Towards Discretization by Piecewise Pseudoholomorphic Curves

Von der Fakultät für Mathematik und Informatik
der Universität Leipzig
angenommene

D I S S E R T A T I O N

zur Erlangung des akademischen Grades

DOCTOR RERUM NATURALIUM
(Dr. rer. nat.)

im Fachgebiet
Mathematik
vorgelegt

von Diplommathematiker David Bauer
geboren am 20.06.1984 in Freiberg

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Die Verleihung des akademischen Grades erfolgt mit Bestehen
der Verteidigung am 04.12.2013 mit dem Gesamtprädikat magna cum laude
Bibliographische Daten

Towards Discretization by Piecewise Pseudoholomorphic Curves
(Zur Diskretisierung durch stückweise pseudoholomorphe Kurven)
Bauer, David
Universität Leipzig, Dissertation, 2012
114 Seiten, 9 Abbildungen, 47 Referenzen
This thesis is dedicated to my grandfather, Fritz Rupprecht.
Acknowledgements

I would like to express my gratitude to my advisor Prof. Matthias Schwarz. Without his guidance and his insight this work would have been impossible. Many key ideas like exploiting the vicinity of boundary punctured disks to Morse theoretic objects can be traced back to his intuition and knowledge. I also thank him for continuing support and encouragement to participate in conferences and in getting to know the symplectic community.

Further thanks go to my former office mates Matti Schneider, Felix Schmäschke and Murat Sağlam. I gained a large part of my knowledge on symplectic geometry in discussions with them. I also like to acknowledge the support of Frol Zapolsky, Barney Bramham, Marco Mazzucchelli and Joel Fish who were present as postdocs during my stay at the Max-Planck-Institute. For proofreading and many helpful comments I thank Nadine Große, Joe Johns and Richard Siefring. Finally, I am grateful for the stimulating working environment provided by the IMPRS and the financial support of the Klaus Tschira foundation.

I thank all of my friends and family and everyone else who has made this possible. In particular, I am grateful for the support of my parents and my sister. I warmly thank Julia Stelter for her love and for cheering me up during times when progress was slow.
Contents

1 Introduction and Main Results 1
   1.1 J-holomorphic Curves in Symplectic Geometry .......................... 1
   1.2 Circle Packing ................................................. 2
   1.3 Moduli Spaces of Piecewise J-holomorphic Curves ...................... 3
   1.4 Organization .................................................. 8

2 On the Geometry of the Tangent Bundle 11
   2.1 Overview ..................................................... 11
   2.2 The Sasaki Metric .............................................. 11
   2.3 Expressions in Local Coordinates .................................. 17
   2.4 Distinguished Complex Structures and Symplectic Forms ................ 20
   2.5 Unbounded Geometry ............................................. 26
   2.6 J-holomorphic Curves ............................................ 32
   2.7 Relation to Harmonic Maps ......................................... 34

3 Existence of Lifted Disks 37
   3.1 Overview ..................................................... 37
   3.2 Admissible Boundary Conditions ..................................... 37
   3.3 Applying the Implicit Function Theorem ............................... 40
   3.4 Existence Results in the Tangent Bundle .............................. 49
   3.5 Diameter and Gradient Bounds ...................................... 57

4 Existence of Punctured Disks 67
   4.1 Overview ..................................................... 67
   4.2 Index Calculations .............................................. 67
   4.3 Conformal Models ................................................. 78
   4.4 Morse Flow Trees in $S\mathbb{R}^n$ .................................. 80
   4.5 From Trees to Disks .............................................. 86
   4.6 Existence Results in $SQ \times \mathbb{R}$ ................................ 93

5 Future Directions 97
   5.1 From Lifted to Punctured Disks .................................... 97
   5.2 Suitable Choices of Negative Punctures and Meshes ................. 98
   5.3 Connections to String Topology .................................. 99

A Appendix 101
   A.1 A General Index Formula for Punctured Disks ......................... 101
   A.2 The Holomorphic Sectional Curvature of the Sasaki Metric .......... 103

Bibliography 109
# List of Figures

1.1 Front and back view of a circle packing having the combinatorics of the icosahedron. 2
1.2 Boundary and asymptotic conditions for punctured disks. 7

2.1 A complete Riemannian manifold with bounded curvature and zero injectivity radius. 31

3.1 A family of boundary curves on $S^2$. 54

4.1 A slit domain with corresponding weighted source tree. 78
4.2 A Morse flow tree on $S^2$. 83
4.3 Partition of a slit domain. 90

5.1 A capped domain with three ends. 97
5.2 A refinement step for a triple of circles. 98
Chapter 1

Introduction and Main Results

1.1 J-holomorphic Curves in Symplectic Geometry

J-holomorphic curves had a profound impact on symplectic geometry since their introduction by M. Gromov in [Gro85]. As a probe exploring the structure of a symplectic manifold, they form the basic ingredient in order to define powerful tools like Floer Homology, Quantum Cohomology or Symplectic Field Theory.

Given a symplectic manifold $(M, \omega)$, a J-holomorphic or pseudoholomorphic curve consists of a $(j, J)$-holomorphic map $u : \Sigma \to M$ from a Riemann surface $(\Sigma, j)$ into $M$. Here $J$ denotes an almost complex structure which by definition is an automorphism of the tangent bundle $TM$ satisfying $J^2 = -\text{Id}$. In addition, $J$ satisfies a compatibility or taming condition, implying that in regions where $u$ is injective its image must be a 2-dimensional symplectic submanifold. In this way $J$-curves are closely tied to the symplectic geometry of $M$. A key feature of the Cauchy-Riemann equation

$$du \circ j = J \circ du \quad (1.1)$$

is that its linearization leads to a Fredholm problem. In many cases it can be shown that for generic choice of $J$ the linearized operator $D_u$ is surjective, hence J-holomorphic curves occur in finite-dimensional families with dimension given by the Fredholm index. In particular, by putting additional constraints such as marked points or factoring out the automorphism groups on the domain, it becomes possible to define finite counts of isolated $J$-curves. These counts can be used to define various new differentials on chain complexes and algebraic product structures culminating in the invariants mentioned above.

Notably, in the case where $(\Sigma, j)$ is the Riemann sphere $\mathbb{C}P^1 = S^2$ and $(M, \omega)$ a closed symplectic manifold of dimension $2n$, the moduli space $\mathcal{M}(A, J)$ of J-holomorphic spheres in prescribed homology class $A \in H_2(M, \mathbb{Z})$ is a smooth manifold of dimension $2n + 2c_1(A)$. Here $c_1$ denotes the first Chern class of the complex vector bundle $(TM, J)$. In order to compactify this space one needs to factor by the noncompact 6-dimensional reparametrization group $PSL(2, \mathbb{C})$ of conformal automorphisms on the sphere. Gromov invented a standard way to compactify this quotient space which then forms the crucial ingredient in the definition of the Gromov-Witten invariants.
1.2 Circle Packing

Coincidentally in the same year that Gromov published his seminal work on $J$-holomorphic curves, William Thurston conjectured in a talk a new approach to discretize complex analysis. In his view, packings of circles could be used to provide a discrete version of the Riemann mapping theorem which contains the continuous setting in the limit. The statement is today known as the Rodin-Sullivan theorem and was established two years after Thurston’s talk. The key idea behind this was that an analytic function $f : \mathbb{C} \to \mathbb{C}$ maps a small circle $z_0 + re^{it}$ infinitesimally to the circle $f(z_0) + f'(z_0)r e^{it}$. The quantization then consists in replacing infinitesimal small circles by real circles with prescribed tangencies. These packings of circles mimic the behavior of analytic functions in a very faithful way. A nice introduction into the subject is given in [Ste05].

While defined for arbitrary Riemann surfaces, we will just consider circle packings on the Riemann sphere $\mathbb{C}P^1 = S^2$.

**Definition 1.1.** Let $K$ be a simplicial 2-complex which is equivalent to a triangulation of the oriented 2-sphere, namely a combinatorial 2-sphere. A collection $C = \{c_v\}$ of circles on $S^2$ is said to be a circle packing for $K$ if the following conditions are satisfied:

(i) There is a 1-1 correspondence $v \mapsto c_v$ between circles of $C$ and the vertices $v$ of $K$.

(ii) Two circles $c_u$, $c_v$ are externally tangent whenever $(u, v)$ is an edge of $K$.

(iii) Three circles $c_u$, $c_v$, $c_w$ form a positively oriented triple on $S^2$ whenever $(u, v, w)$ forms a positively oriented face of $K$.

To each circle packing we associate its carrier $\text{carr}(C)$, the geometric 2-complex formed by connecting the centers of neighboring circles by geodesic segments. One may think of the carrier as a geometric realization of the 2-complex $K$. Each vertex is thus identified with the center of the corresponding circle, each edge with the geodesic between the centers and so on. A circle packing $C$ is called univalent or locally univalent if $\text{carr}(C)$ provides an embedding or immersion, respectively, of $K$ in $S^2$.

![Figure 1.1: Front and back view of a circle packing having the combinatorics of the icosahedron.](image)

Each univalent circle packing induces a circle packing domain on $S^2$ consisting of the various closed disks bounded by the circles. In the following the open triangles between the circles are referred to as interstices.

The study of circle packings in the context of conformal mappings was already initiated in the 1930s by Paul Koebe, but dropped out of sight until Thurston brought it up to light again.
1.3. Moduli Spaces of Piecewise $J$-Holomorphic Curves

The fundamental existence and uniqueness result of univalent circle packings on the sphere was first established by Koebe in [Koe36]. Independent proofs were found later by Andreev and Thurston. For an elementary approach see chapters 6, 7 in [Ste05].

**Theorem 1.2.** Let $K$ be a combinatorial 2-sphere. Then there exists a univalent circle packing $\mathcal{C}_K$ for $K$ on $S^2$. Moreover, $\mathcal{C}_K$ is unique up to conformal automorphisms.

Any conformal automorphism, i.e. any M"{o}bius transformation $\sigma \in \text{Aut}(\mathbb{C}P^1) = \text{PSL}(2, \mathbb{C})$, 

$$\sigma(z) = \frac{az + b}{cz + d} \quad \text{with} \quad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1$$ \hspace{1cm} (1.2)

maps circles to circles and preserves tangencies between them.

### 1.3 Moduli Spaces of Piecewise $J$-Holomorphic Curves

In this work we study moduli spaces of piecewise $J$-holomorphic curves on triangulated or circle packing domains. The main scheme is to consider a subdivision of $S^2$ into a collection of small domains $\{\Delta_j\}$ and to study collections of maps $\{u_j\}$ into a symplectic manifold $(M, \omega)$ which are coupled by Lagrangian boundary conditions and such that each piece $u_j : \Delta_j \to M$ is $J$-holomorphic.

It should be seen as a 2-dimensional analogue of the finite-dimensional path space approximation. Namely, given a Riemannian manifold $(Q, g)$, denote by $\Omega_{p,q}$ the set of piecewise $C^\infty$-paths connecting two points $p, q \in Q$. The subset $\Omega^c_{p,q} \subset \Omega_{p,q}$ then consists of paths having energy less than $c$. For some subdivision $0 = t_0 < t_1 < \ldots < t_k = 1$ of the unit interval, let $\Omega_{p,q}(t_0, t_1, \ldots, t_k) \subset \Omega_{p,q}$ be the subspace of paths $\gamma : [0, 1] \to Q$ satisfying

(i) $\gamma(0) = p$ and $\gamma(1) = q$,

(ii) $\gamma|_{[t_{i-1}, t_i]}$ is a geodesic for $i = 1, \ldots, k$.

Finally, let

$$\Omega^c_{p,q}(t_0, \ldots, t_k) = \Omega^c_{p,q} \cap \Omega_{p,q}(t_0, \ldots, t_k)$$ \hspace{1cm} (1.3)

be the subset of piecewise geodesic paths with energy less than $c$. We summarize its properties:

- $\Omega^c_{p,q}(t_0, \ldots, t_k)$ carries the structure of a smooth finite-dimensional manifold whenever the subdivision is sufficiently fine.

- All critical points of the restriction of the energy $E$ to $\Omega^c_{p,q}(t_0, \ldots, t_k)$ coincide with critical points of $E$ in $\Omega_{p,q}$. In particular, $\Omega^c_{p,q}$ deformation retracts onto $\Omega^c_{p,q}(t_0, \ldots, t_k)$.

In that sense $\Omega^c_{p,q}(t_0, \ldots, t_k)$ yields a finite-dimensional approximation to $\Omega^c_{p,q}$. For proofs of these results we refer to Lemma 16.1 and Theorem 16.2 in [Mil69].

In order to pass from piecewise geodesics to piecewise $J$-holomorphic curves, the following issues have to be resolved:

(A) We require local existence and uniqueness results to hold true on a small scale. In particular, the pseudoholomorphic pieces are solutions of an index zero Fredholm problem.
(B) We need to choose suitable Lagrangians to serve as boundary conditions. For this purpose, it is helpful to choose reasonable simple target manifolds \((M, \omega)\) where Lagrangian submanifolds are well-understood.

(C) There is a suitable notion of refinement which admits passing to the case in (A).

Note that (A) reflects the local existence and uniqueness of geodesics. For a nice class of target manifolds we consider tangent bundles of Riemannian manifolds and symplectizations of unit tangent bundles. Via polarization they provide a rich set of Lagrangians which can be used to define appropriate boundary value problems for the \(J\)-holomorphic pieces. Finally, (C) addresses the issue of replacing the unit interval \([0, 1]\) by a Riemann surface domain such as \(S^2\). The most natural answer here would be a triangulation. However, the circle packing domains from the previous section should be contemplated as well. They have the considerable advantage of being invariant under the action of \(\text{PSL}(2, \mathbb{C})\).

A large part of our work deals with existence theory. Further points include compactness issues for the piecewise moduli spaces. Once overcome, global questions such as combinatorial refinement and the quality of the approximation can be attacked. We start by defining appropriate triangulations.

**Definition 1.3.** Fix a conformal metric \(g\) on a Riemann surface \((\Sigma, j)\). A triangular domain with angles \(\alpha_1, \alpha_2, \alpha_3\) on \(\Sigma\) is a domain bounded by a geodesic triangle with respect to \(g\) having interior angles \(\alpha_1, \alpha_2, \alpha_3 \in (0, \pi)\). A triangulation then consists of a subdivision of \(\Sigma\) into triangular domains.

Note that the carrier of any univalent circle packing on \(S^2\) provides a triangulation with respect to the round metric. In the next step we consider functions on a polytopal 2-complex into a Riemannian manifold such that the image of the vertex set is sufficiently close.

**Definition 1.4.** Let \(K\) be a polytopal 2-complex and \((Q, g)\) a Riemannian manifold. Denote the set of vertices of \(K\) by \(V\). A function \(\Phi : V \rightarrow Q\) is called meshed if for any pair of adjacent vertices \(v_i, v_j \in V\) the image point \(\Phi(v_j)\) lies within the injectivity radius of \(\Phi(v_i)\).

The reason why we use polytopal rather than simplicial 2-complexes is that later on we like to consider meshed functions also on the dual of a simplicial 2-complex. A meshed function \(\Phi\) on the vertex set \(\{v_1, \ldots, v_n\}\) of a triangulation can be canonically extended to its 1-skeleton by requiring that the geodesic edge connecting \(v_i\) and \(v_j\) is mapped totally geodesically to the unique geodesic connecting \(\Phi(v_i)\) and \(\Phi(v_j)\) within the injectivity radius of \(\Phi(v_i)\). In other words, whenever \(v_i\) and \(v_j\) are vertices of a triangular domain \(F_k\) and \(\gamma_{ij} : [0, 1] \rightarrow \Sigma\) is the geodesic edge connecting \(v_i\) and \(v_j\), then we expect \(\Phi \circ \gamma_{ij}\) to be the (locally) unique geodesic in \(Q\) connecting \(\Phi(v_i)\) and \(\Phi(v_j)\). We denote the restriction of the described extension to \(\partial F_k\) by \(\gamma_{ik}\).

While defined for arbitrary Riemann surfaces, we will restrict in the following to triangulations of \(S^2\) with respect to the round metric. Suppose such a triangulation consists of vertices \(v_1, \ldots, v_n\) and faces \(F_1, \ldots, F_m\). Since in any combinatorial 2-sphere twice the number of edges equals \(3m\), by invoking the formula for the Euler characteristic on \(S^2\)

\[
m + n = \# \text{edges} + 2,
\]

one easily obtains the relation \(m = 2n - 4\). We consider the following moduli space of piecewise lifted \(J\)-holomorphic curves.
1.3. MODULI SPACES OF PIECEWISE J-HOLOMORPHIC CURVES

Definition 1.5. Fix a triangulation of $S^2$ with vertex set $V$ and faces $F_1,\ldots,F_m$. Moreover, fix a point $p_k \in F_k$ in the interior of each face. Let further $\pi : TQ \to Q$ be the tangent bundle of a Riemannian manifold $(Q,g)$ and choose an almost complex structure $J$ on $TQ$ tamed by the canonical or magnetic symplectic form $\tilde{\omega}$ or $\tilde{\omega}_e$, respectively. Let $\Phi : V \to Q$ denote a meshed function, canonically extended to the 1-skeleton and fix a smooth section $s \in \Gamma(TQ)$. The moduli space of lifted type associated to $\Phi$ and $s$ is given by

$$\mathcal{M}_\Phi = \left\{ u^k : F_k \to TQ \text{ for } 1 \leq k \leq m : \tilde{\partial}_J u^k = 0, \ u^k(p_k) \in s(Q), \ \pi \circ u^k |_{\partial F_k} = \gamma^k \right\}. \ (1.5)$$

Thus the moduli space consists of piecewise $J$-holomorphic disks defined on the triangular domains. The loops $\gamma^k$ induce parametrized loops of fiberwise Lagrangian boundary conditions in $TQ$ with Maslov index zero. Together with the normalization condition $u^k(p_k) \in s(Q)$, which decreases the index by $n$, we obtain a Fredholm problem of index zero for each triangular face. Considering the meshed function $\Phi$ as parameter yields a finite-dimensional space akin to the path space approximation. The existence result for the lifted pieces holds whenever the image points of the meshed function are sufficiently close to each other. It applies to any almost complex structure tamed by the standard symplectic form $\tilde{\omega} = -d\lambda$ or a magnetic symplectic form $\tilde{\omega}_e$.

Theorem 1.6. Given a triangulation on $S^2$, a closed Riemannian manifold $(Q,g)$, a smooth section $s : Q \to TQ$ and an almost complex structure $J$ on $TQ$ tamed by $\tilde{\omega}$ or $\tilde{\omega}_e$, there is a constant $C > 0$ with the following property. If the distance of the image points $\Phi(v_i)$ and $\Phi(v_j)$ for any edge $e_k = \langle v_i, v_j \rangle$ of the triangulation is less than $C$, then the moduli space of lifted type $\mathcal{M}_\Phi$ is nonempty.

The constant $C$ depends on the choice of almost complex structure, the choice of normalization points $p_k \in F_k$ and the $C^2$-norm of the section $s$. In particular, if one uses the canonical almost complex structure $J_{LC}$ induced by the Levi-Civita connection and varies the metric $g$ on the base, then $C$ depends on bounds of $\nabla^j R$ for $j = 0, 1, 2$. In the proof, which is accomplished in Chapter 3, we apply a quantitative implicit function theorem. It turns out to be crucial to obtain a dependency between $C^1$-bounds of the almost complex structure and the size of the boundary condition. By choosing the latter sufficiently small, that is the image points of $\Phi$ close to each other, the existence of a $J$-holomorphic solution can be guaranteed.

For a heuristic example, let $(Q, \omega)$ be itself a symplectic manifold admitting $J$-holomorphic spheres for some compatible almost complex structure. The induced metric $g_J$ provides a Riemannian structure on $Q$. We may then consider the tangent bundle $TQ$ equipped with the magnetic symplectic form

$$\tilde{\omega}_e = -d\lambda + \varepsilon \pi^* \omega \quad (1.6)$$

with $\varepsilon \neq 0$. After extending $J$ to a compatible almost complex structure $\tilde{J}$ on $(TQ, \tilde{\omega}_e)$, any $J$-holomorphic sphere in $Q$ can now be seen as a $\tilde{J}$-holomorphic sphere in $TQ$ contained in the image of the zero section $s_0 : Q \to TQ$. One may then expect to $C^0$-approximate these spheres by piecewise $\tilde{J}$-holomorphic curves in $\mathcal{M}_{s_0}^\Phi$ with suitable choices of $\Phi$ and by successive refinements of the triangulation.

We further consider compactness properties of the piecewise moduli space of lifted type. This is interesting since the tangent bundle as well as the fiberwise boundary condition are noncompact. Moreover, the canonical almost Kähler structure on $TQ$ consisting of the canonical 2-form $\tilde{\omega} = -d\lambda$, an almost complex structure $J_{LC}$ induced by the Levi-Civita connection of $g$ and a Riemannian metric $\tilde{g}$ known as the Sasakian metric exhibits unbounded geometry. We establish a new result which controls the growth of curvature and injectivity radius.
Theorem 1.7. Let \((Q, g)\) be a closed Riemannian manifold. Then there is a constant \(C > 0\) such that the sectional curvature of \((TQ, \tilde{g})\) satisfies
\[
\sup \left\{ |\tilde{K}_{sec}(\xi)| : \xi \in TQ \quad \text{with} \quad \|\xi\| \leq \rho \right\} \leq C(\rho^2 + 1) \tag{1.7}
\]
and the injectivity radius decreases according to
\[
\inf \left\{ \rho(\xi) : \xi \in TQ \quad \text{with} \quad \|\xi\| \leq \rho \right\} \geq \frac{1}{C\rho}. \tag{1.8}
\]

The nontrivial part is the estimate of the injectivity radius which involves a volume comparison technique due to Cheeger, Gromov and Taylor ([CGT82]). By employing similar arguments, we further fix a gap in the proof of the statement that (magnetic) tangent bundles are geometrically bounded, meaning that there exists a compatible almost complex structure such that the induced metric has bounded sectional curvature and injectivity radius. This is a prerequisite to show that symplectic homology or Rabinowitz-Floer homology is well-defined which may then be used to prove existence results for periodic orbits, see [CGK04] or [Mer11]. Theorem 1.7 is exploited in section 3.5 in order to deduce diameter and gradient bounds for the lifted pieces in the Levi-Civita case.

We will next consider a second type of moduli space which will be defined on a circle packing domain of a univalent circle packing \(C_K\) on \(S^2\) rather than on a triangulation. At all points where circles are tangent to each other the domain will contain boundary punctures where corresponding \(J\)-holomorphic disks are asymptotic to certain Reeb chords. Let us briefly recall the notion.

Definition 1.8. Let \(\Sigma\) be a Riemann surface with boundary puncture \(z'\) and \(Y\) a contact manifold with Legendrian submanifold \(L\). Further let \(c\) be a Reeb chord of length \(T\) with endpoints in \(L\).

\begin{itemize}
\item A map \(f : (\Sigma, \partial\Sigma) \to (Y \times \mathbb{R}, L \times \mathbb{R})\) is positively asymptotic to \(c\) at \(z'\) if \(f = (u, a)\) satisfies:
\begin{enumerate}
\item \(\lim_{z \to z'} a(z) = +\infty\)
\item In holomorphic polar coordinates \((\rho, \theta)\) centered at \(z'\) such that \(\theta \in [-\pi, 0]\) and within \([-\pi, 0]\) along \(\partial\Sigma\) one has
\[
\lim_{\rho \to 0} u(\rho, \theta) = c \left( -\frac{T}{\pi} \theta \right).
\]
\end{enumerate}

\item A map \(f : (\Sigma, \partial\Sigma) \to (Y \times \mathbb{R}, L \times \mathbb{R})\) is negatively asymptotic to \(c\) at \(z'\) if \(f = (u, a)\) satisfies:
\begin{enumerate}
\item \(\lim_{z \to z'} a(z) = -\infty\)
\item In holomorphic polar coordinates \((\rho, \theta)\) centered at \(z'\) such that \(\theta \in [0, \pi]\) and within \([0, \pi]\) along \(\partial\Sigma\) one has
\[
\lim_{\rho \to 0} u(\rho, \theta) = c \left( \frac{T}{\pi} \theta \right).
\]
\end{enumerate}
\end{itemize}

In order to make sense of this, we consider for a given Riemannian manifold \((Q, g)\) the symplectization of the unit tangent bundle \(SQ \times \mathbb{R}\) as target manifold. The Reeb flow on \((SQ, \lambda)\) coincides with the geodesic flow, thus we actually require disks to be asymptotic to geodesics at boundary punctures. Here \(\lambda\) is shorthand for the restriction of the canonical 1-form \(\lambda\) on \(TQ\)
to $SQ$. On $SQ \times \mathbb{R}$ we consider cylindrical almost complex structures which are $\mathbb{R}$-invariant, restrict to an almost complex structure on the contact hyperplanes and map the Reeb vector field $\overline{X}_R$ onto the symplectization direction $\frac{\partial}{\partial \tau}$.

Let us fix a univalent circle packing $\mathcal{C}_K$ for a combinatorial 2-sphere on $S^2$. We denote the vertices of $K$ by $\{v_1, \ldots, v_n\}$, the edges by $\{e_1, \ldots, e_p\}$ and the faces by $\{f_1, \ldots, f_m\}$. We identify any vertex $v_i$ with its geometric realization in the circle packing which is the center of the circle $c_{v_i}$. We further denote by $\mathcal{D}_i^0$ the boundary punctured disk which is obtained by removing all points of tangency of the bounding circle $c_{v_i}$ with its neighboring circles. $\mathcal{D}_i^0$ is invariant under the action of $\text{PSL}(2, \mathbb{C})$ on the circle packing $\mathcal{C}_K$. For the boundary condition, we associate points in $Q$ to the interstices of $\mathcal{C}_K$. Note that the latter are in 1-1 correspondence to faces of $K$. Each boundary component of $\mathcal{D}_i^0$ is then mapped to the fiberwise Lagrangian $S_{q_i}Q \times \mathbb{R}$ where $q_i \in Q$ denotes the point associated to the adjacent interstice. At a boundary puncture with neighboring boundary components mapped to $S_{q_i}Q \times \mathbb{R}$ and $S_{q_j}Q \times \mathbb{R}$ the disk is required to be asymptotic to the locally unique geodesic $\overline{q_i q_j}$. For this we have to ensure that $q_i$ and $q_j$ are sufficiently close to each other.

![Figure 1.2: Boundary and asymptotic conditions for punctured disks.](image)

In order to make this precise, let

$$\beta^i : \partial \mathcal{D}_i^0 \to \{f_1, \ldots, f_m\}$$

be the locally constant function which assigns each boundary component of $\mathcal{D}_i^0$ the face $f_j$ corresponding to the adjacent interstice in the circle packing. We next prescribe the geodesics to which disks converge and the signs of the punctures. For this we choose a meshed function $\Phi : \{f_1, \ldots, f_m\} \to Q$ on the dual 2-complex of $K$. Here we identify dual vertices with faces of $K$. For the signum of convergence we fix a binary function

$$\text{sign} : \{\overline{e_1}, \ldots, \overline{e_p}\} \to \{+, -\}$$

on the set of directed edges of $K$. We are now prepared to attach a moduli space of piecewise boundary punctured disks.

**Definition 1.9.** Consider a circle packing domain on $S^2$ induced by a univalent circle packing $\mathcal{C}_K$. Let $(SQ \times \mathbb{R}, d(e^z\lambda))$ be the symplectization of the unit tangent bundle equipped with a cylindrical almost complex structure $J$. The moduli space of punctured type associated to a meshed function $\Phi : \{f_1, \ldots, f_m\} \to Q$ and sign as in (1.10) is given by

$$\mathcal{M}_{\Phi, \text{sign}} = \left\{ u^i : \mathcal{D}_i^0 \to SQ \times \mathbb{R} : \partial_J u^i = 0, \ u^i(z) \in S_{\Phi_0 \beta(z)}Q \times \mathbb{R} \text{ for } z \in \partial \mathcal{D}_i^0 \right\}.$$
Here $1 \leq i \leq n$ and we additionally require that $\pi_\mathbb{R} \circ u^i(v_i) = 0$ and $u^i$ converges at any boundary puncture corresponding to an edge $(f_k, f_l)$ of the dual complex to the locally unique geodesic connecting $\Phi(f_k)$ and $\Phi(f_l)$. The convergence is positive or negative depending on $\text{sign}(\overrightarrow{e_j})$ where $\overrightarrow{e_j}$ denotes the directed edge starting at $v_i$ and bounding $f_k$ and $f_l$.

Roughly speaking, the moduli space of punctured type is defined on $S^2$ with open interstices of $\mathcal{C}_K$ and points of tangencies removed. The piecewise problem is again Fredholm with index given as follows.

**Theorem 1.10.** Denote the number of negative punctures of the disk $u^i$ by

$$\# \text{neg}(i) = \# \left\{ j : \overrightarrow{v_i v_j} \text{ is an oriented edge of } K \text{ such that } \text{sign}\left(\frac{v_i}{v_j}\right) = -1 \right\}. \quad (1.12)$$

Then the Fredholm index of $u^i$ is given by

$$\text{index } D_{u^i} = (\dim(Q) - 1) (1 - \# \text{neg}(i)). \quad (1.13)$$

In particular, we obtain an index zero Fredholm problem if the piecewise punctured disk $u^i$ contains exactly one negative puncture. A binary function sign : $\{\overrightarrow{e_1}, \ldots, \overrightarrow{e_M}\} \to \{+,-\}$ satisfying $\# \text{neg}(i) = 1$ for $1 \leq i \leq n$ will be called balanced. In this case we derive an existence result for the punctured pieces, similar as for the moduli space of lifted type. It is established only for a particular cylindrical almost complex structure.

**Theorem 1.11.** Given a circle packing domain on $S^2$ and a closed Riemannian manifold $(Q, g)$, there exists a constant $C > 0$ and a cylindrical almost complex structure $J$ on $SQ \times \mathbb{R}$ with the following property. If $J$ is balanced and the distance of the image points $\Phi(f_k)$ and $\Phi(f_l)$ for any edge $(f_k, f_l)$ is less than $C$, then the moduli space of punctured type $\mathcal{M}^\circ_{\Phi, \text{sign}}$ is nonempty.

The proof of Theorem 1.11 strongly takes advantage of the work of Ekholm, who established connections between boundary punctured $J$-holomorphic disks in 1-jet bundles and so-called Morse flow trees in [Ekh07]. The argument, which is accomplished in Chapter 4, involves translating the fiberwise Lagrangians $S_q, Q \times \mathbb{R}$ into the 1-jet bundle picture via the strong contactomorphism $\mathbb{S}^n \cong TS^{n-1} \times \mathbb{R}$. On the way, we also deduce the index formula in Theorem 1.10. We then study the moduli space of Morse flow trees in the translated picture and obtain a boundary punctured $J$-holomorphic disk in the vicinity of such a tree.

Besides the finite-dimensional family of meshed functions, the moduli space of lifted type depends additionally on the choice of the section $s$ and the points $p_k \in \mathcal{F}_k$ in the interior of each face of the triangulation. For the moduli space of punctured type one has to pick the balanced binary function sign. How to make these choices canonical is a delicate question and will be further pursued in the final chapter of this work.

### 1.4 Organization

Chapter 2 contains a survey on the geometry of tangent bundles of Riemannian manifolds, both from the viewpoint of Riemannian and symplectic geometry. Most of the results in there are known, such as the construction of integrable adapted complex structures. The canonical almost Kähler structure $(\tilde{\omega}, J_{\mathcal{LC}}, \tilde{g})$ is studied in detail. We particularly focus on the interplay between the geometry of $TQ$ and the geometry of its base. For instance, section 2.7 studies the relationship between $J_{\mathcal{LC}}$-holomorphic disks in $TQ$ and harmonic maps into $Q$. In general, $(TQ, \tilde{g})$ lacks special Riemannian geometric properties (Theorem 2.2.11) and we prove the estimates in Theorem 1.7.
1.4. ORGANIZATION

For a given curve $\gamma : S^1 \to Q$ and a section $s \in \Gamma(TQ)$, chapter 3 is devoted to the moduli space of lifted disks:

$$\mathcal{M}^s_\gamma = \{ u : \mathbb{D} \to TQ : \bar{\partial}_J u = 0, \ u(0) \in s(Q), \ \pi \circ u \mid_{\partial \mathbb{D}} = \gamma \}.$$  

(1.14)

An implicit function theorem is used to show existence results when $\gamma$ is sufficiently small in $W^{1,p}$ for some $p > 1$. Instead of interpolating the boundary condition we vary $J$ and the main difficulty is to ensure that any tamed almost complex structure $J$ can be reached by choosing $\gamma$ small. From there it is easy to deduce Theorem 1.6. The final section 3.5 considers compactness properties of $\mathcal{M}^s_\gamma$. By taking advantage of the growth rates in Theorem 1.7, we deduce diameter bounds for $J_{LC}$-holomorphic lifted disks with given energy. Here the standard argument using convexity fails by the noncompactness of the fiberwise boundary condition.

Chapter 4 is devoted to the study of punctured disks in $SQ \times \mathbb{R}$. We first establish the index formula in Theorem 1.10 by calculating the relative Maslov index of the Lagrangian boundary loop in section 4.2. The existence result for the boundary punctured pieces in Theorem 1.11 is shown by applying an implicit function theorem in the vicinity of a Morse flow tree. Such connections between Morse theory and holomorphic disks in tangent bundles were first exploited in the setting of Lagrangian Floer homology in [Flo89] and [FO97]. The appropriate concept for Legendrian contact homology of 1-jet bundles was developed by Ekholm in [Ekh07] where it is shown that the count of boundary punctured pseudoholomorphic disks may be replaced by a count of rigid Morse flow trees. Our argument is based on this work of Ekholm, but adjustments are necessary due to the prescribed conformal structure on the domain stemming from the prescription of a circle packing on $S^2$.

Chapter 5 addresses directions for further research revolving mainly around how the choices of punctures and meshed functions can be eliminated in the moduli spaces. We also show up connections between disks of lifted and punctured type as well as connections to string topology.
Chapter 2
On the Geometry of the Tangent Bundle

2.1 Overview

We survey the geometry of tangent bundles $TQ$ of Riemannian manifolds $(Q, g)$. Section 2.2 introduces the canonical almost Kähler structure $(\tilde{\omega}, J_{LC}, \tilde{g})$ with $\tilde{g}$ known as the Sasaki metric. We consider the interaction of Riemannian geometric properties of $(TQ, \tilde{g})$ and its base manifold $(Q, g)$. There will be an abundant list of properties \((\ast)\) such that

\[ \tilde{g} \text{ satisfies } (\ast) \implies g \text{ is flat.} \]

Section 2.3 expresses the geometric quantities in local coordinates. By calculating the Reeb vector field on the unit tangent bundle $SQ$, we see that the geodesic flow coincides with the Reeb dynamics. Various other classes of (almost) complex structures and symplectic forms are reviewed in section 2.4. In particular, we introduce magnetic tangent bundles and adapted complex structures. In section 2.5 we turn back to the Sasaki metric and prove the growth result in Theorem 1.7. We further discuss the geometric boundedness of (magnetic) tangent bundles from a symplectic viewpoint. Section 2.6 introduces $J$-holomorphic curves. The fundamental energy identity is stated and Cauchy-Riemann equations are expressed for $J_{LC}$. Furthermore, in section 2.7 we calculate the tension field of the projection of $J_{LC}$-holomorphic disks in $Q$, which measures the degree of harmonicity.

2.2 The Sasaki Metric

We construct the natural metric on the tangent bundle of a Riemannian manifold and survey Riemannian geometric properties of it. In particular, we express its curvature in terms of the curvature of the base. Together with the metric there comes a natural almost complex structure which turns the tangent bundle into an almost Kähler manifold. From a different viewpoint, the tangent bundle can thus be seen as a noncompact symplectic manifold with a convex end.

Let $(Q, g)$ be an $n$-dimensional Riemannian manifold without boundary and $\pi : TQ \to Q$ its tangent bundle of dimension $2n$. Via the metric $g$ the tangent bundle can be identified with the cotangent bundle $T^*Q$. The isomorphism $\Phi : TQ \to T^*Q$ of vector bundles is given by

\[ \Phi(X)(Y) = g(X, Y). \]  \hspace{1cm} (2.2.1)
With respect to the dual metric

$$\|\eta\| = \sup \{\eta(X) : X \in T_xQ, \ g(X, X) = 1\} \quad \text{for} \ \eta \in T^*_xQ$$

\(\Phi\) becomes a fiberwise isometry. More importantly, if \(\nabla\) denotes the Levi-Civita connection on the tensor bundle of \(Q\), we have for vectors field \(X, Y \in \Gamma(TQ)\) and the dual 1-form \(\eta \in \Gamma(T^*Q)\) of \(\nabla\)

$$(\nabla_X\eta)(Z) = X(\eta(Z)) - \eta(\nabla_X Z) = X(g(Y, Z)) - g(Y, \nabla_X Z) = g(\nabla_X Y, Z).$$

Hence the following diagram commutes:

\[
\begin{array}{ccc}
\Gamma(TQ) & \xrightarrow{\nabla_X} & \Gamma(TQ) \\
\downarrow{\Phi} & & \downarrow{\Phi} \\
\Gamma(T^*Q) & \xrightarrow{\nabla_X} & \Gamma(T^*Q)
\end{array}
\]

In the sequel we will identify the cotangent bundle with the tangent bundle via \(\Phi\). This lifts to an identification \(T(T^*Q) \cong T(TQ)\) and we denote by \(\bar{\pi} : T(TQ) \to TQ\) the canonical projection.

**Remark 2.2.1.** In the following we will work frequently with the Riemannian manifold \(Q\) and the associated tangent bundles \(TQ\) and \(T(TQ)\). Our notational convention will be that we denote points on \(Q\) by \(p, q, r\). Greek letters will be used to denote points on \(TQ\) or \(T(TQ)\). Vector fields, i.e. sections into these bundles, will be denoted by \(X, Y, Z\) and \(\bar{X}, \bar{Y}, \bar{Z}\), respectively.

The Levi-Civita connection \(\nabla\) induces a decomposition

$$T_\xi(TQ) = \mathcal{H}_\xi(TQ) \oplus \mathcal{V}_\xi(TQ)$$

(2.2.3)

of the tangent space at \(\xi \in TQ\) into a direct sum of horizontal and vertical subspaces. The 2n-dimensional vertical distribution \(\mathcal{V} \subset T(TQ)\) can be characterized as the kernel of the differential \(d\pi : T(TQ) \to TQ\). The horizontal distribution \(\mathcal{H}\) similarly can be viewed as the kernel of the connection map \(\kappa : T(TQ) \to TQ\). Restricted to \(\bar{\pi}^{-1}(\xi)\) with \(\xi \in TQ\), \(\pi(\xi) = p\) it turns out to be an epimorphism to \(T_pQ\). For a precise definition of \(\kappa\) we pick a sufficiently small neighborhood \(U\) of \(\pi(\xi) = p \in Q\). Let \(\tau : \pi^{-1}(U) \to \pi^{-1}(p)\) denote the parallel transport of \(\eta \in \pi^{-1}(U)\) along the unique geodesic ar in \(U\) from \(\pi(\eta)\) to \(p\). If \(\tilde{\xi} \in \bar{\pi}^{-1}(\xi)\) is represented by a curve \(\gamma : (-\varepsilon, \varepsilon) \to \pi^{-1}(U), \gamma(0) = \tilde{\xi}\), then

$$\kappa(\gamma(t)) = \lim_{t \to 0} \frac{\tau(\gamma(t)) - \xi}{t}. $$

The splitting (2.2.3) induces a unique Riemannian metric \(\bar{g}\) on \(TQ\) by demanding that the projection \(\pi : (TQ, \bar{g}) \to (Q, g)\) becomes a Riemannian submersion. This metric is known as the Sasaki metric and was first introduced in [Sas58] and [Sas62].

**Definition 2.2.2.** For vector fields \(\bar{X}, \bar{Y} \in \Gamma(T(TQ))\) and \(\xi \in TQ\), \(\pi(\xi) = p\) the Sasaki metric is determined by the formula

$$\bar{g}_\xi(\bar{X}_\xi, \bar{Y}_\xi) = g_p(\pi_*\bar{X}_\xi, \pi_*\bar{Y}_\xi) + g_p(\kappa(\bar{X}_\xi), \kappa(\bar{Y}_\xi)).$$

(2.2.4)

To any pair of vectors \(\xi, \eta \in T_pQ\) we may assign the horizontal lift \(\eta^H_\xi \in T_\xi(TQ)\) of \(\eta\) at \(\xi\), being the unique vector such that \(\kappa(\eta^H_\xi) = 0\) and \(\pi_*\eta^H_\xi = \eta\) holds. This yields a map

$$H : \Gamma(TQ) \to \Gamma(T(TQ))$$

\[X \mapsto X^H, \quad (X^H)_\xi = (X_{\pi(\xi)})^H_\xi.\]
In the same fashion we define the vertical lift $\eta^V \in T_\xi(TQ)$ by $\kappa(\eta^V) = \eta$ and $\pi_*\eta^V = 0$. On the level of vector fields this yields a map $V: \Gamma(TQ) \to \Gamma(T(TQ))$, $X \mapsto X^V$.

The covariant derivatives and curvature tensor $\tilde{R}$ of the Levi-Civita connection $\tilde{\nabla}$ on $(TQ, \tilde{g})$ were already computed in Theorem 1 of [Kow71]. For a more recent discussion see section 9.1 in [Bla02].

**Proposition 2.2.3.** For vector fields $X, Y \in \Gamma(TQ)$ and $\xi \in TQ$ satisfying $\pi(\xi) = p$ we have for the covariant derivatives with respect to $\tilde{\nabla}$

\[
\tilde{\nabla}_X Y^V = 0, \tag{2.2.5}
\]

\[
(\tilde{\nabla}_X Y^V)_\xi = \frac{1}{2} (R_p(\xi, Y_p) X_p)_{\xi}^H + (\nabla_X Y)^V_{\xi}, \tag{2.2.6}
\]

\[
(\tilde{\nabla}_X Y^H)_\xi = \frac{1}{2} (R_p(\xi, X_p) Y_p)_{\xi}^H, \tag{2.2.7}
\]

\[
(\tilde{\nabla}_X Y^H)_\xi = (\nabla_X Y)^H_{\xi} - \frac{1}{2} (R_p(X_p, Y_p) \xi)^V_{\xi}. \tag{2.2.8}
\]

The curvature $\tilde{R}$ can be expressed in terms of the curvature $R$ and its covariant derivative $\nabla R$.

**Proposition 2.2.4.** For vector fields $X, Y, Z \in \Gamma(TQ)$ and $\xi \in TQ$ satisfying $\pi(\xi) = p$ the curvature tensor $\tilde{R}$ is determined by

\[
\tilde{R}(X^V, Y^V) Z^V = 0, \tag{2.2.9}
\]

\[
(\tilde{R}(X^V, Y^V) Z^H)_\xi = \left( R_p(X_p, Y_p) Z_p + \frac{1}{4} R_p(\xi, X_p) (R_p(\xi, Y_p) Z_p) \right)^H_{\xi} \tag{2.2.10}
\]

\[
- \left( \frac{1}{4} R_p(\xi, Y_p) (R_p(\xi, X_p) Z_p) \right)^H_{\xi},
\]

\[
(\tilde{R}(X^H, Y^V) Z^V)_\xi = - \left( \frac{1}{2} R_p(Y_p, Z_p) X_p + \frac{1}{4} R_p(\xi, Y_p) (R_p(\xi, Z_p) X_p) \right)^H_{\xi}, \tag{2.2.11}
\]

\[
(\tilde{R}(X^H, Y^V) Z^H)_\xi = \left( \frac{1}{2} \nabla_X R \right)_{\xi} (\xi, Y_p) Z_p \left( \xi, Z_p X_p \right)^V_{\xi} + \left( \frac{1}{4} R_p(\xi, Y_p) Z_p, X_p \right)^V_{\xi} \tag{2.2.12}
\]

\[
+ \left( \frac{1}{2} R_p(X_p, Z_p) Y_p \right)^V_{\xi},
\]

\[
(\tilde{R}(X^H, Y^H) Z^V)_\xi = \frac{1}{2} ((\nabla_X R)_{\xi} (\xi, Z_p) Y_p - (\nabla_Y R)_{\xi} (\xi, Z_p) X_p)_{\xi}^H + (R_p(X_p, Y_p) Z_p)^V_{\xi} \tag{2.2.13}
\]

\[
+ \frac{1}{4} (R_p(\xi, X_p) Z_p, X_p) \xi - R_p(\xi, Z_p X_p, Y_p) \xi^V_{\xi},
\]

\[
(\tilde{R}(X^H, Y^H) Z^H)_\xi = \left( R_p(X_p, Y_p) Z_p + \frac{1}{2} R_p(\xi, R_p(X_p, Y_p) \xi) Z_p \right)^H_{\xi} \tag{2.2.14}
\]

\[
+ \frac{1}{4} (R_p(\xi, R_p(Z_p, Y_p) \xi) X_p + R_p(\xi, R_p(X_p, Z_p) \xi Y_p) \xi_{\xi}^H + \frac{1}{2} ((\nabla_Z R)_{\xi} (X_p, Y_p) \xi)^V_{\xi}.
\]

The vanishing of (2.2.5) and (2.2.9) expresses the fact that fibers of the tangent bundle are totally geodesic and intrinsically flat submanifolds with respect to $\tilde{g}$. The next step is to relate the scalar curvatures $\sigma$ and $\tilde{\sigma}$ of $g$ and $\tilde{g}$, respectively.
**Proposition 2.2.5.** Suppose \( \{e_1, \ldots, e_n\} \subset T_pQ \) is an orthonormal frame. Then the scalar curvature \( \tilde{\sigma} \) at \( \xi \in \pi^{-1}(p) \) is given by the formula

\[
\tilde{\sigma}(\xi) = \sigma(p) - \frac{1}{4} \sum_{i,j=1}^{n} \|R_p(e_i, e_j)\xi\|^2. \tag{2.2.15}
\]

The formula appeared first as Lemma 3.1 in [MT88]. The authors concluded that if \( \tilde{g} \) has constant scalar curvature, then \( g \) must be flat. But in fact, more is possible.

**Corollary 2.2.6.** Suppose \((TQ, \tilde{g})\) has scalar curvature bounded below. Then \((Q, g)\) must be flat.

**Proof.** Let \( \tilde{\sigma} \geq C \) and suppose that \( R \) does not vanish at some point \( p \in Q \). Given an orthonormal basis \( \{e_1, \ldots, e_n\} \subset T_pQ \), we can find \( e_k, e_l, e_m \) such that \( R_p(e_k, e_l)e_m \neq 0 \). Pick \( \lambda \in \mathbb{R} \) such that

\[
\lambda > \frac{2\sqrt{\|\sigma(p)\| + |C|}}{\|R_p(e_k, e_l)e_m\|}.
\]

Then by (2.2.15) we obtain the contradiction

\[
\tilde{\sigma}(\lambda e_m) = \sigma(p) - \lambda^2 \sum_{i,j=1}^{n} \|R_p(e_i, e_j)e_m\|^2 \leq \sigma(p) - \frac{\lambda^2}{4} \|R_p(e_k, e_l)e_m\|^2 < C.
\]

\( \square \)

The distributions \( \mathcal{H} \) and \( \mathcal{V} \) lead to a natural almost complex structure \( J_{LC} \) on the tangent bundle which interchanges them. The map \( J_{LC} : T(TQ) \to T(TQ) \) is characterized by

\[
\pi_* \circ J_{LC} = -\kappa, \quad \kappa \circ J_{LC} = \pi_* . \tag{2.2.16}
\]

It was calculated in section 5 of [Dorn62] that the Nijenhuis tensor

\[
N_{LC}(\tilde{X}, \tilde{Y}) = [\tilde{X}, \tilde{Y}] + J_{LC}[J_{LC}\tilde{X}, \tilde{Y}] + J_{LC}[\tilde{X}, J_{LC}\tilde{Y}] - [J_{LC}\tilde{X}, J_{LC}\tilde{Y}], \quad \tilde{X}, \tilde{Y} \in \Gamma(T(TQ))
\]

associated to \( J_{LC} \) vanishes if and only if \( Q \) is flat. This follows from the relation

\[
(N_{LC}(X^V, Y^V))_\xi = (R_p(X_p, Y_p)\xi)_\xi^H ,
\]

where \( X, Y \in \Gamma(TQ) \) and \( \xi \in TQ \) with \( \pi(\xi) = p \).

On the cotangent bundle \( T^*Q \) there is a canonical 1-form \( \lambda_{can} \) given by

\[
(\lambda_{can})_\eta(\tilde{\eta}) = \eta(\pi_*(\tilde{\eta})) \quad \text{for} \quad \eta \in T^*Q, \quad \tilde{\eta} \in T(T^*Q) . \tag{2.2.17}
\]

Via our isomorphism \( \Phi \) this pulls back to a 1-form \( \tilde{\lambda} = \Phi^*\lambda_{can} \) on \( TQ \). The exact 2-form

\[
\tilde{\omega} = -d\tilde{\lambda} , \tag{2.2.18}
\]

induces an almost Kähler structure on the tangent bundle.

**Proposition 2.2.7.** \( (\tilde{\omega}, J_{LC}, \tilde{g}) \) forms a compatible triple of structures on \( TQ \), i.e.

\[
\tilde{\omega}(\cdot, \cdot) = \tilde{g}(J_{LC}\cdot, \cdot).
\]
2.2. THE SASAKI METRIC

Proof. For $\xi \in TQ$, $\eta \in T(TQ)$ we have $\tilde{\lambda}(\eta) = g(\xi, \pi_\ast \eta)$. Hence for $X, Y \in \Gamma(TQ)$ and $\xi \in TQ$ with $\pi(\xi) = p$ we obtain

$$
-\ddot{\lambda}(X^H, Y^H) = -\left( X^H \dot{\lambda}(Y^H) \right) + \left( Y^H \dot{\lambda}(X^H) \right) + \ddot{\lambda}(X^H, Y^H) = -g_p(\xi, (\nabla X)p) + g_p(\xi, (\nabla Y)p) + g_p(\xi, [X, Y]_p) = 0.
$$

In the second step we used that the vector fields $X^H, Y^H$ are horizontal and $\pi_\ast([X^H, Y^H]) = [X, Y]_p$. A similar calculation in light of the identity $\pi_\ast([X^H, Y^V]) = 0$ yields

$$
-\ddot{\lambda}(X^H, Y^V) = -\left( X^H \dot{\lambda}(Y^V) \right) + \left( Y^V \dot{\lambda}(X^H) \right) + \ddot{\lambda}(X^H, Y^V) = g_p(Y_p, X_p) = (\bar{g}(X^H, Y^V))_p = (\bar{g}(J_{\Gamma}X^H, Y^V))_p
$$

and $-\ddot{\lambda}(X^V, Y^V) = 0$ is quite easy to see.

It was shown in Theorem 3.1 of [Sat07] that if one constructs the Sasaki metric and almost complex structure with respect to an arbitrary affine connection on $(Q, g)$, then the resulting almost Hermitian structure becomes almost Kähler if and only if the form $\bar{g}(J, \cdot, \cdot)$ coincides with $\tilde{\omega}$.

We next provide a formula for the holomorphic sectional curvature of the Sasaki metric. If no confusion arises, we denote both the Sasaki metric $\bar{g}$ and $g$ by $\langle \cdot, \cdot \rangle$. The statement is obtained by a tedious calculation involving the curvature formulas in Proposition 2.2.4. We therefore defer its proof to appendix A.2.

**Theorem 2.2.8.** Suppose $\tilde{Z} \in \Gamma(T(TQ))$ splits into $\tilde{Z} = X^H + Y^V$ with $X, Y \in \Gamma(TQ)$. For $\xi \in TQ$ with $\pi(\xi) = p$ we have

$$
\left\langle \tilde{R}(\tilde{Z}, J_{\Gamma} \tilde{Z})J_{\Gamma} \tilde{Z}, \tilde{Z} \right\rangle_\xi = 4 \langle R(X, Y)Y, X \rangle_p + \frac{1}{4} \| R_p(\xi, X_p)X_p \|^2 + \frac{1}{4} \| R_p(\xi, Y_p)Y_p \|^2 + \frac{1}{2} \| R_p(\xi, X_p)Y_p \|^2 \| \| R_p(\xi, Y_p)X_p \|^2 \| - \frac{1}{4} \| R_p(\xi, X_p)X_p \|^2 \| \| R_p(\xi, Y_p)Y_p \|^2 \| \| R_p(\xi, X_p)Y_p \|^2 \| - \frac{1}{2} \langle R_p(\xi, X_p)X_p, R_p(\xi, Y_p)Y_p \rangle.
$$

(2.2.19)

It was calculated in [Sas58] that geodesics $\hat{c} : [0, 1] \to TQ$ are characterized by the equations

$$
\nabla_{\hat{c}} \hat{c} = R(\hat{c}, \nabla_{\hat{c}} \hat{c}) \hat{c} \quad \text{and} \quad \nabla_{\hat{c}} \nabla_{\hat{c}} \hat{c} = 0,
$$

(2.2.20)

where $c = \pi \circ \hat{c}$ denotes the projected curve. From this, it is easily seen that the image of the zero section $s_0 : Q \to TQ$ is totally geodesic in $TQ$.

**Remark 2.2.9.** Sasaki referred to projections $c = \pi \circ \hat{c}$ of geodesics in $TQ$ as submarine geodesics. Note that if $c : [0, 1] \to Q$ is a geodesic, then $\hat{c} : [0, 1] \to TQ$ is a geodesic as well with respect to $\bar{g}$. Hence any geodesic in $Q$ is a submarine geodesic. However, the converse of this statement is not true in general.

We continue by establishing a result on the injectivity radius of the Sasaki metric.

**Proposition 2.2.10.** Suppose the injectivity radius of the tangent bundle $(TQ, \bar{g})$ satisfies $\bar{\rho} > 0$. Then $(Q, g)$ must be flat.
Proof. Assuming that \((TQ, \tilde{g})\) is not flat allows us to pick a point \(p \in Q\) and tangent vectors \(X_p, Y_p, Z_p \in T_pQ\) such that \(R_p(X_p, Y_p)Z_p \neq 0\). For given \(\delta > 0\) it follows that we can find a small closed loop \(\gamma : [0, 1] \rightarrow Q\) based at \(p\) of length \(\delta\) and \(\lambda > 0\) sufficiently large such that the horizontal lift \(\tilde{\gamma}^H\) of \(\gamma\) at \(\lambda Z_p\) satisfies

\[
\delta < \|\tilde{\gamma}^H(1) - \tilde{\gamma}^H(0)\| = \|\tilde{\gamma}^H(1) - \lambda Z_p\| < 2\delta.
\]

By (2.2.4) the length of \(\tilde{\gamma}^H\) with respect to \(\tilde{g}\) equals \(L(\gamma) = \delta\). Consequently, the linear geodesic in the fiber \(T_pQ\) connecting \(\tilde{\gamma}^H(0)\) and \(\tilde{\gamma}^H(1)\) cannot be globally length minimizing. Therefore the injectivity radius at \(\lambda Z_p\) is bounded above by \(2\delta\). 

If \(Q\) is flat, then the injectivity radius will be positive, but it can become arbitrarily small. For an easy example, take \(Q\) to be a circle of small radius.

The next theorem is an accumulation of results due to many people. In summary, it states that assuming special Riemannian geometric properties for \(\tilde{g}\) will usually imply flatness of the metric.

**Theorem 2.2.11.** Suppose one of the following conditions is satisfied:

(a) \((TQ, \tilde{g})\) is Kähler, i.e. \(J_{LC}\) is integrable.

(b) \((TQ, \tilde{g})\) is locally symmetric, i.e. \(\nabla \tilde{R} = 0\).

(c) \((TQ, \tilde{g})\) has bounded sectional curvature.

(d) \((TQ, \tilde{g})\) has bounded holomorphic sectional curvature.

(e) \((TQ, \tilde{g})\) is Einstein.

(f) \((TQ, \tilde{g})\) has constant scalar curvature.

(g) \((TQ, \tilde{g})\) has scalar curvature bounded below.

(h) \((TQ, \tilde{g})\) is conformally flat.

(i) \((TQ, \tilde{g})\) has injectivity radius bounded from below.

Then \((Q, g)\) and thus also \((TQ, \tilde{g})\) must be flat.

Of course, we have \((c) \Rightarrow (d)\) and \((e) \Rightarrow (f) \Rightarrow (g)\). In chronological order, that \((a)\) implies flatness was shown in [Dom62]. The implication from \((b)\) was done in [Kow71] after computing the formulas for \(\tilde{R}\) for the first time. \((c)\) is attributed to [Aso81], \((f)\) was deduced in [MT88], \((h)\) is contained in [Ban94] and \((d)\) was shown in [PPK95]. While \((f)\) also includes the implication from \((e)\), this was proved in a more general context in [Sat07]. Finally, \((g)\) is Corollary 2.2.6 and \((i)\) is Proposition 2.2.10.

Convexity properties of the tangent bundle can be obtained from the metric Hamiltonian \(\tilde{f} : TQ \rightarrow \mathbb{R}\) given by

\[
\tilde{f}(\xi) = \frac{1}{2} \|\xi\|_{\tilde{g}}^2.
\]
2.3. Expressions in Local Coordinates

Proposition 2.2.12. The function $\tilde{f}$ is plurisubharmonic. This means that $\tilde{f}: TQ \to [0, \infty)$ is a proper smooth function satisfying

$$\omega_f(\tilde{X}, J_{LC}\tilde{X}) \geq 0 \quad \text{with} \quad \omega_f = -d(d\tilde{f} \circ J_{LC})$$

for any $\tilde{X} \in \Gamma(T(TQ))$.

Proof. We refer to (2.3.12) from Lemma 2.3.1 which is proved using local coordinates below. With this we obtain

$$\omega_f = -d(d\tilde{f} \circ J_{LC}) = -d\tilde{\lambda} = \tilde{\omega}.$$ 

Thus $\omega_f(\tilde{X}, J_{LC}\tilde{X}) \geq 0$ follows from Proposition 2.2.7. \qed

The level sets of $\tilde{f}$ are the tangent sphere bundles. In the following, we denote by

$$T_rQ = \{ \xi \in TQ : \|\xi\|_g = r \}.$$  \hspace{1cm} (2.2.22)

the tangent sphere bundle of radius $r > 0$. The unit tangent bundle will be referred to as $SQ = T_1Q$. $(T_rQ, \tilde{\lambda} |_{T_rQ})$ is a contact manifold for any $r > 0$. Its Reeb vector field $\tilde{X}_R$ is uniquely determined by the equations

$$\iota(\tilde{X}_R)\tilde{\omega} |_{T_rQ} = 0 \quad \text{and} \quad \tilde{\lambda}(\tilde{X}_R) = 1.$$ \hspace{1cm} (2.2.23)

A formula for $\tilde{X}_R$ in local coordinates is given in the next section.

2.3 Expressions in Local Coordinates

We like to present the geometric quantities introduced in the previous section in coordinates. In order to keep formulas as simple as possible, we employ the Einstein summation convention.

Suppose $(x^1, \ldots, x^n)$ are local coordinates defined on an open subset $U \subset Q$. Every vector field $X \in \Gamma(TQ)$ can be expressed locally with respect to the basis $\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \}$. This allows to extend the given chart in a canonical way to local coordinates

$$(x^1, \ldots, x^n, \xi^1, \ldots, \xi^n)$$
on $\pi^{-1}(U) \subset TQ$, such that $\xi^i(X) = dx^i(X)$. The connection map $\kappa$ is given in coordinates by

$$\left( a^i \frac{\partial}{\partial x^i} + b^i \frac{\partial}{\partial \xi^i} \right)_{(x^1, \ldots, x^n, \xi^1, \ldots, \xi^n)} \mapsto \left( b^i + \Gamma^i_{jk}(x)a^j \xi^k \right) \frac{\partial}{\partial x^i}_{(x^1, \ldots, x^n)}.$$ \hspace{1cm} (2.3.1)

Written in the canonical coordinates for $TQ$, the latter becomes $(x^1, \ldots, x^n, \eta^1, \ldots, \eta^n)$ with $\eta^i = b^i + \Gamma^i_{jk}(x)a^j \xi^k$ for $1 \leq i \leq n$. The image of $\pi_*$ in this notation is just $(x^1, \ldots, x^n, a^1, \ldots, a^n)$.

The horizontal and vertical lift of $\frac{\partial}{\partial x^i}$ at $\xi$ turn out to be

$$\left( \frac{\partial}{\partial x^i} \right)^H_{(x, \xi)} = \frac{\partial}{\partial x^i} - \Gamma^k_{ij}(x) \xi^j \frac{\partial}{\partial \xi^k}, \quad \left( \frac{\partial}{\partial x^i} \right)^V_{(x, \xi)} = \frac{\partial}{\partial \xi^i}.$$ \hspace{1cm} (2.3.2)

For convenience, we introduce the matrix valued function

$$P : \pi^{-1}(U) \to M_n(\mathbb{R}), \quad P_i^j(x, \xi) = \Gamma^j_{ik}(x) \xi^k.$$ \hspace{1cm} (2.3.3)
If $G = G(x)$ denotes the metric tensor with respect to the basis $\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \}$, then the Sasaki metric $\tilde{G} = \hat{G}(x, \xi)$ with respect to $\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial \xi^1}, \ldots, \frac{\partial}{\partial \xi^n} \}$ can be calculated using (2.2.4) and (2.3.1). We obtain

$$\tilde{G} = \begin{pmatrix} P & I & 0 \\ I & 0 & 0 \\ 0 & G & 0 \end{pmatrix} \begin{pmatrix} P^t & I & 0 \\ I & 0 & 0 \\ 0 & G & 0 \end{pmatrix} = \begin{pmatrix} G + PGP^t & P & G \\ GP^t & I & 0 \\ G & 0 & 0 \end{pmatrix}. \quad (2.3.4)$$

In particular, we have for the volume form

$$\sqrt{\det \tilde{G}(x, \xi)} = \det G(x). \quad (2.3.5)$$

The inverse metric tensor equals

$$\tilde{G}^{-1} = \begin{pmatrix} P^t & I & 0 \\ I & 0 & 0 \\ 0 & G & 0 \end{pmatrix}^{-1} \begin{pmatrix} G^{-1} & 0 & 0 \\ 0 & G^{-1} & 0 \\ 0 & 0 & (I - P)^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & I & 0 \\ I & -P^t & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} 0 & I & 0 \\ I & -P^t & 0 \\ 0 & 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} G^{-1} & -G^{-1}P \\ -P^tG^{-1} & G^{-1} + P^tG^{-1}P \end{pmatrix}. \quad (2.3.6)$$

The natural almost complex structure $J_{LC}$ coincides with respect to the basis (2.3.2) of horizontal and vertical lifts with the standard almost complex structure $J_0$ in $\mathbb{R}^{2n}$. Translating this into the basis $\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial \xi^1}, \ldots, \frac{\partial}{\partial \xi^n} \}$ leads to

$$J_{LC} = \begin{pmatrix} I & 0 \\ -P^t & I \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -P^t & I \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & P^t \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (2.3.7)$$

The 1-form $\tilde{\lambda}$ is given by

$$\tilde{\lambda}(x, \xi) = g_{ij}(x) \xi^j \, dx^i. \quad (2.3.8)$$

Employing the identities $g_{ij,k} = g_{ik} \Gamma^i_{jk} + g_{ji} \Gamma^i_{ik}$ and $\Gamma^k_{ij} \, dx^i \wedge dx^j = 0$, we deduce the formula

$$\tilde{\omega}(x, \xi) = -d\tilde{\lambda}(x, \xi) = -g_{ij,k} \xi^j \, dx^i \wedge dx^k + g_{ij} \xi^j \, dx^i \wedge d\xi^j = g_{ik} \Gamma^i_{jk} \xi^j \, dx^i \wedge dx^k + g_{ji} \Gamma^i_{ik} \xi^j \, dx^i \wedge dx^k + g_{ij} \xi^i \wedge d\xi^j = g_{ik} P^k_j \xi^j \, dx^i \wedge dx^j + g_{ij} \xi^i \wedge d\xi^j. \quad (2.3.9)$$

In matrix notation with respect to the basis $\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial \xi^1}, \ldots, \frac{\partial}{\partial \xi^n} \}$ this becomes

$$\tilde{W} = \begin{pmatrix} GP^t & -PG & G \\ -G & 0 \end{pmatrix}. \quad (2.3.9)$$

Proposition 2.2.7 is now reflected in the identity $\tilde{W} = J_{LC}^t \tilde{G}$.

The metric Hamiltonian $\tilde{f}$ is given in coordinates by

$$\tilde{f}(x, \xi) = \frac{1}{2} g_{jk}(x) \xi^j \xi^k. \quad (2.3.10)$$
2.3. **Expressions in Local Coordinates**

**Lemma 2.3.1.** The Reeb vector field $\tilde{X}_R$ is given in local coordinates by

$$\tilde{X}_R(x, \xi) = \frac{1}{||\xi||^2_g} \left( \xi^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk}(x) \xi^j \xi^k \frac{\partial}{\partial \xi^i} \right).$$

(2.3.11)

In particular, the Reeb flow on $SQ$ is identical with the geodesic flow. Moreover, we have

$$\lambda = d \tilde{f} \circ J_{LC} \quad \text{and} \quad \iota(\tilde{X}_R) \tilde{\omega} = \frac{1}{2} d \left( \log \tilde{f} \right).$$

(2.3.12)

**Proof.** Differentiating (2.3.10) yields

$$d \tilde{f}(x, \xi) = \frac{1}{2} g_{ij,k}(x) \xi^i \xi^j \, dx^k + g_{ij}(x) \xi^i \, d\xi^j.$$  

It suffices to check $\lambda = d \tilde{f} \circ J_{LC}$ on the basis $(\frac{\partial}{\partial x^i})^H, (\frac{\partial}{\partial x^i})^V$. Via (2.3.2) and (2.3.8) we obtain

$$\tilde{\lambda}_{(x, \xi)} \left( \left( \frac{\partial}{\partial x^i} \right)^H \right) = \tilde{\lambda}_{(x, \xi)} \left( \frac{\partial}{\partial x^i} - \Gamma^i_{jk}(x) \xi^j \frac{\partial}{\partial \xi^k} \right) = g_{ij}(x) \xi^j$$

$$= (d \tilde{f})_{(x, \xi)} \left( \frac{\partial}{\partial x^i} \right) = (d \tilde{f})_{(x, \xi)} \left( J_{LC} \left( \frac{\partial}{\partial x^i} \right)^H \right)$$

and

$$\tilde{\lambda}_{(x, \xi)} \left( \left( \frac{\partial}{\partial x^i} \right)^V \right) = \tilde{\lambda}_{(x, \xi)} \left( \frac{\partial}{\partial \xi^i} \right) = 0 = -\frac{1}{2} g_{jk,i}(x) \xi^i \xi^j \xi^k + g_{kl}(x) \Gamma_{ij}^l(x) \xi^i \xi^j \xi^k$$

$$= (d \tilde{f})_{(x, \xi)} \left( \frac{\partial}{\partial \xi^i} - \Gamma^i_{jk}(x) \xi^j \frac{\partial}{\partial \xi^k} \right) = (d \tilde{f})_{(x, \xi)} \left( J_{LC} \left( \frac{\partial}{\partial \xi^i} \right)^V \right).$$

The second identity follows from

$$\left( \iota(2 \tilde{f} \tilde{X}_R) \tilde{\omega} \right)_{(x, \xi)} = \iota \left( \xi^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk}(x) \xi^j \xi^k \frac{\partial}{\partial \xi^i} \right) \left( g_{ij,k}(x) \xi^j \, dx^k \wedge dx^k + g_{ij}(x) \, dx^i \wedge d\xi^j \right)$$

$$= g_{ij,k} \xi^i \xi^j dx^k - g_{jk,i} \xi^i \xi^j dx^k + g_{ij} \xi^i \, d\xi^j + g_{kl} \Gamma^i_{jk}(x) \xi^j \xi^k dx^k$$

$$= g_{ij,k} \xi^i \xi^j dx^k - g_{jk,i} \xi^i \xi^j dx^k + g_{ij} \xi^i \, d\xi^j + \left( g_{jk,i} - \frac{1}{2} g_{ij,k} \right) \xi^i \xi^j dx^k$$

$$= \frac{1}{2} g_{ij,k}(x) \xi^i \xi^j dx^k + g_{ij}(x) \xi^i \, d\xi^j = (d \tilde{f})_{(x, \xi)}.$$  

Consequently, we must have

$$\iota(\tilde{X}_R) \tilde{\omega} = \frac{1}{2f} d \tilde{f} = \frac{1}{2} d \left( \log \tilde{f} \right).$$

Since $T_rQ$ corresponds to a level set of $\tilde{f}$, we obtain $\iota(\tilde{X}_R) \tilde{\omega} |_{T_rQ} = 0$. In order to verify that $\tilde{X}_R$ is indeed the Reeb vector field, we calculate

$$\tilde{\lambda}_{(x, \xi)}(\tilde{X}_R) = \frac{1}{||\xi||^2_g} g_{ij}(x) \xi^i \xi^j = 1.$$  

In particular, the Reeb flow satisfies on $SQ$

$$\dot{x}^i = \xi^i \quad \text{and} \quad \dot{\xi}^i + \Gamma^i_{jk}(x) \xi^j \xi^k = 0$$

which is just a disguise of the geodesic equation. ∎
Last but not least we consider the vector field

\[ \tilde{X}_L = 2\tilde{f} J_{LC} \tilde{X}_R, \]  

locally given by

\[ \left( \tilde{X}_L \right)_{(x, \xi)} = \xi^i \frac{\partial}{\partial \xi^i}. \]  

Taking advantage of the Lemma above, we deduce the property

\[ \mathcal{L}_{\tilde{X}_L} \tilde{\omega} = d \left( \iota(\tilde{X}_L) \tilde{\omega} \right) = d \left( 2\tilde{f} \tilde{\omega}(J_{LC} \tilde{X}_R, \cdot) \right) = -d \left( 2\tilde{f} \tilde{\omega}(\tilde{X}_R, J_{LC} \cdot) \right) = -d\tilde{\lambda} = \tilde{\omega}. \]

Because of this \( \tilde{X}_L \) is also referred to as the Liouville vector field. Note that the Reeb vector field is purely horizontal while the Liouville vector field belongs to the vertical distribution.

### 2.4 Distinguished Complex Structures and Symplectic Forms

Besides the Levi-Civita almost complex structure \( J_{LC} \) there are various other distinguished almost complex structures available. For a different class, we present adapted complex structures which are known to exist in a neighborhood of the zero section on tangent bundles of compact real analytic Riemannian manifolds. In particular, we provide an explicit computation for the unique adapted complex structure on the tangent bundle of the round sphere. Moreover, we consider variations of the standard symplectic form \( \tilde{\omega} \) to obtain what is known as magnetic tangent bundles.

Let us start by describing classes of almost complex structures which can be associated to any symplectic manifold.

**Definition 2.4.1.** Let \((M, \omega)\) be a symplectic manifold. The space of smooth almost complex structures \( \mathcal{J}(M) \) contains the subsets

\[ \mathcal{J}^\varepsilon(M, \omega) = \{ J \in \mathcal{J}(M) \mid \omega(X, JX) > 0 \text{ for any non-zero } X \in \Gamma(TM) \}, \] 

\[ \mathcal{J}(M, \omega) = \{ J \in \mathcal{J}(M) \mid g_J(X, Y) = \omega(X, JY) \text{ is a Riemannian metric} \} \]

of structures tamed by and compatible with \( \omega \), respectively.

For \( J \in \mathcal{J}^\varepsilon(M, \omega) \) we have the corresponding metric

\[ g_J(X, Y) = \frac{1}{2} (\omega(X, JY) + \omega(Y, JX)). \]  

According to Lemma 4.15 in [MS98], a compatible almost complex structure \( J \in \mathcal{J}(M, \omega) \) is integrable if and only if \( J \) is covariantly constant with respect to the Levi-Civita connection induced by \( g_J \). The following proposition provides an identity involving \( \nabla J \). It is part of Lemma C.7.1 in [MS04]. While the second equality in this identity just follows from covariantly differentiating \( J^2 = \text{Id} \), the first equality indeed relies on the fact that \( \omega \) is closed and \( J \) an isometry.

**Proposition 2.4.2.** Let \((M, \omega)\) be a symplectic manifold and \( J \in \mathcal{J}(M, \omega) \). Denote the Levi-Civita connection of \( g_J \) by \( \nabla \). Then for \( X \in \Gamma(TM) \) one has

\[ (\nabla_{JX} J) = -J (\nabla_X J) = (\nabla_X J) J. \]  

2.4. DISTINGUISHED COMPLEX STRUCTURES AND SYMPLECTIC FORMS

An almost complex structure turns $TM$ into a complex vector bundle. Since each of $\mathcal{J}^+(M, \omega)$ and $\mathcal{J}(M, \omega)$ is contractible within $\mathcal{J}(M)$ (see Proposition 2.50 and 2.51 in [MS98]), there are well-defined Chern classes

$$c_j(M, \omega) \in H^{2j}(M; \mathbb{Z}) \quad \text{for} \quad 1 \leq j \leq \frac{\dim M}{2}. \quad (2.4.5)$$

In particular, the first Chern class will play a prominent role in index formulas for associated $J$-holomorphic curves later on.

**Proposition 2.4.3.** Let $(Q, g)$ be an $n$-dimensional Riemannian manifold. Then

$$2c_j(TQ, \bar{\omega}) = 0 \quad (2.4.6)$$

holds for odd $1 \leq j \leq n$.

**Proof.** The zero section $s_0 : Q \rightarrow TQ$ induces an isomorphism on cohomology. Hence it suffices to consider the odd Chern classes of the pullback $s_0^*(TQ)$. For $\xi = s_0(p)$ with $p \in Q$ we have the splitting

$$T_{\xi} (TQ) = \mathcal{H}_\xi(TQ) \oplus \mathcal{V}_\xi(TQ) = T_pQ \oplus T_pQ$$

and the action on the right hand side of the compatible almost complex structure $J_{LC}$ is according to (2.3.7) given by

$$J_{LC} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$ 

Now following p. 78 in [MS74] we see that $(TQ \oplus TQ, J_{LC})$ is conjugate equivalent to itself. Therefore its total Chern class satisfies

$$\sum_{j=1}^n c_j(TQ \oplus TQ, J_{LC}) = c(TQ \oplus TQ, J_{LC})$$

$$= c(TQ \oplus TQ, -J_{LC}) = \sum_{j=1}^n (-1)^j c_j(TQ \oplus TQ, J_{LC}),$$

implying $2c_j(TQ \oplus TQ, J_{LC}) = 0$ for odd $1 \leq j \leq n$. \qed

We have seen in Section 2.2 that $J_{LC}$ lacks integrability if the base manifold $(Q, g)$ is curved. From there it seems natural to ask whether it is always possible to put a Kähler structure on $TQ$ by using a different integrable complex structure. The answer is affirmative. One way to prove this is via real algebraic geometry, another possibility is Eliashberg’s existence theorem for Stein structures (see [Eli90]). However, the output of these powerful machines are complex structures on $TQ$ which cannot be formally expressed. For a compromise, we like to describe so-called adapted complex structures which allow a geometric characterization. The prize one has to pay is that their definition is constrained to a tubular neighborhood of the zero section.

The geometric significance of adaptedness is that complex structures with this property preserve a certain foliation in the tangent bundle. To make this precise, let us denote by $sc_\lambda : TQ \rightarrow TQ$ with $\lambda \in \mathbb{R}$ the fiber scaling function, given by

$$sc_\lambda(x, \xi) = (x, \lambda \xi) \quad (2.4.7)$$

in local coordinates. Moreover, we denote the tangent disk bundle of radius $r > 0$ by

$$T_{\leq r}Q = \{ \xi \in TQ : \|\xi\|_g \leq r \}. \quad (2.4.8)$$
Definition 2.4.4. Let \((Q, g)\) be a complete Riemannian manifold and \(s_0 : Q \to TQ\) the zero section in the tangent bundle. To any geodesic \(\gamma : \mathbb{R} \to Q\) we may associate the immersion 
\[
\psi_\gamma(s + it) = sc_t(\gamma(s)).
\] (2.4.9)

For varying geodesics, the images \(\psi_\gamma(C \setminus \mathbb{R})\) define a smooth foliation of \(TQ \setminus s_0(Q)\) known as the Riemann foliation. Its leaves carry a complex structure obtained by a pushforward of the standard complex structure on \(C\) via \(\psi_\gamma\). A complex structure on the tangent disk bundle \(T_{\leq} Q\) is called adapted if the leaves of the Riemann foliation are complex submanifolds.

The next theorem summarizes existence and uniqueness results for adapted complex structures (Theorem 2.2 and Theorem 2.5 in [Sze91]).

Theorem 2.4.5. Let \((Q, g)\) be a real analytic Riemannian manifold.

(a) If \(Q\) is compact, then there is \(\varepsilon > 0\) such that \(T_{\leq \varepsilon} Q\) carries an adapted complex structure \(J_{AD}\). This turns the tangent disk bundle into a Kähler manifold whose Kähler form coincides with the standard symplectic form \(\tilde{\omega}\).

(b) Suppose \((Q, g)\) is complete and locally symmetric with sectional curvature bounded below by \(-K\), \(K \geq 0\). Then an adapted complex structure \(J_{AD}\) exists on \(T_{\leq R} Q\) with
\[
R = \frac{\pi}{2\sqrt{K}}.
\]

In particular, the adapted complex structure exists on the whole tangent bundle if \(Q\) has nonnegative sectional curvature.

(c) An adapted complex structure is unique, if it exists.

According to [HK11], the adapted complex structure \(J_{AD}\) can be geometrically described as follows. Let \(\Phi_t : TQ \to TQ\) be the time-\(t\) geodesic flow on the tangent bundle. For \(\xi \in TQ\) denote by
\[
V^C_\xi \subset T^C\xi(TQ)
\] (2.4.10)
the complexification of the vertical tangent space at \(\xi\). For \(t \in \mathbb{R}\) we may consider the map
\[
\Upsilon_\xi(t) = (\Phi_t)_* \left(V^C_{\Phi_t^{-1}(\xi)}\right).
\] (2.4.11)
Whenever \(\xi\) lies in the domain of \(J_{AD}\), the map \(\Upsilon_\xi\) can be analytically continued to a holomorphic map from the unit disk into the Grassmannian of \(n\)-dimensional complex subspaces of \(T^C\xi(TQ)\). Now \(J_{AD}\) is characterized as the unique complex structure whose eigenspace corresponding to the eigenvalue \(i\) equals \(\Upsilon_\xi(i)\). Roughly speaking, \(J_{AD}\) is given by the imaginary time geodesic flow.

We like to employ this characterization to calculate \(J_{AD}\) on the tangent bundle of the round sphere \((S^n, g)\). Theorem 2.4.5 ensures that the complex structure is globally defined. Let us consider \(TS^n\) as a submanifold of \(\mathbb{R}^{2n+2}\)
\[
TS^n = \{(x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : \|x\| = 1 \quad \text{and} \quad \langle x, y \rangle = 0\}.
\] (2.4.12)
2.4. *DISTINGUISHED COMPLEX STRUCTURES AND SYMPLECTIC FORMS*

The tangent space at \((x_0, y_0) \in TS^n\) equals
\[
T_{(x_0, y_0)}TS^n = \{ (\xi, \eta) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : \langle x_0, \xi \rangle = 0 \text{ and } \langle x_0, \eta \rangle + \langle y_0, \xi \rangle = 0 \}. \tag{2.4.13}
\]

Let \(\{w_2, \ldots, w_n\}\) be an orthonormal basis of the hyperplane
\[
\{ x \in \mathbb{R}^{n+1} : \langle x, x_0 \rangle = \langle x, y_0 \rangle = 0 \}.
\]

An orthogonal basis of \(T_{(x_0, y_0)}TS^n\) is then given by
\[
e_1 = \left( \frac{y_0}{\|y_0\|}, -\frac{y_0}{\|y_0\|} \cdot x_0 \right), \quad e_j = (w_j, 0) \quad \text{for } 2 \leq j \leq n, \tag{2.4.14}
e_{n+1} = \left( 0, \frac{y_0}{\|y_0\|} \right), \quad e_{n+j} = (0, w_j) \quad \text{for } 2 \leq j \leq n.
\]

**Lemma 2.4.6.** *The unique adapted complex structure on \(TS^n\) satisfies*
\[
J_{AD}(x_0, y_0) e_1 = e_{n+1}, \quad J_{AD}(x_0, y_0) e_j = \|y_0\| \coth \|y_0\| e_{n+j} \quad \text{for } 2 \leq j \leq n.
\]

*Proof.* The time-\(t\) geodesic flow on \(TS^n\) can be explicitly described by the formula
\[
\Phi_t(x_0, y_0) = \left( \cos(||y_0||t) x_0 + \frac{\sin(||y_0||t)}{||y_0||} y_0, -\|y_0\| \sin(||y_0||t) x_0 + \cos(||y_0||t) y_0 \right). \tag{2.4.15}
\]

Let \((x_t, \|y_0\| y_t) = \Phi_{-t}(x_0, y_0)\), that is
\[
x_t = \cos(||y_0||t) x_0 - \frac{\sin(||y_0||t)}{||y_0||} y_0 \quad \text{and} \quad y_t = \sin(||y_0||t) x_0 + \frac{\cos(||y_0||t)}{||y_0||} y_0. \tag{2.4.16}
\]

Then \(T_{x_t}S^n\) is spanned by the orthonormal basis \(\{y_t, w_2, \ldots, w_n\}\). Consequently, the tangent vectors corresponding to \(V_{\Phi_{-t}}(x_0, y_0)\) are exactly those which may be represented as curves \(\gamma^t_\lambda : \mathbb{R} \to TS^n\) of the form
\[
\gamma^t_\lambda(s) = (x_t, y^t_\lambda(s)) = \left( x_t, (||y_0|| + \lambda s) y_t + \sum_{k=2}^{n} \lambda_k w_k s \right)
\]
where \(\lambda \in \mathbb{R}^n\) parametrizes \(T_{x_t}S^n\). Therefore
\[
V_{\Phi_{-t}}(x_0, y_0) = \left\{ \frac{d}{ds} \gamma^t_\lambda(s) \big|_{s=0} : \lambda \in \mathbb{R}^n \right\}.
\]

Considering (2.4.11) yields
\[
\Upsilon_{(x_0, y_0)}(t) = (\Phi_t)_* \left( V_{\Phi_{-t}}(x_0, y_0) \right) = \left\{ \frac{d}{ds} (\Phi_t \circ \gamma^t_\lambda)(s) \big|_{s=0} : \lambda \in \mathbb{R}^n \right\}^C.
\]

Note that
\[
\left. \frac{d}{ds} \|y^t_\lambda(s)\| \right|_{s=0} = \frac{d}{ds} \sqrt{(||y_0|| + \lambda_1 s)^2 + (\lambda_2^2 + \ldots + \lambda_n^2) s^2} \bigg|_{s=0} = \frac{2\lambda_1 ||y_0||}{2||y_0||} = \lambda_1.
\]
Using this and (2.4.15), (2.4.16) we can perform the calculation

\[
\frac{d}{ds} \left( \Phi_t \circ \gamma^1 \right) (s)_{|s=0} = \frac{d}{ds} \left( \cos(\|y_\lambda^1(s)\|t) x_t + \frac{\sin(\|y_\lambda^1(s)\|t)}{\|y_\lambda^1(s)\|} y_\lambda^1(s), \right. \\
\left. - \|y_\lambda^1(s)\| \sin(\|y_\lambda^1(s)\|t) x_t + \cos(\|y_\lambda^1(s)\|t) y_\lambda^1(s) \right) \bigg|_{s=0}
\]

\[
= - \lambda_1 t \sin(\|y_0\|t) x_t + \lambda_1 t \cos(\|y_0\|t) y_t - \lambda_1 \frac{\sin(\|y_0\|t)}{\|y_0\|} y_t + \frac{\sin(\|y_0\|t)}{\|y_0\|} \left( \lambda_1 y_t + \sum_{k=2}^n \lambda_k w_k \right) ,
\]

\[
- \lambda_1 \sin(\|y_0\|t) (x_t + t\|y_0\|y_t) - \lambda_1 \|y_0\| \cos(\|y_0\|t) x_t + \cos(\|y_0\|t) \left( \lambda_1 y_t + \sum_{k=2}^n \lambda_k w_k \right) ,
\]

\[
= \left( \lambda_1 t \frac{y_0}{\|y_0\|} + \frac{\sin(\|y_0\|t)}{\|y_0\|} \left( \sum_{k=2}^n \lambda_k w_k \right) \right) , \lambda_1 \frac{y_0}{\|y_0\|} - \lambda_1 \|y_0\| x_t + \cos(\|y_0\|t) \left( \sum_{k=2}^n \lambda_k w_k \right) \right) .
\]

It can be easily checked via (2.4.13) that this vector indeed belongs to \( T_{(x_0, y_0)}TS^n \) for arbitrary \( \lambda \in \mathbb{R}^n \). The involved functions sine and cosine are entire. Using the characterization of \( J_{AD} \) by the imaginary time geodesic flow, we may plug in \( t = i \) in order to obtain the eigenspace at \( (x_0, y_0) \) corresponding to the eigenvalue \( i \). It is spanned by

\[
\left( -\frac{y_0}{\|y_0\|} , i \frac{y_0}{\|y_0\|} + \|y_0\| x_0 \right), \quad (\sin(\|y_0\|i) w_j, \cos(\|y_0\|i) \|y_0\| w_j) \quad \text{for} \quad 2 \leq j \leq n.
\]

With the help of the identity

\[
i \cot(i z) = \coth z
\]

and (2.4.14) we conclude the statement. \( \square \)

The embedded tangent bundle \( TS^n \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \) can be identified with the affine hyperquadric

\[
QD^n = \left\{ (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : \sum_{k=1}^{n+1} z_k^2 = 1 \right\} \quad (2.4.17)
\]

using the diffeomorphism \( v : TS^n \to QD^n \) given by

\[
v(x, y) = \cosh(\|y\|) x + i \frac{\sinh(\|y\|)}{\|y\|} y. \quad (2.4.18)
\]

**Proposition 2.4.7.** The pushforward of the adapted complex structure via \( v \) coincides with the complex structure of \( QD^n \) induced from \( \mathbb{C}^{n+1} \), i.e.

\[
v_* J_{AD} = i.
\]

**Proof.** It suffices to calculate the pushforward of the basis (2.4.14) of \( T_{(x_0, y_0)}TS^n \). It is given by

\[
v_* e_1 = \frac{\cosh(\|y_0\|)}{\|y_0\|} y_0 - i \sinh(\|y_0\|) x_0, \quad v_* e_j = \cosh(\|y_0\|) w_j \quad \text{for} \quad 2 \leq j \leq n,
\]

\[
v_* e_{n+1} = \sinh(\|y_0\|) x_0 + i \frac{\cosh(\|y_0\|)}{\|y_0\|} y_0, \quad v_* e_{n+j} = i \frac{\sinh(\|y_0\|)}{\|y_0\|} w_j \quad \text{for} \quad 2 \leq j \leq n.
\]

The claim then follows from Lemma 2.4.6. \( \square \)
2.4. DISTINGUISHED COMPLEX STRUCTURES AND SYMPLECTIC FORMS

We finally turn to modifications of the standard symplectic structure $\tilde{\omega}$ and introduce magnetic tangent bundles. As the name suggests this is motivated by physics. Namely, the motion of a charged particle in a Riemannian manifold $(Q,g)$ can be described as the Hamiltonian flow of the standard metric Hamiltonian $\tilde{f}$ given in (2.2.21) with respect to a twisted symplectic form on $TQ$. For the reader interested in magnetic flows on tangent bundles we recommend the survey article [Gin96].

**Definition 2.4.8.** For a given closed 2-form $\sigma$ on a Riemannian manifold $(Q,g)$, the magnetic tangent bundle $(TQ, \tilde{\omega}_c)$ carries the symplectic structure

$$\tilde{\omega}_c = \tilde{\omega}_c(\sigma) = \tilde{\omega} + \varepsilon \pi^* \sigma. \quad (2.4.19)$$

Here $\sigma$ may be interpreted as a magnetic field on $Q$ whereas the real parameter $\varepsilon$ displays its magnitude. We will check that $\tilde{\omega}_c$ is indeed nondegenerate and observe that Proposition 2.2.7 does not necessarily hold for magnetic tangent bundles. At least the Levi-Civita almost complex structure remains tamed by the twisted symplectic form whenever $Q$ is compact and $\varepsilon$ sufficiently close to zero. These statements are shown in the next Lemma.

**Lemma 2.4.9.** Let $(Q,g)$ be a Riemannian manifold and $\sigma$ a closed 2-form on $Q$. The following holds:

(a) $\tilde{\omega}_c$ is nondegenerate for any $\varepsilon \in \mathbb{R}$.
(b) $J_{LC} \notin J(TQ, \tilde{\omega}_c)$ for $\sigma \neq 0$ and $\varepsilon \neq 0$.
(c) Suppose $\sigma$ is uniformly bounded by $g$, i.e. there is a constant $C > 0$ satisfying

$$\sigma(X,Y) \leq C \|X\| \|Y\|.$$  

Then $J_{LC} \in J^+(TQ, \tilde{\omega}_c)$ for $|\varepsilon| < 4C^{-1}$.

**Proof.**

(a) Assume there is $\tilde{Z} \in \Gamma(TQ)$) and $\xi \in TQ$ satisfying $\tilde{\omega}_c(\tilde{Z}, \cdot) = 0$. For any $X \in \Gamma(TQ)$ we obtain

$$\tilde{\omega}(\tilde{Z}, X^H) = \tilde{\omega}(\tilde{Z}, J_{LC} X^H) = \tilde{\omega}(\tilde{Z}, X^V) = \tilde{\omega}_c(\tilde{Z}, \cdot) = 0.$$

Consequently, $\tilde{Z}_\xi$ belongs to the vertical distribution. Therefore $\tilde{\omega}_c(\tilde{Z}, \cdot) = \tilde{\omega}(\tilde{Z}, \cdot)$ and the nondegeneracy of $\tilde{\omega}$ implies $\tilde{Z}_\xi = 0$.

(b) Pick $\tilde{Z}_1, \tilde{Z}_2 \in \Gamma(TQ)$) and split these into $\tilde{Z}_i = X_i^H + Y_i^V$, $i = 1, 2$ with $X_i, Y_i \in \Gamma(TQ)$. We calculate using (2.2.16)

$$\pi^* \sigma(\tilde{Z}_1, J_{LC} \tilde{Z}_2) = \sigma(\pi_* \tilde{Z}_1, (\pi_* \circ J_{LC}) \tilde{Z}_2) = -\sigma(\pi_* \tilde{Z}_1, \kappa \tilde{Z}_2) = -\sigma(X_1, Y_2). \quad (2.4.20)$$

Thus $\pi^* \sigma(\cdot, J_{LC} \cdot)$ is not symmetric in its arguments. On the other hand $\tilde{\omega}(\cdot, J_{LC} \cdot)$ is symmetric by Proposition 2.2.7, hence $\tilde{\omega}_c(\cdot, J_{LC} \cdot)$ cannot be symmetric as well.

(c) We have to verify $\tilde{\omega}_c(\tilde{Z}, J_{LC} \tilde{Z}) > 0$ for $\tilde{Z} \in \Gamma(TQ)$, $\xi \in T_pQ$ with $\tilde{Z}_\xi \neq 0$. Using the splitting $\tilde{Z} = X^H + Y^V$ with $X, Y \in \Gamma(TQ)$, this follows via (2.4.20) from the estimate

$$\tilde{\omega}_c(\tilde{Z}, J_{LC} \tilde{Z}) = \omega(\tilde{Z}, J_{LC} \tilde{Z}) + \varepsilon \pi^* \sigma(\tilde{Z}, J_{LC} \tilde{Z})$$

$$= \tilde{\omega}(\tilde{Z}, \tilde{Z}) + \varepsilon \pi^* \sigma(\tilde{Z}, J_{LC} \tilde{Z})$$

$$= \tilde{\omega}(\tilde{Z}, \tilde{Z}) + \varepsilon \sigma(X, Y)_p$$

$$= g(X, X)_p + g(Y, Y)_p - \varepsilon \sigma(X, Y)_p$$

$$\geq 4 \|X\|_p \|Y\|_p - |\varepsilon| \|\sigma(X, Y)_p| > 0.$$
Note that the zero section in \((TQ, \tilde{\omega})\) is not Lagrangian anymore, but fibers are. Magnetic tangent bundles will further appear in the next section in the context of Riemannian metrics admitting bounded geometry.

### 2.5 Unbounded Geometry

The tangent bundle equipped with the Sasaki metric is a noncompact manifold with unbounded geometry. Nevertheless, we want to have control on the growth of the sectional curvature and injectivity radius in fiber direction. We also describe a construction due to Cieiebak, Ginzburg and Kerman which circumvents this issue by changing the Sasaki metric at infinity. In particular, we fix a gap in their argument that bounded geometry can be obtained at the convex end.

**Definition 2.5.1.** A Riemannian manifold \(M\) is said to have bounded geometry if it has injectivity radius

\[
\rho(M) = \inf_{p \in M} \rho(p) > 0
\]

bounded from below and bounded sectional curvature.

Theorem 2.2.11 tells us that \((TQ, \tilde{\omega})\) admits bounded geometry only if \(Q\) is flat. In order to gain a better understanding of this, we like to study the growth functions

\[
\tilde{\rho}(Q, r) = \inf \{ \tilde{\rho}(\xi) : \xi \in TQ \text{ with } \|\xi\| \leq r \}
\]

and

\[
\tilde{K}_{sec}(Q, r) = \sup \left\{ |\tilde{K}_{sec}(\xi)| : \xi \in TQ \text{ with } \|\xi\| \leq r \right\}.
\]

Our aim is to show that for \(r \to \infty\) we have

\[
\tilde{K}_{sec}(Q, r) = O(r^2) \quad \text{and} \quad \tilde{\rho}(Q, r) = o\left(\frac{1}{r}\right).
\]

That is, the sectional curvature of the Sasaki metric grows quadratically in the fiber and the injectivity radius decreases linearly.

While the statement concerning the growth rate of sectional curvature is rather straightforward, the decay of the injectivity radius requires some work. In the case of a general complete Riemannian manifold, it is known via heat kernel estimates that the injectivity radius decays at most exponentially with distance, see [CLY81]. In order to get prepared for the proof of the stronger linear decay rate in (2.5.4), we need what is known in the literature as the Klingenberg Lemma. It takes advantage of the fact that the injectivity radius at a point \(p\) is bounded from below by the minimum of the convexity radius at \(p\) and half the length of the shortest closed geodesic running through \(p\).

**Lemma 2.5.2.** Let \((M, g)\) be a complete Riemannian manifold and \(B_r(p) \subset M\) an open metric ball. Suppose the curvature is bounded on \(B_r(p)\) by

\[
\sup \{|K_{sec}(q)| : q \in B_r(p)\} \leq K
\]
and any geodesic loop in $B_r(p)$ through $p$ has length bounded below by $L$. Then the injectivity radius at $p$ is bounded below by

$$\rho(p) \geq \min \left\{ \frac{\pi}{\sqrt{K}}, \frac{L}{2}, r \right\}. \quad (2.5.5)$$

In order to estimate the length of short geodesic loops in $(TQ, \tilde{g})$ from below, we will use the following result (Theorem 4.3 in [CGT82]).

**Proposition 2.5.3.** Let $(M, g)$ be a complete Riemannian manifold of dimension $n$. For $H > 0$ denote the volume of a ball of radius $r$ in the simply connected $n$-dimensional space of constant curvature $-H$ by

$$\text{Vol}_n(H, r) = \text{Vol}\left(S^{n-1}\right) \int_0^r \left(\frac{\sinh(\sqrt{H}s)}{\sqrt{H}}\right)^{n-1} ds.$$ 

Further let $\text{Vol}_M(p, r)$ be the volume of the open metric ball $B_r(p) \subset M$. Fix such a ball together with $K > 0$ such that the curvature on $B_r(p)$ is bounded by

$$\sup \left\{ |K_{sec}(p)| : p \in B_r(p) \right\} \leq K.$$

Suppose $\gamma$ is a geodesic loop through $p$ of length $L$. Then

$$L \geq \frac{r_0}{4} \left(1 + \frac{\text{Vol}_n(K, 3r_0)}{\text{Vol}_M(p, 2r_0)}\right)^{-1} \quad (2.5.6)$$

whenever

$$r_0 \leq \frac{1}{4} \min \left\{ \frac{\pi}{\sqrt{K}}, r \right\}.$$

The final ingredient will be a lower estimate for the volume of metric balls in $(TQ, \tilde{g})$. The crucial point is that this cannot be achieved by a direct comparison geometry argument, since the volume comparison theorem of P. Günther ([Jos05], p. 206) just applies within the injectivity radius. We somehow have to exploit the structure of the Sasaki metric.

**Lemma 2.5.4.** Let $(Q, g)$ be an $n$-dimensional closed Riemannian manifold with bound

$$\sup \left\{ K_{sec}(p) : p \in Q \right\} \leq K$$
on sectional curvature. For $\xi \in TQ$ and

$$r \leq \min \left(\rho(Q), \frac{\pi}{2\sqrt{K}}\right) \quad (2.5.7)$$

consider the open metric ball $B_{2r}(\xi)$ in $(TQ, \tilde{g})$. Then there is a constant $C(n)$ such that the volume of $B_{2r}(\xi)$ can be estimated via

$$\text{Vol}_{TQ} (\xi, 2r) \geq Cr^{2n}. \quad (2.5.8)$$

**Proof.** For $p = \pi(\xi) \in Q$ take the open metric ball $B_r(p)$ and lift any geodesic $\gamma : [0, r) \to Q$ with $\gamma(0) = p$ horizontally into $TQ$ such that $\tilde{\gamma}(0) = \xi$. This provides a horizontal lift

$$H : B_r(p) \to B_r^H(\xi) \subset TQ$$
satisfying \( H(p) = \xi \) which is a diffeomorphism onto its image. Let us now define the polydisk

\[ P_{r,r}(\xi) = \{ \eta \in TQ : \eta - \eta^V \in B_r^H(\xi) \text{ for some } \eta^V \in T_{\pi(\eta)}Q \text{ with } \| \eta^V \| < r \text{ and } \pi_* \eta^V = 0 \}. \]

By (2.3.5) the volume form of the Sasaki metric does not depend on the fiber variable and we obtain

\[
\text{Vol}_{TQ}(P_{r,r}(\xi)) = \text{Vol}(B_r) \int_{B_r(p)} \text{dvol}_Q^2 \geq \left( \int_{B_r(p)} \text{dvol}_Q \right)^2 = (\text{Vol}_Q(p,r))^2 \tag{2.5.9}
\]

where \( B_r \) denotes an Euclidean ball of radius \( r \) and the estimate holds due to the Cauchy-Schwarz inequality. Using \( r \leq \rho(Q) \), the volume comparison theorem of P. Günther together with (2.5.7) implies

\[
\text{Vol}_Q(p,r) \geq \text{Vol}(S^{n-1}) \int_0^r \left( \frac{\sin(\sqrt{Ks})}{\sqrt{K}} \right)^{n-1} \text{ds} \tag{2.5.10}
\]

By construction, any point in \( P_{r,r}(\xi) \) has distance \(< 2r \) to \( \xi \). Therefore \( P_{r,r}(\xi) \subset B_{2r}(\xi) \) and the statement follows from (2.5.9) and (2.5.10). \( \square \)

We are now ready to prove the asymptotic formulas in (2.5.4).

**Theorem 2.5.5.** Let \((Q,g)\) be a closed Riemannian manifold. Then there is a constant \( C > 0 \) depending only on bounds of \( R, \nabla R \) and \( \rho(Q) \) such that the sectional curvature of \((TQ,\tilde{g})\) satisfies

\[
\tilde{K}_{sec}(Q,r) \leq C(r^2 + 1) \tag{2.5.11}
\]

and the injectivity radius decreases according to

\[
\tilde{\rho}(Q,r) \geq \frac{1}{Cr}. \tag{2.5.12}
\]

**Proof.** For the estimate of sectional curvature pick \( \tilde{W}, \tilde{Z} \in \Gamma(T(TQ)) \) and \( U,V,X,Y \in \Gamma(TQ) \) such that \( \tilde{W} = U^H + V^V \) and \( \tilde{Z} = X^H + Y^V \) holds. By (2.2.4) we have for \( \xi \in TQ \) with \( \pi(\xi) = p \)

\[
\| \tilde{W}_\xi \|^2_g = \| U_p \|^2_g + \| V_p \|^2_g \quad \text{and} \quad \| \tilde{Z}_\xi \|^2_g = \| X_p \|^2_g + \| Y_p \|^2_g.
\]

Therefore

\[
\sup_{\| \tilde{W}_\xi \|=1, \| \tilde{Z}_\xi \|=1} \left| \left\langle \tilde{R}(\tilde{W},\tilde{Z})\tilde{Z},\tilde{W} \right\rangle \right| \leq \sup_{\| U_p \|,\| V_p \| \leq 1, \| X_p \|,\| Y_p \| \leq 1} \left| \left\langle \tilde{R}(U^H + V^V, X^H + Y^V) X^H + Y^V, U^H + V^V \right\rangle \right| \xi
\]

and one continues by expanding the right-hand side into 16 terms and estimating each one of them via the curvature formulas in Proposition 2.2.4. For example, with

\[
C_1 = \sup_{\| X \|,\| Y \|,\| Z \| = 1} \| R(X,Y)Z \|.
\]
and (2.2.12) one obtains
\[
\sup_{\|U_p\|,\|V_p\|,\|X_p\|,\|Y_p\| \leq 1} \left| \left\langle \tilde{R}(U^H, Y V^H) X^H, V^V \right\rangle \right| \\
= \sup_{\|U_p\|,\|V_p\|,\|X_p\|,\|Y_p\| \leq 1} \left| \frac{1}{4} \langle R_p (R_p(\xi, Y_p) X_p, U_p) \xi, V_p \rangle + \frac{1}{2} \langle R_p (U_p, X_p) Y_p, V_p \rangle \right| \\
\leq \sup_{\|U_p\|,\|V_p\|,\|X_p\|,\|Y_p\| \leq 1} \left( \frac{1}{4} \langle R_p (\xi, Y_p) (R_p(\xi, Y_p) X_p, U_p) \rangle + \frac{1}{2} \langle R_p (U_p, X_p) Y_p, V_p \rangle \right) \\
\leq \frac{1}{4} C_1^4 \|\xi\|^2 + \frac{1}{2} C_4.
\]

Analog estimates for the other terms imply
\[
\sup_{\|\tilde{W}_\xi\| = \|\tilde{Z}_\xi\| = 1} \left| \left\langle \tilde{R}(\tilde{W}, \tilde{Z}) \tilde{Z}, \tilde{W} \right\rangle \xi \right| \leq C_2 \|\xi\|^2 + C_3 \|\xi\| + C_4 \leq C_2 \|\xi\|^2 + \frac{1}{2} C_3 (\|\xi\|^2 + 1) + C_4
\]
with constants $C_2, C_3, C_4$ just depending on curvature bounds of $R$ and $\nabla R$. Hence for
\[
A = \max \left( C_2 + \frac{1}{2} C_3, \frac{1}{2} C_3 + C_4 \right)
\]
we have $\tilde{K}_{sec}(Q, r) \leq A(r^2 + 1)$, as desired. A very similar and more refined argument estimating the holomorphic sectional curvature of the Sasaki metric is presented in Proposition 3.5.9.

Let us now turn to the estimate of the injectivity radius. By the previous argument we know that the sectional curvature on $B_1(\xi) \subset TQ$ is bounded by
\[
\sup \{|K_{sec}(\eta)| : \eta \in B_1(\xi)\} \leq A \left( \|\xi\| + 1 \right)^2 \leq (C_5(\|\xi\| + 1))^2
\]
with $C_5$ just depending on curvature bounds of $R$ and $\nabla R$. With $K$ referring to an upper bound for the sectional curvature on $Q$, we set
\[
r_0 = \min \left\{ \frac{\pi}{4 C_5 (\|\xi\| + 1)}, \frac{1}{4}, \frac{\pi}{2 \sqrt{K}} \right\}.
\]
Hence Proposition 2.5.3 applied to $B_1(\xi)$ tells us that any geodesic loop with respect to the Sasaki metric and passing through $\xi$ has length at least
\[
L \geq \frac{r_0}{4} \left( 1 + \frac{\text{Vol}_{2n}(C_5(\|\xi\| + 1))^2, 3r_0)}{\text{Vol}_{TQ}(\xi, 2r_0)} \right)^{-1}.
\]
(2.5.13)

By the choice of $r_0$ we have
\[
C_5(\|\xi\| + 1) \cdot 3r_0 \leq \frac{3\pi}{4}.
\]
We may pick a constant $C_6$ such that $\sinh x \leq C_6 x$ holds for $0 \leq x \leq \frac{3\pi}{4}$. This allows the estimate
\[
\text{Vol}_{2n}(C_5(\|\xi\| + 1))^2, 3r_0) = \text{Vol} \left( S^{2n-1} \right) \int_0^{3r_0} \left( \frac{\sinh(C_5(\|\xi\| + 1)s)}{C_5(\|\xi\| + 1)} \right)^{2n-1} ds \leq \text{Vol} \left( S^{2n-1} \right) \int_0^{3r_0} (C_6 s)^{2n-1} ds = \frac{1}{2n} \text{Vol} \left( S^{2n-1} \right) C_6^{2n-1} (3r_0)^{2n}.
\]
Denoting the constant in Lemma 2.5.4 by \( C_7 \) yields
\[
\frac{\text{Vol}_{2n}(C_5(\|\xi\| + 1)^2, 3r_0)}{\text{Vol}_{TQ}(\xi, 2r_0)} \leq \frac{1}{2n} \text{Vol}(S^{2n-1}) C_6^{2n-1} \frac{(3r_0)^{2n}}{C_7 r_0^{2n}} \leq C_8.
\]
Plugging this into (2.5.13) and applying Klingenbergs Lemma 2.5.2 to \( B_1(\xi) \) leads to
\[
\tilde{\rho}(\xi) \geq \min \left\{ \frac{\pi}{C_5(\|\xi\| + 1)^4}, \frac{r_0}{4(1 + C_8)}, 1 \right\}
\geq \frac{1}{4(1 + C_8)} \min \left\{ \frac{\pi}{4 C_5(\|\xi\| + 1)^4}, \frac{1}{4}, \rho(Q), \frac{\pi}{2\sqrt{R}} \right\} \geq \frac{1}{B \|\xi\|}.
\]
Again, the constants \( C_8 \) and \( B \) only depend on bounds of \( R, \nabla R \) and \( \rho(Q) \). The inequalities (2.5.11) and (2.5.12) then hold with \( C = \max(A, B) \).

For good reasons without proof, it is misstated in [AL94] on p. 96 and p. 286 that \((TQ, \bar{g})\) has bounded geometry. The discussion above shows that this is not the case. However, one might wonder whether it is possible to find an almost complex structure \( J \in \mathcal{J}(TQ) \) such that the corresponding metric \( g_J \) has bounded geometry. This is particularly desirable to obtain the necessary compactness results when studying symplectic homology, Rabinowitz-Floer homology or Gromov-Witten theory on cotangent bundles, as done for instance in [CGK04], [Mer11] and [Lu06]. Let us recall the following notion of geometrical boundedness.

**Definition 2.5.6.** A symplectic manifold \((M, \omega)\) is said to be geometrically bounded, if \( M \) admits an almost complex structure \( J \) and a complete Riemannian metric \( g \) such that

(i) \( J \) is uniformly \( \omega \)-tame, that is for some constants \( C_1 \) and \( C_2 \) one has
\[
\omega(X, JX) \geq C_1 \|X\|^2 \quad \text{and} \quad |\omega(X, Y)| \leq C_2 \|X\| \|Y\|
\]
for all \( X, Y \in \Gamma(TM) \).

(ii) \((M, g)\) has bounded geometry.

Note that (i) is automatically satisfied when \( J \in \mathcal{J}(M, \omega) \) with \( M \) compact and \( g = g_J \). It seems to be a well-known statement in the community that tangent bundles and also magnetic tangent bundles are geometrically bounded. However, the proofs appearing in the literature (i.e. Proposition 4.1 in [Lu96] and Proposition 2.2 in [CGK04]) contain gaps in that they do not establish property (ii) for the metric in consideration. While [Lu96] chooses the Sasaki metric which does not admit bounded geometry, the argument in [CGK04] can be completed. Let us first recall their statement.

**Proposition 2.5.7.** Let \((Q, g)\) be a closed Riemannian manifold and \( \sigma \) a closed 2-form on \( Q \). Then the magnetic tangent bundle \((TQ, \tilde{\omega}_\sigma)\) is geometrically bounded.

Its proof is based on an idea of J.-C. Sikorav. Let us briefly sketch the construction of the metric in there. The authors denote by \( \varphi_t \) the flow on \( TQ \) formed by fiberwise dilations by the factor \( e^t \) and choose a fiberwise convex hypersurface \( \Sigma \) in \( TQ \) enclosing the zero section. On \( TQ |_\Sigma \) they pick a fiberwise metric \( g_{GB} \) and extend it to the unbounded component
\[
U = \bigcup_{t \geq 0} \varphi_t(\Sigma) \subset TQ
\]
2.5. **UNBOUNDED GEOMETRY**

via

$$\varphi_t^* g_{GB} = e^t g_{GB} \quad \text{for } t \geq 0.$$  \hfill (2.5.14)

On the bounded component $TQ \setminus U$ the metric is extended arbitrarily. The authors correctly argue that $g_{GB}$ is complete and its sectional curvature remains bounded on $TQ$. However, this does not necessarily imply that the injectivity radius of $g_{GB}$ is bounded from below, as claimed in the paper. For a simple counterexample consider a surface with one cylindrical end $\mathbb{R}_+ \times S^1$ as shown in Figure 2.1. Suppose it comes with a metric which can be written as a warped product

$$ds \otimes ds + e^{-2s} d\varphi \otimes d\varphi$$

with $s \in \mathbb{R}_+$ and $\varphi \in [0, 2\pi)$ on the end. Then the sectional curvature is constantly $-1$ on the end, hence remains bounded on the surface. On the other hand the curves in $S^1$-direction form closed geodesics of length $2\pi e^{-s}$. Thus the injectivity radius must be zero.

![Figure 2.1](image)

Figure 2.1: A complete Riemannian manifold with bounded curvature and zero injectivity radius.

In order to prove that $g_{GB}$ has injectivity radius bounded from below, one has to find lower bounds for the length of small geodesic loops in $U$ which may be achieved via Proposition 2.5.3 and a volume estimate of geodesic balls akin to Lemma 2.5.4.

**Lemma 2.5.8.** Let $(Q, g)$ be an $n$-dimensional closed Riemannian manifold and $TQ$ its tangent bundle equipped with the metric $g_{GB}$. Let

$$r \leq \min \left( \rho(\Sigma), \frac{\pi}{2\sqrt{K}} \right)$$

where $\rho(\Sigma) > 0$ denotes the injectivity radius of the compact hypersurface $(\Sigma, g_{GB}|_{\Sigma})$ and $K$ bounds its sectional curvature. Then there is a constant $C(n) > 0$ such that the volume of any
open metric ball $B_{2r}(\xi) \subset U$ can be estimated via

$$\text{Vol}_{TQ}(\xi, 2r) \geq Cr^{2n}.$$  

**Proof.** By substituting $s = e^t$ one may identify $U$ with $\Sigma \times [1, \infty)$. The metric then assumes the form

$$g_{GB} = \frac{1}{s} ds \otimes ds + s g_{GB}|_{\Sigma}$$
on $U$. For $\xi = (x, a) \in \Sigma \times [1, \infty)$ we may choose an open metric ball $B_r(a) \subset [1, \infty)$ with respect to the metric $\frac{1}{s} ds \otimes ds$. Moreover, for any $y \in B_r(a)$ let $B_{2r}(y) \subset \Sigma$ be an open metric ball of radius $r$ with respect to the metric $y g_{GB}|_{\Sigma}$. Employing the volume comparison theorem of P. Günther on $\Sigma$ yields that the polydisk

$$P_{r, r}(\xi) = \bigcup_{y \in B_r(a)} B^y_r(x)$$
satisfies the volume estimate

$$\text{Vol}_{TQ}(P_{r, r}(\xi)) = \int_{B_r(a)} \sqrt{g^{2n-1}} \text{Vol}_{\Sigma} \left( x, \frac{r}{\sqrt{y}} \right) \text{dvol}_{[1, \infty)} \geq \int_{B_r(a)} \sqrt{g^{2n-1}} C \left( \frac{r}{\sqrt{y}} \right)^{2n-1} \text{dvol}_{[1, \infty)} \geq Cr^{2n}.$$  

The statement then follows from $P_{r, r}(\xi) \subset B_{2r}(\xi)$. \hfill \qed

From here one proceeds as before by establishing lower bounds on the length of closed geodesics in $(TQ, g_{GB})$. In combination with the bounds on sectional curvature, this implies that the injectivity radius of $g_{GB}$ is bounded from below. Hence we are able to fill the gap in the proof of Proposition 2.5.7 presented in [CGK04].

### 2.6 J-holomorphic Curves

This section is devoted to some basic properties of J-holomorphic curves. We present the Cauchy-Riemann equations, the energy identity and implications of convexity.

**Definition 2.6.1.** Let $(\Sigma, j)$ be a Riemann surface and $(M, J)$ an almost complex manifold. A J-holomorphic (or pseudoholomorphic) curve is a smooth map $u : (\Sigma, j) \to (M, J)$ such that its differential is complex linear, i.e.

$$Tu \circ j = J \circ Tu.$$  

(2.6.1)

We note that (2.6.1) is a nonlinear first-order partial differential equation, usually referred to as the Cauchy-Riemann equation. Locally, in conformal coordinates it can be rewritten as

$$\partial_{\bar{z}} u + J(u) \partial_z u = 0.$$  

(2.6.2)

J-holomorphic curves can also be understood as the zero section of the Cauchy-Riemann operator $\partial_J$ which assigns to a smooth map $u : \Sigma \to M$ the 1-form

$$\partial_J(u) = \frac{1}{2} (du + J \circ du \circ j) \in \Omega^{0,1}(\Sigma, u^* TM).$$  

(2.6.3)
Given a metric \( g \) on \( M \), each \( J \)-curve can be assigned its Dirichlet energy
\[
E(u) = \frac{1}{2} \int_{\Gamma} \|du\|_{g}^2 \, d\vol.
\]  
(2.6.4)

One of the crucial observations making \( J \)-holomorphic curves beneficial is the following energy identity (Lemma 2.2.1 in [MS04]). It allows to control the energy by expressing it in terms of purely topological data.

**Proposition 2.6.2.** Let \((M, \omega)\) be a symplectic manifold.

(a) If \( J \in \mathcal{J}^r(M, \omega) \), then each \( J \)-holomorphic curve \( u : \Sigma \to M \) satisfies
\[
E(u) = \frac{1}{2} \int_{\Sigma} \|du\|_{g_J}^2 \, d\vol = \int_{\Sigma} u^* \omega.
\]  
(2.6.5)

(b) If \( J \in \mathcal{J}(M, \omega) \), then every smooth map \( u : \Sigma \to M \) satisfies
\[
E(u) = \frac{1}{2} \int_{\Sigma} \|du\|_{g_J}^2 \, d\vol = \int_{\Sigma} \|\partial J(u)\|_{g_J}^2 + \int_{\Sigma} u^* \omega.
\]  
(2.6.6)

In our case the manifold of interest will be the tangent bundle \( M = TQ \) of a Riemannian manifold \((Q, g)\) with its canonical almost complex structure \( J_{LC} \). In local coordinates, as described in section 2.3, a \( J_{LC} \)-holomorphic curve can be seen as a map \( u = (x, \xi) \) with \( x, \xi : \Sigma \to \mathbb{R}^n \).

**Lemma 2.6.3.** The Cauchy-Riemann equations of a \( J_{LC} \)-holomorphic curve into \( TQ \) expressed in local conformal coordinates are
\[
\frac{\partial x^i}{\partial s} = -\frac{\partial x^i}{\partial t} - \Gamma^i_{jk}(x) \xi^j \frac{\partial x^j}{\partial s} \quad \text{for } i = 1, \ldots, n.
\]  
(2.6.7)
\[
\frac{\partial \xi^i}{\partial t} = \frac{\partial x^i}{\partial s} - \Gamma^i_{jk}(x) \xi^j \frac{\partial x^j}{\partial t}
\]  
(2.6.8)

for \( i = 1, \ldots, n \). Here \( \Gamma^i_{jk} \) are the Christoffel symbols of the Levi-Civita connection on \((Q, g)\). For convenience, we employed the Einstein summation convention.

**Proof.** With \( J_{LC} \) in local coordinates given by (2.3.7), we rewrite (2.6.2) to obtain
\[
\left( \begin{array}{c} \partial_s x \\ \partial_s \xi \end{array} \right) + \left( \begin{array}{cc} -P^t \\ I + P^t \end{array} \right) \left( \begin{array}{c} \partial_t x \\ \partial_t \xi \end{array} \right) = 0.
\]

This can be reformulated as
\[
\partial_s \xi = -\partial_t x - P^t \partial_t \xi \quad \partial_t \xi = \partial_s x - P^t \partial_s x.
\]

Using the second equation, the first one is equivalent to
\[
\partial_s \xi = -\partial_t x - P^t \partial_t x - P^t (\partial_s x - P^t \partial_s x) = -\partial_t x - P^t \partial_s x.
\]

Plugging in the definition of \( P \), we have reached the desired formulas. \( \Box \)

We have seen in Proposition 2.2.12 that the metric Hamiltonian \( \tilde{f} : TQ \to \mathbb{R} \) is plurisubharmonic. The consequences on \( J \)-curves are explained in the next proposition. It is taken from [MS04], Lemma 9.2.9.

**Proposition 2.6.4.** For a closed set \( \Omega \subset Q \) and \( J \in \mathcal{J}(TQ, \tilde{\omega}) \) consider a \( J \)-holomorphic curve \( u : \Omega \to TQ \). Then \( f \circ u \) is subharmonic. In particular, \( f \circ u \) assumes its maximum on \( \partial \Omega \).
2.7 Relation to Harmonic Maps

The energy identity implies that \( J \)-holomorphic curves are harmonic with respect to the compatible metric \( g_J \). In particular, by projecting \( J_{LC} \)-holomorphic disks in the tangent bundle down to \( (Q, g) \), we obtain the following diagram of maps

\[
\begin{array}{c}
\Sigma \xrightarrow{f} (Q, g) \\
\downarrow \pi \\
(TQ, \tilde{g}) \xrightarrow{u}
\end{array}
\]

with \( u \) and \( \pi \) harmonic. In this section we determine the degree of harmonicity of \( f \) by calculating its tension field.

For the generalized notion of harmonic maps we consider Riemannian manifolds \((M, g)\) and \((N, h)\) of dimension \(m\) and \(n\), respectively. Suppose local coordinates \( (x^1, \ldots, x^m) \) on \( M \) and \( (f^1, \ldots, f^n) \) on \( N \) are given. The differential

\[
df = \frac{\partial f^i}{\partial x^\alpha} dx^\alpha \otimes \frac{\partial}{\partial f^i}
\]

of a smooth map \( f : M \to N \) can be interpreted as a section into the bundle \( T^*M \otimes f^*TN \). On this we have a natural bundle metric and Levi-Civita connection \( \nabla \) induced from the metrics \( g \) and \( h \). With respect to this one considers its Dirichlet energy

\[
E(f) = \frac{1}{2} \int_M \|df\|^2 \, \text{dvol}_M. \tag{2.7.1}
\]

The density is given in local coordinates by

\[
\|df\|^2(x) = g^{\alpha \beta}(x) h_{ij}(f(x)) \frac{\partial f^i(x)}{\partial x^\alpha} \frac{\partial f^j(x)}{\partial x^\beta}.
\tag{2.7.2}
\]

Critical points of \( E \) are called harmonic maps.

Intrinsically, the Euler-Lagrange equations for \( E \) are given by

\[
\tau(f) = \text{trace} \nabla df = 0, \tag{2.7.3}
\]

where \( \tau \in \Gamma(f^*TN) \) is called the tension field of \( f \). In local coordinates the components of the tension field

\[
\tau(f) = \tau^i(f) \frac{\partial}{\partial f^i}
\]

are given by

\[
\tau^i(f) = g^{\alpha \beta} \frac{\partial^2 f^i}{\partial x^\alpha \partial x^\beta} - g^{\alpha \gamma} \Gamma^i_{\alpha \beta} \frac{\partial f^i}{\partial x^\gamma} + g^{\alpha \beta} \Gamma^i_{jk} \frac{\partial f^j}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta} \tag{2.7.4}
\]

\[
= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} \left( \sqrt{g} g^{\alpha \beta} \frac{\partial f^i}{\partial x^\beta} \right) + g^{\alpha \beta} \Gamma^i_{jk} \frac{\partial f^j}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta}
\]

with greek indices denoting the metric tensor and Christoffel symbols on \( M \) and latin indices used for \( N \). Moreover, \( g = \det(g_{\alpha \beta}) \). Note that the first part in the last expression corresponds
2.7. RELATION TO HARMONIC MAPS

to the negative of the Laplace-Beltrami operator $\Delta_M$ applied to $f^i$, while the second part comes from the curvature of $N$. This splitting reveals the fact that harmonic functions $f : M \to \mathbb{R}$, i.e. $\Delta_M f = 0$, are harmonic in this generalized setting as well. For a nice introduction into harmonic maps between Riemannian manifolds we recommend chapter 8 of [Jos05]. The following formula for the tension field of concatenated maps will be useful for us, see Lemma 8.7.2 in this book.

**Proposition 2.7.1.** Let $(M, g)$, $(N, h)$ and $(Q, k)$ be Riemannian manifolds. For smooth maps $u : M \to N$ and $v : N \to Q$ the tension field satisfies the chain rule

$$\tau(v \circ u) = \text{trace} \nabla dv(du, du) + dv \circ \tau(u).$$

(2.7.5)

In particular, if $u$ is harmonic and $v$ is totally geodesic (i.e. $\nabla dv \equiv 0$), then $v \circ u$ is harmonic as well.

We continue by considering a $J_{LC}$-holomorphic disk $u : \mathbb{D} \to TQ$ into the tangent bundle of a Riemannian manifold $(Q, g)$. As a $J$-curve $u$ must be harmonic with respect to the Sasaki metric $\tilde{g}$ on $TQ$. We like to study under which conditions $f = \pi \circ u$ becomes harmonic as well. In view of the chain rule (2.7.5) this would necessarily be the case if $\pi$ was totally geodesic. However, we can only ensure the following.

**Proposition 2.7.2.** The projection $\pi : (TQ, \tilde{g}) \to (Q, g)$ is always harmonic.

**Proof.** As argued in [GK02], Corollary 7.3, this follows from the fact that $\pi : TQ \to Q$ is a Riemannian submersion with totally geodesic fibers. For a proof in local coordinates write $\pi$ as $(x^1, \ldots, x^n, \xi^1, \ldots, \xi^n) \mapsto (x^1, \ldots, x^n)$. Plugging this into (2.7.4) yields with the help of (2.3.5) and (2.3.6)

$$\tau^i(\pi) = \frac{1}{\det G} \frac{\partial}{\partial x^j}(\det G \cdot \tilde{g}^{ij}) + \frac{1}{\det G} \frac{\partial}{\partial \xi^j}(\det G \cdot \tilde{g}^{i,j+n}) + \tilde{g}^{jk} \Gamma^i_{jk}$$

$$= \frac{g^{ij}}{\det G} \frac{\partial}{\partial x^j}(\det G) + \frac{g^{ij}}{\det G} - \frac{\partial}{\partial \xi^j} \left( g^{ik} \nabla_i \xi^l + g^{ij} \nabla_j \xi^l \right) + \tilde{g}^{jk} \Gamma^i_{jk}$$

with all summations running from 1 to $n$. The latter expression turns out to be zero, which is easiest seen by assuming $(x^1, \ldots, x^n)$ to be normal coordinates centered at the point of consideration.

Finally, we determine the tension field of $f$ in terms of the curvature $R$ of $(Q, g)$.

**Theorem 2.7.3.** Let $u : \mathbb{D} \to TQ$ be a $J_{LC}$-holomorphic disk. Then the projection $f = \pi \circ u$ satisfies

$$\tau(f) = R(\partial_s f, \partial_t f) u.$$  

(2.7.6)

**Proof.** Consider an immersed curve $\tilde{\gamma} : (-\varepsilon, \varepsilon) \to TQ$ with projection $\gamma = \pi \circ \tilde{\gamma}$. The tangent vector field $\tilde{Z}_\gamma$ along $\tilde{\gamma}$ splits into $\tilde{Z}_\gamma = X^H_\gamma + Y^V_\gamma$ with $X_\gamma, Y$ vector fields along $\gamma$ and $X_\gamma$ tangent to it. We employ Proposition 2.2.3 and (2.2.16) to calculate

$$\nabla \tau(\tilde{Z}_\gamma, \tilde{Z}_\gamma) = \nabla_{X_\gamma} X_\gamma - \pi_* \left( \tilde{\nabla}_{\tilde{Z}} \tilde{Z} \right)$$

$$= \nabla_{X_\gamma} X_\gamma - \pi_* \left( \tilde{\nabla}_{Y^V_\gamma} Y^V + \tilde{\nabla}_{X^H_\gamma} X^H_\gamma + \tilde{\nabla}_{Y^H_\gamma} X^H_\gamma + \tilde{\nabla}_{X^V_\gamma} X^H_\gamma \right)$$

$$= \nabla_{X_\gamma} X_\gamma - R(\gamma, Y) X_\gamma - \nabla_{X_\gamma} X_\gamma$$

$$= -R \left( \gamma, \kappa(\tilde{Z}_\gamma) \right) X_\gamma = R \left( \gamma, \pi_*(J_{LC} \tilde{Z}_\gamma) \right) X_\gamma.$$
We may apply the chain rule given in Proposition 2.7.1 and use the fact that \( u \) is harmonic. Taking into account the Cauchy-Riemann equation \( \partial_s u + J(u)\partial_t u = 0 \) as well, leads to

\[
\tau(f) = \tau(\pi \circ u) = \text{trace} \, \nabla d\pi(du, du) = \nabla d\pi(\partial_s u, \partial_s u) + \nabla d\pi(\partial_t u, \partial_t u) = R(u, \pi_\ast \partial_t u) \pi_\ast \partial_s u - R(u, \pi_\ast \partial_s u) \pi_\ast \partial_t u = R(\pi_\ast \partial_s u, \pi_\ast \partial_t u) u = R(\partial_s f, \partial_t f) u.
\]

\[\square\]

In particular, we conclude that projections of \( J_{LC} \)-holomorphic disks are harmonic, whenever \( (Q, g) \) is flat.
Chapter 3

Existence of Lifted Disks

3.1 Overview

For a given curve $\gamma : \partial \mathbb{D} \rightarrow Q$ into a Riemannian manifold and a section $s \in \Gamma(TQ)$ we consider the moduli space of lifted disks

$$\mathcal{M}_s = \{ u : \mathbb{D} \rightarrow TQ : \bar{\partial} u = 0, \ u(0) \in s(Q), \ \pi \circ u \mid_{\partial \mathbb{D}} = \gamma \}.$$  \hfill (3.1.1)

We employ a quantitative implicit function theorem in order to show existence of solutions when $\gamma$ is sufficiently small. In particular, we will usually assume that the image of $\gamma$ lies in the domain of a normal chart. If $\Delta$ denotes a triangular domain on $S^2$, we like to apply the result afterwards to the curves

$$\gamma_{q_1,q_2,q_3} : \partial \Delta \rightarrow Q$$  \hfill (3.1.2)

mapping the vertices of $\partial \Delta$ to points $q_1, q_2, q_3 \in Q$ and the connecting segments totally geodesically into $Q$. Thus the image of the geodesic triangle $\partial \Delta$ will be a geodesic triangle in $Q$. In the end, we will obtain a proof of Theorem 1.6.

In section 3.2 we determine the required regularity for $\gamma$ such that $J_\omega$-holomorphic solutions belong to $W^{1,p}$ with $p > 2$. We next explain the Banach space setup in section 3.3. To obtain solutions for different almost complex structures, we apply the implicit function theorem. The required closeness of $J$ to $J_0$ will depend on the smallness of $\gamma$ in $W^{1,5}$. Section 3.4 shows that this works fine for any given almost complex structure tamed by the standard or magnetic symplectic form. Finally, section 3.5 studies compactness properties of $\mathcal{M}_s$. As always, we particularly focus on the Levi-Civita case and explain how the gentle unbounded geometry can be exploited to deduce diameter bounds. It turns out that the geometric growth rates determined in Theorem 1.7 are critical for our argument to work.

3.2 Admissible Boundary Conditions

In local coordinates, we study a boundary value problem for pseudoholomorphic disks with boundary condition described by a curve $\gamma \in L^p(\partial \mathbb{D}, \mathbb{R}^n)$, $p > 2$. In order to get existence results we put the following admissibility condition.

**Definition 3.2.1.** A curve $\gamma \in L^p(\partial \mathbb{D}, \mathbb{R}^n)$ is called admissible if its analytic continuation, uniquely determined by $\bar{\partial} u = 0$, $\text{Re} \ u \mid_{\partial \mathbb{D}} = \gamma$ and $\text{Im} \ u(0) = 0$, belongs to $W^{1,p}(\mathbb{D}, \mathbb{C}^n)$.
The analytic continuation can be described componentwise by the Schwarz operator

\[ u^i(z) = S(\gamma^i) = \frac{1}{2\pi i} \int_{\partial D} \frac{t + z}{t - z} \gamma^i(t) \frac{dt}{t}. \]  

(3.2.1)

Alternatively, writing \( \gamma \) as a Fourier series

\[ \gamma(e^{it}) = \sum_{k \geq 0} a_k \cos kt + \sum_{l > 0} b_l \sin lt, \]

the power series expansion of \( u \) turns out to be

\[ u(z) = \sum_{n \geq 0} (a_n - ib_n)z^n. \]

The aim of this section is to provide sufficient conditions for \( \gamma \) to be admissible. In particular, we will apply this to the curves \( \gamma_{q_1,q_2,q_3} : \partial \Delta \to Q \).

In the course of the discussion we will employ the classical Banach spaces of holomorphic functions, i.e. Hardy and Bergman spaces. For \( u : D \to \mathbb{C} \) analytic and \( 1 < p < \infty \) we define the integral mean

\[ M_p(r,u) = \left( \frac{1}{2\pi} \int_0^{2\pi} |u(re^{i\theta})|^p |d\theta| \right)^{\frac{1}{p}}. \]

The Hardy space \( H^p(D) \) then consists of holomorphic functions \( u : D \to \mathbb{C} \) of bounded mean value. Thus we require the Hardy norm

\[ \|u\|_{H^p} = \lim_{r \to 1} M_p(r,u) \]  

(3.2.2)

to be finite. Each function \( u \in H^p(D) \) has a boundary function \( \tilde{u} \in L^p(\partial D, \mathbb{C}) \) and vice versa the analytic continuation of each \( \tilde{u} \in L^p(\partial D, \mathbb{R}) \) belongs to \( H^p(D) \). The Hardy spaces are studied in much detail in the book [Dur70].

We will need the following inequality due to Hardy-Littlewood (Theorem 5.11 in [Dur70]).

**Proposition 3.2.2.** If \( 1 < p < q \leq \infty \), \( u \in H^p(D) \), \( \lambda \geq p \) and \( \alpha = \frac{1}{p} - \frac{1}{q} \), then

\[ \int_0^1 (1 - r)^{\lambda \alpha - 1} (M_q(r,u))^\lambda \, dr < \infty. \]

Of the two sufficient admissibility conditions presented in the next theorem we will usually exploit the first one.

**Theorem 3.2.3.** Suppose \( \gamma \in L^p(\partial D, \mathbb{R}^n) \), \( p > 2 \) has an analytic continuation \( u : D \to \mathbb{C}^n \) with power series expansion

\[ u^i(z) = \sum_{m \geq 0} c_{im}z^m. \]

Then \( \gamma \) is admissible, i.e. \( u \in W^{1,p}(D, \mathbb{C}^n) \), if one of the following conditions is satisfied:

(a) One has

\[ \gamma \in W^{1,\frac{p}{2}}(\partial D, \mathbb{R}^n). \]
3.2. ADMISSIBLE BOUNDARY CONDITIONS

(b) One has \( p \geq 4 \) and for \( q = \frac{p}{p-2} \) and each \( 1 \leq i \leq n \)

\[ \{ m c_{im} \}_m \in l^q \]

holds.

Proof. It suffices to show this for \( n = 1 \). For \((a)\) consider \( \gamma \in W^{1,\frac{p}{2}}(\partial \mathbb{D}, \mathbb{R}) \) with analytic continuation

\[ u(z) = \sum_{m \geq 0} c_m z^m. \]

The Schwarz operator takes \( \gamma' \in L^{\frac{p}{2}}(\partial \mathbb{D}, \mathbb{R}) \) into \( H^p(\mathbb{D}) \). Since \( S(\gamma') = iu' \) we conclude that \( u' \in H^\frac{p}{2}(\mathbb{D}) \). We now have

\[ \|u'\|_{L^p(\mathbb{D}, \mathbb{C})}^p = \int_0^1 r \left( \int_0^{2\pi} |u'(re^{i\theta})|^p \, d\theta \right) \, dr \leq 2\pi \int_0^1 (M_p(r, u'))^p \, dr. \]

Taking advantage of Proposition 3.2.2 with \( 1 < \frac{p}{2} < p, \lambda = p \) and \( \alpha = \frac{p}{p-1} = \frac{1}{p} \) shows that the right hand side above is finite. Therefore \( u' \in L^p(\mathbb{D}, \mathbb{C}) \) follows. The same argument applied to \( \gamma \in L^{\frac{p}{2}}(\partial \mathbb{D}, \mathbb{R}) \) shows that \( u \in L^p(\mathbb{D}, \mathbb{C}) \) as well. Hence \( \gamma \) must be admissible. Part \((b)\) implies \((a)\) and thus admissibility of \( \gamma \) by the discrete Hausdorff-Young inequality. This can be used whenever \( q \leq 2 \), i.e. \( p \geq 4 \). \( \square \)

Corollary 3.2.4. If \( \gamma : \partial \mathbb{D} \to \mathbb{R}^n \) is \( C^1 \), then \( \gamma \) is admissible for any \( p > 2 \).

To get a different perspective we introduce the weighted Bergman spaces \( A^{p,\alpha}(\mathbb{D}) \) for \( \alpha, p > 0 \). They consist of holomorphic functions \( u \) such that the corresponding norm

\[ \|u\|_{A^{p,\alpha}} = \left( \int_{\mathbb{D}} |u(z)|^p \left( 1 - |z| \right)^{\alpha-1} \, dz \right)^{\frac{1}{p}} \]

is finite. The Hardy space \( H^p(\mathbb{D}) \) can be seen as the limiting case \( A^{p,0}(\mathbb{D}) \). The following proposition is a special case of the main theorem in [CK04] which deals with general domains in \( \mathbb{C}^n \).

Proposition 3.2.5. Let \( 0 < p \leq q < \infty \) and \( \alpha, \beta \geq 0 \) such that \( \frac{1+\alpha}{p} \leq \frac{1+\beta}{q} \). Then there is a continuous embedding \( A^{p,\alpha}(\mathbb{D}) \hookrightarrow A^{q,\beta}(\mathbb{D}) \).

Applying this with \( 0 < \frac{p}{2} < p < \infty \) and \( \alpha = 0, \beta = 1 \) shows that \( H^\frac{p}{2}(\mathbb{D}) \) embeds into \( A^{p,1}(\mathbb{D}) = L^p(\mathbb{D}) \). This leads to a different proof of Theorem 3.2.3 and shows additionally that for some constant \( C \)

\[ \|S(\gamma)\|_{W^{1,p}(\mathbb{D}, \mathbb{C}^n)} \leq C \|\gamma\|_{W^{1,\frac{p}{2}}(\partial \mathbb{D}, \mathbb{R}^n)}. \tag{3.2.3} \]

Let us finish this section by extending the notion of admissibility to any simply connected Jordan domain \( \Omega \subset \mathbb{C} \). By the Riemann mapping theorem there exists a biholomorphic map \( \Phi : \mathbb{D} \to \Omega \) which extends continuously onto the boundary. A map \( \gamma : \partial \Omega \to \mathbb{R}^n \) is then called admissible for \( p > 2 \) whenever \( \gamma \circ \Phi |_{\partial \mathbb{D}} \) has this property.
Lemma 3.2.6. Let $\Omega \subset \mathbb{C}$ be a domain whose boundary is a convex Euclidean polygon with interior angles $\alpha_k \pi$. Moreover, let $\gamma : \partial \Omega \to \mathbb{R}^n$ be a continuous map which is $C^1$ in the interior of any polygon side. Then $\gamma$ is admissible for $p > 2$ whenever

$$p < \frac{2}{1 - \alpha_k}$$

holds for all $k$.

Proof. The biholomorphism $\Phi : \mathbb{D} \to \Omega$ takes according to the Schwarz-Christoffel formulas the form

$$\Phi(z) = C \int_0^z \prod_{k=1}^n (z - z_k)^{\alpha_k - 1} \, dz + C'$$

where $z_k \in \partial \mathbb{D}$ are the points which are mapped to the corresponding polygon vertices. Let us see under which conditions $\gamma \circ \Phi |_{\partial \mathbb{D}}$ becomes an element of $W^{1,\frac{p}{2}}(\partial \mathbb{D}, \mathbb{R}^n)$. The only regions where integrability may fail are the vertices, i.e. in neighborhoods of the $z_k$. Taking advantage of the chain rule and since $\gamma$ is $C^1$, this reduces to the question whether

$$\int_0^\delta \left( r^{\alpha_k - 1} \right)^\frac{p}{2} \, dr$$

is finite for some $\delta > 0$. Of course, this is the case whenever $(\alpha_k - 1)\frac{p}{2} > -1$.

Corollary 3.2.7. Let $\Delta \subset S^2$ be a triangular domain with angles $\alpha_1, \alpha_2, \alpha_3 \in (0, \pi)$. Pick points $q_1, q_2, q_3$ on the Riemannian manifold $(Q, g)$ such that $q_2, q_3$ lie within the injectivity radius of $q_1$. Then $\gamma_{q_1,q_2,q_3} : \partial \Delta \to Q$ is admissible for

$$p < \frac{2\pi}{\pi - \min\{\alpha_1, \alpha_2, \alpha_3\}}.$$  \hspace{1cm} (3.2.4)

3.3 Applying the Implicit Function Theorem

We next describe the Banach space setup of the boundary value problem $u \in \mathcal{M}_+^*$ in coordinates. Starting with the observation that this problem can easily be solved for $J_0$, we like to apply the implicit function theorem in order to get solutions for nearby almost complex structures. It still might happen that a given almost complex structure $J$ is not contained in the neighborhood where the implicit function is defined. To get over this difficulty we have to employ a quantitative version. It will induce lower bounds on the size of the aforementioned neighborhood in terms of the size of $\gamma$ and thus allows us to conclude that $J$ is incorporated if $\gamma$ is small enough.

Fix $p > 2$ and a $C^2$-function $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $s : \mathbb{R}^n \to \mathbb{C}^n$ given by $s(x) = (x, h(x))$ locally describes the section $s$. Further let $\gamma \in L^p(\partial \mathbb{D}, \mathbb{R}^n)$ be an admissible curve such that its analytic continuation $u_0$ given by

$$\bar{\partial} u_0 = 0, \quad \text{Re} \, u_0 |_{\partial \mathbb{D}} = \gamma, \quad \text{Im} \, u_0(0) = h(\text{Re} \, u_0(0))$$

belongs to $W^{1,p}(\mathbb{D}, \mathbb{C}^n)$. Fix a radius $r > 0$ such that the image of $u_0$ is contained in the open ball $B_r(z_0) \subset \mathbb{C}^n$ with $z_0 = x_0 + iy_0 = u_0(0)$.

We consider the Banach space

$$W^{1,p}_{\text{im}}(\mathbb{D}, \mathbb{C}^n) = \{ u \in W^{1,p}(\mathbb{D}, \mathbb{C}^n) \mid \pi_1 \circ u(z) = 0 \text{ for } z \in \partial \mathbb{D} \}.$$  \hspace{1cm} (3.3.2)
3.3. APPLYING THE IMPLICIT FUNCTION THEOREM

Here \( \pi_1, \pi_2 : \mathbb{C}^n \to \mathbb{R}^n \) are the projections \( z \mapsto \text{Re} \, z, \ z \mapsto \text{Im} \, z \) onto real and imaginary part, respectively. With the identification \( \mathbb{C}^n \cong T\mathbb{R}^n \), \( \pi_1 \) can also be seen as the projection onto the zero section. Define the open subset

\[
\Omega_2 = \left\{ u \in W^{1,p}_{\text{loc}}(\mathbb{D}, \mathbb{C}^n) \mid u(\mathbb{D}) \subset B_r \right\}
\]

where we employ the shorthand notation \( B_r = B_r(0) \subset \mathbb{C}^n \).

On the open subset

\[
\Omega_1 = \{ X \in C^1_{\text{End}}(z_0, 2r) : \| X \|_{C^0} < 1 \}
\]

of the Banach space

\[
C^1_{\text{End}}(z_0, 2r) = C^1 \left( \overline{B_{2r}(z_0)}, \text{End}_\mathbb{R}(\mathbb{C}^n) \right)
\]

there is a map into the Banach manifold \( J(B_{2r}(z_0)) \) of almost complex structures on \( \overline{B_{2r}(z_0)} \) defined by

\[
X \mapsto J_X = \left( I + \frac{1}{2} J_0 X \right) J_0 \left( I + \frac{1}{2} J_0 X \right)^{-1}.
\]

It takes \( \Omega_1 \) surjectively to a neighborhood of \( J_0 \). Via the identification \( \text{End}_\mathbb{R}(\mathbb{C}^n) \cong \text{End}(\mathbb{R}^{2n}) \), the norm on \( C^1_{\text{End}}(z_0, 2r) \), which is the induced operator norm, is given by

\[
\| X \|_{C^1(\overline{B_{2r}(z_0)})} = \sup_{z \in \overline{B_{2r}(z_0)}} \left\{ \| X(z) \|_\infty + \sum_{j=1}^{2n} \left\| \frac{\partial X}{\partial z^j}(z) \right\|_\infty \right\}
\]

with \( \| \cdot \|_\infty \) denoting the maximum absolute row sum of a matrix.

Now let us define the Cauchy-Riemann map

\[
\Phi : \Omega_1 \times \Omega_2 \to L^p(\mathbb{D}, \mathbb{C}^n) \oplus \mathbb{R}^n
\]

given with \( u_\xi = u_0 + \xi \) by

\[
\Phi(X, \xi) = (\partial_\xi u_\xi + J_X(u_\xi) \partial_\xi u_\xi, \ \pi_2 \circ u_\xi(0) - h \circ \pi_1 \circ u_\xi(0))
\]

Note that due to the Sobolev embedding \( W^{1,p}(\mathbb{D}) \hookrightarrow C^0(\mathbb{D}) \) evaluation at zero is well-defined.

We continue by stating the quantitative implicit function theorem we are going to use. It yields explicit bounds on the size of the neighborhood where the implicit function is defined. The formulation and its proof are adapted from [Hol70].

**Theorem 3.3.1.** Let \( X, Y, Z \) be Banach spaces and

\[
F : \Omega_1 \times \Omega_2 \subset X \times Y \to Z
\]

a map of class \( C^k \) for some \( k \geq 1 \). Here \( \Omega_1 \times \Omega_2 \) denotes some open subset containing \((0,0)\).

Suppose the following conditions are satisfied:

(a) \( F(0,0) = 0 \).
(b) The operator $F_Y(0,0) \in \mathcal{L}(Y, Z)$ admits a bounded linear inverse $F_Y^{-1}(0,0) \in \mathcal{L}(Z, Y)$. Fix a constant $C_1 > 0$ such that $\|F_Y^{-1}(0,0)\| \leq C_1$.

(c) There are $\delta, \varepsilon > 0$ such that

$$S = \Omega_1 \times \overline{B_\varepsilon(0)} \subset \Omega_1 \times \Omega_2 \subset B_\delta(0) \times \Omega_2.$$

(d) There is a nonnegative function $g : [0, \delta] \times [0, \varepsilon] \to \mathbb{R}$ such that

$$\|F_Y(x, y) - F_Y(0,0)\| \leq g(\|x\|, \|y\|) \leq g(\delta, \varepsilon)$$

for $(x, y) \in S$.

(e) For $(x, 0) \in S$ and some constant $C_2$ we have

$$\|F(x, 0)\| \leq C_2 \|x\|.$$

(f) For some $\alpha \in (0, 1)$ one has

$$C_1 g(\delta, \varepsilon) \leq \alpha \quad \text{and} \quad C_1 C_2 \delta \leq (1 - \alpha) \varepsilon.$$

Then there exists a $C^k$-map $G : \Omega_1 \subset X \to Y$ such that $G(0) = 0$, $G(\Omega_1) \subset \overline{B_\varepsilon(0)}$ and $F(x, G(x)) = 0$ holds for $x \in \Omega_1$. The function $G$ is unique in the sense that if $(x, y_0) \in S$ satisfies $F(x, y_0) = 0$, then $y_0 = G(x)$.

**Proof.** We just show that $G$ is well-defined and unique by applying the contraction mapping principle. For $x \in \Omega_1$ let $P^x(y) : \overline{B_\varepsilon(0)} \subset Y \to Y$ be the map

$$P^x(y) = y - (F_Y^{-1}(0,0) \circ F)(x, y).$$

Consequently,

$$P_Y^x(y) = \text{Id} - F_Y^{-1}(0,0) \circ F_Y(x, y) = F_Y^{-1}(0,0) (F_Y(0,0) - F_Y(x, y))$$

and thus

$$\|P^x_Y(y)\| \leq \|F_Y^{-1}(0,0)\| \cdot \|F_Y(0,0) - F_Y(x, y)\| \leq C_1 g(\|x\|, \|y\|) \leq C_1 g(\delta, \varepsilon) \leq \alpha.$$ 

Moreover, we have

$$\|P^x(y)\| \leq \|(F_Y^{-1}(0,0) \circ F)(x,0)\| + \|P^x(y) - P^x(0)\| < C_1 C_2 \delta + \alpha \|y\| \leq (1 - \alpha) \varepsilon + \alpha \varepsilon = \varepsilon.$$

Therefore $P^x$ is a contraction which maps $\overline{B_\varepsilon(0)}$ into itself. We conclude that there must be a unique fixed point $G(x)$ within this ball. The equation $P^x(G(x)) = G(x)$ simplifies to $F(x, G(x)) = 0$. \qed

Let us denote the derivative of $J_X$ with respect to $\mathbb{R}^{2n} \cong \mathbb{C}^n$ by $dJ_X$. Moreover, let

$$D_Y J_X = \lim_{t \to 0} \frac{J_{X + ty} - J_X}{t}.$$

The following estimates of $J_X$ and its derivatives will be useful.
3.3. APPLYING THE IMPLICIT FUNCTION THEOREM

Lemma 3.3.2. For $X \in \Omega_1$ we have

$$\|J_X - J_0\|_{C^0(B_{2r}(x_0))} < 2 \|X\|_{C^0(B_{4r}(x_0))}, \quad (3.3.8)$$
$$\|dJ_X\|_{C^0(B_{2r}(x_0))} < 4 \|X\|_{C^1(B_{4r}(x_0))}, \quad (3.3.9)$$
$$\|D_Y J_X\|_{C^0(B_{2r}(x_0))} < 4 \|Y\|_{C^1(B_{4r}(x_0))}. \quad (3.3.10)$$

Proof. We express $J_X$ by a Neumann series. Taking advantage of

$$\|J_0X\|_{C^0} = \|X\|_{C^0} < 1,$$

we obtain

$$J_X - J_0 = \left( I + \frac{1}{2} J_0 X \right) J_0 \left( I + \frac{1}{2} J_0 X \right)^{-1} - J_0 = \left( J_0 + \frac{1}{2} J_0 X J_0 \right) \sum_{n=0}^{\infty} \left( -\frac{1}{2} J_0 X \right)^n - J_0$$

$$= \frac{1}{2} J_0 X J_0 + J_0 \left( I + \frac{1}{2} X J_0 \right) \sum_{n=1}^{\infty} \left( -\frac{1}{2} J_0 X \right)^n.$$

Therefore

$$\|J_X - J_0\|_{C^0} < \frac{1}{2} \|X\|_{C^0} + 3 \sum_{n=1}^{\infty} \left\| J_0 X \right\|_{C^0} \left( \frac{1}{2} \right)^n \|X\|_{C^0} = 2 \|X\|_{C^0}.$$

Next we calculate

$$dJ_X = \frac{1}{2} J_0 (dX) J_0 \sum_{n=0}^{\infty} \left( -\frac{1}{2} J_0 X \right)^n$$

$$+ \left( J_0 + \frac{1}{2} J_0 X J_0 \right) \sum_{n=1}^{\infty} \sum_{l=1}^{n} \left( -\frac{1}{2} \right)^n (J_0 X)^{l-1} (J_0 dX) (J_0 X)^{n-l}.$$

Consequently,

$$\|dJ_X\|_{C^0} < \left( \frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n + 3 \sum_{n=1}^{\infty} n \left( \frac{1}{2} \right)^n \right) \|X\|_{C^0} = \left( \frac{1}{2} \cdot 2 + \frac{3}{2} \cdot 2 \right) \|X\|_{C^1} = 4 \|X\|_{C^1}. \quad (3.3.10)$$

A similar estimate applied to

$$D_Y J_X = \frac{1}{2} J_0 Y J_0 \sum_{n=0}^{\infty} \left( -\frac{1}{2} J_0 X \right)^n + \left( J_0 + \frac{1}{2} J_0 X J_0 \right) \sum_{n=1}^{\infty} \sum_{l=1}^{n} \left( -\frac{1}{2} \right)^n (J_0 X)^{l-1} (J_0 Y) (J_0 X)^{n-l}$$

yields (3.3.10).

As expected, the map $\Phi$ turns out to be $C^1$ between the respective Banach spaces and we calculate its differential. This will be a first step in verifying the assumptions of the implicit function theorem.

Proposition 3.3.3. The map $\Phi$ is $C^1$ and its differential is given by

$$d\Phi(X, \xi)(Y, \eta) = \left( \partial_h \eta + D_Y J_X(u_\xi) \cdot \partial_h u_\xi + (dJ_X \circ u_\xi) \eta \cdot \partial_h u_\xi + (J_X \circ u_\xi) \partial_h \eta, \right.$$  
$$\pi_2 \circ \eta(0) - dh(\pi_1 \circ u_\xi(0)) \cdot (\pi_1 \circ \eta(0)) \right).$$
Proof. To verify the formula for the derivative we calculate
\[
\|J_{X+Y}(u_{\xi+\eta})\partial_t u_{\xi} - J_X(u_{\xi})\partial_t u_{\xi} - D_Y J_X(u_{\xi}) \cdot \partial_t u_{\xi} - (dJ_X \circ u_{\xi}) \eta \cdot \partial_t u_{\xi}\|_{L^p(D)}
\]
\[
\leq \|J_{X+Y}(u_{\xi+\eta}) - J_X(u_{\xi+\eta}) - D_Y J_X(u_{\xi+\eta})\|_{C^0(B_{2r}(z_0))} \cdot \|u_{\xi}\|_{W^{1,p}(D)}
\]
\[
+ \|D_Y J_X(u_{\xi}) - D_Y J_X(u_{\xi+\eta})\|_{C^0(D)} \cdot \|u_{\xi}\|_{W^{1,p}(D)}
\]
\[
+ \|J_X(u_{\xi+\eta}) - J_X(u_{\xi}) - (dJ_X \circ u_{\xi}) \eta\|_{C^0(D)} \cdot \|u_{\xi}\|_{W^{1,p}(D)}
\]
We may assume \(\|\eta\|_{W^{1,p}(D)} < \varepsilon\) where \(\varepsilon\) is chosen small enough to ensure \(\xi + \eta \in \Omega_2\). We then obtain for the first term
\[
\|J_{X+Y}(u_{\xi+\eta}) - J_X(u_{\xi+\eta}) - D_Y J_X(u_{\xi+\eta})\|_{C^0(D)} \cdot \|u_{\xi}\|_{W^{1,p}(D)}
\]
\[
\leq \|J_{X+Y} - J_X - D_Y J_X\|_{C^0(B_{2r}(z_0))} \cdot \|u_{\xi}\|_{W^{1,p}(D)}
\]
\[
= \|u_{\xi}\|_{W^{1,p}(D)} \cdot o \left( \|Y\|_{C^1(B_{2r}(z_0))} \right).
\]
To estimate the second term we use (3.3.10). Let \(C\) be the constant from the Sobolev embedding \(W^{1,p}(D) \hookrightarrow C^0(D)\). We have
\[
\|D_Y J_X(u_{\xi+\eta}) - D_Y J_X(u_{\xi})\|_{C^0(D)} \cdot \|u_{\xi}\|_{W^{1,p}(D)}
\]
\[
\leq C \|D_Y J_X\|_{C^0(B_{2r}(z_0))} \cdot \|\eta\|_{W^{1,p}(D)} \cdot \|u_{\xi}\|_{W^{1,p}(D)}
\]
\[
\leq 4C \|u_{\xi}\|_{W^{1,p}(D)} \cdot \|Y\|_{C^1(B_{2r}(z_0))} \cdot \|\eta\|_{W^{1,p}(D)}
\]
\[
= 4C \|u_{\xi}\|_{W^{1,p}(D)} \cdot o \left( \|(Y, \eta)\|_{C^1(B_{2r}(z_0))} \right).\]
Finally, for the last term we need a Lipschitz bound \(L\) of \(dJ_X\) on \(B_{2r}(z_0)\). With this we get
\[
\|J_X(u_{\xi+\eta}) - J_X(u_{\xi}) - (dJ_X \circ u_{\xi}) \eta\|_{C^0(D)} \cdot \|u_{\xi}\|_{W^{1,p}(D)}
\]
\[
\leq \left\| \int_0^1 dJ_X(u_{\xi+\eta}) dt \cdot \eta - (dJ_X \circ u_{\xi}) \eta \right\|_{C^0(D)} \cdot \|u_{\xi}\|_{W^{1,p}(D)}
\]
\[
\leq C \left\| \int_0^1 dJ_X(u_{\xi+\eta}) - dJ_X(u_{\xi}) dt \right\|_{C^0(D)} \cdot \|\eta\|_{W^{1,p}(D)} \cdot \|u_{\xi}\|_{W^{1,p}(D)}
\]
\[
\leq C L \int_0^1 t \|\eta\|_{C^0(D)} dt \cdot \|\eta\|_{W^{1,p}(D)} \cdot \|u_{\xi}\|_{W^{1,p}(D)}
\]
\[
\leq \frac{C^2 L}{2} \|u_{\xi}\|_{W^{1,p}(D)} \cdot \|\eta\|_{W^{1,p}(D)}^2 = \frac{C^2 L}{2} \|u_{\xi}\|_{W^{1,p}(D)} \cdot o \left( \|\eta\|_{W^{1,p}(D)} \right).
\]
Now putting everything together, let
\[
A(X, Y, \xi, \eta) = \partial_t \eta + D_Y J_X(u_{\xi}) \cdot \partial_t u_{\xi} + (dJ_X \circ u_{\xi}) \eta \cdot \partial_t u_{\xi} + (J_X \circ u_{\xi}) \partial_{\xi} \eta.
\]
Then we have shown
\[
\|\Phi(X + Y, \xi + \eta) - \Phi(X, \xi) - (A(X, Y, \xi, \eta), \pi_2 \circ \eta(0) - dh(\pi_1 \circ u_{\xi}(0)) (\pi_1 \circ \eta(0)))\|_{L^p(D) \oplus \mathbb{R}^n}
\]
\[
= o \left( \|\eta\|_{C^1(B_{2r}(z_0))} \right).\]
\[
\square
\]
For the partial derivative \(\Phi_{\xi}\) the proposition above implies the formula
\[
\Phi_{\xi}(X, \xi) \eta = (\partial_t \eta + (dJ_X \circ u_{\xi}) \eta \cdot \partial_t u_{\xi} + (J_X \circ u_{\xi}) \partial_{\xi} \eta, \pi_2 \circ \eta(0) - dh(\pi_1 \circ u_{\xi}(0)) (\pi_1 \circ \eta(0))).\quad (3.3.11)
\]
3.3. APPLYING THE IMPLICIT FUNCTION THEOREM

**Proposition 3.3.4.** The operator

\[ \Phi_\xi(0, 0) : W^{1,p}_\text{im}(\mathbb{D}, \mathbb{C}^n) \rightarrow L^p(\mathbb{D}, \mathbb{C}^n) \oplus \mathbb{R}^n \]

is an isomorphism.

**Proof.** From (3.3.11) we see that

\[ \Phi_\xi(0, 0) \eta = (\tilde{\partial} \eta, \pi_2 \circ \eta(0) - dh(\pi_1(z_0)) \cdot (\pi_1 \circ \eta(0))) . \]

The Cauchy-Riemann operator

\[ \tilde{\partial} : W^{1,p}_\text{im}(\mathbb{D}, \mathbb{C}^n) \rightarrow L^p(\mathbb{D}, \mathbb{C}^n) \]

is Fredholm of index \( n \). The kernel of this operator is given by holomorphic functions with vanishing real part on \( \partial \mathbb{D} \). Applying the Schwarz operator (3.2.1) shows that the kernel consists of constant functions \( u(z) = ih, h \in \mathbb{R}^n \). Since this is an \( n \)-dimensional subspace of \( W^{1,p}_\text{im}(\mathbb{D}, \mathbb{C}^n) \), we conclude that \( \tilde{\partial} \) is surjective. The statement now follows from the observation that the second component of \( \Phi_\xi(0, 0) \) parametrizes the kernel of \( \tilde{\partial} \).

We next establish the estimates described in (d) and (e) of the implicit function theorem.

**Lemma 3.3.5.** Recalling \( y_0 = \pi_2(z_0) \), we have

\[ \| \Phi(X, 0) \|_{L^p(\mathbb{D}, \mathbb{C}^n) \oplus \mathbb{R}^n} < 2 \| u_0 - iy_0 \|_{W^{1,p}(\mathbb{D}, \mathbb{C}^n)} \cdot \| X \|_{C^1(\overline{B_2(z_0)})} \]

**Proof.** With (3.3.7) in mind we calculate

\[
\begin{align*}
\Phi(X, 0) &= (\partial_s u_0 + J_X(u_0) \partial_s u_0, \pi_2(z_0) - h \circ \pi_1(z_0)) \\
&= (\partial_s u_0 + J_0(u_0) \partial_s u_0 + (J_X - J_0)(u_0) \partial_s u_0, 0) \\
&= ((J_X - J_0)(u_0) \partial_s u_0, 0).
\end{align*}
\]

Hence employing (3.3.8) yields

\[
\begin{align*}
\| \Phi(X, 0) \|_{L^p(\mathbb{D}, \mathbb{C}^n) \oplus \mathbb{R}^n} &\leq \| (J_X - J_0)(u_0) \partial_s u_0 \|_{L^p(\mathbb{D})} \leq \| J_X - J_0 \|_{C^0(\overline{B_2(z_0)})} \cdot \| \partial_s u_0 \|_{L^p(\mathbb{D})} \\
&\leq \| J_X - J_0 \|_{C^0(\overline{B_2(z_0)})} \cdot \| u_0 - iy_0 \|_{W^{1,p}(\mathbb{D})} \\
&< 2 \| X \|_{C^1(\overline{B_2(z_0)})} \cdot \| u_0 - iy_0 \|_{W^{1,p}(\mathbb{D})}.
\end{align*}
\]

**Lemma 3.3.6.** Let \( C \) be the constant from the Sobolev embedding \( W^{1,p}(\mathbb{D}) \rightarrow C^0(\mathbb{D}) \). Then

\[
\| \Phi_\xi(X, \xi) - \Phi_\xi(0, 0) \|_{L^p(W^{1,p}(\mathbb{D}), L^p(\mathbb{D}) \oplus \mathbb{R}^n)} < 4C \left( \| u_0 - iy_0 \|_{W^{1,p}(\mathbb{D})} + \| \xi \|_{W^{1,p}(\mathbb{D})} \right) \| X \|_{C^1(\overline{B_2(z_0)})} \\
+ 2 \| X \|_{C^0(\overline{B_2(z_0)})} + C^2 \| h \|_{C^2(\mathbb{R}^n)} \| \xi \|_{W^{1,p}(\mathbb{D})}.
\]

**Proof.** By (3.3.11) we see

\[
\begin{align*}
(\Phi_\xi(X, \xi) - \Phi_\xi(0, 0)) \eta &= (\partial_s \eta + (dJ_X \circ u_\xi) \eta \cdot \partial_s u_\xi + (J_X \circ u_\xi) \partial_s \eta - \tilde{\partial} \eta, \\
&- dh(\pi_1 \circ u_\xi(0)) \cdot (\pi_1 \circ \eta(0)) + dh(\pi_1 \circ u_0(0)) \cdot (\pi_1 \circ \eta(0))) \\
&= ((dJ_X \circ u_\xi) \eta \cdot \partial_s u_\xi + (J_X \circ u_\xi - J_0 \circ u_0) \partial_s \eta, \\
&- dh(\pi_1 \circ u_\xi(0)) \cdot (\pi_1 \circ \eta(0)) + dh(\pi_1 \circ u_0(0)) \cdot (\pi_1 \circ \eta(0))).
\end{align*}
\]
For any $\eta \in W^{1,p} (\mathbb{D}, \mathbb{C}^n)$ we have
\[
\|(dJ_X \circ u_\xi) \eta \cdot \partial_t u_\xi + (J_X \circ u_\xi - J_0 \circ u_0) \partial_t \eta\|_{L^p(\mathbb{D})} \\
\leq \|(dJ_X \circ u_\xi) \eta\|_{C^0(\mathbb{D})} \|\partial_t u_\xi\|_{L^p(\mathbb{D})} + \|(J_X \circ u_\xi - J_0 \circ u_0) \partial_t \eta\|_{L^p(\mathbb{D})} \\
\leq \|(dJ_X \circ u_\xi) \eta\|_{C^0(\mathbb{B}_{2r}(z_0))} \|\eta\|_{C^0(\mathbb{D})} \left(\|u_0 - iy_0\|_{W^{1,p}(\mathbb{D})} + \|\xi\|_{W^{1,p}(\mathbb{D})}\right) + \|((J_X - J_0) \circ u_\xi) \partial_t \eta\|_{L^p(\mathbb{D})} \\
\leq \|dJ_X\|_{C^0(\mathbb{B}_{2r}(z_0))} \|\eta\|_{C^0(\mathbb{D})} \left(\|u_0 - iy_0\|_{W^{1,p}(\mathbb{D})} + \|\xi\|_{W^{1,p}(\mathbb{D})}\right) \\
+ \|J_X - J_0\|_{C^0(\mathbb{B}_{2r}(z_0))} \|\eta\|_{W^{1,p}(\mathbb{D})} \\
< 4C \|X\|_{C^1(\mathbb{B}_{2r}(z_0))} \|\eta\|_{W^{1,p}(\mathbb{D})} \left(\|u_0 - iy_0\|_{W^{1,p}(\mathbb{D})} + \|\xi\|_{W^{1,p}(\mathbb{D})}\right) \\
+ 2\|X\|_{C^0(\mathbb{B}_{2r}(z_0))} \|\eta\|_{W^{1,p}(\mathbb{D})}.
\]

In the course we employed the estimates from Lemma 3.3.2. Moreover,
\[
\|dh(\pi_1 \circ u_0(0)) \cdot (\pi_1 \circ \eta(0)) - dh(\pi_1 \circ u_\xi(0)) \cdot (\pi_1 \circ \eta(0))\|_{\mathbb{R}^n} \\
\leq \|h\|_{C^2(\mathbb{R}^n)} \cdot \|\pi_1(0)\| \cdot \|\eta(0)\| \leq C^2 \|h\|_{C^2(\mathbb{R}^n)} \|\pi_1\|_{W^{1,p}(\mathbb{D})} \|\eta\|_{W^{1,p}(\mathbb{D})}.
\]

\[
\square
\]

We will now apply the implicit function theorem to the Cauchy-Riemann map $\Phi$.

**Theorem 3.3.7.** Let $p > 2$, $h : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^2$-function and $\gamma \in L^p (\partial \mathbb{D}, \mathbb{R}^n)$ an admissible curve in the zero section of $T\mathbb{R}^n$. Denote its analytic continuation by $u_0 \in W^{1,p}(\mathbb{D}, \mathbb{C}^n)$ such that $\text{Im} \ u_0(0) = h(\text{Re} \ u_0(0))$. Set $z_0 = x_0 + iy_0 = u_0(0)$ and fix $r > 0$ such that $u_0(\mathbb{D}) \subset B_r(z_0)$.

Let us fix in addition the following notation:

(a) Consider the isomorphism $\Psi : W^{1,p}_{\text{Im}}(\mathbb{D}, \mathbb{C}^n) \to L^p (\mathbb{D}, \mathbb{C}^n) \oplus \mathbb{R}^n$ given by

\[
\Psi(u) = \left(\partial U, \pi_2 \circ u(0) - dh(\pi_1(z_0)) \cdot (\pi_1 \circ u(0))\right).
\]

Let

\[
C_1 = \|\Psi^{-1}\|_{L^p(\mathbb{D}, \mathbb{C}^n) \oplus \mathbb{R}^n, W^{1,p}(\mathbb{D}, \mathbb{C}^n)}.
\]

(b) Let $C_2$ be the constant from the Sobolev embedding $W^{1,p}(\mathbb{D}) \hookrightarrow C^0(\mathbb{D})$.

(c) Choose $\varepsilon > 0$ such that

\[
\varepsilon < \min \left(\frac{1}{4C_1C_2^2 \|h\|_{C^2(\mathbb{R}^n)}}, \frac{r}{C_2}\right)
\]

and set

\[
\delta = \left(4C_1 \left(2C_2 \left(\|u_0 - iy_0\|_{W^{1,p}(\mathbb{D})} + \varepsilon\right) + \frac{\|u_0 - iy_0\|_{W^{1,p}(\mathbb{D})}}{\varepsilon}\right)\right)^{-1}.
\]
3.3. APPLYING THE IMPLICIT FUNCTION THEOREM

Then for every \( X \in C^1 \left( \overline{B_{2r}(z_0)}, \operatorname{End}_\mathbb{R}(\mathbb{C}^n) \right) \) with

\[
\|X\|_{C^1(\overline{B_{2r}(z_0)})} < \delta \quad \text{and} \quad \|X\|_{C^0(\overline{B_{2r}(z_0)})} < \min \left( \frac{1}{8C_1}, 1 \right)
\]

there is a unique \( \xi \in \overline{B_{r}(0)} \subset W^{1,p}_{\text{im}}(\mathbb{D}, \mathbb{C}^n) \) such that \( u = u_0 + \xi \) satisfies \( \pi_2 \circ u(0) = h \circ \pi_1 \circ u(0) \), the Cauchy-Riemann equation

\[
\partial_x u + J_X(u) \partial_{\xi} u = 0
\]

and the boundary condition \( \pi_1 \circ u |_{\partial \mathbb{D}} = \gamma \).

Proof. Set

\[
\Omega_\delta = \left\{ X \in C^1_{\operatorname{End}}(z_0, 2r) : \|X\|_{C^0} < \min \left( \frac{1}{8C_1}, 1 \right) \quad \text{and} \quad \|X\|_{C^1} < \delta \right\}.
\]

The general idea will be to apply Theorem 3.3.1 to the map

\[\Phi : \Omega_\delta \times \Omega_2 \subset C^1_{\operatorname{End}}(z_0, 2r) \times W^{1,p}_{\text{im}}(\mathbb{D}, \mathbb{C}^n) \to L^p(\mathbb{D}, \mathbb{C}^n) \oplus \mathbb{R}^n\]

defined in (3.3.7). In order to do so, we have to verify the various assumptions.

(i) The map \( \Phi \) is of class \( C^1 \), \( \Phi(0, 0) = 0 \) and

\[\Phi_{\xi}(0, 0) \in \mathcal{L} \left( W^{1,p}_{\text{im}}(\mathbb{D}, \mathbb{C}^n), L^p(\mathbb{D}, \mathbb{C}^n) \oplus \mathbb{R}^n \right)\]

admits a bounded linear inverse with norm \( C_1 \).

First of all, we have

\[
\Phi(0, 0) = (\partial_x u_0 + J_0(u_0) \partial_{\xi} u_0, \pi_2 \circ u_0(0) - h \circ \pi_1 \circ u_0(0)) = (\partial u_0, \pi_2(z_0) - h \circ \pi_1(z_0)) = (0, 0).
\]

The rest follows from Proposition 3.3.3 and 3.3.4.

(ii) \( S = \Omega_\delta \times \overline{B_{r}(0)} \subset \Omega_\delta \times \Omega_2 \subset B_{\delta}(0) \times \Omega_2 \).

The second inclusion is obvious. Now recall the definition of \( \Omega_2 \) from (3.3.3) and let \( \xi \in \overline{B_{r}(0)} \). Then

\[
\|\xi\|_{C^0(\mathbb{D})} \leq C_2 \|\xi\|_{W^{1,p}(\mathbb{D})} \leq C_2 \varepsilon < r
\]

and therefore \( \xi \in \Omega_2 \).

(iii) Define the nonnegative function \( g : [0, \delta] \times [0, \varepsilon] \to \mathbb{R} \) by

\[
g(x, y) = \frac{1}{4C_1} + 4C_2 x \left( \|u_0 - iy_0\|_{W^{1,p}(\mathbb{D})} + y \right) + C_2^2 \|h\|_{C^2(\mathbb{R}^n)} y. \tag{3.3.13}
\]

Then

\[
\|\Phi_{\xi}(X, \xi) - \Phi_{\xi}(0, 0)\|_{\mathcal{L}(W^{1,p}(\mathbb{D}), L^p(\mathbb{D}) \oplus \mathbb{R}^n)} \leq g \left( \|X\|_{C^1(\overline{B_{2r}(z_0)}), \|\xi\|_{W^{1,p}(\mathbb{D})}} \right) \leq g(\delta, \varepsilon)
\]

for \( (X, \xi) \in S \). Moreover,

\[
\|\Phi(X, 0)\|_{L^p(\mathbb{D}) \oplus \mathbb{R}^n} < 2 \|u_0 - iy_0\|_{W^{1,p}(\mathbb{D})} \|X\|_{C^1(\overline{B_{2r}(z_0)})}
\]

for \( (X, 0) \in S \).

The first inequality follows from Lemma 3.3.6 and the obvious fact that \( g \) is monotone increasing in each argument. The second inequality is Lemma 3.3.5.
(iv) With $C_3 = 2\|u_0 - iy_0\|_{W^{1,p}}$ and $g$ defined as in (3.3.13) there is $\alpha \in (0, 1)$ such that

$$C_1 g(\delta, \varepsilon) \leq \alpha \quad \text{and} \quad C_1 C_3 \delta \leq (1 - \alpha) \varepsilon.$$ 

A quick calculation leads to

$$\frac{C_1 C_3 \delta}{\varepsilon} = \frac{\|u_0 - iy_0\|}{4 C_2 \varepsilon (\|u_0 - iy_0\| + \varepsilon) + 2 \|u_0 - iy_0\|} < \frac{1}{2}.$$ 

Hence the latter inequality will be satisfied by the choice

$$\alpha = 1 - \frac{C_1 C_3 \delta}{\varepsilon} > 0.$$ 

The first inequality follows from

$$C_1 g(\delta, \varepsilon) + \frac{C_1 C_3 \delta}{\varepsilon} = \frac{1}{4} + 4 C_1 C_2 \delta (\|u_0 - iy_0\| + \varepsilon) + C_1 C_2^2 \|h\| \varepsilon + \frac{2 C_1 \|u_0 - iy_0\| \delta}{\varepsilon} < \frac{1}{4} + 2 C_1 \delta \left( 2 C_2 (\|u_0 - iy_0\| + \varepsilon) + \frac{\|u_0 - iy_0\|}{\varepsilon} \right) + \frac{1}{4} = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1.$$ 

In summary, all assumptions of Theorem 3.3.1 are satisfied. Hence we may conclude that for every $X \in C^1(B_{2r}(z_0), \text{End}_\mathbb{R}(\mathbb{C}^n))$ with

$$\|X\|_{C^1(B_{2r}(z_0))} < \delta \quad \text{and} \quad \|X\|_{C^0(B_{2r}(z_0))} < \min\left(\frac{1}{8 C_1}, 1\right)$$

there is a unique $\xi \in \mathbb{B}_r(0) \subset W^{1,p}(\mathbb{D}, \mathbb{C}^n)$ satisfying $\Phi(X, \xi) = 0$. Consequently, we have $\pi_2 \circ u_\xi(0) = h \circ \pi_1 \circ u_\xi(0)$ and

$$\partial_s u_\xi + J_X(u_\xi) \partial_t u_\xi = 0.$$ 

Since $\xi \in W^{1,p}(\mathbb{D}, \mathbb{C}^n)$ we also deduce the boundary condition

$$\pi_1 \circ u_\xi(z) = \pi_1 \circ u_0(z) + \pi_1 \circ \xi(z) = \pi_1 \circ u_0(z) = \gamma(z)$$

for $z \in \partial \mathbb{D}$.

As already mentioned, it will be crucial to get $\delta$ sufficiently large. This can be accomplished by picking $\varepsilon = \sqrt{\|u_0 - iy_0\|}$. The point is that

$$\delta = \frac{1}{4 C_1 \left( 2 C_2 \left( \|u_0 - iy_0\| + \sqrt{\|u_0 - iy_0\|} \right) + \sqrt{\|u_0 - iy_0\|} \right)} \quad (3.3.14)$$

can be made arbitrary large by choosing $\|u_0 - iy_0\|_{W^{1,p}(\mathbb{D}, \mathbb{C}^n)}$ sufficiently small.
3.4 Existence Results in the Tangent Bundle

Given a point \( q \in Q \) and a curve \( \gamma : \partial \mathbb{D} \to Q \) in the range of the exponential map \( \exp_q \) we show existence results for pseudoholomorphic curves \( u : \mathbb{D} \to TQ \) satisfying \( \pi \circ u \mid_{\partial \mathbb{D}} = \gamma \). Taking advantage of the preparations done in the previous section, the existence result will hold for arbitrary almost complex structures tamed by the standard or magnetic symplectic form under the assumption that \( \gamma \) is sufficiently small. We will particularly focus on the Levi-Civita case.

For convenience, we introduce normal coordinates around \( q \) and thus obtain a bundle chart defined on \( \pi^{-1} \circ \exp_q(T_qQ) \) as explained in section 2.3. With Theorem 3.3.7 in mind it is easy to understand that we will need \( C^1 \)-estimates of the involved almost complex structure. In the Levi-Civita case the almost complex structure \( J_{LC} \) was described in local coordinates by (2.3.7). Comparing with (3.3.5) yields

\[
X(x,\xi) = 2J_0 \begin{pmatrix}
0 & 0 \\
P^t(x,\xi) & 0
\end{pmatrix} = -2 \begin{pmatrix}
P^t(x,\xi) & 0 \\
0 & 0
\end{pmatrix},
\]

recalling \( P^j_i(x,\xi) = \Gamma^j_{ik}(x) \xi^k \). Hence \( C^1 \)-estimates of \( J_{LC} \) require \( C^1 \)-bounds of the Christoffel symbols.

This delicate question was first studied in [Kau76] where it was shown that one needs bounds on the curvature tensor \( R \) and its derivative \( \nabla R \) in order to obtain \( C^k \)-bounds for the Christoffel symbols. Later Jost (see chapter 10 in [Jos84]) used harmonic coordinates to deduce \( C^\infty \)-bounds for the Christoffel symbols depending only on bounds of \( R \). However, the disadvantage of this approach was that in high dimension existence of harmonic coordinates can be shown only on small balls. Finally, Eichhorn answered the general question of \( C^k \)-bounds in normal coordinates. The next proposition follows immediately from Theorem A and Theorem 4.2 in [Eic91].

**Proposition 3.4.1.** Let \((Q, g)\) be a Riemannian manifold with a normal chart defined on a metric ball \( B_r(q) \) and centered at \( q \). Assume that the curvature tensor \( R \) is bounded by

\[
\|\nabla^j R\|_{B_r(q)} \leq \mu_j
\]

for \( 0 \leq j \leq n \). Then there is a constant \( C(r, \mu_0, \ldots, \mu_n) \) such that the Christoffel symbols in normal coordinates satisfy

\[
\|\Gamma^k_{ij}\|_{C^{n-1}(B_r)} \leq C.
\]

Moreover, \( C \) is of power series character in \( r \) and of polynomial character in \( \mu_0, \ldots, \mu_n \).

Note that curvature prevents us from getting arbitrary small \( C^1 \)-bounds for \( J_{LC} \). Hence the best we can hope for is to employ the above result in order to deduce \( C^1 \)-bounds depending on curvature and sufficiently small \( C^0 \)-bounds.

**Lemma 3.4.2.** Let \( TQ \) be the tangent bundle of a Riemannian manifold \((Q, g)\). For a point \( q \in Q \) consider a normal chart on a geodesic ball and its induced bundle chart on \( \pi^{-1}(B_r(q)) \). Moreover, let \( \xi = z = (0, h) \) be a point in the fiber over \( q \) and suppose we have curvature bounds

\[
\|\nabla^j R\|_{B_r(q)} \leq \mu_j
\]

for \( j = 0, 1, 2 \). Then the following holds:
(a) There is a constant \( C(r, \mu_0, \mu_1, \mu_2, \|h\|) \) such that
\[
\|P\|_{C^1(B_r(z))} < C.
\]

(b) For \( \varepsilon > 0 \) there is a constant \( \rho(r, \mu_0, \mu_1, \|h\|, \varepsilon) \leq r \) such that
\[
\|P\|_{C^0(B_r(z))} < \varepsilon.
\]

Proof.

(A) Let \( C_1(r, \mu_0, \mu_1, \mu_2) \) be the constant for the \( C^1 \)-bound in Proposition 3.4.1. Then with
\[
C(r, \mu_0, \mu_1, \mu_2, \|h\|) = 2n(2n + 1) \max \left\{ 1, \sqrt{n} (\|h\| + r) \right\} \ C_1(r, \mu_0, \mu_1, \mu_2)
\]
we obtain by the Cauchy-Schwarz inequality with \( n = \dim Q \)
\[
\left\| P_i^j \right\|_{C^0(B_r(z))} \leq \sqrt{n} \left( \max_{1 \leq k \leq n} \left\| \Gamma_{ik}^j \right\|_{C^0(B_r)} \right) (\|h\| + r) \leq \sqrt{n} (\|h\| + r) C_1 < \frac{C}{n(2n + 1)}.
\]
Similarly,
\[
\left\| \frac{\partial P_i^j}{\partial x^l} \right\|_{C^0(B_r(z))} \leq \sqrt{n} \left( \max_{1 \leq k \leq n} \left\| \Gamma_{ik}^j \right\|_{C^1(B_r)} \right) (\|h\| + r) \leq \sqrt{n} (\|h\| + r) C_1 < \frac{C}{n(2n + 1)}
\]
and
\[
\left\| \frac{\partial P_i^j}{\partial \xi^l} \right\|_{C^0(B_r(z))} = \left\| \Gamma_{il}^j \right\|_{C^0(B_r)} \leq C_1 \leq \frac{C}{n(2n + 1)}.
\]

Recalling the norm formula (3.3.6) and summing up the entrywise estimates implies the statement.

(B) In a first step we will show that we can choose \( \rho(r, \mu_0, \mu_1, \|h\|) < r \) such that
\[
\left\| \Gamma_{ij}^k \right\|_{C^0(B_r)} \leq \frac{\varepsilon}{n\sqrt{n} (\|h\| + r)}.
\]
(3.4.2)

The inequality follows easily for small \( \rho \) from \( \Gamma_{ij}^k (0) = 0 \). The difficulty consists in choosing \( \rho \) just in terms of \( r, \mu_0, \mu_1 \). To achieve this we will invoke a scaling argument.

Let \( C_0(r, \mu_0, \mu_1) \) be the constant for the \( C^0 \)-bound in Proposition 3.4.1. Looking carefully at [Eic91] (in particular, the initial step of the induction on p. 148–151), one realizes that Eichhorn shows indeed
\[
C_0(r, \mu_0, \mu_1) = A + B(r, \mu_0, \mu_1) \cdot r
\]
where \( A \) is a positive constant independent of \( r, \mu_0, \mu_1 \) and \( B \) is of power series character in \( r \) and of polynomial character in \( \mu_0, \mu_1 \). For
\[
\lambda = \lambda(r, \|h\|, \varepsilon) = \max \left\{ 1, \frac{2n\sqrt{n}}{\varepsilon} (\|h\| + r) A \right\}
\]
we may define the metric \( \tilde{g}_{ij}(x) = g_{ij}(\lambda x) \) on the ball \( B_{\tilde{r}}(0) \) of radius \( \tilde{r} = \frac{r}{\lambda} \). Note that the corresponding Christoffel symbols satisfy \( \tilde{\Gamma}^k_{ij}(x) = \lambda \Gamma^k_{ij}(\lambda x) \) and the curvature scales with \( \nabla^j \tilde{R}(x) = \lambda^{j+2} \nabla^j R(\lambda x) \).

Hence we obtain for \( \rho < r \leq \lambda r \)
\[
\| \Gamma^k_{ij} \|_{C^0(B_r)} = \frac{1}{\lambda} \| \tilde{\Gamma}^k_{ij} \|_{C^0(B_{\lambda r})} \leq \frac{1}{\lambda} C_0 \left( \frac{\rho}{\lambda}, \lambda^2 \mu_0, \lambda^3 \mu_1 \right) \leq \frac{A}{\lambda} + \frac{B \left( \frac{\rho}{\lambda}, \lambda^2 \mu_0, \lambda^3 \mu_1 \right)}{\lambda^2}.
\]

For
\[
M = M(r, \mu_0, \mu_1, \| h \|, \varepsilon) = \max_{t \in [0, r]} \left| B \left( \frac{t}{\lambda}, \lambda^2 \mu_0, \lambda^3 \mu_1 \right) \right|
\]
and
\[
\rho = \rho(r, \mu_0, \mu_1, \| h \|, \varepsilon) = \min \left\{ r, \frac{\lambda^2 \varepsilon}{2n \sqrt{n} (\| h \| + r) M} \right\}
\]
we conclude
\[
\| \Gamma^k_{ij} \|_{C^0(B_r)} \leq \frac{A}{\lambda} + \frac{\varepsilon}{2n \sqrt{n} (\| h \| + r)} \leq \frac{\varepsilon}{n \sqrt{n} (\| h \| + r)}.
\]

Consequently, (3.4.2) holds as desired.

Now, by the Cauchy-Schwarz inequality we have
\[
\| P^j_i \|_{C^0(B_r)} \leq \sqrt{n} \left( \max_{1 \leq k \leq n} \| \Gamma^j_{ik} \|_{C^0(B_r)} \right) (\| h \| + \rho) \leq \frac{\varepsilon \sqrt{n} (\| h \| + \rho)}{n \sqrt{n} (\| h \| + r)} < \frac{\varepsilon}{n}.
\]

Summing up the entrywise estimates on rows of \( P \) proves the claim.

\[\square\]

These preliminaries allow us to prove the following existence result in the Levi-Civita case. In particular, it can be applied to the piecewise geodesic curves \( \gamma_{q_1,q_2,q_3} \).

**Theorem 3.4.3.** Let \( TQ \) be the tangent bundle of a Riemannian manifold. Consider a normal chart on a metric ball and its induced bundle chart on \( \pi^{-1}(B_r(q)) \). Suppose the curvature is bounded by
\[
\| \nabla^j R \|_{B_r(q)} \leq \mu_j \quad (3.4.3)
\]
for \( j = 0, 1, 2 \). Moreover, fix \( p > 2 \) and let \( s : Q \to TQ \) be a \( C^2 \)-section.

Then there is a constant \( C > 0 \) just depending on \( p, r, \mu_0, \mu_1, \mu_2, \| s \|_{C^2} \) such that for any \( \gamma : \partial \mathbb{D} \to \mathbb{R}^n \) in this chart with
\[
\| \gamma \|_{W^{1,\frac{p}{2}}(\partial \mathbb{D})} < C \quad (3.4.4)
\]
there is a \( J_{LC} \)-holomorphic disk \( u : \mathbb{D} \to TQ \) with \( \pi \circ u \mid_{\partial \mathbb{D}} = \gamma \) and \( u(0) \in s(Q) \).
Proof. The section can be expressed in coordinates as $s : \mathbb{R}^n \to \mathbb{C}^n$ with $s = (\text{id}, h)$. We set $h_0 = h(0)$ such that $\|h_0\| = \|s(q)\|$ by the properties of normal coordinates. Further denote by $u_0$ the holomorphic extension of $\gamma$ satisfying $\pi_2 \circ u_0(0) = 0$ and let $x_0 = \pi_1 \circ u_0(0)$. Define the constants $C_1, C_2$ as in Theorem 3.3.7 and note that both constants depend on $p$. According to (3.2.3) there is a constant $C_3$ such that

$$\|u_0\|_{W^{1,p}(\mathbb{D})} \leq C_3 \|\gamma\|_{W^{1,\frac{p}{2}}(\partial \mathbb{D})}.$$  

Let $C_4 = C_4(r, \mu_0, \mu_1, \mu_2, \|h_0\|)$ be the constant in Lemma 3.4.2 (a) corresponding to the geometric assumptions. In the following we will always measure the size of $h$ in $C^2$, of $u_0$ in $W^{1,p}(\mathbb{D})$ and $\gamma$ in $W^{1,\frac{p}{2}}(\partial \mathbb{D})$.

Denote by $C_5 = C_5(p, r, \mu_0, \mu_1, \mu_2, \|h\|)$ the largest positive constant such that

$$2C_4 < \frac{1}{4C_1 \left( 2C_2 \left( \|u_0\| + \sqrt{\|u_0\|} \right) + \sqrt{\|u_0\|} \right)}$$

holds whenever $\|u_0\| < C_5$. With

$$C_6 = \min \left( \frac{1}{8C_1}, 1 \right)$$

and $\rho = \rho(r, \mu_0, \mu_1, \|h_0\|, \frac{1}{2}C_6)$ from Lemma 3.4.2 (b) define

$$C = \frac{1}{C_3} \min \left\{ C_5, \frac{1}{(1 + \|h\|)^2}, \frac{1}{17C^2_4 C^2_2 \|h\|^2}, \left( \frac{\rho}{8C_2} \right)^2 \right\}.$$  

Since $\rho, C_5$ and $C_6$ only depend on $p, r, \mu_0, \mu_1, \mu_2, \|h\|$, $C$ does as well. We will prove that $C$ is the desired constant by applying Theorem 3.3.7. Note that the shifted function $u_0 - iy_0$ in there corresponds to $u_0$ in our situation here.

Consider

$$X(x, y) = -2 \begin{pmatrix} P^l(x, y) & 0 \\ 0 & 0 \end{pmatrix}$$

and set $\varepsilon = \sqrt{\|u_0\|}$ and $\sigma = 2C_2 \sqrt{\|u_0\|}$. This choice guarantees

$$\varepsilon < \min \left( \frac{1}{4C_1 C_2^2 \|h\|}, \frac{\sigma}{C_2} \right)$$

whenever $\|u_0\| < C_3 C$. Further let

$$\delta = \frac{1}{4C_1 \left( 2C_2 \left( \|u_0\| + \varepsilon \right) + \frac{\|u_0\|}{\varepsilon} \right)} = \frac{1}{4C_1 \left( 2C_2 \left( \|u_0\| + \sqrt{\|u_0\|} \right) + \sqrt{\|u_0\|} \right)}.$$  

We then have

$$\|u_0\|_{C^0(\mathbb{D})} \leq C_2 \|u_0\|_{W^{1,p}(\mathbb{D})} \leq \frac{C_2}{1 + \|h\|} \sqrt{\|u_0\|_{W^{1,p}(\mathbb{D)}}} \leq \frac{\sigma}{2 \left( 1 + \|h\| \right)}$$ (3.4.7)
3.4. EXISTENCE RESULTS IN THE TANGENT BUNDLE

whenever \( \|u_0\|_{W^{1,p}(\mathbb{D})} < C_3 C \leq \left( \frac{1}{1 + \|h\|} \right)^2 \) and hence with \( z_0 = u_0(0) + ih \circ u_0(0) = x_0 + ih(x_0) \) we obtain

\[
\|z_0 - ih_0\| \leq \|x_0\| + \|h(x_0) - h_0\| \leq \|x_0\| (1 + \|h\|) \leq \|u_0\|_{C^0(\mathbb{D})} (1 + \|h\|) \leq \frac{\sigma}{2}.
\]

Employing

\[
2\sigma = 4C_2 \sqrt{\|u_0\|} < 4C_2 \sqrt{C_3 C} \leq \frac{\rho}{2}
\]

for \( \|u_0\| < C_3 C \) leads to

\[
\overline{B}_{2\sigma}(z_0) \subset \overline{B}_{\rho}(ih_0).
\]

Consequently, by definition of \( C \) and Lemma 3.4.2 we have for \( \|\gamma\| < C \) and thus \( \|u_0\| < C_3 C \)

\[
\|X\|_{C^1(\overline{B}_{2\sigma}(z_0))} = 2 \|P\|_{C^1(\overline{B}_{2\sigma}(z_0))} \leq 2 \|P\|_{C^1(\overline{B}_\rho(ih_0))} \leq 2 \|P\|_{C^1(\overline{B}_\rho(ih_0))} < 2C_4 \leq \delta.
\]

In the course we also used (3.4.5). Similarly,

\[
\|X\|_{C^0(\overline{B}_{2\sigma}(z_0))} = 2 \|P\|_{C^0(\overline{B}_{2\sigma}(z_0))} \leq 2 \|P\|_{C^0(\overline{B}_\rho(ih_0))} < 2 \frac{1}{2}C_6 = \min \left( \frac{1}{8C_1}, 1 \right).
\]

It remains to check that \( u_0(\mathbb{D}) \subset B_\sigma(x_0) \) holds for \( \|u_0\| < C_3 C \). Indeed, using (3.4.7) again leads to

\[
\|u_0 - x_0\|_{C^0(\mathbb{D})} \leq \|u_0\|_{C^0(\mathbb{D})} + \|x_0\| \leq 2 \|u_0\|_{C^0(\mathbb{D})} \leq \frac{\sigma}{1 + \|h\|} \leq \sigma.
\]

Hence by Theorem 3.3.7 there is a pseudoholomorphic curve \( u : \mathbb{D} \to \mathbb{C}^n \) satisfying the Cauchy-Riemann equation \( \partial_\zeta u + J_X \partial_\bar{\zeta} u = 0 \), the boundary condition \( \pi \circ u \mid_{\partial \mathbb{D}} = \gamma \) and \( \text{Im} u(0) = h \left( \text{Re} u(0) \right) \). The latter is equivalent to \( u(0) \in s(Q) \).

\[\square\]

**Corollary 3.4.4.** Let \( \Delta \subset S^2 \) be a triangular domain with angles \( \alpha_1, \alpha_2, \alpha_3 \in (0, \pi) \). Moreover, let \( TQ \) be the tangent bundle of a compact Riemannian manifold \( (Q, g) \) and \( s : Q \to TQ \) a \( C^2 \)-section. There is a constant \( C > 0 \) depending only on the geometry of \( Q \) and \( \|s\|_{C^2} \) such that for any points \( q_1, q_2, q_3 \in Q \) with mutual distances less than \( C \) there exists a \( J_{LC} \)-holomorphic curve \( u : \Delta \to TQ \) satisfying \( u \mid_{\partial \mathbb{D}} = \gamma_{q_1, q_2, q_3} \) and \( u(0) \in s(Q) \).

**Proof.** For fixed \( q \in Q \) consider a normal chart on a metric ball centered at \( q \) and its induced bundle chart on \( \pi^{-1}(B_r(q)) \). Essentially, we will prove a local version of the result when \( q_1, q_2, q_3 \in B_r(q) \). We may assume that \( r \) is small enough such that \( q_2, q_3 \) must lie within the injectivity radius of \( q_1 \). Fix

\[
p < \frac{2\pi}{\pi - \min \{ \alpha_1, \alpha_2, \alpha_3 \}}
\]

and let

\[
M(\rho) = \sup \left\{ \|\gamma_{q_1, q_2, q_3}\|_{W^{1, p}} : q_1, q_2, q_3 \in B_\rho(q) \right\}
\]

for \( 0 \leq \rho < r \).

From the argument in Lemma 3.2.6 we see that \( M \) is continuous, bounded from above and obviously \( M(0) = 0 \). Consequently, we may pick a radius \( 0 < C(q) < r \) such that the existence of the desired \( J_{LC} \)-holomorphic disk can be guaranteed by Theorem 3.4.3. Using compactness of the manifold \( Q \), the statement then holds with

\[
C = \min_{q \in Q} C(q) > 0.
\]

\[\square\]
Let us dwell for a moment on an explicit example where the theorem above applies as well. Consider the 2-sphere $S^2$ with round metric $g$ and a family of circles $\gamma_\lambda$ centered at the north pole and parametrized by arc length. In spherical polar coordinates $\tau \in [0, \pi]$ and $\varphi \in (0, 2\pi)$ the circles are given by $\tau = \lambda$. Denote the zero section in $TS^2$ by $s_0$.

![Figure 3.1: A family of boundary curves on $S^2$.](image)

Now let us start with a constant $J_{\text{LC}}$-holomorphic curve into $TS^2$ whose image is the north pole. The theorem ensures the existence of a constant $C_\gamma$ and a family of $J_{\text{LC}}$-holomorphic curves $u_\lambda : \mathbb{D} \to TS^2$ such that $u_\lambda(0) \in s_0(S^2)$ and $\pi \circ u_\lambda |_{\partial \mathbb{D}} = \gamma_\lambda$ as long as $\lambda < C_\gamma$. It is impossible to find a family of such curves for all $\lambda \in [0, \pi]$. The projection of the final curve $u_\pi$ onto $S^2$ would be surjective and its boundary would project entirely on the south pole. In particular, $u_\pi$ would be nonconstant with energy

$$E(u_\pi) = \int_{\mathbb{D}} u^* \tilde{\omega} = - \int_{\partial \mathbb{D}} u^* \tilde{\lambda} = 0$$

by $\pi \circ u_\pi |_{\partial \mathbb{D}} = \text{const}$ and the local expression (2.3.8). This is a contradiction.

In order to get a grasp on the constant $C_\gamma$ which is guaranteed by Theorem 3.4.3, consider the 2-sphere as a submanifold

$$S^2 = \{ y \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1 \}.$$

One may use stereographic projection at the north pole to introduce normal coordinates. Define a chart $\psi : S^2 \setminus (0,0,-1) \to \mathbb{R}^2$ via

$$(x_1, x_2) = \psi(y_1, y_2, y_3) = \left( \frac{y_1}{1 + y_3}, \frac{y_2}{1 + y_3} \right).$$
The metric tensor becomes

\[ g_{ij}(x) = \frac{4 \delta_{ij}}{(1 + \|x\|^2)^2}. \]

Calculating the Christoffel symbols gives

\[ \Gamma^j_{hi}(x) = \frac{2x_j}{1 + \|x\|^2}, \quad \Gamma^i_{j} = \frac{2x_j}{1 + \|x\|^2}, \quad \Gamma^i_{i} = \frac{2x_i}{1 + \|x\|^2} \]

with \( i \neq j \) in the first two cases. A rough estimate implies

\[ \|\Gamma^k_{ij}\|_{C^0(B_r)} < 2r \quad \text{and} \quad \|\Gamma^k_{ij}\|_{C^1(B_r)} < 2(1 + r^2). \]

Hence in this normal chart on the 2-sphere we obtain

\[ \|P\|_{C^0(B_{r}(0))} < 8r^2 \quad \text{and} \quad \|P\|_{C^1(B_{r}(0))} < 24r (1 + r^2). \]

Following the proof of Theorem 3.4.3, the constant \( C_\gamma \) can be constructed in four steps:

(i) Choose the constants \( C_1, C_2 \) as in Theorem 3.3.7. Fix a radius \( r > 0 \) and choose \( C_5 \) such that

\[ 192 C_1 r (1 + 2r^2) < \frac{1}{2 C_2 (C_5 + \sqrt{C_5}) + \sqrt{C_5}}. \]

(ii) Choose a radius \( \rho \in (0, r) \) such that

\[ 8\rho^2 < \min \left( \frac{1}{16 C_1}, \frac{1}{2} \right). \]

(iii) Set

\[ C = \min \left\{ 1, C_5, \left( \frac{\rho}{8C_2} \right)^2 \right\}. \]

(iv) For some fixed \( p > 2 \) let

\[ C_\gamma = 2 \arctan \frac{C}{C_p} \quad \text{with} \quad C_p = \left( 2\pi \left( 1 + \frac{1}{p + 2} \right) \right)^{\frac{1}{2}}. \]

Only the last steps require some explanation. In contrast to (3.4.6) we do not scale \( C \) by \( C_3 \), since we directly estimate the \( W^{1,p} \)-norm of \( u_\lambda \). The curve \( \gamma_\lambda \) translates in the normal chart into

\[ \gamma_\lambda(x^r) = R(\lambda)(\cos \tau, \sin \tau) \]

with

\[ R(\lambda) = \sqrt{x_1^2 + x_2^2} = \frac{1}{1 + y_2} \sqrt{y_1^2 + y_2^2} = \frac{\sin \lambda}{1 + \cos \lambda} = \tan \frac{\lambda}{2}. \]

The analytic continuation of \( \gamma_\lambda \) in coordinates is given by \( u_\lambda(z) = R(\lambda)z \). Its \( W^{1,p} \)-norm has to be less than \( C \). We calculate

\[ \|u_\lambda\|_{W^{1,p}(D,C)} = \left( \int_D |u(z)|^p \, dz + 2 \int_D |u'(z)|^p \right)^{\frac{1}{p}} = \left( 2\pi R(\lambda)^p \left( \int_0^1 r^{p+1} \, dr + 1 \right) \right)^{\frac{1}{p}} = C_p R(\lambda). \]
Therefore $C_\gamma$ can be characterized by $C_p R(C_\gamma) = C$.

Finally, we will turn our attention to existence results for arbitrary tamed almost complex structures $J$ on the tangent bundle. The approach will be again to choose a small ball around the point of consideration where the $C^0$-norm of $J$ becomes sufficiently small. Analogous to Corollary 3.4.4 we apply the result to piecewise geodesic boundary curves.

**Theorem 3.4.5.** Let $TQ$ be the tangent bundle of a Riemannian manifold and $J$ an almost complex structure tamed by $\tilde{\omega}$ or $\tilde{\omega}_e$. Consider a normal chart on a metric ball and its induced bundle chart on $\pi^{-1}(B_r(q))$. Fix $p > 2$ and let $s : Q \to TQ$ be a $C^2$-section.

Then there is a constant $C > 0$ just depending on $p, r, J, \|s\|_{C^2}$ such that for any curve $\gamma : \partial \mathbb{D} \to \mathbb{R}^n$ in this chart with

$$\|\gamma\|_{W^{1,2}(\partial \mathbb{D})} < C \tag{3.4.8}$$

there is a $J$-holomorphic curve $u : \mathbb{D} \to TQ$ with $\pi \circ u \big|_{\partial \mathbb{D}} = \gamma$ and $u(0) \in s(Q)$.

**Proof.** We consider two cases.

**Case 1.** In coordinates at $z_0 = s(q)$ we have $J(z_0) = J_0$. Here the argument is very similar to the proof of Theorem 3.4.3. Fix a radius $\tilde{r} < r$ such that $J$ can be written as $J_X$ with $X \in \Omega_1$ on $B_{\tilde{r}}(z_0)$. Express $s = (\text{id}, h)$ in coordinates with $h : \mathbb{R}^n \to \mathbb{R}^n$ and set $h_0 = h(0) = \text{Im} z_0$ such that $\|h_0\| = \|s(q)\|$. Pick the constants $C_1, C_2$ as in Theorem 3.3.7 and $C_3$ as in (3.2.3). With

$$C_4 = C_4(r, J, \|h_0\|) = \|X\|_{C^1(B_{\tilde{r}}(z_0))},$$

define $C_5 = C_5(p, r, J, \|h\|)$ as in (3.4.5). Let $\rho = \rho(r, J, \|h_0\|) < \tilde{r}$ be small enough such that

$$\|X\|_{C^0(B_{\rho}(z_0))} < \min \left( \frac{1}{8C_1}, 1 \right).$$

Here we need the assumption $J(z_0) = J_0$, that is $X(z_0) = 0$. Then

$$C = \frac{1}{C_3} \min \left\{ C_5, \frac{1}{(1 + \|h\|^2)^2}, \frac{1}{17C_1^2C_2^2} \frac{\rho}{8C_2} \right\}$$

does the job by the same reasoning as in the proof of Theorem 3.4.3.

**Case 2.** At $z_0 = s(q)$ we have $J(z_0) \neq J_0$.

We first construct $A \in \text{End}_{\mathbb{R}}(\mathbb{C}^n)$ satisfying

$$AJ(z_0) = J_0 A. \tag{3.4.9}$$

Each tangent fibre is Lagrangian with respect to $\tilde{\omega}$ and $\tilde{\omega}_e$. Since

$$J \in \mathcal{J}^+(TQ, \tilde{\omega}) \cup \mathcal{J}^+(TQ, \tilde{\omega}_e)$$

we deduce that each fibre is totally real with respect to $J$. In particular, in our chart $i\mathbb{R}^n$ is totally real with respect to $J$. Denoting the Euclidean standard basis of $i\mathbb{R}^n \subset \mathbb{C}^n$ by $e_{n+1}, \ldots, e_{2n}$, we see that $e_{n+1}, \ldots, e_{2n}, Je_{n+1}, \ldots, Je_{2n}$ represents a real basis of $\mathbb{C}^n$. Now $A$ is defined by

$$Ae_{n+j} = e_{n+j} \quad \text{and} \quad A(Je_{n+j}) = -e_j = J_0e_{n+j}.$$
for $1 \leq j \leq n$. Thus it is possible to find $A \in \text{End}_\mathbb{R}(\mathbb{C}^n)$ such that (3.4.9) holds and $A$ restricts to the identity on $i\mathbb{R}^n$. When composing our bundle chart with $A$ the fibre $z + i\mathbb{R}^n$ is mapped to the fibre $Az + i\mathbb{R}^n$. This makes it possible to apply Case 1 with the curve $\tilde{\gamma} = \pi_1 \circ A \circ \gamma$. We just have to check that $\tilde{\gamma}$ is admissible whenever $\gamma$ is. However, this is clear since the holomorphic extensions of the two curves are related by $\tilde{u}_0 = Au_0$.

**Corollary 3.4.6.** Let $\Delta \subset S^2$ be a triangular domain with angles $\alpha_1, \alpha_2, \alpha_3 \in (0, \pi)$. Moreover, let $TQ$ be the tangent bundle of a compact Riemannian manifold $(Q,g)$ equipped with an almost complex structure $J$ tamed by $\tilde{\omega}$ or $\tilde{\omega}_z$ and $s : Q \to TQ$ a $C^2$-section. There is a constant $C > 0$ depending only on the geometry of $Q$, $J$ and $\|s\|_{C^2}$ such that for any points $q_1, q_2, q_3 \in Q$ with mutual distances less than $C$ there exists a $J$-holomorphic curve $u : \Delta \to TQ$ satisfying $u|_{\partial \Delta} = \gamma_{q_1,q_2,q_3}$ and $u(0) \in s(Q)$.

### 3.5 Diameter and Gradient Bounds

In this section we like to explore compactness properties of the moduli space $\mathcal{M}_k^\tau$. Due to exactness of $\omega$ and the boundary condition, bubbling can be excluded immediately. However, an additional source of noncompactness may appear from the fact that the fiber of the tangent bundle is open. Hence it would be desirable to derive bounds on the diameter of the $J$-holomorphic disks in terms of their area. This is a well-known result when working in a setup of bounded geometry with fixed Lagrangian boundary condition. It was discussed in section 2.5 that the most prominent almost complex structure $J_{LC}$ fails to have this property. Additionally, we have a parametrized Lagrangian boundary condition. The aim of this section will be to show that diameter bounds of $J$-holomorphic curves can be obtained if the ambient manifold has unbounded geometry with gentle growth of geometric quantities like curvature, injectivity radius and covariant derivative of $J$. We will then apply this to the moduli space $\mathcal{M}_k^\tau$ in the Levi-Civita case and outline how diameter bounds imply gradient bounds under the assumption that the length of the loop $\gamma$ is sufficiently small.

In order to specify the class of open manifolds we are interested in, we introduce the following notion.

**Definition 3.5.1.** Let $(M,g)$ be an open connected Riemannian manifold. A continuous function $F : M \to [0, \infty)$ is called unbounded exhausting whenever the following holds:

(i) For any $c \geq 0$ the preimage $F^{-1}(c)$ is a nonempty submanifold of $M$ and $F^{-1}([0,c])$ is compact.

(ii) For any $c_1, c_2 \geq 0$ one has

$$\inf \{ d(p,q) : p \in F^{-1}(c_1), q \in F^{-1}(c_2) \} \geq |c_1 - c_2|. \quad (3.5.1)$$

The latter condition (3.5.1) ensures that infinity cannot be reached in finite distance. Of course, the tangent bundle $(TQ, \tilde{g})$ of a compact Riemannian manifold $(Q,g)$ admits an unbounded exhausting function. We may choose $F = \| \cdot \|_g$ then. In order to verify (3.5.1) one notices that the horizontal distribution is tangent to the level sets of $F$ and uses the local expression of the vertical vectors from (2.3.2) to deduce

$$\inf \{ d(p,q) : \|p\|_g = c_1, \|q\|_g = c_2 \} = |c_1 - c_2|$$

with distance measured with respect to $\tilde{g}$.
The key for our argument will be the following monotonicity lemma. It can be traced back to [Gro85], but we use the version stated in [Fis11], Proposition 3.4. The boundary case with a fixed totally geodesic Lagrangian boundary condition is treated in section 3.3 of [Fis12].

**Proposition 3.5.2.** Let $(\Sigma, j)$ be a Riemann surface with boundary and $(M, J, g)$ an almost Hermitian manifold. Suppose $L \subset M$ is a compact embedded totally geodesic Lagrangian. For $p \in M$ consider the open metric ball $B_r(p)$ of radius $r > 0$ and suppose we have geometric bounds

\[
\inf \{ \rho(q) : q \in B_r(p) \} \geq \frac{1}{C}, \quad (3.5.2)
\]

\[
\sup \{ \| \nabla J(q) \|_g : q \in B_r(p) \} \leq \frac{1}{4} C, \quad (3.5.3)
\]

\[
\sup \{ \| K_{sec}(q) \|_g : q \in B_r(p) \} \leq \frac{1}{4} C^2 \quad (3.5.4)
\]

for some constant

\[0 < C < \frac{1}{8r} .\]

Here $\nabla$ denotes the Levi-Civita connection and $\rho$ the injectivity radius associated to $g$. Then any $J$-holomorphic curve $u : \Sigma \to M$ satisfying $p \in u(\Sigma)$ and $u(\partial \Sigma) \cap B_r(p) \subset L$ has energy bounded below by

\[E(u) \geq \frac{\pi}{2} r^2. \quad (3.5.5)\]

To control the sum of the areas guaranteed by the monotonicity lemma we will invoke the following elementary estimate.

**Lemma 3.5.3.** Given $a_0 > 0$ and $B > 0$, define a monotone increasing and divergent sequence $(a_n)_{n \in \mathbb{N}}$ by the recursive relation

\[a_{n+1} = a_n + \frac{B}{a_n}.\]

Then we have

\[\sum_{n=0}^{N} \frac{1}{a_n^2} > \frac{B}{a_0} \ln \frac{a_N}{a_0 + 2} - \left(1 + \frac{1}{B}\right) .\]

**Proof.** We observe that for any integer $m \geq \lceil a_0 \rceil$ there are at least $\lceil \frac{m}{B} \rceil$ elements in the sequence satisfying $a_n \in [m, m + 1)$. This allows us to estimate

\[\sum_{n=0}^{N} \frac{1}{a_n^2} > \sum_{m=\lceil a_0 \rceil}^{\lceil a_N \rceil - 1} \frac{m}{\lceil a_n \rceil + 1} \left(1 + \frac{1}{B}\right) - \frac{1}{m^2} - \left(1 + \frac{1}{B}\right)
\]

\[> \frac{1}{B} \sum_{m=\lceil a_0 \rceil}^{\lceil a_N \rceil} \frac{1}{m} \left(1 + \frac{1}{B}\right) - \left(1 + \frac{1}{B}\right)
\]

\[\quad > \frac{1}{B} \left( \ln \left( \lceil a_N \rceil + 1 \right) - \ln \left( \lceil a_0 \rceil + 1 \right) \right) - \left(1 + \frac{1}{B}\right)
\]

\[> \frac{1}{B} \ln \frac{a_N}{a_0 + 2} - \left(1 + \frac{1}{B}\right) .\]

In the second line we interpreted the partial sum as a step function and compared it to the lower integral. \qed
3.5. DIAMETER AND GRADIENT BOUNDS

We will now show that diameter bounds of \( J \)-holomorphic curves can be established on a class of open manifolds with unbounded geometry if the growth rates of curvature, injectivity radius and \( \nabla J \) are mild enough.

**Theorem 3.5.4.** Let \((\Sigma, j)\) be a Riemann surface with boundary and \((M, J, g)\) an open almost Hermitian manifold equipped with an unbounded exhausting function \( F : M \to [0, \infty) \). Suppose \( L \subset M \) is a compact embedded totally geodesic Lagrangian. Assume there is a constant \( C > 0 \) such that the injectivity radius, the sectional curvature and the covariant derivative of \( J \) with respect to the Levi-Civita connection \( \nabla \) of \( g \) satisfy

\[
\rho(p) \geq (C F(p))^{-1},
\]

\[
\|K_{\text{sec}}(p)\|_g \leq C (F(p) + 1)^2,
\]

\[
\|\nabla J(p)\|_g \leq C (F(p) + 1).
\]

Consider a \( J \)-holomorphic curve \( u : \Sigma \to M \) with \( u(\partial \Sigma) \subset L \) and energy \( E(u) < a \). Then there is a constant \( \tilde{C} \) depending only on \( a, C \) and \( L \) such that

\[
\|F \circ u\|_{C^0} < \tilde{C}.
\]

**Proof.** Set

\[
C_1 = \max (256 C, 1).
\]

By (3.5.1) we know that any open metric ball \( B_r(p) \) in \( M \) satisfies

\[
B_r(p) \subset F^{-1}([0, F(p) + r]).
\]

In particular, we obtain from (3.5.6)-(3.5.8)

\[
\inf \{\rho(q) : q \in B_r(p)\} \geq \frac{1}{C(F(p) + r)} > \frac{8}{C_1(F(p) + r + 1)}
\]

as well as

\[
\sup \{\|K_{\text{sec}}(q)\| : q \in B_r(p)\} \leq C (F(p) + r + 1)^2 \leq \frac{1}{4} \left( \frac{C_1}{8} (F(p) + r + 1) \right)^2
\]

and

\[
\sup \{\|\nabla J(q)\| : q \in B_r(p)\} \leq C (F(p) + r + 1) \leq \frac{1}{4} \frac{C_1}{8} (F(p) + r + 1).
\]

Hence we may apply Proposition 3.5.2 to \( B_r(p) \) whenever

\[
C_1 r (F(p) + r + 1) < 1.
\]

Now suppose \( u \) satisfies additionally

\[
\|F \circ u\|_{C^0} \geq C_2 = \max \left( \|F|_L\|_{C^0}, 2 \right).
\]

Then we may choose a finite sequence of open metric balls \((B_{r_j}(p_j))_{0 \leq j \leq m}\) such that the following holds.
(i) The balls are centered on increasing level sets of $F$. Starting at $F(p_0) = C_2$ the level sets are defined recursively via

$$F(p_{j+1}) = F(p_j) + 2r_j \quad \text{with radii} \quad r_j = \frac{1}{2C_1 F(p_j)}$$  \hspace{1cm} (3.5.11)$$

for $j \geq 0$.

(ii) The centers $p_j \in M$ lie on the image $u(\mathbb{D})$. In order to ensure this the sequence terminates at $B_{r_m}(p_m)$ iff

$$F(p_m) + 2r_m > \|F \circ u\|_{C^0}.$$  \hspace{1cm} (3.5.12)$$

From (3.5.11) it follows that the radii $r_j$ are strictly monotone decreasing. In particular,

$$C_1 r_j (F(p_j) + r_j + 1) = \frac{1}{2 F(p_j)} \left( F(p_j) + \frac{1}{2C_1 F(p_j)} + 1 \right) \leq \frac{1}{2} \left( 1 + \frac{1}{2C_1 F(p_j)} + \frac{1}{4(F(p_j))^2} \right)$$

$$\leq \frac{1}{2} + \frac{1}{2F(p_0)} + \frac{1}{4(F(p_0))^2} \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{16} < 1.$$  

Thus condition (3.5.10) is satisfied for each ball. Considering the statement in the monotonicity lemma, we must have $r_j \leq \frac{1}{8} \tilde{\rho}(p_j)$. Employing (3.5.1) again yields

$$d(p_j, p_{j+1}) \geq F(p_{j+1}) - F(p_j) = 2r_j > r_j + r_{j+1}.$$  

Therefore we deduce that the balls $B_{r_j}(p_j)$ are pairwise disjoint. Invoking the lower bound (3.5.5) for each of them allows us to conclude

$$E(u) \geq \frac{\pi}{2} \sum_{j=0}^{m} r_j^2.$$  

Note that the sequence of inverse radii $a_j = r_j^{-1}$ satisfies the recurrence relation

$$a_{j+1} = \frac{1}{a_{j+1}} = 2C_1 F(p_{j+1}) = 2C_1 (F(p_j) + 2r_j) = \frac{1}{r_j} + 4C_1 r_j = a_j + 4C_1.$$  

Hence we may take advantage of Lemma 3.5.3 with $B = 4C_1$ to estimate further

$$a > E(u) > \frac{\pi}{2} \left( \frac{1}{4C_1} \ln \frac{a_m}{a_0 + 2} - \left( 1 + \frac{1}{4C_1} \right) \right) = \frac{\pi}{8C_1} \ln \frac{r_0}{r_m(1 + 2r_0)} - \frac{\pi}{2} \left( 1 + \frac{1}{4C_1} \right).$$  

This leads to

$$\frac{r_0}{r_m(1 + 2r_0)} < C_3 = \exp \left( \frac{8C_1}{\pi} a + 4C_1 + 1 \right)$$  

and finally the terminating condition (3.5.12) implies

$$\|F \circ u\|_{C^0} < F(p_m) + 2r_m = \frac{1}{2C_1 r_m} + 2r_m < \frac{C_3(1 + 2r_0)}{2C_1 r_0} + 2r_0.$$  

Plugging in $r_0 = \frac{1}{4C_1}$ yields the desired constant $\tilde{C}$ on the right hand side. \hfill \square
Note that the theorem cannot be generalized to steeper growth rates of geometry. If we replace (3.5.6)-(3.5.8) by polynomial rates
\[\rho(p) \geq (C F(p))^{-N},\]
\[\|K_{sec}(p)\|_g \leq C (F(p) + 1)^{2N},\]
\[\|\nabla J(p)\|_g \leq C (F(p) + 1)^N\]
for some integer \(N > 1\), then the argument does not go through. This is due to the fact that for the corresponding sequence
\[a_{n+1} = \left(\frac{\sqrt[n]{a_n} + B}{a_n}\right)^N\]
the sum \(\sum_{n=0}^{\infty} a_n^{-2}\) converges.

We continue by showing that the Levi-Civita almost complex structure \(J_{LC}\) in the tangent bundle \((TQ, \bar{g})\) satisfies the growth rates in Theorem 3.5.4. While the estimates (3.5.6) and (3.5.7) of the injectivity radius and sectional curvature will be dealt with via Theorem 2.5.5, in order to cope with the estimate of the covariant derivative \(\nabla J_{LC}\) we will employ the subsequent lemma.

**Lemma 3.5.5.** Let \((Q, g)\) be a Riemannian manifold with curvature bound
\[C = \sup_{\|X\| = \|Y\| = \|Z\| = 1} \|R(X, Y)Z\|^2.\]
Then we have at \(\xi \in TQ\) the following estimate for the Levi-Civita almost complex structure
\[\left\|\nabla J_{LC}(\xi)\right\|_{\bar{g}} \leq 2\sqrt{C} \|\xi\|_{g}.\]

**Proof.** Let \(p = \pi(\xi)\) and pick \(\bar{W}, \bar{Z} \in \Gamma(T(TQ))\) as well as \(U, V, X, Y \in \Gamma(TQ)\) such that \(\bar{W} = U^H + V^V\) and \(\bar{Z} = X^H + Y^V\) holds. Taking advantage of Proposition 2.2.3 we may compute
\[
\left(\left(\nabla_{\bar{W}} J_{LC}\right) \bar{Z}\right)_{\xi} = \left(J_{LC} \left(\nabla_{\bar{W}} \bar{Z}\right)\right)_{\xi} - \left(\nabla_{\bar{W}} \left(J_{LC} \bar{Z}\right)\right)_{\xi}
\]
\[
\begin{align*}
&= J_{LC} \left(\frac{1}{2} (R_{p}(\xi, Y_{p})U_{p})^H + (\nabla U Y)^{V}_{\xi} + \frac{1}{2} (R_{p}(\xi, V_{p})X_{p})^H + (\nabla U X)^{H}_{\xi} - \frac{1}{2} (R_{p}(U_{p}, X_{p})\xi)^V \right) \\
&\quad - \nabla_{\bar{W}} \left((X^V_{\xi} - Y^H_{\xi})\right) \\
&= \frac{1}{2} (R_{p}(\xi, Y_{p})U_{p})^V - (\nabla U Y)^{H}_{\xi} + \frac{1}{2} (R_{p}(\xi, V_{p})X_{p})^V + (\nabla U X)^{V}_{\xi} + \frac{1}{2} (R_{p}(U_{p}, X_{p})\xi)^H \\
&\quad - \frac{1}{2} (R_{p}(\xi, X_{p})U_{p})^H - (\nabla U X)^{H}_{\xi} + \frac{1}{2} (R_{p}(\xi, V_{p})Y_{p})^H + (\nabla U Y)^{H}_{\xi} - \frac{1}{2} (R_{p}(U_{p}, Y_{p})\xi)^V \\
&= \frac{1}{2} \left((R_{p}(\xi, V_{p})Y_{p} - R_{p}(\xi, U_{p})X_{p})^H + (R_{p}(\xi, V_{p})X_{p} + R_{p}(\xi, U_{p})Y_{p})^V\right).
\end{align*}
\]
Consequently, using the definition of the Sasaki metric given in (2.2.4) we obtain
\[
\sup_{\|\bar{W}\|_{g}, \|\bar{Z}\|_{g} \leq 1} \left\|\left(\nabla_{\bar{W}} J_{LC}\right) \bar{Z}\right\|_{g}
\leq \sup_{\|U\|_{g}, \|V\|_{g}, \|X\|_{g}, \|Y\|_{g} \leq 1} \frac{1}{2} \sqrt{C} \|\xi\| \left(\|V\|_{g} \|Y\|_{g} + \|U\|_{g} \|X\|_{g} + \|V\|_{g} \|X\|_{g} + \|U\|_{g} \|Y\|_{g}\right) = 2\sqrt{C} \|\xi\|.
\]
□
Corollary 3.5.6. Let \((TQ, \tilde{g})\) be the tangent bundle of a closed Riemannian manifold \((Q, g)\), equipped with the Levi-Civita almost complex structure \(J_{\LC}\). Then there is a constant \(C > 0\) depending only on bounds of \(R\), \(\nabla R\) and \(\rho(Q)\) such that
\[
\tilde{\rho}(\xi) \geq (C \|\xi\|_g)^{-1}, \quad \|\tilde{K}_{\sec}(\xi)\|_{\tilde{g}} \leq C \left(\|\xi\|_g + 1\right)^2, \quad \|\tilde{\nabla} J_{\LC}(\xi)\|_{\tilde{g}} \leq C \left(\|\xi\|_g + 1\right)
\]
hold for any \(\xi \in TQ\).

Due to convexity properties it is fairly easy to see that the statement of Theorem 3.5.4 holds in the case of the tangent bundle \((TQ, J_{\LC}, \tilde{g})\). Indeed, Proposition 2.6.4 tells us that
\[
\bigg\| \tilde{f} \circ u \bigg\|_{C^0} \leq \bigg\| \tilde{f} \circ u|_{\partial \Sigma} \bigg\|_{C^0}
\]
holds for any \(J_{\LC}\)-holomorphic curve \(u : \Sigma \to TQ\) where \(\tilde{f}\) denotes the metric Hamiltonian defined in (2.2.21). This already implies a diameter bound when \(u(\partial \Sigma) \subset L\) with \(L\) compact which is independent of the energy of \(u\).

In contrast, any lifted disk \(u \in \mathcal{M}_s\) satisfies a parametrized boundary condition consisting of noncompact Lagrangian fibers. Hence (3.5.13) does not help much and we have to employ an argument akin to Theorem 3.5.4 above to bound the diameter of \(u\) in terms of its energy. Fix a small \(\delta > 0\) and define the tubular neighborhood
\[
T^\gamma_\delta = \bigcup_{t \in S^1} B_\delta(\gamma(t))
\]
of \(\gamma\). Here \(B_\delta(p)\) with \(p \in Q\) denotes an open metric ball in \((Q, g)\). In fact, we will derive the diameter bound on
\[
TQ^\gamma_\delta = TQ \setminus \pi^{-1} \left( T^\gamma_\delta \right).
\]

Theorem 3.5.7. Suppose \(u \in \mathcal{M}_s\) is a \(J_{\LC}\)-holomorphic lifted disk with energy \(E(u) < a\). Then there is a constant \(C\) depending only on \(a\), \(\delta\), \(\|\tilde{f} \circ s\|_{C^0}\) and the geometry of \(Q\) such that
\[
\bigg\| \tilde{f} \mid_{u(\mathcal{D}) \cap TQ^\gamma_\delta} \bigg\|_{C^0} < C.
\]

Proof. One applies the argument in the proof of Theorem 3.5.4 to the unbounded exhausting function \(\tilde{l} : TQ \to [0, \infty)\), \(\tilde{l}(\xi) = \|\xi\|\). The only difference is that the initial level for the ball sequence will be
\[
C_2 = \max \left( \|\tilde{l} \circ s\|_{C^0}, \frac{1}{2C_1^2} \right)
\]
and the terminating condition becomes
\[
\|\xi_m\| + 2r_m > \bigg\| \tilde{l} \mid_{u(\mathcal{D}) \cap TQ^\gamma_\delta} \bigg\|_{C^0}.
\]

Then
\[
r_0 = \frac{1}{2C_1C_2} \leq \delta
\]
implies \(r_j \leq \delta\) and thus \(u(\partial \mathcal{D}) \cap B_{r_j}(\xi_j) = \emptyset\) for \(0 \leq j \leq m\). \(\square\)
3.5. DIAMETER AND GRADIENT BOUNDS

It would be desirable to obtain the diameter bound on the whole tangent bundle $TQ$. In order to prove this one would need a boundary version of the monotonicity lemma for parametrized paths of totally geodesic Lagrangians. We expect this to be true with a constant depending additionally on a bound of $\nabla \tilde{\gamma}$.

We continue by showing how diameter bounds imply interior gradient bounds whenever the length of $\gamma$ is sufficiently small. To get started, we state the mean value inequality from bubbling analysis concerning solutions of the partial differential inequality $\Delta w \geq -aw^2$. For a proof consult [MS04], Lemma 4.3.2.

**Proposition 3.5.8.** Let $r > 0$, $a \geq 0$ and $B_{r} = \{ z \in \mathbb{C} : \| z \| \leq r \}$. If $w : B_{r} \to \mathbb{R}$ is a $C^2$-function satisfying the inequalities

$$\Delta w \geq -aw^2, \quad w \geq 0, \quad \int_{B_{r}} w < \frac{\pi}{8a},$$

then

$$w(0) \leq \frac{8}{\pi r^2} \int_{B_{r}} w.$$  

Furthermore, we like to refine the quadratic estimate for the growth of the sectional curvature of the Sasaki metric in section 2.5 for the holomorphic sectional curvature, thus generalizing the inequality provided in Corollary A.2.4 for the tangent bundle of the round sphere.

**Proposition 3.5.9.** Suppose $(Q, g)$ has the following bounds

$$C = \sup_{\|X\|=\|Y\|=\|Z\|=1} \|R(X, Y)Z\|^2, \quad C_{\nabla} = \sup_{\|W\|=\|X\|=\|Y\|=\|Z\|=1} \|\nabla_{W}R(X, Y)Z\|$$

on curvature. Then for $\tilde{Z} \in \Gamma(T(TQ))$ and $\xi \in TQ$ we have the estimate

$$\left| \langle \tilde{R} (\tilde{Z}, J_{LC} \tilde{Z}) J_{LC} \tilde{Z}, \tilde{Z} \rangle \xi \right| \leq \left( C \| \xi \| \right)^2 + \frac{1}{2} C_{\nabla} \| \xi \| + \sqrt{C} \| \tilde{Z} \|.$$  

**Proof.** Let $\pi(\xi) = p$ and suppose we have the splitting $\tilde{Z} = X^{H} + Y^{V}$ with $X, Y \in \Gamma(TQ)$. Plugging in the curvature bounds and the estimate

$$\left[ \frac{1}{2} \left( \| R_{p}(\xi, X_{p}) Y_{p} \|^2 + \| R_{p}(X_{p}) Y_{p} \| \| X_{p} \| \| \xi \| \| Y_{p} \| \right) \right] = \| R_{p}(\xi, X_{p}) Y_{p}, R_{p}(\xi, Y_{p}) X_{p} \| \leq \| R_{p}(\xi, X_{p}) Y_{p} \| \| R_{p}(\xi, Y_{p}) X_{p} \| \leq C \| X_{p} \| \| Y_{p} \| \| \xi \| \| Y_{p} \| \| X_{p} \| \| \xi \|$$

into Theorem 2.2.8 yields

$$\left| \langle \tilde{R} (\tilde{Z}, J_{LC} \tilde{Z}) J_{LC} \tilde{Z}, \tilde{Z} \rangle \xi \right| \leq 4 \sqrt{C} \| X_{p} \| \| Y_{p} \| \| \xi \| \| Y_{p} \| \| X_{p} \| \| \xi \| + \frac{C}{4} \left( \| X_{p} \|^4 + \| Y_{p} \|^4 \right) \| \xi \| \| Y_{p} \| \| X_{p} \| \| \xi \|$$

$$+ C_{\nabla} \left( \| X_{p} \|^3 \| Y_{p} \| + \| X_{p} \| \| Y_{p} \|^3 \| \xi \| \| Y_{p} \| \| X_{p} \| \| \xi \| \| Y_{p} \| \right.$$

$$+ \frac{3}{4} \| R_{p}(X_{p}) Y_{p} \| \| X_{p} \| \| Y_{p} \| \| \xi \|$$

$$\leq 4 \sqrt{C} \| X_{p} \| \| Y_{p} \| \| \xi \| \| Y_{p} \| \| X_{p} \| \| \xi \|$$

$$+ \frac{3}{4} \| R_{p}(X_{p}) Y_{p} \| \| X_{p} \| \| Y_{p} \| \| \xi \|$$

$$\leq 4 \sqrt{C} \| X_{p} \| \| Y_{p} \| \| \xi \| \| Y_{p} \| \| X_{p} \| \| \xi \|$$

$$+ \frac{3}{4} \| R_{p}(X_{p}) Y_{p} \| \| X_{p} \| \| Y_{p} \| \| \xi \|$$

$$\leq \left( C \| \xi \| \right)^2 + \frac{1}{2} C_{\nabla} \| \xi \| + \sqrt{C} \left( \| X_{p} \| \| Y_{p} \| \right)^2 \left( \| X_{p} \|^2 + \| Y_{p} \|^2 \right)^2.$$
In the last line we used the inequalities \(4a^2b^2 \leq (a^2 + b^2)^2\) and \(a^3b + ab^3 \leq \frac{1}{2} (a^2 + b^2)^2\). This completes the proof, since \(\|Z\|^2 = \|X_p\|^2 + \|Y_p\|^2\).

It is well-known that the curvature tensor can be expressed solely in terms of sectional curvature. Hence it should be possible to formulate the statement above by incorporating a bound on sectional curvature instead of \(C\). Indeed, this can be accomplished by the following result.

**Lemma 3.5.10.** Suppose \((Q, g)\) has the curvature bounds

\[
C = \sup_{\|X\| = \|Y\| = \|Z\| = 1} \|R(X, Y)Z\|^2, \quad K = \sup_{\|X\| = \|Y\| = 1} \|(R(X, Y)Y, X)\|.
\]

Then

\[
\sqrt{C} \leq 68 K. \tag{3.5.20}
\]

**Proof.** Taking advantage of Lemma 3.3.3 in [Jos05] leads to the estimate

\[
\|R(X, Y)Z\| = \sup_{\|W\| = 1} \langle R(X, Y)Z, W \rangle
\]

\[
\leq \sup_{\|W\| = 1} K \left( \|X + W\|^2 \left( \|Y + Z\|^2 + \|Y\|^2 + \|Z\|^2 \right) + \left( \|X\|^2 + \|W\|^2 \right) \|Y + Z\|^2 \right)
\]

\[
+ \sup_{\|W\| = 1} K \left( \|X\|^2 \|Z\|^2 + \|W\|^2 \|Y\|^2 + \|Y + W\|^2 \left( \|X + Z\|^2 + \|X\|^2 + \|Z\|^2 \right) \right)
\]

\[
+ \sup_{\|W\| = 1} K \left( \left( \|Y\|^2 + \|W\|^2 \right) \|X + Z\|^2 + \|Y\|^2 \|Z\|^2 + \|W\|^2 \|X\|^2 \right).
\]

After using the inequality \((a + b)^2 \leq 2(a^2 + b^2)\) and plugging in \(\|X\| = \|Y\| = \|Z\| = \|W\| = 1\), (3.5.20) follows. \(\square\)

Putting the pieces together yields the interior gradient estimate.

**Theorem 3.5.11.** With the bounds

\[
C = \sup_{\|X\| = \|Y\| = \|Z\| = 1} \|R(X, Y)Z\|^2, \quad C venerable = \sup_{\|W\| = \|X\| = \|Y\| = \|Z\| = 1} \|\langle \nabla W R \rangle (X, Y)Z\|
\]

of \((Q, g)\) on curvature define the quadratic function

\[
P(x) = C x^2 + \frac{1}{2} C venerable x + \sqrt{C}.
\]

Suppose \(u \in \mathcal{M}_\gamma^d\) is a \(J_{LC}\)-holomorphic lifted disk with diameter bound

\[
\|j \circ u\|_{C_0} < \frac{1}{2} d^2
\]

for some \(d > 0\). If the length of \(\gamma\) satisfies

\[
L(\gamma) < \frac{\pi}{16dP(d)},
\]

then \(u\) admits the interior gradient bound

\[
\|d u (r e^{id})\|_{\bar{g}} \leq \sqrt{\frac{dL(\gamma)}{\pi}} \frac{4}{1 - r} < \frac{1}{\sqrt{P(d)} (1 - r)} \tag{3.5.21}
\]

for \(r < 1\).
3.5. DIAMETER AND GRADIENT BOUNDS

Proof. We compute the Laplacian of the energy density

$$ e(u) = \frac{1}{2} \| du \|^2 = \| \partial_s u \|^2 $$

with the help of the Levi-Civita connection $\tilde{\nabla}$ and obtain

$$ \Delta e(u) = \Delta \langle \partial_s u, \partial_s u \rangle = 2\| \tilde{\nabla}_s \partial_s u \|^2 + 2\| \tilde{\nabla}_t \partial_s u \|^2 + 2 \langle \partial_s u, \tilde{\nabla}_s \tilde{\nabla}_s \partial_s u + \tilde{\nabla}_t \tilde{\nabla}_t \partial_s u \rangle. \quad (3.5.22) $$

By applying the Cauchy-Riemann equation, we have

$$ \tilde{\nabla}_s \partial_s u + \tilde{\nabla}_t \partial_t u = \tilde{\nabla}_t (J_{LC} \partial_s u) - \tilde{\nabla}_s (J_{LC} \partial_t u) = \left( \tilde{\nabla}_t J_{LC} \right) \partial_s u - \left( \tilde{\nabla}_s J_{LC} \right) \partial_t u. $$

We observe that the term on the right hand side vanishes by Proposition 2.4.2. Consequently,

$$ \tilde{\nabla}_s \tilde{\nabla}_s \partial_s u + \tilde{\nabla}_t \tilde{\nabla}_t \partial_t u = \tilde{\nabla}_s \left( \tilde{\nabla}_s \partial_s u + \tilde{\nabla}_t \partial_t u \right) + \tilde{\nabla}_t \tilde{\nabla}_t \partial_t u - \tilde{\nabla}_s \tilde{\nabla}_t \partial_t u \\
= \tilde{\nabla}_t \tilde{\nabla}_t \partial_t u - \tilde{\nabla}_s \tilde{\nabla}_t \partial_t u \\
= \left( \tilde{\nabla}_t \tilde{\nabla}_s - \tilde{\nabla}_s \tilde{\nabla}_t \right) \partial_t u \\
= - \tilde{R}(\partial_s u, \partial_t u) \partial_t u. $$

Plugging this into (3.5.22) and applying Proposition 3.5.9 yields

$$ \Delta e(u) = 2\| \tilde{\nabla}_s \partial_s u \|^2 + 2\| \tilde{\nabla}_t \partial_t u \|^2 - 2 \langle \tilde{R}(\partial_s u, \partial_t u) \partial_t u, \partial_s u \rangle \\
\geq - 2 \left| \tilde{R}(\partial_s u, \partial_t u) \partial_t u, \partial_s u \right| \\
= - 2 \left| \tilde{R}(\partial_s u, J_{LC} \partial_s u) J_{LC} \partial_t u, \partial_s u \right| \\
\geq - 2 \left( C d^2 + \frac{1}{2} C d + \sqrt{C} \right) \| \partial_s u \|^4 \\
= - 2 P(d) \cdot (e(u))^2. $$

Thus the partial differential inequality in Proposition 3.5.8 holds with $w = e(u)$ and $a = 2 P(d)$. We further observe via Stokes theorem

$$ E(u) = \int_D u^* \tilde{\omega} = \int_{\partial D} u^* \tilde{\lambda} \leq d L(\gamma) < \frac{\pi}{16 P(d)}. $$

Hence we may apply the mean value inequality to a disk of radius $(1 - r)$ around $re^{it}$ obtaining

$$ \frac{1}{2} \| du(re^{it}) \|^2 \leq \frac{8}{\pi(1 - r)^2} E(u) \leq \frac{8 d L(\gamma)}{\pi(1 - r)^2} \leq \frac{1}{2 P(d)(1 - r)^2}. $$

$\square$
Chapter 4

Existence of Punctured Disks

4.1 Overview

We study boundary punctured $J$-holomorphic disks in the symplectization $SQ \times \mathbb{R}$ of unit tangent bundles which assemble the pieces of the moduli space $\mathcal{M}_{g,h}^3$ of punctured type. We frequently employ the equivalence $S^n \approx T^S \times \mathbb{R}$ to obtain the benefiting situation of a 1-jet bundle. From there we may resort to the work of Ekholm who established connections between pseudoholomorphic disks and Morse flow trees. In section 4.2 we verify Reeb chord genericity and prove the index formula in Theorem 1.10. The prescription of a conformal structure on the domain requires an adequate conformal model which is presented in section 4.3. The moduli space of Morse flow trees with corresponding boundary condition is considered in section 4.4. By adjusting the argument of Ekholm in ([Ekh07]) we prove existence results for boundary punctured $J$-holomorphic disks in the vicinity of such a Morse flow tree in section 4.5. The final step is to extend the argument to $SQ \times \mathbb{R}$ in section 4.6 which culminates in a proof of Theorem 1.11.

4.2 Index Calculations

We calculate the Fredholm index for boundary punctured $J$-holomorphic disks in the symplectization $M = SQ \times \mathbb{R}$ of a unit tangent bundle of a Riemannian manifold $(Q, g)$. Our standard domain will be the closed unit disk $D_m$ with $m$ fixed boundary punctures $p_1, \ldots, p_m$. At punctures the $J$-holomorphic disks converge to Reeb chords and their boundary is mapped onto $\Lambda \times \mathbb{R}$ with a Legendrian in $SQ$. Given $m$ points $q_1, \ldots, q_m \in Q$ close enough to each other such that any two of them can be connected by a locally unique geodesic, the Legendrian of interest will be the disconnected union of fibers

$$\Lambda = \bigcup_{j=1}^m S_{q_j}Q.$$  \hfill (4.2.1)

The symplectization $M = SQ \times \mathbb{R}$ is symplectomorphic to $TQ \setminus s_0(Q)$, that is the tangent bundle with zero section removed. Similarly,

$$\Lambda \times \mathbb{R} \approx \bigcup_{j=1}^m (T_{q_j}Q \setminus \{q_j\}).$$

We start by determining the index in the case $Q = \mathbb{R}^n$. Our approach will be to transfer the problem into a situation where the relative Chern class vanishes. We then explicitly calculate the Maslov index of the Lagrangian boundary loop in a symplectic trivialization. Finally, by
employing a continuity argument we deduce the index formula for unit tangent bundles of curved Riemannian manifolds.

In order to obtain a Fredholm problem, we prove Reeb chord genericity for fiberwise Legendrians. For the terminology we refer to Definition A.1.1 in the appendix. As shown in Lemma 2.3.1 the Reeb flow on $SQ$ coincides with the geodesic flow.

**Proposition 4.2.1.** Let $q_j, q_{j+1}$ be points in $Q$ within the injectivity radius, i.e. 
\[0 < \text{dist} (q_j, q_{j+1}) < \text{inj} (q_j).\]

Then the Reeb chord connecting the Legendrians $S_{q_j}Q$ and $S_{q_{j+1}}Q$ in $SQ$ and corresponding to the locally unique geodesic between $q_j$ and $q_{j+1}$ is generic. In particular, the Legendrian $\Lambda$ given in (4.2.1) is admissible.

**Proof.** The tangent space of fibrewise Legendrians consists exactly of the vertical vectors, that is the kernel of the linearized projection $\pi : SQ \to Q$. We denote the time $t$ Reeb flow on $SQ$ by $\varphi^t$. Moreover, let $\beta : [0, L] \to SQ$ be the locally unique geodesic connecting $\pi \circ \beta(0) = q_j$ and $\pi \circ \beta(L) = q_{j+1}$ where $L = \text{dist} (q_j, q_{j+1})$. The pushforward of any nonzero vertical tangent vector $\gamma : (-\varepsilon, \varepsilon) \to SQ$, 
\[
\gamma(s) = \left( q_j, \cos (\|v\|s) \beta(0) + \sin (\|v\|s) \frac{v}{\|v\|} \right) \quad \text{with} \quad \langle \beta(0), v \rangle = 0, \, v \in T_{q_j}Q
\]
along $\beta$ is vertical again if and only if 
\[
(\pi \circ \varphi^L \circ \gamma)(s) = \text{const} = q_{j+1}.
\] 

However, we may calculate 
\[
(\pi \circ \varphi^L \circ \gamma)(s) = \pi \circ \varphi^1 \left( q_j, \, L \left( \cos (\|v\|s) \beta(0) + \sin (\|v\|s) \frac{v}{\|v\|} \right) \right)
\]
\[
= \exp_{q_j} \left( L \left( \cos (\|v\|s) \beta(0) + \sin (\|v\|s) \frac{v}{\|v\|} \right) \right)
\]
and by definition $\exp_{q_j}$ is a diffeomorphism on 
\[
\{ w \in T_{q_j}Q : \|w\| < \text{inj} (q_j) \}. 
\]

Now suppose the pushforward of $\gamma$ along $\beta$ is vertical. Since 
\[
\left\| L \left( \cos (\|v\|s) \beta(0) + \sin (\|v\|s) \frac{v}{\|v\|} \right) \right\| = L = \text{dist} (q_j, q_{j+1}) < \text{inj} (q_j),
\]
(4.2.2) allows us to conclude the contradiction $\gamma(s) = \text{const}$ for $s \in (-\varepsilon, \varepsilon)$. \hfill \Box

In the sequel we will first determine the index for unit tangent bundles of Euclidian space. However, it will be more convenient for us to replace $S\mathbb{R}^n$ with the 1-jet space 
\[
J^1(Q) = T^*Q \times \mathbb{R}
\] 
(4.2.3) of spheres, i.e. $Q = S^{n-1}$. 

Proposition 4.2.2. The unit tangent bundle of \( \mathbb{R}^n \) is strongly contactomorphic to the 1-jet bundle of \( S^{n-1} \),

\[
(S\mathbb{R}^n, \lambda |_{S\mathbb{R}^n}) \cong (J^1(S^{n-1}), \alpha).
\] (4.2.4)

Here \( \alpha = dz - \lambda \) where \( \lambda \) always denotes the canonical 1-form and \( z \) corresponds to the coordinate in the \( \mathbb{R} \)-factor. Identifying \( T^*S^{n-1} \) via

\[
T^*S^{n-1} \cong TS^{n-1} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \|x\| = 1 \text{ and } \langle x, y \rangle = 0\},
\]

the contactomorphism \( \Phi : S\mathbb{R}^n \to J^1(S^{n-1}) \) is given by

\[
\Phi(q, p) = (p, q - \langle q, p \rangle p, \langle q, p \rangle).
\] (4.2.5)

Proof. Note that

\[
\langle p, q - \langle q, p \rangle p \rangle = \langle p, q \rangle - \langle q, p \rangle \|p\|^2 = 0.
\]

Hence \( \Phi \) is well-defined. The inverse

\[
\Phi^{-1}(x, y, z) = (y + z x, x)
\]
is smooth and bijective. Consequently, \( \Phi \) is a diffeomorphism.

Denote the embedding \( S\mathbb{R}^n \hookrightarrow T\mathbb{R}^n \) by \( \iota \) and suppose \( \hat{\Phi} : T\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \) satisfies (4.2.5). The composition \( \Phi = \hat{\Phi} \circ \iota \) implies

\[
\Phi^* (dz - \lambda) = \iota^* \hat{\Phi}^* \left( dz - \sum_{j=1}^n y_j \, dx_j \right) = \iota^* \left( \sum_{j=1}^n (p_j \, dq_j + q_j \, dp_j - (q_j - \langle q, p \rangle p_j) \, dp_j) \right)
\]

\[
= \iota^* \left( \sum_{j=1}^n p_j \, dq_j + \langle q, p \rangle \sum_{j=1}^n p_j \, dp_j \right) = \iota^* \left( \sum_{j=1}^n p_j \, dq_j \right) = \lambda |_{S\mathbb{R}^n}.
\]

Here we employed the fact that the image of \( \iota_* \) belongs to the kernel of the radial 1-form

\[
\sum_{j=1}^n p_j \, dp_j = dF(p) \quad \text{with} \quad F(p) = \frac{1}{2}\|p\|^2,
\]
since \( SQ \) corresponds to a level set of \( F \).

The advantage of working with \( J^1(S^{n-1}) \) instead of \( S\mathbb{R}^n \) is that the Reeb vector field on \( J^1(S^{n-1}) \) is simply given by \( \frac{\partial}{\partial x} \). Pulling a cylindrical almost complex structure on \( S\mathbb{R}^n \times \mathbb{R} \) via \((\Phi, \text{Id})\) back to the symplectization

\[
M = J^1(S^{n-1}) \times \mathbb{R} = TS^{n-1} \times \mathbb{R} \times \mathbb{R}
\] (4.2.6)
yields an almost complex structure \( J \in \mathcal{J}(M, d(e^t \alpha)) \) which restricts to \( J_0 \) on the two \( \mathbb{R} \)-factors. The Legendrian fibre \( S_q \mathbb{R}^n \) over \( q \in \mathbb{R}^n \) corresponds in \( J^1(S^{n-1}) \) to the Legendrian sphere

\[
\Lambda_q = \{(x, q - \langle q, x \rangle x, \langle q, x \rangle) \in TS^{n-1} \times \mathbb{R} : x \in S^{n-1}\}.
\] (4.2.7)

It can be seen as a section over \( S^{n-1} \) given by a specific function on the sphere.
Lemma 4.2.3. Let \( f_q : S^{n-1} \to \mathbb{R} \) be the function given by
\[
f_q(x) = \langle q, x \rangle. \tag{4.2.8}
\]
Then the Legendrian sphere \( \Lambda_q \) satisfies via the identification \( T^*S^{n-1} \cong TS^{n-1} \)
\[
\Lambda_q = \left\{ (x, \mathbf{d}f_q(x), f_q(x)) : x \in S^{n-1} \right\}. \tag{4.2.9}
\]

Proof. The projection of \( \Lambda_q \) to the tangent bundle intersects the fiber \( T_xS^{n-1} \) according to (4.2.7) in the tangent vector
\[
\xi_q(x) = \sum_{j=1}^{n} (q_j - \langle q, x \rangle x) \frac{\partial}{\partial x_j}.
\]
For another tangent vector \( \eta = \sum y_j \frac{\partial}{\partial x_j} \in T_xS^{n-1} \) we obtain
\[
\langle \xi_q(x), \eta \rangle = \langle q, y \rangle - \langle q, x \rangle \langle x, y \rangle = \langle q, y \rangle = \mathbf{d}f_q(y),
\]
taking advantage of \( \langle x, y \rangle = 0 \). \( \square \)

Any two Legendrian spheres \( \Lambda_{q_i} \) and \( \Lambda_{q_j} \) with \( q_i \neq q_j \) are disjoint and share exactly two connecting Reeb chords. In order to find their endpoints one has to solve the equation
\[
q_i - \langle q_i, x \rangle x = q_j - \langle q_j, x \rangle x
\]
for \( x \). It turns out that the forwards chord \( c^{i,j} \) from \( \Lambda_{q_i} \) to \( \Lambda_{q_j} \) starts at
\[
c^{i,j}_{-} = \left( \frac{q_j - q_i}{\|q_j - q_i\|}, q_i - \frac{\langle q_i, q_j - q_i \rangle (q_j - q_i)}{\|q_j - q_i\|^2}, \frac{\langle q_i, q_j - q_i \rangle}{\|q_j - q_i\|} \right) \in \Lambda_{q_i}, \tag{4.2.10}
\]
and the backwards chord \( c^{j,i} \) from \( \Lambda_{q_j} \) to \( \Lambda_{q_i} \) ends at
\[
c^{j,i}_{+} = \left( - \frac{q_j - q_i}{\|q_j - q_i\|}, q_i - \frac{\langle q_i, q_j - q_i \rangle (q_j - q_i)}{\|q_j - q_i\|^2}, - \frac{\langle q_i, q_j - q_i \rangle}{\|q_j - q_i\|} \right) \in \Lambda_{q_i}. \tag{4.2.11}
\]
Given pairwise disjoint \( q_1, \ldots, q_m \in \mathbb{R}^n \), for the Legendrian
\[
\Lambda = \bigcup_{j=1}^{m} \Lambda_{q_j} \tag{4.2.12}
\]
there are exactly \( m(m - 1) \) connecting Reeb chords. We are considering \( J \)-holomorphic curves \( u : \mathbb{D}_m^0 \to M \) such that the component of \( \partial \mathbb{D}_m^0 \) between \( p_j \) and \( p_{j+1} \) is mapped onto \( \Lambda_{q^{j+1}} \times \mathbb{R} \) and the curve is at the boundary puncture \( p_j \) asymptotic to \( c^{j,j+1} \) or \( c^{j+1,j} \) depending on whether the puncture is positive or negative, respectively. Thus we define
\[
c(j) = \left\{ \begin{array}{ll}
c^{i,j+1} & \text{if } p_j \text{ is a positive puncture}, \\
c^{j+1,j} & \text{if } p_j \text{ is a negative puncture}. \end{array} \right. \tag{4.2.13}
\]
Let \( \pi : M \to J^1(S^{n-1}) \) denote the canonical projection. Moreover, we consider the Lagrangian projection
\[
\Pi : J^1(S^{n-1}) \to TS^{n-1}, \quad \Pi(x, y, z) = (x, y). \tag{4.2.14}
\]
4.2. INDEX CALCULATIONS

The Legendrian sphere $\Lambda_q$ projects onto

$$L_q = \Pi(\Lambda_q) = \{(x, q - (q, x)x) \in TS^{n-1} \subset \mathbb{R}^n \times \mathbb{R}^n : x \in S^{n-1}\}$$

which is a Lagrangian section in $(TS^{n-1}, -d\lambda)$. In particular,

$$L = \bigcup_{j=1}^{m} L_{q_j} = \Pi(\Lambda).$$

Each connecting Reeb chord $c^{i,j}$ projects onto a double point

$$\tilde{c}^{i,j} = \Pi(c^{i,j}) = \left(\frac{q_j - q_i}{\|q_j - q_i\|}, q_i - \frac{\langle q_j, q_j - q_i \rangle (q_j - q_i)}{\|q_j - q_i\|^2}\right) \in L_{q_i} \cap L_{q_j}.$$  

Moreover, let us denote

$$\tilde{c}(j) = \Pi(c(j)).$$

One easily checks that

$$L_{q_i} \cap L_{q_j} = \{\tilde{c}^{i,j}, \tilde{c}^{j,i}\},$$

i.e. all self intersections of $L$ are double points which are transverse by Proposition 4.2.1. Given that the boundary of $\mathcal{D}^o_m$ in the neighborhood of a boundary puncture $p_j$ is mapped to the Lagrangians $\Lambda_{q_j} \times \mathbb{R}$ and $\Lambda_{q_{j+1}} \times \mathbb{R}$, exactly one of the connecting Reeb chords $c^{j+1,j}, c^{j+1,j}$ allows for positive/negative convergence at $p_j$.

Hence we may consider two types of curves:

- For $J \in \mathcal{J}(J^1(S^{n-1}) \times \mathbb{R}, d(e^{t}\alpha))$ one may look at $J$-holomorphic curves in the symplectization with $J$ cylindrical and invariant under the Reeb flow.

- For $J \in \mathcal{J}(TS^{n-1}, \tilde{\omega})$ one may also look at $J$-holomorphic disks $\tilde{u} : \mathcal{D}_m^o \to TS^{n-1}$ asymptotic to double points at boundary punctures.

Recall that $J$ being cylindrical means that the restriction to the hyperplane field $\xi$ is an almost complex structure itself, $J$ maps the Reeb vector field onto $\frac{\partial}{\partial t}$ and is invariant under translations in $t$. Restricting to cylindrical almost complex structures which are additionally invariant under the Reeb flow is possible since the index of the linearized Cauchy-Riemann operator $D_u$ does not depend on the choice of compatible $J$. The following lemma shows that both types of curves are in 1-1 correspondence up to a shift in the symplectization direction. In particular, indices differ by one.

**Lemma 4.2.4.** Suppose

$$J \in \mathcal{J}(M, d(e^{t}\alpha)) = \mathcal{J}(TS^{n-1} \times \mathbb{R} \times \mathbb{R}, d(e^{t}\alpha))$$

is invariant under $\frac{\partial}{\partial t}, \frac{\partial}{\partial \lambda}$ and restricts to $J_0$ on $\mathbb{R} \times \mathbb{R}$ and an almost complex structure on $\xi$. Then the following is true:

(a) The pushforward

$$\tilde{J} = (\Pi \circ \pi)_* J$$

is a well-defined almost complex structure belonging to $\mathcal{J}(TS^{n-1}, \tilde{\omega})$. 

(b) Any $J$-holomorphic curve $u : \mathbb{D}_m^o \to M$ with boundary on $\Lambda \times \mathbb{R}$ and asymptotic to Reeb chords $c(1), \ldots, c(m)$ at boundary punctures projects via $\Pi \circ \pi$ onto a $\tilde{J}$-holomorphic curve $\tilde{u} : \mathbb{D}_m^o \to T^*S^{n-1}$ with boundary on $L$ and asymptotic to the respective double points $\tilde{c}(1), \ldots, \tilde{c}(m)$ at boundary punctures.

(c) Any $\tilde{J}$-holomorphic curve $\tilde{u} : \mathbb{D}_m^o \to T^*S^{n-1}$ with boundary on $L$ and asymptotic to double points $\tilde{c}(1), \ldots, \tilde{c}(m)$ at boundary punctures lifts to a $J$-holomorphic curve $u : \mathbb{D}_m^o \to M$ with boundary on $\Lambda \times \mathbb{R}$ and asymptotic to the respective Reeb chords $c(1), \ldots, c(m)$ at boundary punctures. The lifted curve $u$ is uniquely determined up to a shift in the symplectization direction $\frac{\partial}{\partial \xi}$.

(d) The indices of $u$ and $\tilde{u}$ are related by

$$\text{index } D_u = \text{index } D_{\tilde{u}} + 1.$$  

Proof.

(a) By translation invariance of $J$, the pushforward $(\Pi \circ \pi)_* J$ is a well-defined endomorphism of $T^*S^{n-1}$. Since $J|_\xi$ is an almost complex structure and $\Pi_*\xi = T^*S^{n-1}$, we obtain that $\tilde{J}$ is indeed an almost complex structure. In order to show compatibility, it suffices to check that $J|_\xi$ is compatible with $-d\tilde{\lambda}$. This holds because of

$$d(e^t a)|_\xi = d\left(e^t\left(dz - \tilde{\lambda}\right)\right)|_\xi = e^t dt \wedge \left(dz - \tilde{\lambda}\right)|_{\xi} - e^t d\tilde{\lambda}|_{\xi} = -e^t d\tilde{\lambda}|_{\xi}.$$  

(b) This follows from (a) in combination with (4.2.16) and (4.2.18).

(c) Since each Legendrian sphere $\Lambda_q$ may be seen as a section, there is a unique $\tilde{a} : \partial\mathbb{D}_m^o \to \mathbb{R}$ such that

$$\left(\tilde{u}\big|_{\partial\mathbb{D}_m^o}, \tilde{a}\right) : \partial\mathbb{D}_m^o \to T^*S^{n-1} \times \mathbb{R} = J^1(S^{n-1})$$

maps into $\Lambda$. We may extend $\tilde{a}$ analytically to $a : \mathbb{D}_m^o \to \mathbb{R} \times \mathbb{R}$ such that

$$\text{Re } a|_{\partial\mathbb{D}_m^o} = \tilde{a}.$$  

The extension is determined uniquely up to an imaginary constant. We know that $J$ restricts to $J_0$ on $\mathbb{R} \times \mathbb{R}$, hence the tuple $(u, a) : \mathbb{D}_m^o \to T^*S^{n-1} \times \mathbb{R} \times \mathbb{R}$ yields the desired $J$-holomorphic map.

(d) Consider $M$ as a complex vector bundle over $S^{n-1}$. Any unitary trivialization of $T^*S^{n-1}$ lifts to a unitary trivialization of

$$\iota^* (TM|_\xi)$$

via $\Pi_*$ with $\iota : T^*S^{n-1} \to M$, $\iota(x, y) = (x, y, 0, 0)$ denoting the zero section and thus to a trivialization of $TM|_\xi$ by translation invariance of $J$ in $\mathbb{R} \times \mathbb{R}$. Finally, we obtain a unitary trivialization of $TM$ by adding the $\mathbb{C}$-factor corresponding to the Reeb and symplectization direction. Hence the indices of $u$ and $\tilde{u}$ are equal up to a difference of one due to possible translations of $u$ in the symplectization direction.

$\square$

The boundary of $\tilde{u}$ traces out a Lagrangian loop $\Lambda^{\tilde{u}}$ in $(T^*S^{n-1}, \tilde{\omega})$ as follows:
4.2. INDEX CALCULATIONS

• The boundary component of $\mathbb{D}_m^n$ between the punctures $p_j$ and $p_{j+1}$ is mapped to a path in the Lagrangian $L_{q_{j+1}}$ connecting $\tilde{c}(j)$ and $\tilde{c}(j+1)$.

• At a boundary puncture $p_j$ the Lagrangian paths are closed by a negative rotation from $T_{\tilde{c}(j)}L_{q_j}$ to $T_{\tilde{c}(j)}L_{q_{j+1}}$.

The next proposition implies that the relative Maslov index $\mu(\Lambda^\varpi)$ is independent of the choice of symplectic trivialization.

Proposition 4.2.5. The first Chern class $c_1(TS^{n-1}, \omega)$ always vanishes.

Proof. For $n \neq 3$ the statement is obvious since $H^2(S^{n-1}, \mathbb{Z}) = 0$. For $n = 3$ the claim follows from Proposition 2.4.3 and the fact that $H^2(S^2, \mathbb{Z}) = \mathbb{Z}$ does not contain nontrivial elements of order 2.

We conclude

$$\text{index } D_\varpi = (n - 1) + \mu(\Lambda^\varpi) \quad (4.2.19)$$

and continue by considering suitable trivializations of $TS^{n-1}$.

Lemma 4.2.6. Suppose $S = (0, \ldots, 0, -1)$ denotes the south pole of the embedded tangent bundle $TS^{n-1} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \|x\| = 1 \text{ and } \langle x, y \rangle = 0\}$.

Then $\varphi : TS^{n-1}\setminus \{S\} \times \mathbb{R}^n \to \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ given by

$$\varphi(x, y) = \left(\frac{x_1}{1 + x_n}, \ldots, \frac{x_{n-1}}{1 + x_n}, y_1 + (x_ny_1 - y_nx_1), \ldots, y_{n-1} + (x_ny_{n-1} - y_nx_{n-1})\right) \quad (4.2.20)$$

is a symplectic trivialization. In particular, we have

$$\varphi^* \lambda = \lambda |_{TS^{n-1}} \quad (4.2.21)$$

where $\lambda$ denotes the canonical 1-form.

Proof. Note that

$$x \mapsto \left(\frac{x_1}{1 + x_n}, \ldots, \frac{x_{n-1}}{1 + x_n}\right)$$

is a diffeomorphism between $S^{n-1}\setminus \{S\}$ and $\mathbb{R}^{n-1}$. Its linearization

$$(x, y) \mapsto \left(\frac{x_1}{1 + x_n}, \ldots, \frac{x_{n-1}}{1 + x_n}, \frac{y_1}{1 + x_n} - \frac{y_nx_1}{(1 + x_n)^2}, \ldots, \frac{y_{n-1}}{1 + x_n} - \frac{y_nx_{n-1}}{(1 + x_n)^2}\right)$$

coincides with (4.2.20) up to fiberwise multiplication with

$$\frac{4}{\left(1 + \sum_{j=1}^{n-1} \frac{x_j}{1 + x_n}\right)^2} = \frac{4(1 + x_n)^4}{\left((1 + x_n)^2 + x_1^2 + \ldots + x_{n-1}^2\right)^2} = \frac{4(1 + x_n)^4}{(2 + 2x_n)^2} = (1 + x_n)^2$$

which corresponds to identifying $T^*S^{n-1}$ and $TS^{n-1}$ via the metric. In particular, $\varphi$ is a trivialization of vector bundles.
In order to verify (4.2.21), let \( \iota : T S^{n-1} \hookrightarrow \mathbb{R}^n \times \mathbb{R}^n \) denote the embedding of the tangent bundle. Moreover, suppose \( \hat{\varphi} : (\mathbb{R}^n \setminus \{ S \}) \times \mathbb{R}^n \to \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \) satisfies (4.2.20) such that the composition rule \( \varphi = \hat{\varphi} \circ \iota \) is satisfied. A simple calculation yields

\[
\varphi^* \lambda = \sum_{j=1}^{n-1} (y_j + x_n y_j - y_n x_j) \, d \left( \frac{x_j}{1 + x_n} \right)
= \sum_{j=1}^{n-1} (y_j (1 + x_n) - y_n x_j) \left( \frac{dx_j}{1 + x_n} - \frac{x_j}{(1 + x_n)^2} \, dx_n \right)
= \sum_{j=1}^{n-1} \left( y_j \, dx_j - \frac{x_j y_n \, dx_j}{1 + x_n} - \frac{x_j y_n \, dx_n}{1 + x_n} + \frac{x_j^2 y_n \, dx_n}{(1 + x_n)^2} \right)
= \sum_{j=1}^{n} y_j \, dx_j - \frac{y_n}{1 + x_n} \sum_{j=1}^{n} x_j \, dx_j - \frac{x_j y_n}{1 + x_n} \, dx_n + \frac{x_j^2 y_n}{(1 + x_n)^2} \, dx_n
- y_n \, dx_n + \frac{2 x_n y_n}{1 + x_n} \, dx_n - \frac{x_n^2 y_n}{(1 + x_n)^2} \, dx_n.
\]

Since \( S^{n-1} \) is given by \( \| x \| = 1 \), the 1-form \( \sum x_j \, dx_j \) pulls back by \( \iota^* \) to zero. Plugging in \( \| x \| = 1 \) and \( \langle x, y \rangle = 0 \) leads to

\[
\varphi^* \lambda = \iota^* \hat{\varphi}^* \lambda = \sum_{j=1}^{n} y_j \, dx_j + \frac{1}{1 + x_n} \left( \frac{y_n}{1 + x_n} - y_n + x_n y_n - \frac{x_n^2 y_n}{1 + x_n} \right) \, dx_n = \sum_{j=1}^{n} y_j \, dx_j.
\]

Applying the differential \( d \) to both sides of (4.2.21) shows that \( \varphi \) is indeed a symplectomorphism. \( \square \)

Let us denote by \( \mathcal{L}(n-1) \) the set of Lagrangian planes in \( \mathbb{C}^{n-1} \). The part of the Lagrangian section \( L_q \) over \( S^{n-1} \setminus \{ S \} \) is mapped via \( \varphi \) to a Lagrangian in \( \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \cong \mathbb{C}^{n-1} \). The point

\[ L_q(x) = (x, q - \langle q, x \rangle x) \]  

(4.2.22)

satisfying \( \| x \| = 1 \) and \( x \neq S \) goes to

\[ \varphi \circ L_q(x) = \left( \frac{x_1}{1 + x_n}, \ldots, \frac{x_{n-1}}{1 + x_n}, v_1, \ldots, v_{n-1} \right), \quad v_j = q_j + (q_j x_n - q_n x_j) - \langle q, x \rangle x_j. \]  

(4.2.23)

For convenience, let us denote the tangent space of \( L_q \) at \( L_q(x) \) by \( T_x L_q \). The next lemma determines the pushforward of this Lagrangian space under \( \varphi \).

**Lemma 4.2.7.** Suppose the vectors \( y^1, \ldots, y^{n-1} \in \mathbb{R}^n \) complete \( x \in \mathbb{R}^n \) with \( \| x \| = 1 \), \( x \neq S \) to an orthonormal basis. Then the Lagrangian tangent space \( \varphi_* (T_x L_q) \in \mathcal{L}(n-1) \) is spanned by the \( n-1 \) vectors

\[
\left( \begin{array}{c}
\frac{1}{(1+x_n)^2} (y^1_1 + x_n y^1_1 - x_1 y^1_n) \\
\vdots \\
\frac{1}{(1+x_n)^2} (y^k_1 + x_n y^k_1 - x_1 y^k_n) \\
q_1 y^1_n - q_n y^1_1 - \langle q, x \rangle y^1_n - \langle q, y^1 \rangle x_1 \\
\vdots \\
q_{n-1} y^{n-1}_n - q_n y^{n-1}_1 - \langle q, x \rangle y^{n-1}_n - \langle q, y^{n-1} \rangle x_{n-1}
\end{array} \right) \quad \text{with} \quad 1 \leq k \leq n-1. \]  

(4.2.24)
4.2. INDEX CALCULATIONS

Proof. The tangent space of $TS^{n-1}$ at $x$ is spanned by the tangent vectors to the curves

$$\gamma^k(t) = (\cos t) x + (\sin t) y^k$$

at $t = 0$. Since $L_q$ is a section of $TS^{n-1}$ and $\varphi$ is a symplectomorphism by Lemma 4.2.6, it suffices to consider the curves $\varphi \circ L_q \circ \gamma^k$. Their derivatives at $t = 0$ are easily computed from (4.2.22) and (4.2.23), the result is (4.2.24). By definition these vectors span the Lagrangian tangent space $\varphi_* (T_x L_q)$.

The Maslov index of a loop in $\mathcal{L}(n-1)$ can be seen as the intersection number with the Maslov cycle $\Sigma(n-1)$ of Lagrangian subspaces intersecting the vertical $\{0\} \times \mathbb{R}^{n-1}$ nontransversally.

Lemma 4.2.8. Any Lagrangian tangent space $\varphi_* (T_x L_q) \in \mathcal{L}(n-1)$ with $\|x\| = 1$, $x \neq S$ and $q \in \mathbb{R}^n$ is not an element of the Maslov cycle $\Sigma(n-1)$.

Proof. Let $\pi_1 : \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be the projection onto the first factor. According to Lemma 4.2.7, we have

$$\pi_1 \circ \varphi_* (T_x L_q) = \left\{ \left( \begin{array}{c} \frac{1}{1+x_n^2} (y_1 + x_n y_1 - x_1 y_n) \\ \vdots \\ \frac{1}{1+x_n^2} (y_{n-1} + x_n y_{n-1} - x_{n-1} y_n) \\ x_1 y_1 + \ldots + x_n y_n \end{array} \right) : y \in \mathbb{R}^n \text{ and } \langle x, y \rangle = 0 \right\}.$$  

It suffices to show that this equals $\mathbb{R}^{n-1}$ or, equivalently, that

$$\left( \begin{array}{c} \frac{1}{1+x_n^2} (y_1 + x_n y_1 - x_1 y_n) \\ \vdots \\ \frac{1}{1+x_n^2} (y_{n-1} + x_n y_{n-1} - x_{n-1} y_n) \\ x_1 y_1 + \ldots + x_n y_n \end{array} \right) = 0$$

implies $y = 0$. Multiplying the first $n-1$ rows by $(1 + x_n)^2$, this may be rewritten as

$$\left( \begin{array}{cccc} 1 + x_n & 0 & \ldots & 0 \\ 0 & 1 + x_n & \ldots & 0 \\ \vdots \\ 0 & 0 & 0 & 1 + x_n \end{array} \right) \left( \begin{array}{c} -x_1 \\ -x_2 \\ \vdots \\ -x_{n-1} \\ x_n \end{array} \right) = 0.$$

We are done by applying Laplace's formula to the last row, because it follows that the matrix on the left-hand side has positive determinant

$$\sum_{k=1}^{n-1} (-1)^{n+k} (-1)^{n-k} x_k^2 (1 + x_n)^{n-2} + x_n (1 + x_n)^{n-1} = (\|x\|^2 + x_n) (1 + x_n)^{n-2} = (1 + x_n)^{n-1}.$$

The Lemma shows that the only intersections of the Lagrangian loop $\varphi \circ \Lambda^2$ with the Maslov cycle may occur at boundary punctures. Let us denote the standard basis of $\mathbb{R}^n$ by $\{e_1, \ldots, e_n\}$.

In order to see what happens at a puncture $p_j$, we consider the special case where the adjacent boundary components are mapped to Lagrangians $L_{q_j}, L_{q_{j+1}}$ with $q_j$ and $q_{j+1}$ a scalar multiple of $e_1$. 

\(\square\)
Lemma 4.2.9. Suppose \( q_j = \lambda e_1 \) and \( q_j+1 = \mu e_1 \) with \( \lambda \neq \mu \). Then the negative rotation of the Lagrangian loop \( \varphi \circ \Lambda^0 \) at the puncture \( p_j \) intersects the Maslov cycle \( \Sigma (n-1) \) only if the puncture is negative. In this case the multiplicity of the intersection equals \(-n+1\).

Proof. Let

\[
\sigma = \text{sign}(j) \frac{\mu - \lambda}{|\mu - \lambda|}.
\]

Thus \( \sigma \) takes values according to the following table:

<table>
<thead>
<tr>
<th>( p_j ) is positive</th>
<th>( \lambda &lt; \mu )</th>
<th>( \lambda &gt; \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_j ) is negative</td>
<td>(-1)</td>
<td>(1)</td>
</tr>
</tbody>
</table>

(4.2.25)

By (4.2.17) and (4.2.18) the double point \( \tilde{c}(j) \in L_{q_j} \cap L_{q_{j+1}} \) to which \( \tilde{u} \) converges at \( p_j \) lies in the fibre \( T_x S^{n-1} \) with \( x = \sigma e_1 \).

Let us now determine the Lagrangian space

\[
\varphi_* (T_{(\sigma e_1)} L_{q_j})
\]

tangent to \( \varphi(L_{q_j}) \). By Lemma 4.2.7 it is spanned by

\[
\begin{pmatrix}
y_1 - \sigma y_n \\
y_2 \\
\vdots \\
y_{n-1} \\
\lambda y_n - 2 \lambda \sigma y_1 \\
-\lambda \sigma y_2 \\
\vdots \\
-\lambda \sigma y_{n-1}
\end{pmatrix}
\]

where \( y \) runs through \( e_2, \ldots, e_n \). Plugging this in, we obtain

\[
\varphi_* (T_{(\sigma e_1)} L_{q_j}) = \begin{pmatrix} \text{Id} \\ -\lambda \sigma \text{Id} \end{pmatrix}
\]

(4.2.26)

and similarly

\[
\varphi_* (T_{(\sigma e_1)} L_{q_{j+1}}) = \begin{pmatrix} \text{Id} \\ -\mu \sigma \text{Id} \end{pmatrix}.
\]

(4.2.27)

We see that the orthogonal decomposition of \( \mathbb{C}^{n-1} \) induced by these two Lagrangians subspaces is given by \( W^1 = \mathbb{C}^{n-1} \). Moreover, the negative rotation taking (4.2.26) to (4.2.27) intersects the Maslov cycle \( \Sigma (n-1) \) if and only if

\[
-\lambda \sigma < -\mu \sigma.
\]

(4.2.28)

In this case the multiplicity of intersection is \(-(n-1)\). Now (4.2.25) implies that (4.2.28) is satisfied exactly in the case of \( p_j \) being negative. \( \square \)
4.2. INDEX CALCULATIONS

Let us denote the number of positive and negative punctures by
\[
\#\text{pos} = \# \{ \text{sign}^{-1}(+) \} \quad \text{and} \quad \#\text{neg} = \# \{ \text{sign}^{-1}(-) \},
\]
where sign : \{p_1, \ldots, p_m\} \to \{+, -\} prescribes whether punctures are positive or negative. The previous lemma allows us to calculate the index of \( \tilde{u} \) through a deformation of \( L \).

**Lemma 4.2.10.** The index of \( \tilde{u} \) equals
\[
\text{index } D_{\tilde{u}} = (n - 1) (1 - \#\text{neg}).
\]

**Proof.** The space of \( m \)-tuples
\[
\{(q_1, \ldots, q_m) \in (\mathbb{R}^n)^m : q_i \neq q_j \text{ for } 1 \leq i, j \leq m\}
\]
is connected. Hence we may find a path of admissible Legendrians \( \Lambda \) and corresponding Lagrangians \( L \) of the form (4.2.12), (4.2.16) starting at any given \( m \)-tuple and ending at \( (q_1, \ldots, q_m) \) such that \( q_j = j e_1 \) for \( 1 \leq j \leq m \). The Maslov index \( \mu(\Lambda^{\tilde{u}}) \) does not change along this path. According to Lemma 4.2.9, it equals
\[
\mu(\Lambda^{\tilde{u}}) = -\#\text{neg} (n - 1).
\]
The vanishing of \( c_1(TS^{n-1}) \) implies that the index of \( \tilde{u} \) is given by (4.2.30). Thus the statement follows. \( \square \)

The symplectic area of \( \tilde{u} \) equals
\[
\int_{D^0_m} \tilde{u}^*(\lambda) = \sum_{j=1}^{m} \text{sign}(j) l(c(j))
\]
where \( l(c(j)) \) denotes the length of the Reeb chord / geodesic \( c(j) \). Thus (4.2.30) displays a certain monotonicity between the index and the symplectic area of the considered disks. In particular, disks with at least two negative punctures have a negative index. In case the total number of punctures is three such disks cannot exist, since they would also have a negative area by the triangle inequality.

We are now able to deduce the desired index formula.

**Theorem 4.2.11.** Given points \( q_1, \ldots, q_m \) on a Riemannian manifold \((Q, g)\) within their injectivity radii and in the domain of a normal chart \( \varphi : U \to V \), the index of a boundary punctured J-holomorphic curve \( u : D^0_m \to SQ |_{\pi^{-1}(U)} \times \mathbb{R} \) with boundary on
\[
\Lambda \times \mathbb{R} = \bigcup_{j=1}^{m} S_{t_j} Q \times \mathbb{R}
\]
and asymptotic to connecting Reeb chords \( c(1), \ldots, c(m) \) is given by
\[
\text{index } D_u = (n - 1) (1 - \#\text{neg}) + 1.
\]

**Proof.** In case \( Q = \mathbb{R}^n \) this is an immediate consequence of Lemma 4.2.4 and Lemma 4.2.10. For a curved Riemannian manifold \((Q, g)\) we consider a linear path of metrics \( g_t = t g + (1 - t) g_0 \) on \( U \) deforming the problem through a continuous path of Fredholm operators to the case \( S\mathbb{R}^n \times \mathbb{R} \). The index does not change during such a deformation and hence we end up with the same formula. \( \square \)
4.3 Conformal Models

Let $D_m^o$ be the closed unit disk with $m \geq 3$ boundary punctures such that one of them is distinguished. We consider various representations of the space $C_m$ of conformal structures on $D_m^o$. In particular, using coordinates from slit domains, we identify $C_m$ with the space of weighted source trees with $m$ ends.

The standard way to introduce coordinates on $C_m$ is to fix the distinguished puncture at $1 \in \partial D$ and the two following punctures at $i$ and $-1$, respectively. One may then use the positions of the remaining $m - 3$ punctures on the lower hemicycle for parametrization. In particular, this implies that topologically $C_m$ is a simplex of dimension $m - 3$. Different coordinates on $C_m$ can be obtained via slit domains. Here we follow section 2.1.A in [Ekh07].

**Definition 4.3.1.** Let $a = (a_1, \ldots, a_{m-2}) \in \mathbb{R}^{m-2}$. A slit domain $\Delta_m(a)$ is defined as the subset of $\mathbb{R} \times [0, m-1]$ obtained by removing $m - 2$ horizontal slits of width $\varepsilon$, $0 < \varepsilon \ll 1$ starting at $(a_j, j)$ and going to $\infty$. All slits have the same shape ending in a half-circle.

Endowing $\Delta_m(a)$ with the flat metric yields a conformal structure $\kappa(a)$ on $D_m^o$ where the distinguished puncture corresponds to the end of $\Delta_m(a)$ at $-\infty$. Consider the action of $t \in \mathbb{R}$ on $\mathbb{R}^{m-2}$ by $t(a) = a + t(1, \ldots, 1)$. The orbit space is $\mathbb{R}^{m-3}$. Since translations are biholomorphic, one obtains $\kappa(a) = \kappa(t(a))$. In fact, the following is true ([Ekh07], Lemma 2.2).

**Proposition 4.3.2.** The map $a \mapsto \kappa(a)$ induces a diffeomorphism between the orbit space $\mathbb{R}^{m-2}/\mathbb{R} = \mathbb{R}^{m-3}$ and $C_m$.

The goal of this section is to interpret slit domains as weighted source trees. The picture one should have in mind is presented in Figure 4.1. The idea is to prescribe the weight of the inner edges (connecting black vertices) by the horizontal length of the corresponding strip regions.

![Figure 4.1: A slit domain with corresponding weighted source tree.](image)

**Definition 4.3.3.** A source tree $\Gamma$ is a rooted tree having no vertices of degree 2 together with the following structure. At each vertex $v$ the edges adjacent to $v$ are cyclically ordered. A weighted source tree $(\Gamma, w)$ is a source tree $\Gamma$ together with a weight function $w : E \to (0, \infty]$ defined on its edge set such that

$$w(e) = \infty \iff e \text{ is adjacent to an end.} \tag{4.3.1}$$

The space of weighted source trees with $m$ ends will be denoted by $G_m$. 

4.3. CONFORMAL MODELS

We will next define a map \( \Phi : \mathcal{G}_m \to \mathbb{R}^{m-2}/\mathbb{R} \) where the latter is the orbit space considered above. For this pick any weighted source tree \((\Gamma, w) \in \mathcal{G}_m\). Its ends are cyclically ordered. Denote the ends following the root in this ordering by \(v_1, \ldots, v_{m-1}\). By (4.3.1) any interior edge \(e\) has a well-defined length \(w(e) < \infty\). In particular, we may assign any interior path \(P\) in \(\Gamma\) a finite length

\[
w(P) = \sum_{e \in P} w(e) .
\]

For \(1 \leq j \leq m - 2\) denote by \(P_j\) the maximal interior path which lies both on the unique path connecting the root with \(v_j\) and the path connecting the root with \(v_{j+1}\). We then define

\[
\Phi(\Gamma, w) = [(w(P_1), \ldots, w(P_{m-2}))].
\]  

(4.3.2)

**Lemma 4.3.4.** The map \( \Phi : \mathcal{G}_m \to \mathbb{R}^{m-2}/\mathbb{R} \) is bijective.

**Proof.** Let us start with the observation

\[
\min \{w(P_1), \ldots, w(P_{m-2})\} = 0 .
\]  

(4.3.3)

Indeed, the vertex \(u_1\) adjacent to the root \(v_0\) has degree \(\geq 3\). Hence their must be ends \(v_j, v_{j+1}\) whose connecting paths to the root only have the edge \(v_0u_1\) in common. Consequently, \(w(P_j) = 0\) follows. We further see that the statement of the lemma obviously holds for \(m = 3\).

Any weighted source tree with three ends must be a star graph with all weights \(\infty\). We then proceed via induction on \(m\).

Pick \([(a_1, \ldots, a_{m-2})] \in \mathbb{R}^{m-2}/\mathbb{R}\). By removing the end \(v_{m-1}\) and possibly the vertex adjacent to it (if its valence equals \(3\)), we may see any \((\Gamma, w) \in \mathcal{G}_m\) with \(\Phi(\Gamma, w) = [(a_1, \ldots, a_{m-2})]\) as an extension of a weighted source tree \((\Gamma', w') \in \mathcal{G}_{m-1}\) satisfying \(\Phi(\Gamma', w') = [(a_1, \ldots, a_{m-3})]\). By the induction hypothesis there is in fact a unique weighted source tree \((\Gamma', w') \in \mathcal{G}_{m-1}\) such that \(\Phi(\Gamma', w') = [(a_1, \ldots, a_{m-3})]\). According to (4.3.3) we have

\[
w'(P_j') = a_j - \min \{a_1, \ldots, a_{m-3}\} .
\]

Denote its ends following the root \(v_0\) in cyclic order by \(v_1, \ldots, v_{m-2}\) and let \(P'_{m-2} = (u_1, \ldots, u_k)\) such that \(u_1\) is adjacent to \(v_0\) and \(u_k\) is adjacent to \(v_{m-2}\). Set

\[
L_{m-1} = a_{m-2} - \min \{a_1, \ldots, a_{m-3}\} .
\]

Now there are four cases to consider:

- If \(L_{m-1} < 0\), then \(a_{m-2} = \min \{a_1, \ldots, a_{m-2}\}\) and hence \(v_{m-1}\) must be adjacent to a vertex on \(v_0u_1\). We construct \((\Gamma, w)\) from \((\Gamma', w')\) by adding a vertex \(u\) on \(v_0u_1\) and attaching the end \(v_{m-1}\) to it. The cyclic ordering of the neighbors of \(u\) becomes \(v_0, u_1, v_{m-1}\) and the new inner edge \(uu_1\) obtains the weight \(-L_{m-1}\). When passing from \((\Gamma', w')\) to \((\Gamma, w)\), we thus obtain

\[
w(P_j) = w'(P_j') - L_{m-1} = a_j - \min \{a_1, \ldots, a_{m-3}\} - L_{m-1} = a_j - a_{m-2}
\]

for \(1 \leq j \leq m - 3\). Moreover, \(w(P_{m-2}) = 0\) such that

\[
\Phi(\Gamma, w) = [(a_1 - a_{m-2}, \ldots, a_{m-3} - a_{m-2}, 0)] = [(a_1, \ldots, a_{m-3}, a_{m-2})].
\]
• For some $1 \leq l \leq k$ we have $L_{m-1} = w'(u_1, \ldots, u_l)$. Then $v_{m-1}$ must be adjacent to $u_l$. Hence we construct $(\Gamma', w')$ from $(\Gamma', w')$ by attaching $v_{m-1}$ to $u_l$ such that in the cyclic ordering of its neighbors $v_{m-1}$ precedes $u_{l-1}$ or $v_0$ in case $l = 1$. When passing from $(\Gamma', w')$ to $(\Gamma, w)$, the weights do not change and $w(P_{m-2}) = L_{m-1}$. Therefore
\[
\Phi(\Gamma, w) = [(w'(P'_1), \ldots, w'(P'_{m-3}), L_{m-1})] = [(a_1, \ldots, a_{m-3}, a_{m-2})].
\]

• For some $1 \leq l \leq k-1$ we have
\[
w'(u_1, \ldots, u_l) < L_{m-1} < w'(u_1, \ldots, u_{l+1}).
\]

It follows that $v_{m-1}$ is adjacent to a vertex on $u_l u_{l+1}$. We add a vertex $u$ on $u_l u_{l+1}$ and attach the end $v_{m-1}$ to it. The cyclic ordering of the neighbors of $u$ becomes $u_l, u_{l+1}, v_{m-1}$ and the new inner edges obtain the weights
\[
w(u_l u) = L_{m-1} - w'(u_1, \ldots, u_l)
\quad \text{and} \quad
w(u_{l+1} u) = w'(u_1, \ldots, u_{l+1}) - L_{m-1}.
\]

When passing from $(\Gamma', w')$ to $(\Gamma, w)$, the weights do not change and $w(P_{m-2}) = L_{m-1}$. As before, $\Phi(\Gamma, w) = [(a_1, \ldots, a_{m-2})]$ follows.

• If $L_{m-1} > w'(u_1, \ldots, u_k)$, then $v_{m-1}$ is adjacent to a vertex on $u_k v_{m-2}$. We add a vertex $u$ on $u_k v_{m-2}$ and attach the end $v_{m-1}$ to it. The cyclic ordering of the neighbors of $u$ becomes $u_k, v_{m-2}, v_{m-1}$ and the new inner edge $u_k u$ obtains the weight $L_{m-1} - w'(u_1, \ldots, u_k)$.

When passing from $(\Gamma', w')$ to $(\Gamma, w)$, the weights do not change and $w(P_{m-2}) = L_{m-1}$. As before, $\Phi(\Gamma, w) = [(a_1, \ldots, a_{m-2})]$ follows.

In each case we succeeded in constructing a unique extension of $(\Gamma', w')$ whose image under $\Phi$ equals $[(a_1, \ldots, a_{m-2})]$. This completes the induction step.

By transporting the topology and differentiable structure via $\Phi$ onto $G_m$, we obtain the desired identification.

**Corollary 4.3.5.** For any $m \geq 3$ there is a natural diffeomorphism which identifies the spaces $C_m \cong G_m$ of conformal structures and weighted source trees.

We may measure the size of a weighted source tree $(\Gamma, w)$ by
\[
\|w\| = \max \{ w(e) : e \text{ is an inner edge of } \Gamma \}.
\]

In particular, we denote by $(\Gamma, w_0) \in G_m$ the unique weighted source tree with no inner edges and set $\|w_0\| = 0$. It has the combinatorics of a star graph with all weights $\infty$. Note that this norm is continuous with respect to the topology on $G_m$ introduced above.

### 4.4 Morse Flow Trees in $S\mathbb{R}^n$

After recalling the definition of Morse flow trees in 1-jet bundles, we explicitely identify these trees in the case $S\mathbb{R}^n \cong J^1(S^{n-1})$ with Legendrian corresponding to a disconnected union of fibers $S_{\eta_i} \mathbb{R}^n$. Here the Morse flow will be determined by the round metric on $S^{n-1}$. In particular, we prove existence and uniqueness of such Morse flow trees whenever their domain is a weighted source tree with sufficiently small weights. Additionally, we require a transversality assumption on the choice of the fibers.
Let \((J^1(Q), d\z - \lambda)\) be the 1-jet bundle of a smooth \(n\)-dimensional manifold \(Q\), equipped with its standard contact structure. We denote its Lagrangian projection by \(\Pi : J^1(Q) \to T^*Q\). Any Legendrian \(\Lambda \subset J^1(Q)\) without singularities and with a finite number of sheets may be written as an union of sections

\[
\Lambda = \bigcup_{j=1}^s \Lambda_j, \quad \Lambda_j = \{(x, df_j(x), f_j(x)) : x \in Q\}
\]

(4.4.1)
generated by functions \(f_j : Q \to \mathbb{R}\). We fix such a Legendrian and denote its Lagrangian projection by \(L = \Pi(\Lambda)\). We also fix a metric \(g\) on \(Q\). The following definitions stem from [Ekh07], section 2.2.B.

**Definition 4.4.1.** A 1-jet lift of a path \(\gamma : [0, 1] \to Q\) is a pair \((\gamma_1, \gamma_2)\) of continuous lifts \(\tilde{\gamma} : [0, 1] \to \Lambda\), \(j = 1, 2\) of \(\gamma\) into different sheets, i.e. \(\tilde{\gamma}_1(t) \neq \tilde{\gamma}_2(t)\). Write \(\gamma_j = \Pi \circ \tilde{\gamma}_j\) for the cotangent lift of \(\gamma\). Now let \(\gamma : (-\varepsilon, \varepsilon) \to Q\) be a path with 1-jet lift \((\tilde{\gamma}_1, \tilde{\gamma}_2)\) into sheets generated by \(f_1\) and \(f_2\), respectively. Whenever \(\gamma\) satisfies the gradient flow equation

\[
\tilde{\gamma}(t) = -\nabla(f_1 - f_2)(\gamma(t)),
\]

we say that \(\gamma\) is a flow line of \(\Lambda\).

1-jet lifts of flow lines are oriented by requiring that the flow orientation of \(\tilde{\gamma}_1\) at \(p \in \Lambda\) corresponds to the unique lift of the vector field

\[-\nabla(f_1 - f_2)(\pi(p)) \in T_{\pi(p)}Q\]

where \(f_1\) is the local function determined by the sheet of \(p\) and \(\pi : J^1(Q) \to Q\) denotes the canonical projection.

**Definition 4.4.2.** Let \((\Gamma, w) \in \mathcal{G}_m\) be a weighted source tree. A Morse flow tree of \(\Lambda\) is a continuous map \(\phi : \Gamma \to Q\) satisfying the following conditions:

(a) If \(e\) is an edge of \(\Gamma\), then \(\phi : [0, w(e)] \to Q\) is a flow line of \(\Lambda\). In case \(e\) is adjacent to an end, then \(w(e) = \infty\) and the flow line converges to a critical point of the function difference.

(b) Let \(v\) be a \(k\)-valent vertex with cyclically ordered adjacent edges \(e_1, \ldots, e_k\). Let \(\{\tilde{\phi}_j^1, \tilde{\phi}_j^2\}\) be the cotangent lift corresponding to \(e_j\), \(1 \leq j \leq k\). We require that there exists a pairing of lift components such that for every \(1 \leq j \leq k\)

\[
\tilde{\phi}_j^2(v) = \tilde{\phi}_{j+1}^1(v) = \tilde{m}_j \in L
\]

and such that the flow orientation of \(\tilde{\phi}_j^2\) at \(\tilde{m}_j\) is directed toward \(\tilde{m}_j\) iff the flow orientation of \(\tilde{\phi}_{j+1}^1\) at \(\tilde{m}_j\) is directed away from \(\tilde{m}_j\).

(c) The cotangent lifts of the edges of \(\Gamma\) fit together to an oriented curve \(\tilde{\phi}\) in \(L\). We require that this curve is closed.

(d) Let \(v\) be any 1-valent vertex with adjacent edge \(e\) and corresponding 1-jet lifts \(\{\tilde{\phi}^1, \tilde{\phi}^2\}\) on different sheets generated by \(f_1\) and \(f_2\), respectively. The condition \(\tilde{\phi}^1(v) = \tilde{\phi}^2(v)\) implies that \(v\) is mapped to a critical points of \(f_1 - f_2\) in \(Q\). In particular, both points \(\tilde{\phi}^1(v) \neq \tilde{\phi}^2(v)\) must equal Reeb chord end points. We say that the Morse flow tree contains a puncture at \(v\). If \(\tilde{\phi}^1\) is oriented toward (away from) \(\tilde{\phi}^2\), the puncture is positive (negative) if

\[
z(\tilde{\phi}^1(v)) < z(\tilde{\phi}^2(v))
\]

and the puncture is negative (positive) if the opposite inequality holds.
For convenience, we will often suppress the map $\phi$ and denote a Morse flow tree simply by $(\Gamma, w)$ or $\Gamma$. Note that we adjusted Definition 2.10 in [Ekh07] in two points. In our setup punctures only occur at 1-valent vertices, while Ekholm defines punctures at $k$-valent vertices as well. This is compensated by the fact that we allow flow lines to be constant. Thus a puncture at a $k$-valent vertex in Ekholm's setup will occur as an additional edge attached to this vertex with a 1-valent puncture on its end in our setup. The flow line corresponding to this additional edge will be constant, resting on the critical point of the function difference. Secondly, we prescribe the length of the time interval for flow lines via the weight function $w$. The reason for this is that in our case of interest on $J^1(S^{n-1})$ the dimension of the space of Morse flow trees with unspecified time intervals equals the number of inner edges of $\Gamma$, see Proposition 4.4.3. One may then use the length of the time intervals at inner edges to parametrize this moduli space.

It turns out that Morse flow trees and boundary punctured $J$-holomorphic disks with boundary on $L$ and asymptotic to Reeb chords at punctures share similar properties. For instance, it follows from energy considerations that every Morse flow tree has at least one positive puncture ([Ekh07], Lemma 2.13). In some sense prescribing the length of the time intervals for Morse flow trees is analogous to prescribing a conformal structure on $D_m$ for boundary punctured disks.

We will now focus on the case where $(Q, g)$ corresponds to the sphere $S^{n-1}$ equipped with the round metric. In section 4.2 we have already seen how to identify the unit tangent bundle $(S^R^n, \lambda|_{S^R^n})$ with the 1-jet bundle $(J^1(S^{n-1}), dz - \lambda)$. In particular, we calculated the index for boundary punctured $J$-holomorphic disks asymptotic to Reeb chords with boundary on

$$L = \Pi(\Lambda) \quad \text{with} \quad \Lambda = \bigcup_{j=1}^{m} \Lambda_{q_j}.$$ 

Here $q_1, \ldots, q_m \in \mathbb{R}^n$ are pairwise disjoint points and the sheets $\Lambda_{q_j}$ are Legendrian spheres corresponding to the fibers $S_{q_j} \mathbb{R}^n$. According to Lemma 4.2.3 they are generated by functions $f_{q_j}(x) = \langle q_j, x \rangle$ with $S^{n-1}$ considered embedded in $\mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ denoting the Euclidean scalar product. Note that any function difference

$$f_{q_j} - f_{q_i}(x) = \langle q_j - q_i, x \rangle = f_{q_j - q_i}(x)$$

is a perfect Morse function on $S^{n-1}$ and its flow is the height flow towards the rotated south pole

$$S(q_i, q_j) = \frac{q_j - q_i}{\|q_j - q_i\|}. \quad (4.4.2)$$

For a picture of such a Morse flow tree on $S^2$ we refer to Figure 4.2. The index formula for boundary punctured $J$-holomorphic disks suggested to consider disks with exactly one negative puncture.

**Proposition 4.4.3.** Let $q_1, \ldots, q_m \in Q$ be a generic choice of points. The dimension of the moduli space of Morse flow trees $\Gamma$ of $\Lambda \subset J^1(S^{n-1})$ with $m - 1$ positive punctures, exactly one negative puncture and unspecified time intervals of flow lines is then given by

$$\text{gdim}(\Gamma) = \iota(\Gamma).$$

Here $\iota(\Gamma)$ denotes the number of inner edges of $\Gamma$. 
4.4. MORSE FLOW TREES IN $S^R^N$

Figure 4.2: A Morse flow tree on $S^2$.

Proof. Applying the dimension formula in [Ekh07], Definition 3.5 to the Legendrian $\Lambda$ without singularities yields

$$
\text{gdim}(\Gamma) = \sum_{p \in P(\Gamma)} I(p) + \sum_{q \in Q(\Gamma)} (n - 1 - I(q)) - \sum_{r \in R(\Gamma)} (\delta(r) - 1)(n - 1) + \iota(\Gamma)n.
$$

Here $P(\Gamma)$ and $Q(\Gamma)$ denote the set of 1-valent vertices which are positive and negative punctures, respectively. The index $I(v)$ is the Morse index of the critical point of the function difference $f_{q^+} - f_{q^-}$ such that the 1-jet lifts satisfy $z(\tilde{\phi}^+(v)) > z(\tilde{\phi}^-(v))$. It equals the Morse index of $f_{q^+} - f_{q^-}$ at $S(q^-, q^+)$ which is always $n-1$. Moreover, $R(\Gamma)$ denotes the set of vertices $v$ of valence $\delta(v) > 1$. Employing the identities

$$
\sum_{r \in R(\Gamma)} \delta(r) = m + 2 \iota(\Gamma) \quad \text{and} \quad \sum_{r \in R(\Gamma)} 1 = \iota(\Gamma) + 1
$$

leads to

$$
\text{gdim}(\Gamma) = (m - 1)(n - 1) - (m + 2 \iota(\Gamma)) (n - 1) + (\iota(\Gamma) + 1) (n - 1) + \iota(\Gamma)n = \iota(\Gamma).
$$

Any function difference $f_{q^i} - f_{q^j}$ is a perfect Morse function on $S^{n-1}$. Assuming additionally, that the rotated south poles $S(q^i, q^j)$ with $1 \leq i, j \leq m$ and $i \neq j$ are pairwise disjoint ensures that the moduli space is a transversely cut out manifold. This can be ensured by a generic choice of $q_1, \ldots, q_m \in Q$. The dimension of the moduli space is given according to Proposition 3.14 in [Ekh07] by the formula above.

We will next show that the $\iota(\Gamma)$-dimensional degree of freedom is eliminated when prescribing
the length of the time intervals, i.e. a weight function \( w \) on \( \Gamma \). Essentially, we obtain a bijection between \( G_m \) and Morse flow trees passing the sheets of \( \Lambda \) in a fixed order.

**Theorem 4.4.4.** Let \( q_1, \ldots, q_m \in \mathbb{R}^n \) be pairwise disjoint points such that

\[
S(q_{m}, q_1) \notin \{ S(q_1, q_2), \ldots, S(q_{m-1}, q_m) \}.
\]  

(4.4.3)

Then there is a constant \( C(q_1, \ldots, q_m) \) with the following property. For any weighted source tree \( (\Gamma, w) \in G_m \) satisfying \( \|w\| < C \) there exists a unique Morse flow tree of

\[
\Lambda = \bigcup_{j=1}^{m} \Lambda_{q_j} \subset J^1(S^{n-1})
\]  

(4.4.4)

such that the root is the only negative puncture and the oriented 1-jet lift passes through sheets in order \( \Lambda_{q_1}, \Lambda_{q_2}, \ldots, \Lambda_{q_m} \) when starting at the root.

**Proof.** We first observe that the sheets of any flow line of such a Morse flow tree are uniquely determined by the rooted tree \( \Gamma \) and the order \( \Lambda_{q_1}, \Lambda_{q_2}, \ldots, \Lambda_{q_m} \). The reason for this is that the 1-jet lift can switch between sheets only at the 1-valent punctures. Hence the sheets of any flow line can be determined from the following picture. Let \( \iota : \Gamma \rightarrow \mathbb{D} \) be an orientation preserving embedding of the source tree such that the preimage of \( \partial \mathbb{D} \) consists exactly of the 1-valent vertices. Denote the root of \( \Gamma \) by \( v_0 \) and the ends following the root in cyclic order by \( v_1, \ldots, v_{m-1} \). Moreover, for \( 0 \leq i \leq n-1 \) let \( p_i = \iota(v_i) \in \partial \mathbb{D} \) and \( p_m = p_0 \). Then \( \mathbb{D} \setminus \iota(\Gamma) \) splits into \( m \) connectedness components \( U_1, \ldots, U_m \) such that \( U_j \) contains the arc between \( p_{j-1} \) and \( p_j \). The image \( \iota(e) \) of any edge bounds exactly two components \( U_i \) and \( U_j \) with indices independent of the choice of \( \iota \). This implies that the flow line corresponding to \( e \) has 1-jet lifts in the sheets \( \Lambda_{q_i} \) and \( \Lambda_{q_j} \).

We now construct the Morse flow tree inductively. Consider a sequence \( \Gamma_1, \ldots, \Gamma_l \subset \Gamma \) of rooted subtrees such that \( \Gamma_1 \) consists of the root \( v_0 \) and the edge attached to it, \( \Gamma_{k+1} \) is obtained from \( \Gamma_k \) for \( 1 \leq k \leq l - 1 \) by attaching an edge and \( \Gamma_l = \Gamma \). A flow line associated to \( \Gamma_1 \) has 1-jet lifts \( \phi^1_1, \phi^2_1 \) belonging to the sheets \( \Lambda_{q_m}, \Lambda_{q_1} \), respectively. Here \( \phi^1_1 \) is oriented toward the Reeb chord at the puncture. Then \( \phi^1_1 \) oriented in the same direction is a gradient flow line of the height flow towards \( S(q_{m}, q_1) \). Since the puncture is negative, the root \( v_0 \) at the end of the flow line is mapped to the antipodal critical point \( -S(q_{m}, q_1) \). With notation as in (4.2.17) the cotangent lift at \( v_0 \) goes to \( \hat{c}^{1,m} \). This forces the flow line to be constant, i.e. the whole subtree \( \Gamma_1 \) has to be mapped to \( -S(q_{m}, q_1) \).

When extending the construction from \( \Gamma_{k-1} \) onto \( \Gamma_k \) there are two cases to consider:

- \( \Gamma_k \) is obtained from \( \Gamma_{k-1} \) by attaching an inner edge of \( \Gamma \). Let us denote this edge by \( e_k = w_k^1w_k^2 \) with \( w_k^1 \) being the vertex in \( \Gamma_{k-1} \). As discussed above, prescribing the order \( \Lambda_{q_1}, \Lambda_{q_2}, \ldots, \Lambda_{q_m} \) yields a unique pair of indices \( (i, j) \) such that the flow line associated to \( e_k \) and oriented towards \( w_k^2 \) must be a gradient flow trajectory of \( f_{q_j} - f_{q_i} \), i.e. the height flow towards \( S(q_i, q_j) \). Starting at the image of the Morse flow tree at \( w_k^1 \) we follow this gradient flow for time \( w(e_k) \). The resulting end point will be the image of the Morse flow tree at \( w_k^2 \).

- \( \Gamma_k \) is obtained from \( \Gamma_{k-1} \) by attaching an edge adjacent to an end \( v_j \). We denote this edge by \( e_k = w_kv_j \). Since the root is already contained in \( \Gamma_1 \), we must have \( 1 \leq j \leq m - 1 \) such that the puncture at \( v_j \) is positive. A flow line associated to \( e_k \) and oriented towards
4.4. MORSE FLOW TREES IN $S^{\mathbb{R}}^N$

$v_j$ is a gradient flow trajectory of the height flow towards $S(q_j, q_{j+1})$. By positivity of the puncture the end $v_j$ is mapped to the rotated south pole $S(q_j, q_{j+1})$. Starting at the image of the Morse flow tree at $w_k$, we just have to follow this flow until the critical point. Assuming that the image of $w_k$ is different from $-S(q_j, q_{j+1})$, this flow line will be uniquely defined on $[0, \infty]$.

Note that each step in the construction yields a unique flow line. The only obstruction to existence occurs in the latter case when attaching an edge $e_k = w_k v_j$ adjacent to an end with $w_k$ being mapped by the Morse flow tree $\phi$ to $-S(q_j, q_{j+1})$. We somehow have to ensure that this case cannot happen. Define

$$\Delta = \min_{1 \leq j \leq m-1} \{ \text{dist} (S(q_m, q_1), S(q_j, q_{j+1})) \}.$$

If (4.4.3) is satisfied, then $\Delta > 0$. Since $\Gamma_1$ is mapped to $-S(q_m, q_1)$, we can ensure that any vertex of $\Gamma_k$, which is an inner vertex of $\Gamma$, is mapped within distance $\frac{k-1}{l} \Delta$ of $-S(q_m, q_1)$ whenever $\|w\| < C$ is sufficiently small. In particular, the vertex $w_k$ above satisfies

$$\text{dist} (\phi(w_k), -S(q_j, q_{j+1})) \geq \text{dist} (-S(q_m, q_1), -S(q_j, q_{j+1})) - \text{dist} (\phi(w_k), -S(q_m, q_1))$$

$$\geq \Delta - \frac{k-1}{l} \Delta \geq \frac{1}{l} \Delta > 0.$$

Therefore we can exclude the possibility that $\phi(w_k) = -S(q_j, q_{j+1})$. \hfill \Box

Here (4.4.3) should be seen as a transversality condition for the moduli space of Morse flow trees. It obviously holds for a generic choice of points $q_1, \ldots, q_m$.

Let us denote by $\text{MFT}(q_1, \ldots, q_m) \subset C(\Gamma, S^{n-1})$ the set of Morse flow trees of $\Lambda$ as defined in (4.4.4) having only one negative puncture at the root and oriented 1-jet lift passing through sheets in order $\Lambda_{q_1}, \ldots, \Lambda_{q_m}$ when starting at the root. On $\text{MFT}(q_1, \ldots, q_m)$ we consider the topology induced from the compact-open topology. Moreover, let

$$\mathcal{G}^C_m = \{ (\Gamma, w) \in \mathcal{G}_m : \|w\| < C \}.$$

By continuity of the construction with respect to the topology introduced on $\mathcal{G}_m$ in section 4.3, Theorem 4.4.4 yields a continuous map

$$\Psi(q_1, \ldots, q_m) : \mathcal{G}^C_m \to \text{MFT}(q_1, \ldots, q_m), \quad (4.4.5)$$

always assuming that (4.4.3) holds. In particular, we obtain the following corollary.

**Corollary 4.4.5.** For $q_1, \ldots, q_m \in Q$ satisfying (4.4.3), $C(q_1, \ldots, q_m)$ from Theorem 4.4.4 and any $0 \leq \delta < C$ there are subsets $\Omega_\delta \subset S^{n-1}$ such that:

(a) $\Omega_0 = \Psi(q_1, \ldots, q_m) (\Gamma, w_0)$.

(b) $\Psi(q_1, \ldots, q_m) (\Gamma, w) \subset \Omega_\delta$ whenever $\|w\| < \delta$.

(c) The sets $\Omega_\delta$, $0 < \delta < C$ are open and form a neighborhood basis of $\Omega_0$. 


4.5 From Trees to Disks

We construct boundary punctured $J$-holomorphic disks in $T^*S^{n-1}$ with boundary on $L = \Pi(\Lambda)$ corresponding to the Lagrangian projection of a union of fibers $S_{q_j} \mathbb{R}^n$. At a puncture such a disk will be asymptotic to a double point with exactly one puncture being negative. Moreover, the conformal structure on the domain is prescribed. The almost complex structure $J \in \mathcal{J}(T^*S^{n-1}, -d\lambda)$ of consideration will be regular, integrable near the double points and $C^0$-close to $J_{LC}$. In a first step we find an approximate solution in the vicinity of a Morse flow tree. We then apply the Floer-Picard lemma in a suitable bundle space in order to obtain a genuine $J$-holomorphic disk.

The arguments employed in this section are very close to [Ekh07], chapter 6. Ekholm considers for a fixed Legendrian $\Lambda \subset J^1(Q)$ in a 1-jet bundle the fiber scaling

$$s_\lambda : J^1(Q) \to J^1(Q), \quad s_\lambda(x, v, z) = (x, \lambda v, \lambda z)$$

(4.5.1)

with $x \in Q$, $v \in T^*_x Q$ and $z \in \mathbb{R}$ and defines

$$\Lambda_\lambda = s_\lambda(\Lambda).$$

(4.5.2)

The scaling parameter $\lambda$ ranges in $0 < \lambda \leq 1$. For $\lambda$ sufficiently small he establishes existence results for boundary punctured $J_\lambda$-holomorphic disks with boundary on the Lagrangian projection $L_\lambda = \Pi(\Lambda_\lambda)$ by exploiting their vicinity to Morse flow trees. The main difference between his and our setup is that we also prescribe the conformal structure on the domain while Ekholm considers conformal variations. Note that for our Legendrian $\Lambda$ consisting of a union of spherical fibers one has

$$\Lambda_\lambda = \bigcup_{j=1}^m \Lambda_\lambda(\Lambda_{q_j}) = \bigcup_{j=1}^m \Lambda_{\lambda q_j}.$$ 

(4.5.3)

This follows directly from (4.2.7) or Lemma 4.2.3 and the linear dependence of $f_q$ on $q$.

Let us begin by describing the almost complex structure $J_\lambda$ for which our existence result will apply. Starting with the Levi-Civita almost complex structure with respect to the round metric on $S^{n-1}$, we go through the series of perturbations which Ekholm introduces in [Ekh07], chapter 4. They consist of $C^1$-small changes of the metric and $C^k$-small changes of the Legendrian. Both can be seen as $C^0$-small perturbations of $J$, the former since the almost complex structure is tied to the metric via $J_{LC}$ and the latter via the fiber preserving diffeomorphism defined below.

**Definition 4.5.1.** Let $k \geq 0$. Given a $C^k$-small Legendrian isotopy from $\Lambda$ to $\Lambda$ in a 1-jet bundle $J^1(Q)$, we denote by

$$\Phi(\Lambda, \Lambda) : T^*Q \to T^*Q$$

(4.5.4)

an associated fiber preserving diffeomorphism satisfying $\Phi \circ \Pi(\Lambda) = \Pi(\Lambda)$ and $C^k$-close to the identity. Here fiber preserving means that $\Phi$ covers a diffeomorphism on $Q$.

If $\tilde{J}_{LC}$ denotes the Levi-Civita almost complex structure on $S^{n-1}$ induced by the perturbed metric $\tilde{g}$ and $\Phi_\lambda$ the fiber preserving diffeomorphism taking $\Pi(\Lambda_\lambda)$ onto $\Pi(\Lambda_\lambda)$, then $J_\lambda$ will be given by

$$J_\lambda = d\Phi_\lambda^{-1} \circ \tilde{J}_{LC} \circ d\Phi_\lambda.$$ 

(4.5.5)
In particular, $J_\lambda$-holomorphic disks with boundary on $L_\lambda = \Pi(\Lambda_\lambda)$ are in 1-1 correspondence to $J_{LC}$-holomorphic disks with boundary on $\tilde{L}_\lambda = \Pi(\Lambda_\lambda)$ (compare with Remark 4.8 in [Ek07]). Whenever $\Phi_\lambda$ is $C^1$-close to the identity, $J_\lambda$ turns out to be $C^0$-close to $\tilde{J}_{LC}$. We continue by describing the aforementioned perturbations in detail.

**STEP 1:** By a $C^1$-small perturbation of the Legendrian we deform $\Lambda$ into

$$\bigcup_{j=1}^m \Lambda_{\tilde{q}_j}$$

such that the transversality condition

$$S(\tilde{q}_m, \tilde{q}_1) \notin \{ S(\tilde{q}_1, \tilde{q}_2), \ldots, S(\tilde{q}_{m-1}, \tilde{q}_m) \}$$

in Theorem 4.4.4 is satisfied.

**STEP 2:** Ekholm requires that in the neighborhood of any critical point of a local function difference $f_{\tilde{q}_j} - f_{\tilde{q}_i}$ there exist flat coordinates such that the function difference agrees with standard Morse coordinates. This is achieved by a $C^1$-small change of the metric. Lemma 4.5.2 below shows that in our situation this change can be done explicitly by pulling back the Euclidean metric via a standard stereographic chart in a neighborhood of the critical point.

**STEP 3:** With notation chosen as in Corollary 4.4.5 we in fact flatten the metric on $\Omega_{\tilde{\delta}}$ with $\tilde{\delta} < C(\tilde{q}_1, \ldots, \tilde{q}_m)$ sufficiently small. By part (c) in the corollary this again can be obtained as a $C^1$-small change of the metric. The idea is to be able to find explicit local solutions of the $\tilde{\partial}_j$-equation near components of Morse flow trees $(\Gamma, w)$ with sufficiently small weights, i.e. $\|w\| < \tilde{\delta}$. Note that $J_{LC} = J_0$ whenever $g$ is flat. From this it follows that the ultimate almost complex structure $J$ will be integrable in the neighborhood of all connecting Reeb chords.

**STEP 4:** There are further $C^1$-small Legendrian isotopies perturbing (4.5.6) into $\tilde{\Lambda}$ with the property that most sheets of the perturbed Legendrian become covariantly constant on $\Omega_{\tilde{\delta}}$. These deformations preserve gradient differences and hence Theorem 4.4.4 still applies to $\tilde{\Lambda}$. All perturbations due to cusps in the Legendrian (i.e. the cusp rounding procedure described in [Ek07], 4.1.B–4.1.D) can be neglected.

**Lemma 4.5.2.** Let $p \in S^{n-1}$ be a critical point of some local function difference $f_{\tilde{q}_j} - f_{\tilde{q}_i}$, and let $(y_1, \ldots, y_{n-1})$ be stereographic coordinates around $p$ in some neighborhood. Then the gradient flow of the function difference with respect to the round metric equals the gradient flow of

$$F(y) = \pm \frac{1}{2} \sum_{k=1}^{n-1} y_k^2$$

with respect to the flat metric $g_0 = dy_1 \otimes dy_1 + \ldots + dy_{n-1} \otimes dy_{n-1}$.

**Proof.** Since $f_{\tilde{q}_j} - f_{\tilde{q}_i} = f_{\tilde{q}_j} - \tilde{q}_i$, it suffices to prove the statement for any function of the form $f_q$. By rotational symmetry we may assume $p = q = (0, \ldots, 0, 1)$. Stereographic coordinates around $p$ are given by

$$\varphi : S^{n-1} \setminus \{p\} \to \mathbb{R}^{n-1}, \quad \varphi(x_1, \ldots, x_n) = \left( \frac{x_1}{1 + x_n}, \ldots, \frac{x_{n-1}}{1 + x_n} \right).$$

Taking advantage of the special choice of $q$, the function $f_q$ expressed in these coordinates equals

$$f_q(y) = \langle q, x \rangle = x_n = \frac{1 - \|y\|^2}{1 + \|y\|^2}.$$
The round metric is given by
\[ g_{ij}(y) = \frac{4}{(1 + \|y\|^2)^2} \delta_{ij}. \]

Hence the gradient of \( f_\tilde{g} \) with respect to this metric turns out to be
\[
\nabla f_\tilde{g}(y) = \frac{(1 + \|y\|^2)^2}{4} \sum_{k=1}^{n-1} \frac{\partial f_\tilde{g}}{\partial y_k} \frac{\partial}{\partial y_k} = \frac{(1 + \|y\|^2)^2}{4} \sum_{k=1}^{n-1} \left( -\frac{2y_k}{1 + \|y\|^2} \right) \frac{\partial}{\partial y_k}.
\]

The latter is the gradient of \( F \) with respect to the flat metric. \( \Box \)

From now on, we denote by \( \tilde{g} \) the perturbed metric on \( S^{n-1} \) which is flat on \( \Omega_\delta \) in a neighborhood of the Morse flow tree \( \Psi(q_1, \ldots, q_m)(\Gamma, w_0) \). The Levi-Civita almost complex structure associated to \( \tilde{g} \) is \( \tilde{J}_{LC} \). For convenience, we denote the Cauchy-Riemann operator for \( \tilde{J}_{LC} \) by \( \partial_J \). Furthermore, the perturbed Legendrian will be \( \tilde{\Lambda} \) with fiber scaled Lagrangian projection \( \tilde{L}_\lambda = \Pi(\tilde{\Lambda}_\lambda) \).

The next step consists in constructing approximate \( \tilde{J}_{LC} \)-holomorphic disks for a prescribed conformal structure on the domain \( \mathbb{D}^m_m \) with boundary on \( \tilde{L}_\lambda \) and in the vicinity of a Morse flow tree. In [Ekh07], section 6.1 it is described how to construct local solutions in the vicinity of a Morse flow tree \( (\Gamma, w_M) \) with \( \|w_M\| < \delta \). The basic idea is that since \( \tilde{g} \) is flat in the neighborhood \( \Omega_\delta \) of the Morse flow tree, solutions of the Cauchy-Riemann equation \( \partial_J u = 0 \) reduce in the vicinity of the tree to classical holomorphic curves. Hence local solutions can be constructed via complex analysis.

**Proposition 4.5.3.** Let \( (\Gamma, w_M) \in \mathcal{G}_m^\delta \) be a Morse flow tree for \( \tilde{\Lambda} \). Then there is a constant \( 0 < \lambda_0 < 1 \) such that the following holds:

(a) For any edge \( e = w_iw_j \) of \( \Gamma \) and \( 0 < \lambda < \lambda_0 \) there is a local solution \( s_\lambda : [0, T] \times [0, 1] \to TS^{n-1} \) such that \( \partial_J s_\lambda = 0 \), the horizontal boundary \( s_\lambda(\sigma, \tau), \tau \in \{0, 1\} \) is mapped into the sheets of \( \tilde{L}_\lambda \) corresponding to the cotangent lift of \( e \) and the vertical boundary \( s_\lambda(\sigma, \tau), \sigma \in \{0, T\} \) is mapped within distance \( O(\lambda) \) of the Morse flow tree vertices \( w_i \) and \( w_j \), respectively. The strip length \( T \) depends continuously on \( \lambda \) and \( w_M(e) \). In particular, if \( w_M(e) < \infty \), then
\[
\frac{w_M(e)}{C\lambda} \leq T \leq C \frac{w_M(e)}{\lambda}
\] (4.5.7)
holds with some constant \( C > 0 \).

(b) For any end \( v_j \) and \( 0 < \lambda < \lambda_0 \) there is a local solution \( s_\lambda : [0, \infty) \times [0, 1] \to TS^{n-1} \) such that \( \partial_J s_\lambda = 0 \), the horizontal boundary \( s_\lambda(\sigma, \tau), \tau \in \{0, 1\} \) is mapped into the sheets of \( \tilde{L}_\lambda \) corresponding to the cotangent lift of the adjacent edge \( e \) and the image of \( s_\lambda \) is mapped within distance \( O(\lambda) \) of the Morse flow tree vertex \( v_j \). In particular, \( s_\lambda(\sigma, \tau) \) uniformly converges to a double point as \( \sigma \to \infty \).
(c) For any inner vertex $v$ of valence $k \geq 3$ consider the slit domain $\Delta_k(a)$ with $a = (0, \ldots, 0)$. Moreover, let $W \subset \Delta_k(a)$ be the open neighborhood consisting of points which are a horizontal distance $< \varepsilon$ away from the boundary minima. For any $0 < \lambda < \lambda_0$ there is a local solution

$$s_\lambda : W \to TS^{n-1}$$

such that $\tilde{\partial}_j s_\lambda = 0$, the boundary of $s_\lambda$ is mapped into the sheets of $\tilde{L}_\lambda$ corresponding to the cotangent lifts at $v$ and the image of $s_\lambda$ is within distance $O(\lambda)$ of the Morse flow tree vertex $v$.

Proof. While local solutions for edges are constructed in [Ekh07], section 6.1.A, local solutions near punctures are constructed in [Ekh07], section 6.1.B and 6.1.G. Inequality (4.5.7) expresses the fact that the strip length of a local solution near a gradient flow trajectory of fixed length increases at rate $\lambda^{-1}$ as $\lambda \to 0$. For (c) we slightly have to generalize the construction in [Ekh07], section 6.1.E which only treats the case $k = 3$.

For this denote the boundary components of $\Delta_k(a)$ from bottom to top by $B^1, \ldots, B^k$. By the perturbations considered above in step 3 and step 4 we may consider flat coordinates with covariantly constant cotangent sheets of $L_\lambda$, i.e. represented by

$$\{ z \in \mathbb{C}^n : \text{Im } z = \lambda A \} \quad \text{with } A \in \mathbb{R}^n.$$ 

In these coordinates the image of the vertex $v$ is the origin. For given $A^1, \ldots, A^k \in \mathbb{R}^n$ it suffices to find a solution $s_\lambda : \Delta_k(a) \to \mathbb{C}^n$ such that $\tilde{\partial}_j s_\lambda = 0$ and $\text{Im } s_\lambda(B^j) = \lambda A^j$ for $1 \leq j \leq k$. Let $U_j : \Delta_k(a) \to \mathbb{R} \times [0, 1]$ be the unique biholomorphic map taking $B^j$ onto $\mathbb{R} \times \{1\}$ and $B^{j-1}$ onto $(0, \infty) \times \{0\}$. Here we set $B^0 = B^k$. In particular, all boundary components different from $B^{j-1}$ and $B^j$ are mapped to $(-\infty, 0) \times \{0\}$. Define $a_j : \mathbb{R} \times [0, 1] \to \mathbb{C}^n$ by $a_j(z) = A_j z$. Then

$$s_\lambda = \lambda \sum_{j=1}^k a_j \circ U_j$$

is the desired solution. □

We will now patch together the local solutions to an approximate solution with respect to a prescribed conformal structure on the domain $D^\circ_m$. As discussed in section 4.3 we may interpret the conformal structure as a weighted source tree $(\Gamma, w_C) \in \mathcal{G}_m$ with associated slit domain $\Delta_C^\circ$. Fix a small $\varepsilon > 0$ such that

$$2\varepsilon \ll \min\{w_C(e) : e \text{ is an inner edge}\}.$$ 

To each inner vertex $v \in \Gamma$ we assign a connected neighborhood $W(v)$ on the slit domain such that $W(v)$ contains all points a horizontal distance $< \varepsilon$ away from the boundary minima corresponding to $v$. This induces a partition of the slit domain into strip regions $W(e)$ of width $w_C(e) - 2\varepsilon$ corresponding to inner edges of $\Gamma$, strip regions $W(e)$ of infinite width corresponding to edges adjacent to an end and the neighborhoods $W(v)$ corresponding to inner vertices of $\Gamma$. For a visual description of the partition we refer to Figure 4.3.
Proposition 4.5.4. Let $\mathbb{D}^m_\lambda$ be equipped with a fixed conformal structure. Then there is a constant $0 < \lambda_0 < 1$ such that for any $\lambda < \lambda_0$ there is a smooth approximate solution

$$u^0_\lambda : \mathbb{D}^m_\lambda \to T S^{n-1}$$

with boundary on $\tilde{L}_\lambda$ which is $\tilde{J}_{LC}$-holomorphic along most of $\mathbb{D}^m_\lambda$, satisfying

$$\| \tilde{\partial}_J u^0_\lambda \|_{W^{1, p}(\mathbb{D}^m_\lambda)} = O(\lambda). \quad (4.5.8)$$

Proof. For any $\lambda$ sufficiently small we first construct a Morse flow tree $(\Gamma, w^\lambda_M)$ for $\tilde{\Lambda}$ satisfying $\| w^\lambda_M \| < \delta$ and such that the tree has the same combinatorics as the given conformal structure $(\Gamma, w_C) \in \mathcal{G}_m$. The construction of $w^\lambda_M$ is done inductively. Consider a sequence $\Gamma_1, \ldots, \Gamma_l \subset \Gamma$ of rooted subtrees such that $\Gamma_1$ consists of the root and the edge attached to it, $\Gamma_{k+1}$ is obtained from $\Gamma_k$ for $1 \leq k \leq l - 1$ by attaching an edge and $\Gamma_l = \Gamma$. From the inductive construction of Morse flow trees in Theorem 4.4.4 it follows that whenever two weight functions $w_1, w_2$ with $\| w_1 \|, \| w_2 \| < \delta$ coincide on $\Gamma_k$ for some $1 \leq k \leq l - 1$, then the Morse flow trees $\Psi(\tilde{q}_1, \ldots, \tilde{q}_m)(\Gamma, w_1)$ and $\Psi(\tilde{q}_1, \ldots, \tilde{q}_m)(\Gamma, w_2)$ also coincide on $\Gamma_k$. Note that the weight of the edge in $\Gamma_1$ is always $\infty$. We extend $w^\lambda_M$ from $\Gamma_k$ to $\Gamma_{k+1}$ as follows.

Suppose $\Gamma_k$ is obtained from $\Gamma_{k-1}$ by attaching an inner edge of $\Gamma$. This is the only case of interest since otherwise the width of the attached edge must be $\infty$. Denote the attached edge by $e_k = v^1_k v^2_k$ with $v^1_k$ being the vertex in $\Gamma_{k-1}$. Then the width of the strip region $W(e_k)$ in $\Delta^C_m$ equals $w_C(e_k) - 2\varepsilon$. We denote its height by $h_C(e_k)$. By the observation above and with $w^\lambda_M$ so far only defined on $\Gamma_{k-1}$ there is a well-defined position for the Morse flow tree vertex $v^1_k$. For any choice of $w^\lambda_M(e_k) < \delta$ the Morse flow tree can be uniquely extended to $\Gamma_k$ and by Proposition 4.5.3 there is a local solution for $0 < \lambda < \lambda_0$ defined on a strip of length $T = T(\lambda, w^\lambda_M(e_k))$ such that (4.5.7) holds with some constants $C$. In particular, for

$$\lambda < \min \left( \lambda_0, \frac{\delta h_C(e_k)}{C(w_C(e_k) - 2\varepsilon)} \right)$$
we have \( T(\lambda, 0) = 0 \) and
\[
T(\lambda, \delta) \geq \frac{\delta}{C\lambda} > \frac{w_C(e_k) - 2\epsilon}{h_C(e_k)}.
\]
By continuity of \( T \) we may choose \( 0 < w_M^\lambda(e_k) < \delta \) such that
\[
T(\lambda, w_M^\lambda(e_k)) = \frac{w_C(e_k) - 2\epsilon}{h_C(e_k)}. \tag{4.5.9}
\]

After completing the construction of the weight function \( w_M^\lambda \) we may also consider the unique Morse flow tree \((\Gamma, w_M^\lambda)\) for \( \Lambda \) which is guaranteed by Theorem 4.4.4.

We next construct local solutions in the vicinity of the Morse flow tree \((\Gamma, w_M^\lambda)\) via Proposition 4.5.3. These local solutions will be defined on the various parts of the partition of the slit domain \( \Delta_m^C \) associated to \((\Gamma, w_C)\). Condition (4.5.9) ensures that the rescaled local solutions at inner edges \( e \) are defined on strips of the correct length \( w_C(e) - 2\epsilon \) and height \( h_C(e) \). Local solutions for strip regions \( W(e) \) of infinite width such that the image of \( e \) is a critical point in the Morse flow tree correspond to local solutions at punctures as described in Proposition 4.5.3 (b). Whenever the image of \( e \) is a nonconstant gradient flow trajectory, one additionally has to attach a local solution in the vicinity of this flow line as described in Proposition 4.5.3 (a). However, this does not affect the domain of definition since \([0, \infty)\) remains invariant when attaching an interval of finite length. Altogether we obtain a map \( s_\lambda : \Delta_m^C \to TS^{n-1} \) with boundary on \( \bar{L}_\Lambda \) and within distance \( O(\lambda) \) of the Morse flow tree \((\Gamma, w_M^\lambda)\). The map \( s_\lambda \) is smooth and \( J_{LC}\)-holomorphic on most of \( \Delta_m^C \) except at a finite set of vertical line segments where different local solutions (and in particular, partition regions) contact each other.

The final step is to turn \( s_\lambda \) into a smooth map \( u_\lambda^0 \) by interpolating between different local solutions. Here we always have to interpolate between two maps of distance \( O(\lambda) \). It follows from the flow line convergence of holomorphic disks (see Lemma 5.12 and Remark 5.13 in [Ekh07]) that we can choose \( u_\lambda^0 \) so that
\[
\sup_{\Delta_m^C} |D^k \partial_J u_\lambda^0| = O(\lambda) \quad \text{for } k = 0, 1
\]
and \( \partial_J u_\lambda^0 \) is supported only on a compact subset of \( \Delta_m^C \) with bounded area. In particular, we may conclude
\[
\|\partial_J u_\lambda^0\|_{W^{1,p}(\Delta_m^C)} = O(\lambda).
\]

\[ \square \]

We next apply the Banach space setup of Ekholm and consider the Cauchy-Riemann operator \( \bar{\partial}_J \) as a bundle map
\[
\bar{\partial}_J : V_{2,\delta}(\lambda) \oplus V_{sol}(\lambda) \to W_{1,\delta}(\lambda). \tag{4.5.10}
\]
Here \( V_{k,\delta}(\lambda) \) and \( W_{k,\delta}(\lambda) \) denote closed subspaces of the weighted Sobolev space \( S_{k,\delta}(\lambda) \) of sections
\[
s : \Delta_m^C \to (u_\lambda^0)^* T( TS^{n-1} )
\]
with \( k \) derivatives in \( L^2 \). The precise structure of the weight function is given in [Ekh07], 6.3.A. For \( \mathcal{W}_{k,\delta}(\lambda) \) one additionally requires that elements \( v \) are tangent to \( \tilde{L}_\lambda \) along \( \partial \Delta^C_m \) and the restriction of

\[
\tilde{\nabla} j_{LC} v = \tilde{\nabla} v + \tilde{J}_{LC} \circ \tilde{\nabla} v \circ i
\]

to the boundary equals zero, where \( \tilde{\nabla} \) denotes the Levi-Civita connection with respect to the metric \( \tilde{g} \). For elements \( v \) of \( \mathcal{W}_{k,\delta}(\lambda) \) one demands that they vanish on the boundary. Finally, \( \mathcal{V}_{\text{sol}}(\lambda) \) denotes a finite dimensional space of cut-off solutions as described in [Ekh07], 6.3.B. Since we consider a fixed conformal structure on the domain, the bundle setup of Ekholm can be simplified by omitting all variations of the conformal structure.

The ultimate goal will be to apply the Floer-Picard lemma to the Cauchy-Riemann operator \( \bar{\partial}_J \) at \( u_\lambda^0 \). Let us recall the statement (for instance from [Sch93] or [Ekh07], Proposition 6.17).

**Lemma 4.5.5.** Suppose \( F : B_1 \to B_2 \) is a smooth Fredholm map of Banach spaces such that

\[
F(v) = F(0) + dF(v) + N(v)
\]

where \( dF \) has a bounded right inverse \( Q \) and such that the non-linear term \( N \) satisfies a quadratic estimate of the form

\[
\| N(u) - N(v) \|_{B_2} \leq C \left( \| u \|_{B_1} + \| v \|_{B_1} \right) \| u - v \|_{B_1}. \tag{4.5.11}
\]

Assuming

\[
\| Q \circ F(0) \|_{B_1} \leq \frac{1}{8C}
\]

implies that the set

\[
\left\{ x \in F^{-1}(0) : \| x \|_{B_1} < \frac{1}{4C} \right\}
\]

is a smooth submanifold diffeomorphic to a ball in \( \ker (dF(0)) \).

In order to apply the lemma, one needs to establish uniform invertibility of the Cauchy-Riemann operator at \( u_\lambda^0 \) when \( \lambda \to 0 \).

**Proposition 4.5.6.** The linearization of the \( \bar{\partial}_J \)-operator at \( u_\lambda^0 \) is Fredholm of index zero. Moreover, it has a right inverse \( Q_\lambda \) such that \( \| Q_\lambda \| \leq C \) holds with some constant \( C \) for all sufficiently small \( \lambda > 0 \).

The vanishing of the Fredholm index follows from Lemma 4.2.10. The boundary condition for the linearization converges to \( \mathbb{R}^n \)-boundary conditions as \( \lambda \to 0 \), except near punctures where they converge to constant \( \mathbb{R}^n \)-boundary conditions with a \( \pi \)-rotation in one direction. Hence the uniform invertibility of \( \bar{\partial}_J \) at \( u_\lambda^0 \) follows from the invertibility of the standard \( \bar{\partial} \)-operator with the same \( \mathbb{R}^n \)-boundary conditions up to \( \pi \)-rotations and uniformly positive exponential weights. For a rigorous proof of this statement we refer to Proposition 6.20 in [Ekh07].

We are now well prepared to prove the existence result for boundary punctured disks with prescribed conformal structure.
Theorem 4.5.7. For a given slit domain $\Delta_m^C$ and points $q_1, \ldots, q_m \in \mathbb{R}^n$ there is a constant $0 < \lambda_0 < 1$ and almost complex structures $J_\lambda$ for $0 < \lambda < \lambda_0$ with the following property. There exists a boundary punctured $J_\lambda$-holomorphic disk

$$u : \Delta_m^C \to TS^{n-1}$$

such that:

(i) $u(\partial \Delta_m^C) \subset L_\lambda = \bigcup_{j=1}^{m} L_{\lambda q_j}$.

(ii) The disk has exactly one negative puncture and starting from there the boundary passes through sheets in order $L_{\lambda q_1}, L_{\lambda q_2}, \ldots, L_{\lambda q_m}$.

(iii) The almost complex structure $J_\lambda$ is integrable in a neighborhood of the Reeb chords and can be constructed arbitrarily $C^0$-close to $J_{LC}$.

Proof. We apply the Floer-Picard lemma to the approximate solutions $u_\lambda^0$ constructed in Proposition 4.5.4. That the Cauchy-Riemann operator satisfies the quadratic estimate (4.5.11) is shown in [Ekh07], 6.4.D. Hence by uniform invertibility of $\partial J$ at $u_\lambda^0$, Lemma 4.5.5 provides a $J_{LC}$-holomorphic disk with boundary on $L_\lambda$ for sufficiently small $\lambda$. Composing with the fiber preserving diffeomorphism $\Phi_\lambda$ then leads to a $J_\lambda$-holomorphic disk with boundary on $L_\lambda$. The statements concerning the properties of $J_\lambda$ are clear from the construction of the almost complex structure.

\[ \square \]

4.6 Existence Results in $SQ \times \mathbb{R}$

We finally transfer the existence result in $S\mathbb{R}^n \cong J^1(S^{n-1})$ to the symplectization $SQ \times \mathbb{R}$ of a unit tangent bundle via the contactomorphism lift of a normal chart. Again the result will hold under the assumption that in the Lagrangian boundary condition $\Lambda \times \mathbb{R}$ the Legendrian $\Lambda$ consists of fibers which are sufficiently close to each other.

Let us begin by treating the question whether a chart of a Riemannian manifold $(Q, g)$ extends to a contactomorphism of unit tangent bundles and how close such a contactomorphism can be to a fiber preserving map. For a chart $\varphi : U \to V$ we may identify the unit tangent bundle $SQ|_{\pi^{-1}(U)}$ as a contact manifold with the restriction of $\lambda(x, \xi) = g_{ij}(x) \xi^j \, dx^i$ to

$$\{\xi \in T\mathbb{R}^n|_{\pi^{-1}(V)} : \|\xi\|_g = 1\}.$$ 

By fiberwise rescaling to Euclidean unit length we obtain the fiber preserving equivalence of contact manifolds

$$\left(SQ|_{\pi^{-1}(U)}, \tilde{\lambda}\right) \cong \left(S\mathbb{R}^n|_{\pi^{-1}(V)}, \lambda_g\right) \quad (4.6.1)$$

with

$$\lambda_g(x, \xi) = \frac{1}{\|\xi\|_g} g_{ij}(x) \xi^j \, dx^i. \quad (4.6.2)$$

The following lemma exploits Gray's stability theorem in order to construct a contactomorphic embedding of $(S\mathbb{R}^n|_{\pi^{-1}(V)}, \lambda_g)$ into the standard unit tangent bundle $(S\mathbb{R}^n|_{\pi^{-1}(V)}, \lambda_0)$.
Lemma 4.6.1. Let \( \varphi : U \to V \) be a normal chart around \( x \in U \) defined on an open subset of the Riemannian manifold \((Q, g)\). For some open subset \( V' \subset V \) containing \( \varphi(x) \) there is an embedding
\[
\Phi_g : \left( S^n_{\pi^{-1}(V')}, \lambda_g \right) \to \left( S^n_{\pi^{-1}(V)}, \lambda_0 \right)
\] (4.6.3)
which is a contactomorphism onto its image and restricts to the identity in the spherical fiber over \( \varphi(x) \). Moreover,
\[
\| \Phi_g - \text{Id} \|_{C^{k-1}} \leq C \| g - g_0 \|_{C^k(V)}
\] (4.6.4)
holds with some constant \( C \) depending only on \( k > 0 \).

Proof.

The linear path of metrics \( g_t = tg + (1-t)g_0 \) induces a path of contact forms
\[
\lambda_t(x, \xi) = \| \xi \|_g \lambda_{g_t}(x, \xi) = (g_t)_{ij}(x) \xi^i \, dx^i
\]
on \( S^n_{\pi^{-1}(V')} = V \times S^{n-1} \) such that \( \lambda_t \) is equivalent to \( \lambda_{g_t} \). According to Lemma 2.3.1 the Reeb vector field \( R_t \) of \( \lambda_t \) is given by
\[
R_t(x, \xi) = \frac{1}{\| \xi \|^2_{g_t}} \left( \xi^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk} \xi^j \xi^k \frac{\partial}{\partial \xi^i} \right).
\]
Following the lines of Gray’s stability theorem, we are going to construct a path of embeddings
\[
\Phi_t : S^n_{\pi^{-1}(V')} \to S^n_{\pi^{-1}(V)}
\]
and functions \( f_t : \text{im} (\Phi_t) \to \mathbb{R} \) such that \( \Phi_t^* (f_t \lambda_0) = \lambda_t \) holds.

We suppose that \( \Phi_t \) corresponds to the flow of a time-dependent vector field \( Y_t \). Then it satisfies the equation
\[
\dot{\lambda}_t + \mathcal{L}_{Y_t} \lambda_t = \dot{\lambda}_t + d\iota_{Y_t} \lambda_t + \iota_{Y_t} d\lambda_t = \left( \frac{f_t}{f_t} \circ \Phi_t^{-1} \right) \lambda_t.
\]
Imitating the argument in [MS98], p. 112, this can be uniquely solved for \( Y_t \in \ker \lambda_t \) and \( f_t \) by demanding
\[
\dot{\lambda}_t + d\lambda_t(Y_t, \cdot) = \dot{\lambda}_t(R_t) \lambda_t \quad \text{and} \quad \dot{\lambda}_t(R_t) = \frac{f_t}{f_t} \circ \Phi_t^{-1}.
\] (4.6.5)
By the first equation and the formula for \( R_t \) we may express \( Y_t(x, \xi) \) completely in terms of \( (g_t)_{ij}, (g_t)_{ij,k} = (g_t - g_0)_{ij,k} \) and \( \xi \). This leads to the estimate
\[
\| Y_t \|_{C^{k-1}} \leq C \| g_t - g_0 \|_{C^k}
\] (4.6.6)
with \( C \) depending only on \( k \). Since
\[
\dot{\lambda}_t(x, \xi) = (\dot{g}_t)_{ij} \xi^i \, dx^i = (g_t - g_0)_{ij} \xi^i \, dx^i,
\]
we obtain \( \dot{\lambda}_t = 0 \) in the spherical fiber over \( \varphi(x) \). Hence \( Y_t \) has to vanish in this fiber by the first equation in (4.6.5) which means that \( \Phi_t \) is constant there. We may thus choose \( V' \) small enough such that the flow of \( Y_t \) with initial values in \( \pi^{-1}(V') \) does not leave \( \pi^{-1}(V) \) for \( t \in [0,1] \).
Setting \( \Phi_g = \Phi_1 \) completes the argument. Inequality (4.6.4) follows from integrating (4.6.6). \( \square \)
4.6. EXISTENCE RESULTS IN $SQ \times \mathbb{R}$

By choosing $V$ small enough, inequality (4.6.4) implies that $\Phi_p$ is $C^0$-close to the identity. Hence we can pass from $SQ \mid_{x-1(U)}$ to $SR^n \mid_{x-1(V')}$ via a contactomorphism which restricts to the identity on the center of the normal chart and maps fibers of the Legendrian

$$\Lambda = \bigcup_{j=1}^{m} S_{\delta_j} Q.$$

$C^0$-close to fibers in $SR^n$. This allows us to construct $J$-holomorphic curves in $SQ \times \mathbb{R}$.

Theorem 4.6.2. For a given slit domain $\Delta^C_m$ and a compact Riemannian manifold $(Q, g)$ there exists a constant $C > 0$ such that for any points $q_1, \ldots, q_m \in Q$ with mutual distances less than $C$ there is an almost complex structure $J$ on $SQ \times \mathbb{R}$ and a boundary punctured $J$-holomorphic disk

$$u : \Delta^C_m \to SQ \times \mathbb{R}$$

such that:

(i) $u(\partial \Delta^C_m) \subset \Lambda \times \mathbb{R} = \bigcup_{j=1}^{m} S_{\delta_j} Q \times \mathbb{R}$.

(ii) The disk has exactly one negative puncture and starting from there the boundary passes through sheets in order $S_{\delta_1} Q \times \mathbb{R}, S_{\delta_2} Q \times \mathbb{R}, \ldots, S_{\delta_m} Q \times \mathbb{R}$.

(iii) $u$ is normalized in the $\mathbb{R}$-direction by demanding $u(p) \in SQ \times \{0\}$ for some fixed $p \in \Delta^C_m$.

(iv) The almost complex structure $J$ is cylindrical and invariant under the Reeb flow in the fibers of a normal chart containing $q_1, \ldots, q_m$.

Proof. We cover $Q$ by normal charts and pick $C > 0$ small enough such that any $m$-tuple of points $q_1, \ldots, q_m \in Q$ with mutual distances less than $C$ is contained in an open set $U' = \phi^{-1}(V')$ with $\phi : Q \mid_{U'} \to \mathbb{R}^n \mid_V$ a normal chart and $V' \subset V$ as mentioned in Lemma 4.6.1. We then obtain a contactomorphism $\Psi_g$ which embeds $SQ \mid_{x-1(U)}$ into $SR^n \mid_{x-1(V)}$ and such that $\Psi_g(f\lambda_0) = \lambda$ for some function $f : SR^n \to \mathbb{R}$. Moreover, $\Lambda = \Psi_g(\Lambda)$ is $C^0$-close to a fiberwise Legendrian $\Lambda$ as in (4.2.12). Via the identification $\Phi : SR^n \to J^1(S^{n-1})$ given in Proposition 4.2.2, we may consider the Lagrangian projections $\hat{L} = \Pi(\Lambda)$ and $\tilde{L} = \Pi(\Lambda)$. If necessary, we diminish $C$ further such that the fibers in $\hat{L}$ are sufficiently close to each other and Theorem 4.5.7 provides a $J$-holomorphic disk

$$\hat{u} : \Delta^C_m \to TS^{n-1} \quad (4.6.7)$$

with boundary on $\hat{L}$. Composing with a fiber preserving diffeomorphism that takes $\hat{L}$ onto $\tilde{L}$ yields a $\hat{J}$-holomorphic disk $\tilde{u} : \Delta^C_m \to TS^{n-1}$ with boundary on $\tilde{L}$. Consider the open subset

$$W = (\Phi \circ \Psi_g)(SQ \mid_{x-1(U)}) \subset J^1(S^{n-1}).$$

We may extend $\hat{J}$ to a cylindrical almost complex structure $J$ on $W \times \mathbb{R}$ which is invariant under the Reeb flow. Here we take the Reeb vector field which is given by the contact form $(f \circ \Phi^{-1})\alpha$. Note that this form pulls back to the canonical contact form $\tilde{\lambda}$ in $SQ$. In particular, the projection along this Reeb flow to $TS^{n-1}$ is well-defined in $W$ and since $\tilde{u}$ is $C^0$-close to a Morse flow tree in $TS^{n-1}$, we may assume that its image is contained in the image of this projection. Employing the argument in Lemma 4.2.4 (e) lifts $\tilde{u}$ to a $J$-holomorphic curve

$$u : \Delta^C_m \to W \times \mathbb{R}$$

which has boundary on $\Lambda \times \mathbb{R}$, is asymptotic to Reeb chords at boundary
punctures and uniquely determined by $\tilde{u}$ up to a shift in the symplectization direction. Pulling back $u$ via $\Phi \circ \Psi_g$ and normalizing it appropriately gives the desired $J$-holomorphic curve in $SQ \times \mathbb{R}$. It remains to extend the pulled back almost complex structure from $SQ |_{\pi^{-1}(U')} \times \mathbb{R}$ to the whole symplectization of the unit tangent bundle.
Chapter 5

Future Directions

5.1 From Lifted to Punctured Disks

Due to the similarities in the choice of the fiberwise boundary conditions, it would be fascinating to see a connection between disks of lifted and punctured type. For this we like to sketch a homotopy of lifted disks whose limits are punctured disks with only positive punctures. Note that when leaving out the normalization conditions $u^k(p_k) \in s(Q)$ in Definition 1.5 and $\pi_R \circ u^i(p_i) = 0$ in Definition 1.9, both problems are Fredholm of index $n = \dim Q$.

As mentioned before, the symplectization $\mathbb{S} \times \mathbb{R}$ is symplectomorphic to the tangent bundle with zero section removed, that is $TQ \setminus s_0(Q)$. For $r > 0$ sufficiently large we choose a tamed almost complex structure which is cylindrical on the tangent disk bundle $T_{\leq r}Q$ of radius $r$ and equals $J_{LC}$ on $TQ \setminus T_{\leq r}Q$. We will consider lifted disks defined on the unit disk represented as $\mathbb{D}(\lambda)$, $\lambda \geq 0$ corresponding to a standard domain with caps. The standard domain (white area in Figure 5.1) is fixed and may be seen as the neighborhood of a 3-valent vertex of one of the slit domains considered in section 4.5. Attached to it are strips of length $\lambda$ with caps on the end (hatched and grey area, respectively). For given points $q_1, q_2, q_3 \in Q$ we consider a curve

\[ \gamma_{q_1, q_2, q_3}^\lambda : \partial \mathbb{D}(\lambda) \to Q \]  

which is constant on the boundaries of the standard domain and the strips and runs through the geodesic segments $\overrightarrow{q_1 q_2}$, $\overrightarrow{q_2 q_3}$ and $\overrightarrow{q_3 q_1}$ on the boundary of the caps. A preliminary homotopy connects the curve $\gamma_{q_1, q_2, q_3}$ defined in (3.1.2) to $\gamma_{q_1, q_2, q_3}^0$. Given a sequence $\lambda_k \to \infty$, we consider

\[ 97 \]
the limit \( u \) of a \( C^\infty_{\text{loc}} \)-convergent sequence of lifted \( J \)-holomorphic disks

\[
u^\lambda_k : \partial \mathbb{D}(\lambda_k) \to TQ
\]

whose boundary projects onto \( \gamma^\lambda_k \). Since the domain converges to the boundary punctured unit disk \( \mathbb{D}_0 \), the limit curve must break at the Reeb chords \( \bar{q}_1, \bar{q}_2, \bar{q}_3 \). We thus obtain a punctured disk having three positive punctures. This shows that punctured disks may be seen as the boundary of certain families of lifted disks.

### 5.2 Suitable Choices of Negative Punctures and Meshes

There are various possibilities to choose a balanced binary function sign which prescribes the location of the negative punctures for the moduli space of punctured type. For a somewhat canonical strategy we propose the following refinement procedure:

(i) For a given combinatorial 2-sphere \( K \) choose all punctures positive.

(ii) By successive refinement of triples add negative punctures until sign is balanced.

Here each refinement consists of adding an inner vertex to a triangular face of \( K \). The punctures of the additional disk may be chosen according to Figure 5.2. The face on which the refinement takes place is divided into three faces. In this way, the positivity of the Fredholm index for the disks of the original complex \( K \) is deferred to additional parameters of the meshed function \( \Phi \) defined on the refined combinatorics.

![Figure 5.2: A refinement step for a triple of circles.](image)

For lifted as well as for punctured disks one is left with choosing a meshed function \( \Phi \) which is defined on the discrete vertex or face set of the combinatorics. In the spirit of the finite-dimensional path space approximation it is desirable to consider configurations corresponding to critical points of some energy functional restricted to the moduli space of lifted or punctured...
type, respectively. It is conceivable to either pick the Dirichlet energy or the symplectic area. The properties of the critical configurations are subject of further research.

5.3 Connections to String Topology

The existence result for punctured disks in Theorem 1.11 should correspond to an identity relation in string topology. Various connections of string topology to symplectic field theory were proposed in [CL09]. Thereupon Abbondandolo and Schwarz established several ring isomorphisms (see [AS10]), for instance

$$H_*(\Omega_q Q) \cong HF_* (T^* Q, T^*_q Q).$$

(5.3.1)

Here the left hand side denotes the based loop space equipped with the Pontryagin product, while the right hand side corresponds to a Hamiltonian Floer homology with a triangle product counting boundary punctured triangles. The construction of the Pontryagin product extends to a product map

$$\#: H_* (\Omega_{q_1,q_2} Q) \otimes \cdots \otimes H_* (\Omega_{q_{m-1},q_m} Q) \to H_* (\Omega_{q_1,q_m} Q)$$

(5.3.2)

on the homology of the path space with fixed end points. The boundary punctured disks in $\mathcal{M}_{\Phi, \text{sign}}$ then reflect the unit identity $\#(e \otimes \cdots \otimes e) = e.$
Appendix A

A.1 A General Index Formula for Punctured Disks

In this section we present an index formula for boundary punctured \( J \)-holomorphic disks in the symplectization of a contact manifold. The boundary loop of Lagrangian subspaces is closed by introducing negative rotations at the punctures. We follow the appendix of [CEL10] which contains a more general index formula dealing with interior and Lagrangian intersection punctures as well.

Let \((\mathbb{D}^o_m, i)\) be the closed unit disk with \(m \geq 3\) boundary punctures \(p_1, \ldots, p_m\) in counterclockwise order. Thus \(\partial \mathbb{D}^o_m = \partial \mathbb{D} \setminus \{p_1, \ldots, p_m\}\). Moreover, let \((Y, \alpha)\) be a contact manifold of dimension \(2n - 1\) with contact hyperplane field \(\xi = \ker \alpha\). Then the symplectization

\[
(M, \omega) = (Y \times \mathbb{R}, d(e^t \alpha))
\]

is a symplectic manifold. It is noncompact and has one convex and one concave end. Let \(J \in \mathcal{J}(M, \omega)\) be a compatible \(\mathbb{R}\)-invariant almost complex structure which restricts to an almost complex structure on \(\xi\) and maps the Reeb vector field onto the symplectization direction \(\frac{\partial}{\partial t}\).

These structures are also referred to as cylindrical, see [BEH+04]. However, we do not require \(J\) to be invariant under the Reeb flow.

Let \(\Lambda\) be a Legendrian submanifold of \(Y\) such that \(L = \Lambda \times \mathbb{R}\) is Lagrangian in \(M\).

**Definition A.1.1.** Suppose \(c\) is a Reeb chord in \(Y\) with endpoints \(c^-, c^+ \in \Lambda\). Here \(c^-\) denotes the endpoint where the Reeb vector field points into \(c\). The Reeb chord \(c\) is called generic whenever the image \((T_c - \Lambda)'\) of the isotropic subspace \(T_c - \Lambda\) under the linearized Reeb flow is transversal to the isotropic subspace \(T_c \Lambda\). The Legendrian \(\Lambda\) is called admissible if all of its connecting Reeb chords are generic and if there are only finitely many of them.

With respect to the \(C^\infty\)-topology the subset of admissible Legendrians is Baire. In what follows we assume \(\Lambda\) to be admissible. Let

\[
\text{sign} : \{1, \ldots, m\} \to \{+, -\}
\]

be a binary function. We like to provide an index formula for \(J\)-holomorphic disks \(u : (\mathbb{D}^o_m, i) \to (M, J)\) satisfying the boundary condition \(u(\partial \mathbb{D}^o_m) \subset L\) and at each boundary puncture \(p_j\) with \(1 \leq j \leq m\) asymptotic to a Reeb chord \(c_j\) connecting two points in \(\Lambda\). The asymptotic should be positive or negative depending on \(\text{sign}(j)\).

In order to incorporate the Maslov index we have to close the paths \(u(\partial \mathbb{D}^o_m) \subset L\) to get a loop of Lagrangian subspaces. This is done by adding negative rotations at the punctures.
Two Lagrangian subspaces \( V_0 \) and \( V_1 \) in \( \mathbb{C}^k \) define a decomposition \( \mathbb{C}^k = W^1 \oplus \cdots \oplus W^l \) into orthogonal subspaces and an angle \( \tau(V_0, V_1) = (\tau_1, \ldots, \tau_l) \) with \( 0 \leq \tau_1 < \tau_2 < \cdots < \tau_l < \pi \) by the following algorithm. \( \tau_1 \in [0, \pi) \) is the smallest number satisfying
\[
\dim \left( (e^{i\tau_1} \cdot V_0) \cap V_1 \right) \geq 1.
\]
\( W^1 \subset \mathbb{C}^k \) denotes the complex subspace generated by \( (e^{i\tau_1} \cdot V_0) \cap V_1 \) and \( W' \) its orthogonal complement. Let \( V_0' = W' \cap (e^{i\tau_1} \cdot V_0) \) and \( V_1' = W' \cap V_1 \). Next let \( \tau_1' \in (0, \pi) \) be the smallest number satisfying
\[
\dim \left( (e^{i\tau_1'} \cdot V_0') \cap V_1' \right) \geq 1.
\]
Set \( \tau_2 = \tau_1 + \tau_1' \) and let \( W'^2 \subset W' \subset W \) be the complex subspace generated by \( (e^{i\tau_2} \cdot V_0') \cap V_1' \). Repeat this procedure until \( W^1 \oplus \cdots \oplus W^l = \mathbb{C}^k \). The invariance
\[
\tau(AV_0, AV_1) = \tau(V_0, V_1)
\]
for every \( A \in U(k) \) is obvious, since the \( S^1 \)-action given by diagonal multiplication with \( e^{it} \) commutes with the \( U(k) \)-action in \( \mathbb{C}^k \). This implies that the angle \( \tau(V_0, V_1) \) is well-defined for any Lagrangian subspaces \( V_0, V_1 \subset T_xM \) of an almost Hermitian manifold \( (M, \omega, J) \).

**Definition A.1.2.** Let \( V_0, V_1 \subset \mathbb{C}^k \) be Lagrangian subspaces with associated decomposition
\[
\mathbb{C}^k = W^1 \oplus \cdots \oplus W^l
\]
and angle \( \tau(V_0, V_1) = (\tau_1, \ldots, \tau_l) \). The negative rotation taking \( V_0 \) to \( V_1 \) denotes the 1-parameter family of linear transformations acting by multiplication with \( e^{-i(\pi - \tau_j)}t \) on \( W^j \). The parameter range is \( t \in [0, 1] \) and \( j \in \{1, \ldots, l\} \).

For a Reeb chord \( c \) connecting \( c^-, c^+ \in \Lambda \) let
\[
R^{\text{neg}}_{c^+} (c^-, c^+) : \xi_{c^+} \to \xi_{c^+}
\]
be the negative rotation in \( \xi_{c^+} \) which takes the image \( (T_{c^-} \Lambda)' \) of \( T_{c^-} \Lambda \) under the linearized Reeb flow along \( c \) onto \( T_{c^+} \Lambda \). Similarly, let
\[
R^{\text{neg}}_{c^-} (c^+, c^-) : \xi_{c^-} \to \xi_{c^-}
\]
be the negative rotation taking the image \( (T_{c^+} \Lambda)' \) of \( T_{c^+} \Lambda \) under the backwards linearized Reeb flow along \( c \) onto \( T_{c^-} \Lambda \).

We have to fix complex trivializations \( Z_c \) of the contact planes along any connecting Reeb chord \( c \) with the property that the linearized Reeb flow from \( c^- \) to \( c^+ \) is equal to the identity. Together with a \( \mathbb{C} \)-summand composed of the Reeb vector field and the vector field in the symplectization direction we obtain trivializations of \( TM \) along any Reeb chord appearing as asymptotic data. Here we are taking advantage of the fact that \( J \) maps the Reeb vector field onto \( \overline{\partial}/\overline{\partial t} \).

Given a \( J \)-holomorphic curve \( u : \overline{D} \to M \) with boundary on \( L \) and asymptotic to connecting Reeb chords of \( \Lambda \) at boundary punctures, these choices provide trivializations \( Z^u_{\partial D} \) of \( u^*TM \) along the \( j \)-th component of the boundary \( \partial \overline{D}_m \) between \( p_j \) and \( p_{j+1} \) for \( 1 \leq j \leq m \). We denote by
\[
e^1_{\text{rel}} (u^*TM; Z^u_{\partial D})
\]
the obstruction to extend this trivialization over \( \mathbb{D} \). The tangent planes to \( L \) along \( u(\partial \overline{D}_m) \) expressed in the trivializations \( Z^u_{\partial D} \) constitute a collection of paths of Lagrangian subspaces in \( \mathbb{C}^n \). These are closed to a loop as follows:
A.2. THE HOLOMORPHIC SECTIONAL CURVATURE OF THE SASAKI METRIC

- The tangent spaces of \( L = \Lambda \times \mathbb{R} \) at endpoints of a Reeb chord \( c \) at a positive puncture are connected by the product of the linearized Reeb flow along \( c \) in \( \xi \) and the identity in the symplectization direction, followed by the path

\[
\text{im} \left( R_{\xi^+}^{\text{reg}} (c^-, c^+) \right) \oplus \mathbb{R} \subset \xi_{c^+} \oplus \mathbb{C}. \tag{A.1.4}
\]

- The tangent spaces of \( L = \Lambda \times \mathbb{R} \) at endpoints of a Reeb chord \( c \) at a negative puncture are connected by the product of the backwards linearized Reeb flow along \( c \) in \( \xi \) and the identity in the symplectization direction, followed by the path

\[
\text{im} \left( R_{\xi^-}^{\text{reg}} (c^+, c^-) \right) \oplus \mathbb{R} \subset \xi_{c^-} \oplus \mathbb{C}. \tag{A.1.5}
\]

Finally, denote by

\[
\mu \left( \mathbb{D}_m^0, Z_0^B \right) \tag{A.1.6}
\]

the Maslov index of the constructed loop of Lagrangian subspaces in \( \mathbb{C}^n \).

**Theorem A.1.3.** The index of the linearized Cauchy-Riemann operator \( D_u \) of a \( J \)-holomorphic curve \( u : \mathbb{D}_m^0 \to M \) with described boundary and asymptotic conditions is given by the formula

\[
\text{index } D_u = n + \mu \left( \mathbb{D}_m^0, Z_0^B \right) + 2c_1^{\text{rel}}(u^*TM; Z_0^B). \tag{A.1.7}
\]

One may interpret the latter sum of the Maslov index and the relative Chern number as the boundary Maslov index, see [MS04] appendix C.3. For proofs we refer to Proposition 5.14 in [EES07] or Theorem A.1 in [CEL10].

**Remark A.1.4.** We have fixed the conformal structure on \( \mathbb{D}_m^0 \). This is equivalent to fixing the boundary punctures \( p_1, \ldots, p_m \). If we allow the conformal structure to vary, we would get an additional summand of \( m - 3 \) in (A.1.7).

### A.2 The Holomorphic Sectional Curvature of the Sasaki Metric

The aim of this section is to deduce the formula for the holomorphic sectional curvature of the Sasaki metric. The calculation is based on Proposition 2.2.4 and the definition of \( J_{LC} \) via (2.2.16). We use the same notation as in section 2.2. If no confusion arises, we will denote the Sasaki metric \( \tilde{g} \) as well as \( g \) by \( \langle \cdot, \cdot \rangle \). The following is Theorem 2.2.8.

**Theorem A.2.1.** Let \((Q, g)\) be a Riemannian manifold and \((TQ, \tilde{g})\) its tangent bundle. Suppose \( \tilde{Z} \in \Gamma(T(TQ)) \) splits into \( \tilde{Z} = X^H + Y^V \) with \( X, Y \in \Gamma(TQ) \). Then for \( \xi \in TQ \) such that \( \pi(\xi) = p \) one has

\[
\left\langle \tilde{R}(\tilde{Z}, J_{LC} \tilde{Z}) J_{LC} \tilde{Z}, \tilde{Z} \right\rangle \bigg|_{\xi} = 4 \left\langle R(X, Y) X, Y \right\rangle_p + \frac{1}{4} \left\| R_p(\xi, X_p) X_p \right\|^2 + \frac{1}{4} \left\| R_p(\xi, Y_p) Y_p \right\|^2 \\
\quad + \frac{1}{2} \left\| R_p(\xi, Y_p) Y_p \right\|^2 + \frac{1}{2} \left\| R_p(\xi, X_p) X_p \right\|^2 - \frac{5}{4} \left\| R_p(X_p, Y_p) \xi \right\|^2 \\
\quad + \left\langle \left\langle \nabla_X R \right\rangle_p(\xi, X_p) X_p, Y_p \right\rangle - \left\langle \left\langle \nabla_Y R \right\rangle_p(\xi, Y_p) Y_p, X_p \right\rangle - \frac{1}{4} \left\langle R_p(\xi, X_p) X_p, R_p(\xi, Y_p) Y_p \right\rangle. \tag{A.2.1}
\]
Proof. For convenience, we use the shorter notation \( J = J_{LC} \) for the involved almost complex structure. The term

\[
\left\langle \tilde{R} (\tilde{Z}, J \tilde{Z}) J \tilde{Z}, \tilde{Z} \right\rangle_{\xi} = \left\langle \tilde{R} (X^{H} + Y^{V}, J (X^{H} + Y^{V})) J (X^{H} + Y^{V}), X^{H} + Y^{V} \right\rangle_{\xi}
\]

splits into 16 summands, which have to be calculated separately with the help of Proposition 2.2.4, the relations \( JX^{H} = X^{V}, JY^{V} = -Y^{H} \) and formula (2.2.4). To get started, we evaluate

\[
\left\langle \tilde{R} (X^{H}, JX^{H}) JX^{H}, X^{H} \right\rangle_{\xi} = \left\langle \tilde{R} (X^{H}, X^{V}) X^{V}, X^{H} \right\rangle_{\xi}
\]

\[
= - \frac{1}{2} (R_{p}(X_{p}, X_{p})X_{p}, X_{p}) - \frac{1}{4} (R_{p}(\xi, X_{p})R_{p}(\xi, X_{p})X_{p}, X_{p})
\]

\[
= \frac{1}{4} \| R_{p}(\xi, X_{p})X_{p} \|^{2}, \quad (A.2.2)
\]

\[
\left\langle \tilde{R} (X^{H}, JX^{H}) JX^{H}, Y^{V} \right\rangle_{\xi} = \left\langle \tilde{R} (X^{H}, X^{V}) X^{V}, Y^{V} \right\rangle_{\xi} = 0,
\]

\[
\left\langle \tilde{R} (X^{H}, JX^{H}) JY^{V}, X^{H} \right\rangle_{\xi} = - \left\langle \tilde{R} (X^{H}, X^{V}) Y^{H}, X^{H} \right\rangle_{\xi} = \left\langle \tilde{R} (X^{H}, X^{V}) X^{H}, Y^{H} \right\rangle_{\xi}
\]

\[
= \frac{1}{2} \langle (\nabla X R)_{p}(\xi, X_{p})X_{p}, Y_{p} \rangle, \quad (A.2.3)
\]

\[
\left\langle \tilde{R} (X^{H}, JX^{H}) JY^{V}, Y^{V} \right\rangle_{\xi} = - \left\langle \tilde{R} (X^{H}, X^{V}) Y^{H}, Y^{V} \right\rangle_{\xi}
\]

\[
= - \frac{1}{2} (R_{p}(X_{p}, Y_{p})X_{p}, Y_{p}) - \frac{1}{4} (R_{p}(R_{p}(\xi, X_{p})Y_{p}, X_{p}) \xi, Y_{p})
\]

\[
= \frac{1}{2} (R(X, Y)Y, X)_{p} + \frac{1}{4} (R_{p}(\xi, X_{p})Y_{p}, R_{p}(\xi, Y_{p})X_{p}) \cdot \quad (A.2.4)
\]

The next four terms are

\[
\left\langle \tilde{R} (X^{H}, JY^{V}) JX^{H}, X^{H} \right\rangle_{\xi} = - \left\langle \tilde{R} (X^{H}, Y^{H}) X^{V}, X^{H} \right\rangle_{\xi}
\]

\[
= \frac{1}{2} \langle (\nabla Y R)_{p}(\xi, X_{p})X_{p}, X_{p} \rangle - \frac{1}{2} \langle (\nabla X R)_{p}(\xi, X_{p})Y_{p}, X_{p} \rangle
\]

\[
= \frac{1}{2} \langle (\nabla X R)_{p}(\xi, X_{p})X_{p}, Y_{p} \rangle, \quad (A.2.5)
\]

\[
\left\langle \tilde{R} (X^{H}, JY^{V}) JX^{H}, Y^{V} \right\rangle_{\xi} = - \left\langle \tilde{R} (X^{H}, Y^{H}) X^{V}, Y^{V} \right\rangle_{\xi}
\]

\[
= - (R_{p}(X_{p}, Y_{p})X_{p}, Y_{p}) - \frac{1}{4} (R_{p}(R_{p}(\xi, X_{p})Y_{p}, X_{p}) \xi, Y_{p})
\]

\[
+ \frac{1}{4} (R_{p}(R_{p}(\xi, X_{p})X_{p}, Y_{p}) \xi, Y_{p})
\]

\[
= (R(X, Y)Y, X)_{p} + \frac{1}{4} (R_{p}(\xi, X_{p})Y_{p}, R_{p}(\xi, Y_{p})X_{p}) \quad (A.2.6)
\]

\[
- \frac{1}{4} (R_{p}(\xi, X_{p})X_{p}, R_{p}(\xi, Y_{p})Y_{p}),
\]
A.2. THE HOLOMORPHIC SECTIONAL CURVATURE OF THE SASAKI METRIC

\[ \langle \bar{R} \left( X^H, JY^V \right) JY^V, X^H \rangle_\xi = \langle \bar{R} \left( X^H, Y^H \right) Y^H, X^H \rangle_\xi \]
\[ = \langle R_p(X_p, Y_p)Y_p, X_p \rangle + \frac{1}{2} \langle R_p(\xi, R_p(X_p, Y_p)\xi)Y_p, X_p \rangle \]
\[ + \frac{1}{4} \langle R_p(\xi, R_p(X_p, Y_p)\xi)X_p, X_p \rangle + \frac{1}{4} \langle R_p(\xi, R_p(X_p, Y_p)\xi)Y_p, X_p \rangle \]
\[ = \langle R(X, Y)Y, X \rangle_p - \frac{1}{2} \| R_p(X_p, Y_p)\xi \|^2 - \frac{1}{4} \| R_p(X_p, Y_p)\xi \|^2 , \]
(A.2.7)

\[ \langle \bar{R} \left( X^H, JY^V \right) JY^V, Y^V \rangle_\xi = \langle \bar{R} \left( X^H, Y^H \right) Y^H, Y^V \rangle_\xi = \frac{1}{2} \langle (\nabla Y R)_p(X_p, Y_p)\xi, Y_p \rangle \]
\[ = - \frac{1}{2} \langle (\nabla Y R)_p(\xi, Y_p)Y_p, X_p \rangle . \]
(A.2.8)

It is easily seen, that
\[ \langle \bar{R} \left( Y^V, JX^H \right) JX^H, X^H \rangle_\xi = \langle \bar{R} \left( Y^V, JX^H \right) JX^H, Y^V \rangle_\xi = \langle \bar{R} \left( Y^V, JX^H \right) JY^V, Y^V \rangle_\xi = 0. \]
Hence we finish off by calculating
\[ \langle \bar{R} \left( Y^V, JX^H \right) JY^V, X^H \rangle_\xi = - \langle \bar{R} \left( Y^V, X^V \right) Y^H, X^H \rangle_\xi \]
\[ = - \langle R_p(Y_p, X_p)Y_p, X_p \rangle - \frac{1}{4} \langle R_p(\xi, Y_p)R_p(\xi, X_p)Y_p, X_p \rangle \]
\[ + \frac{1}{4} \langle R_p(\xi, X_p)R_p(\xi, Y_p)Y_p, X_p \rangle \]
\[ = \langle R(X, Y)Y, X \rangle_p + \frac{1}{1} \langle R_p(\xi, X_p)Y_p, R_p(\xi, Y_p)X_p \rangle \]
\[ - \frac{1}{4} \langle R_p(\xi, X_p)X_p, R_p(\xi, Y_p)Y_p \rangle , \]
(A.2.9)

\[ \langle \bar{R} \left( Y^V, JY^V \right) JX^H, X^H \rangle_\xi = - \langle \bar{R} \left( Y^V, Y^H \right) X^V, X^H \rangle_\xi \]
\[ = - \frac{1}{2} \langle R_p(Y_p, X_p)Y_p, X_p \rangle - \frac{1}{4} \langle R_p(\xi, Y_p)R_p(\xi, X_p)Y_p, X_p \rangle \]
\[ = \frac{1}{2} \langle R(X, Y)Y, X \rangle + \frac{1}{4} \langle R_p(\xi, X_p)Y_p, R_p(\xi, Y_p)X_p \rangle , \]
(A.2.10)

\[ \langle \bar{R} \left( Y^V, JY^V \right) JX^H, Y^V \rangle_\xi = - \langle \bar{R} \left( Y^V, Y^H \right) X^V, Y^V \rangle_\xi = 0, \]

\[ \langle \bar{R} \left( Y^V, JY^V \right) JY^V, X^H \rangle_\xi = \langle \bar{R} \left( Y^V, Y^H \right) Y^H, X^H \rangle_\xi = - \frac{1}{2} \langle (\nabla Y R)_p(\xi, Y_p)Y_p, X_p \rangle , \]
(A.2.11)

\[ \langle \bar{R} \left( Y^V, JY^V \right) JY^V, Y^V \rangle_\xi = \langle \bar{R} \left( Y^V, Y^H \right) Y^H, Y^V \rangle_\xi \]
\[ = - \frac{1}{4} \langle R_p(R_p(\xi, Y_p)Y_p)\xi, Y_p \rangle - \frac{1}{2} \langle R_p(Y_p)Y_p, Y_p \rangle \]
\[ = \frac{1}{4} \| R_p(\xi, Y_p)Y_p \|^2 . \]  (A.2.12)
Adding up equations (A.2.11) – (A.2.12) yields
\[
\left< \bar{R}(\bar{Z}, J\bar{Z})J\bar{Z}, \bar{Z} \right> = 4 \left< R(X, Y)Y, X \right>_p + \frac{1}{4} \| R_p(\xi, X_p)X_p \|^2 + \frac{1}{4} \| R_p(\xi, Y_p)Y_p \|^2 \\
+ \left< R_p(\xi, X_p)Y_p, R_p(\xi, Y_p)X_p \right> - \frac{3}{4} \| R_p(X_p, Y_p)\xi \|^2 \\
+ \left< (\nabla_X R)_p(\xi, X_p)X_p, Y_p \right> - \left< (\nabla_Y R)_p(\xi, Y_p)Y_p, X_p \right> \\
- \frac{1}{2} \left< R_p(\xi, X_p)X_p, R_p(\xi, Y_p)Y_p \right>.
\]

To conclude the statement, we just have to plug in the identity
\[
\left< R_p(\xi, X_p)Y_p, R_p(\xi, Y_p)X_p \right> = \frac{1}{2} \left( \| R_p(\xi, X_p)Y_p \|^2 + \| R_p(\xi, Y_p)X_p \|^2 - \| R_p(X_p, Y_p)\xi \|^2 \right).
\]

\[ \square \]

We immediately obtain two corollaries, of which the first one already appeared in [PPK95].

**Corollary A.2.2.** Suppose \( \bar{Z} \in \Gamma(T(TQ)) \) is horizontal or vertical, i.e. \( \bar{Z} = X^H \) or \( \bar{Z} = X^V \) with \( X \in \Gamma(TQ) \). Then for \( \xi \in TQ \) with \( \pi(\xi) = p \) we have
\[
\left< \bar{R}(\bar{Z}, J_{LC}\bar{Z})J_{LC}\bar{Z}, \bar{Z} \right> = \frac{1}{4} \| R_p(\xi, X_p)X_p \|^2.
\]

**Corollary A.2.3.** Suppose \( \bar{Z} \in \Gamma(T(TQ)) \) splits into \( \bar{Z} = X^H + X^V \) with \( X, Y \in \Gamma(TQ) \). Then the holomorphic sectional curvature of \( (TQ, \bar{g}) \) is proportional to the sectional curvature of \( (Q, g) \) on the zero section \( s_0 : Q \to TQ \). For \( p \in Q \) we have
\[
\left< \bar{R}(\bar{Z}, J_{LC}\bar{Z})J_{LC}\bar{Z}, \bar{Z} \right>_{s_0(p)} = 4 \left< R(X, Y)Y, X \right>_p.
\]

We further consider the example of the round sphere, that is \( Q = S^n \). Then the holomorphic sectional curvature can be calculated explicitly.

**Corollary A.2.4.** Consider \( S^n \) equipped with its standard metric \( g \). For \( \bar{Z} = X^H + X^V \in \Gamma(T(TS^n)) \) and \( \xi \in TS^n \), \( p = \pi(\xi) \) the holomorphic sectional curvature of \( (TS^n, \bar{g}) \) is given by
\[
\left< \bar{R}(\bar{Z}, J_{LC}\bar{Z})J_{LC}\bar{Z}, \bar{Z} \right> = \frac{1}{4} \left( \| X_p \|^4 + \| Y_p \|^4 + 4 \left< X, Y \right>_p^2 - 2 \| X_p \|^2 \| Y_p \|^2 \right) \| \xi \|^2 - \frac{1}{4} \left( \| X_p \|^2 + \| Y_p \|^2 \right) \left( \| \xi \|^2 + \left< \xi, X_p \right>^2 + \left< \xi, Y_p \right>^2 \right) + 4 \left( \| X_p \|^2 \| Y_p \|^2 - \left< X, Y \right>_p^2 \right) \| \xi \|^2.
\]

In particular, we have the estimate
\[
-\frac{1}{4} \| \bar{Z} \xi \|^4 \| \xi \|^2 \leq \left< \bar{R}(\bar{Z}, J_{LC}\bar{Z})J_{LC}\bar{Z}, \bar{Z} \right> = \| \bar{Z} \xi \|^4 \left( \frac{1}{4} \| \xi \|^2 + 1 \right).
\]

**Proof.** We use that \( S^n \) is a symmetric space, implying \( \nabla R \equiv 0 \). Further plugging in the formula
\[
R(X, Y)Z = \left< Y, Z \right> X - \left< X, Z \right> Y
\]
into (A.2.1) turns (A.2.13) into a straightforward calculation. The estimate follows from repeatedly applying the Cauchy-Schwarz inequality. The lower estimate is obtained from
\[
\frac{1}{4} \left( \| X_p \|^4 + \| Y_p \|^4 + 4 \left< X, Y \right>_p^2 - 2 \| X_p \|^2 \| Y_p \|^2 \right) \| \xi \|^2 - \frac{1}{4} \left( \| X_p \|^2 + \| Y_p \|^2 \right) \left( \| \xi \|^2 + \left< \xi, X_p \right>^2 + \left< \xi, Y_p \right>^2 \right) \left( \| \xi \|^2 + \left< \xi, X_p \right>^2 + \left< \xi, Y_p \right>^2 \right) \\
\geq -\frac{1}{4} \left( \| X_p \|^2 + \| Y_p \|^2 \right) \left( \| \xi \|^2 + \left< \xi, X_p \right>^2 + \left< \xi, Y_p \right>^2 \right) \geq -\frac{1}{4} \left( \| X_p \|^2 + \| Y_p \|^2 \right) \| \xi \|^2 = -\frac{1}{4} \| \bar{Z} \xi \|^4 \| \xi \|^2.
\]
Finally, the upper estimate follows from

\[
\frac{1}{4} (\|X_p\|^4 + \|Y_p\|^4 + 4 \langle X, Y \rangle_p^2 - 2 \|X_p\|^2\|Y_p\|^2) \|\xi\|^2 + 4 \|X_p\|^2\|Y_p\|^2 \\
\leq \frac{1}{4} (\|X_p\|^4 + \|Y_p\|^4 + 2 \|X_p\|^2\|Y_p\|^2) \|\xi\|^2 + (\|X_p\|^2 + \|Y_p\|^2)^2 \\
= \|\tilde{Z}_\xi\|^4 \left( \frac{1}{4} \|\xi\|^2 + 1 \right).
\]

In both directions we have used that \(\|\tilde{Z}_\xi\|^2 = \|X_p\|^2 + \|Y_p\|^2\). \(\square\)
Bibliography


Selbstständigkeitserklärung


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