S$ be a semigroup. Then the following conditions are equivalent:

- $S \in Re$.
- $S \in B$ and for all $a, b, c \in S$ we have $a + b + c = a + c$.
- There exists a semigroup $L \in LZ$ and a semigroup $R \in RZ$ such that $S \cong L \times R$.

Lemma A.12. [PR99, Corollary III.5.3] Let $S \in CoR$. Then the following conditions are equivalent:

- There exists a group $G \in G$ and a rectangular band $B \in Re$ such that $S \cong G \times B$.
- $E(S)^+ \in Re$.
- $S$ satisfies the identity $0_a = 0_a + 0_b + 0_a$.
- $S$ is a simple orthogroup.
- The least group congruence $\varrho_g$ on $S$ is over $Re$. 
Bibliography


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5th July 2013

Dipl.-Math. Rick Schumann
Declaration

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