Amenable groups and a geometric view on unitarisability

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Introduction

After its introduction in the first half of the 20th century, the concept of amenability of groups proved to be quite a powerful property in investigating groups. It has many equivalent definitions and means, in some way, that a group is dominated by its finite subsets. So, many properties one knows to hold for finite groups, translated well into the universe of amenable groups.

Opposed to this world, there are non-abelian free groups (i.e. groups generated by more than one element, which do not fulfill any relation). They do behave quite differently compared to amenable groups and, somehow, live on the “opposite” side of the universe of groups. In fact, it was a long-standing question, known as the von-Neumann conjecture, whether all non-amenable groups contain non-abelian free subgroups. This turned out to be wrong, but nonetheless, there are many properties, one knows for amenable groups to hold, that fail for groups containing non-abelian free subgroups.

It is then natural to ask, whether those properties characterize amenability.

Ulam stability and unitarisability are two examples of such properties, the second of which will be the topic for this thesis. Both those properties look at maps from a group into the space of bounded operators on a Hilbert space.

Ulam stability means, that maps into the space of unitary operators, which are of bounded distance from being homomorphisms, are in fact close to homomorphisms.

Unitarisability deals with true homomorphisms. It says, that linear representations, which map into a bounded neighbourhood of the space of unitary operators, are themselves similar to unitary representations.

The question, whether unitarisability is equivalent to amenability was first asked by Jacques Dixmier and is known as Dixmier’s unitarisability problem.

In this thesis, we will, after introducing necessary details from topology and functional analysis in Chapter 1, try to give some overview about unitarisability and the first constructions of non-unitarisable groups in Chapter 2. Of particular interest for this thesis are two theorems by Gilles Pisier.

The first theorem (Theorem 2.9) associates to a unitarisable group $G$ some universal constants $K$ and $\alpha$ and the second theorem (Theorem 2.10) states an equivalence between the amenability of $G$ and the possibility to choose $K$ and $\alpha$ small.

The proofs of those theorems in [Pis05] and [Pis99] are quite abstract and very
algebraic and it will be two central results of this thesis, to give a more intuitive, geometric proof of Theorem 2.9 and one direction of Theorem 2.10.

Nicolas Monod proposed a different, metric approach to unitarisability, which was brought to my attention by Andreas Thom.

We will introduce this approach by translating unitarisability into a fixed point property of induced actions on the space $\mathcal{P}(H)$ of positive invertible operators on a Hilbert space $H$.

In this setup, we can show, that for every unitarisable representation $\pi$ there are always “optimal” operators conjugating $\pi$ into the space of unitary operators (Lemma 3.4).

On $\mathcal{P}(H)$, there is an additional differential geometric structure, which was investigated by Gustavo Corach, Horacio Porta and Lazaro Recht in a series of papers (e.g. [CPR94]). We will introduce some of their results and investigate its interplay with the topologies, that $\mathcal{P}(H)$ inherits as a subset of the Banach space of bounded operators on $H$.

It will then be possible to translate the the Theorems 2.9 and 2.10 into this more geometric picture and to re-prove Theorem 2.9 in a less algebraic and more geometric setup.

This will then motivate us in Chapter 6 to prove some topological results about the space of positive invertible operators. For example, we show that the metric topology agrees with the restriction of the Banach space topology on $B(H)$ to the space of positive invertible operators (Theorem 6.3).

After this, we will introduce a more general concept of so-called GCB-spaces generalizing the notion of complete CAT(0) spaces and construct barycenters for finite sets in such spaces. Investigating their properties, it will be possible to show a fixed-point property for amenable actions on such spaces (Theorem 8.15), which will then geometrically prove one direction of Theorem 2.10 and the fact, that amenable groups are unitarisable (Corollary 8.16).

From this, we deduce some properties about actions of groups on GCB-spaces which allow for bounded orbits. Namely, we will show, that virtually unitarisable groups are unitarisable (Corollary 8.21) and that every amenable subgroup of a group acting properly on GCB-spaces with at least one bounded orbit is finite (Corollary 8.22).

Eventually, we will give conditions on a GCB space and the group action, such that every group yields a fixed point (Theorem 9.4 and Corollary 9.6). This result generalizes the Ryll-Nardzewski Theorem.
1 Functional analytic background

In order to analyse uniformly bounded representations on Hilbert spaces, a little functional analytic background is needed. The details required in this thesis, are quite basic, but will be collected here nonetheless. This chapter is meant to be introductory and basic, so readers with elementary experience in functional analysis may want to skip to the following chapters.

For a more detailed survey, basic definitions (of terms like separable Hilbert space, topology, continuity...) and proofs of the stated results (or even statements such as the Hahn-Banach, Stone-Weierstrass or Banach-Alaoglu Theorems, which will only be used implicitly), the reader may be referred to standard textbooks in functional analysis and topology such as [Rud91], [KR83], [RS80] and Chapter 1 of [Bre93].

To keep this chapter rather short, we will only state the key facts without proving them.

1.1 Point set topology

In investigating continuity of maps between topological spaces, sequences and nets play an important role.

Definition (sequence, net, directed set).

A sequence \((x_n)_{n \in \mathbb{N}}\) in a topological space \(X\) is a map \(\mathbb{N} \to X, n \mapsto x_n\).

More generally, a net \((x_\alpha)_{\alpha \in I}\) in \(X\) is a map \(I \to X, \alpha \mapsto x_\alpha\), where \(I\) is a directed set.

This, by definition, is a set, which carries a pre-order \(\leq\), such that for every two elements \(i_1, i_2 \in I\) there is a common upper bound, i.e. an element \(j \in I\) such that \(i_1 \leq j\) and \(i_2 \leq j\).

Obviously, sequences are nets. But nets may have “larger” index sets.

As known for \(\mathbb{R}\), there is the concept of convergence for sequences and nets:

Definition (convergent net).

A net \((x_\alpha)_{\alpha \in I} \subset (X, \tau)\) is called convergent to some \(x_0 \in X\), if

\[\forall U \in \tau : x_0 \in U \ \exists \tilde{\alpha} \in I : x_\alpha \in U \ \forall \alpha \geq \tilde{\alpha}\]
In this case, we write
\[ \lim_{\alpha \to \infty} x_\alpha = x_0 \]
and call \( x_0 \) the \( \tau \)-limit (the limit with respect to \( \tau \)) of \((x_\alpha)_{\alpha \in I}\).

Observe, that convergence depends on the topology. If there are many topologies on some space, we will therefore use the term \( \tau \)-convergent, to stress with respect to which topology we mean a net or a sequence to converge.

Often, topologies are defined by defining the “elementary” pieces instead of the whole topology:

**Definition** (subbasis of a topology, neighbourhood basis, first countable).
A subset \( \sigma \subset \tau \) of a topology is called a subbasis, if every element in \( \tau \) is a union of finite intersections from \( \sigma \). Elements from a subbasis are called basic open.

For a point \( x \in (X, \tau) \) in a topological space, a neighbourhood basis is a family \( V_x \subset \tau \) of open sets, such that every open \( U \) containing \( x \), also contains some element from \( V_x \).

The topology \( \tau \) is called locally countable, if every point has a countable neighbourhood basis.

An important property of continuous maps is the following

**Lemma 1.1** ([Bre93], Proposition 6.6, [Rud91], Theorem A6, p. 395).
For topological spaces \((X, \tau)\) and \((Y, \tau')\), one has
\[ f : (X, \tau) \to (Y, \tau') \text{ is continuous} \iff \lim_{\alpha \to \infty} f(x_\alpha) = f \left( \lim_{\alpha \to \infty} x_\alpha \right) \]
for all convergent nets

If \((X, \tau)\) is locally countable, one has
\[ f : (X, \tau) \to (Y, \tau') \text{ is continuous} \iff \lim_{\alpha \to \infty} f(x_\alpha) = f \left( \lim_{\alpha \to \infty} x_\alpha \right) \]
for all convergent sequences

one says, continuity is equivalent to **net-continuity** or to **sequential continuity**.
Another topological term, which is used throughout this thesis, is compactness. As convergence, compactness depends on the ambient topology and we will write $\tau$-compact, if clarification concerning the topology is needed.

Compact sets have a range of nice properties (see, Section 7 of Chapter 1 in [Bre93], for example):

- in Hausdorff spaces, compact sets are closed
- continuous functions on compact sets with values in $\mathbb{R}$ attain a maximum on this set.
- every net in a compact set has a convergent subnet.
- the image of a compact set under a continuous map is compact.
- if $\tau_1$ is finer than $\tau_2$, than any $\tau_1$-compact and $\tau_2$-closed set is $\tau_2$-compact.
- closed subsets of compact sets are compact.

Functional analysis looks at the interplay of topology and the theory of linear spaces. The vector spaces in functional analysis are therefore always topological:

**Definition** (topological vector space).

A topological vector space $(V, \tau)$ over $\mathbb{R}$ or $\mathbb{C}$ is a vector space $V$ over some field $F \in \{\mathbb{R}, \mathbb{C}\}$ together with a topology $\tau$, such that the addition $+: V \times V \to V$ and the scalar multiplication $\cdot : F \times V \to V$ are continuous (here, $F$ is given the standard topology).

**Remark 1.2.**

Sometimes, topological vector spaces are required to be Hausdorff. As far as this thesis is concerned, this will always be the case.

The topology of a topological vector space is invariant under translation (since addition with a particular vector is a homeomorphism). Hence, the topology is already defined by the neighbourhood basis of 0.
**Definition** (Cauchy sequence, complete topological vector space, Banach space). A sequence \((x_n)_{n \in \mathbb{N}}\) in a topological vector space is called a Cauchy sequence, if for any open set \(U\) containing \(0\), there is some \(N \in \mathbb{N}\), such that

\[
x_n - x_m \in U \quad \forall n, m \geq N
\]

A topological vector space \(V\) such that every Cauchy sequence converges, is called complete.

Topological vector spaces \(V\) with a norm, such that the induced topology (in the sense of [KR83], page 35) is complete, is called a Banach space.

**Example** 1 ([KR83], pages 75-80). By definition, a Hilbert space \(H\) with inner product \(\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}\) carries a norm by \(\|v\| = \sqrt{\langle v, v \rangle}\), which turns \(H\) into a Banach space.

**Definition** (unitary equivalence). A map \(U : (H, \langle \cdot, \cdot \rangle) \to (H', \langle \cdot, \cdot \rangle)\) is a unitary equivalence, if it is a linear homeomorphism such that

\[
\langle u, v \rangle = \langle U u, U v \rangle \quad \forall u, v \in H
\]

**Remark 1.3.** Throughout this thesis, we will tacitly assume, that every Hilbert space \(H\) is separable, i.e., that the cardinality of an orthonormal basis (in the sense of [KR83], p.93) of \(H\) is countable. It is shown in [KR83], Theorem 2.2.12, that any two Hilbert spaces of same dimension are unitarily equivalent.

**Definition** (topological group). A group \(G\) together with a topology on \(G\) is called a topological group, if the multiplication

\[
\mu : G \times G \to G, \quad (g, h) \mapsto gh
\]

and the map of taking inverses

\[
\iota : G \to G, \quad g \mapsto g^{-1}
\]

are continuous.
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**Definition** (discrete topology, discrete group).
The finest possible topology (namely \( \tau = P(X) \), the power set containing all subsets) is called discrete topology. Groups equipped with this topology are called discrete.

**Notation.** Unlike otherwise stated, all groups denoted by \( G \) or \( H \) will be countable discrete groups.

### 1.2 Operators and their adjoints

To fix some notation, in this thesis \( H \) shall always denote a separable (real or complex) Hilbert space, \( \langle \cdot , \cdot \rangle \) its inner product and \( B(H) \) the space of bounded linear operators on \( H \) (as defined in [KR83], page 40). Moreover, \( \text{id}_H \) shall denote the identity operator on \( H \).

**Definition** (Aut(\( H \))).
We define \( \text{Aut}(H) \) to be the group of all bounded operators which have bounded inverses:

\[
\text{Aut}(H) = \{ A \in B(H) | \exists B \in B(H) : BA = AB = \text{id}_H \}
\]

As usual, the inverse of an operator will be denoted by \( A^{-1} \). We will always implicitly assume \( A^{-1} \) to be bounded.

On \( B(H) \), a norm can be defined as follows ([KR83], pages 40-41):

\[
\| A \| = \min \{ \lambda \in \mathbb{R}_{\geq 0} | \| Ax \| \leq \lambda \| x \| \ \forall x \in H \}
\]

\[
= \sup_{x \in H} \frac{\| Ax \|}{\| x \|}
\]

\[
= \sup_{x, y \in H} \frac{1}{\| x \| \cdot \| y \|} \langle Ax, y \rangle
\]

Amongst others, this norm has the following properties ([KR83] pages 40-42):

- linear operators are bounded if and only if they are continuous ([KR83], Theorem 1.5.5)
- it is submultiplicative: \( \| AB \| \leq \| A \| \cdot \| B \| \)
- one has \( \| Ax \| \leq \| A \| \cdot \| x \| \)
1.2 Operators and their adjoints

- it turns $B(H)$ into a Banach space ([KR83], Theorem 1.5.6)

For two Hilbert spaces $H_1$ and $H_2$, the direct sum $H_1 \oplus H_2$ forms a Hilbert space with inner product

$$\langle (x_1, x_2), (y_1, y_2) \rangle_{H_1 \oplus H_2} := \langle x_1, y_1 \rangle_{H_1} + \langle x_2, y_2 \rangle_{H_2}$$

If the scalar product on $H_1 \oplus H_2$ is defined in this way, vectors $x = (x_1, 0)$ and $y = (0, x_2)$ coming from $H_1$ and $H_2$ are obviously orthogonal to each other.

One speaks of $(H_1 \oplus H_2, \langle \cdot, \cdot \rangle_{H_1 \oplus H_2})$ to be the orthogonal sum of $H_1$ and $H_2$ (cf [KR83], pages 121-122).

This way, one gets

$$\| (x_1, x_2) \| = \sqrt{\langle (x_1, x_2), (x_1, x_2) \rangle_{H_1 \oplus H_2}}$$

$$= \sqrt{\langle x_1, x_1 \rangle_{H_1} + \langle x_2, x_2 \rangle_{H_2}}$$

$$= \sqrt{\| x_1 \|^2 + \| x_2 \|^2}$$

$$\leq \| x_1 \| + \| x_2 \|$$

**Definition** (direct sum of operators, [KR83], page 124).

For operators $A \in B(H_1)$ and $B \in B(H_2)$, one defines

$$A \oplus B \in B(H_1 \oplus H_2), \ (x_1, x_2) \mapsto (Ax_1, Bx_2)$$

The following lemma states, that the operator norm behaves nicely with respect to the direct sum:

**Lemma 1.4.**

For $A_i \in B(H_i)$, where $i \in \{1, 2\}$, one has

$$\| A_1 \oplus A_2 \| = \max\{\| A_1 \|, \| A_2 \|\}$$

**Proof.** On the one hand

$$\| (A_1 \oplus A_2)(x_1, x_2) \| = \| (A_1 x_1, A_2 x_2) \|$$

$$= \sqrt{\| A_1 x_1 \|^2 + \| A_2 x_2 \|^2}$$

$$\leq \sqrt{\| A_1 \|^2 \cdot \| x_1 \|^2 + \| A_2 \|^2 \cdot \| x_2 \|^2}$$

$$\leq \sqrt{(\max\{\| A_1 \|, \| A_2 \|\}) \cdot (\| x_1 \|^2 + \| x_2 \|^2)}$$

$$= \max\{\| A_1 \|, \| A_2 \|\} \cdot \| (x_1, x_2) \|$$
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and hence \( \| A_1 + A_2 \| \leq \max \{ \| A_1 \|, \| A_2 \| \} \).

On the other hand

\[
\| A_1 + A_2 \| = \sup_{(x_1,x_2) \in H_1 \oplus H_2} \frac{\| A_1 + A_2 (x_1, x_2) \|}{\| (x_1, x_2) \|} \\
\geq \sup_{x_1 \in H_1} \frac{\| (A_1 + A_2)(x_1, 0) \|}{\| x_1 \|} \\
\geq \sup_{x_1 \in H_1} \frac{\| A_1 \| \cdot \| x_1 \|}{\| x_1 \|} \\
= \| A_1 \|
\]

and the analogous result holds for \( A_2 \) instead of \( A_1 \).

Hence,

\[
\| A_1 + A_2 \| \geq \max \{ \| A_1 \|, \| A_2 \| \}
\]

\[\Box\]

Definition (adjoint).

For an operator \( A \in B(H) \), the adjoint \( A^* \) of \( A \) is the unique operator (uniqueness is proven in [Rud91], Theorem 4.10), such that

\[
\langle Ax, y \rangle = \langle x, A^* y \rangle \quad \forall x, y \in H
\]

One easily sees, that the following identities hold ([KR83], Theorem 2.4.2 and Proposition 2.4.5):

- \( (AB)^* = B^* A^* \quad \forall A, B \in B(H) \)
- \( (A^*)^{-1} = (A^{-1})^* \quad \forall A \in \text{Aut}(H) \)
- \( (A + \lambda B)^* = A^* + \overline{\lambda} B^* \quad \forall A, B \in B(H), \lambda \in \mathbb{C} \)
- \( \text{id}^* = \text{id} \)

Furthermore, the norms of \( A \) and \( A^* \) coincide ([Rud91], Theorem 4.10) and ([KR83], Theorem 2.4.2 (IV))

\[
\| AA^* \| = \| A^* A \| = \| A \|^2
\]

Those properties turn the map \( A \mapsto A^* \) into what is called an involution with \( C^*\)-identity (cf [KR83], page 236).
1.2 Operators and their adjoints

**Definition (C*-algebra, ∗-isomorphism [KR83], page 236).**

Let \( A \) be a Banach algebra ([KR83], Definition 3.1.1) with an involution that fulfills the \( C^* \)-identity. Then it is called a \( C^* \)-algebra.

A linear homeomorphism \( \varphi : (A, *) \to (B, \star) \), between \( C^* \)-algebras, such that

\[
\varphi(a^*) = (\varphi(a))^*
\]

is called a ∗-isomorphism.

**Definition (self-adjoint operator, positive operator).**

Operators \( A \in B(H) \), for which \( A^* = A \) holds, are called self-adjoint.

We will denote the space of all self-adjoint operators on \( H \) by \( \mathcal{A}(H) \).

From the properties above, one sees that the space of self-adjoint operators is a real vector space.

Self-adjoint operators \( A \) such that \( \langle Ax, x \rangle \geq 0 \ \forall x \in H \) are called positive.

It will follow from continuous spectral calculus, that invertible positive operators will have \( \langle Ax, x \rangle > 0 \ \forall x \in H \setminus \{0\} \) (actually, since the spectrum is always closed, they will have \( \langle Ax, x \rangle > \varepsilon \ \forall x \in H \setminus \{0\} \) for some \( \varepsilon > 0 \)).

**Example 1.** \( A^*A \) is positive for any \( A \in B(H) \):

\[
\langle A^*Ax, x \rangle = \langle Ax, Ax \rangle = \|Ax\|^2 \geq 0 \ \forall x \in H
\]

Moreover, the space of positive invertible operators is closed under

- taking real multiples \( \lambda A \), where \( \lambda \in \mathbb{R}_+ := \{\lambda \in \mathbb{R} : \lambda > 0\} \) is strictly positive
- taking convex combinations

\[
\mu A + (1 - \mu)B \ \mu \in [0,1]
\]

One therefore says, the positive invertible operators form a cone inside the space of self-adjoint operators. This cone is denoted by \( \mathcal{P}(H) \):

\[
\mathcal{P}(H) := \{A \in B(H) \text{ positive and invertible}\}
\]
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$\mathcal{P}(H)$ and the set of all positive operators are norm-closed in $B(H)$ ([KR83], p. 105).

$\mathcal{P}(H)$ is open in the subspace topology (as defined in [Bre93], Definition 3.1 in Chapter 1) of the space of self-adjoint operators (since $\text{Aut}(H) \subset B(H)$ is open, see [KR83], p. 176). Its closure (i.e. the smallest norm-closed set containing it) both, in the space of self-adjoint operators and in $B(H)$, is the cone of all positive operators.

**Definition** (normal operator).

Operators in general (like matrices) will not commute, even with their own adjoint. If an operator $A$ does commute with its adjoint $A^*$, it is called a normal operator.

An important subgroup of $B(H)$ is $U(H)$, the group of unitary operators.

**Definition** (unitary operator).

An operator $U \in B(H)$ is called unitary, if

$$UU^* = U^*U = \text{id}_H$$

It is the set of linear bijective isometries with respect to $\langle \cdot, \cdot \rangle$.

From this, it is clear that $\|U\| = 1$ for every $U \in U(H)$ and that $U(H)$ is a group.

Another useful tool in functional analysis is the polar decomposition, which exists for any $A \in \text{Aut}(H)$ (see, for example, [Rud91], Theorem 12.35):

**Definition** (polar decomposition).

A factorisation $A = PU$ for an invertible $A \in \text{Aut}(H)$, a positive operator $P$ and a unitary operator $U$ is called polar decomposition.

**Remark 1.5.**

Usually, the polar decomposition is defined as $A = UP$ and our definition is referred to as the “right polar decomposition”, but since $A = UP = (UPU^*)U$ holds, the existence of right and left polar decompositions are equivalent.

We will use the definition above.
1.3 Spectrum and continuous spectral calculus

An important tool in analysing bounded operators on a Hilbert space, is the concept of the spectrum of a bounded operator. This is the generalization of the set of eigenvalues to operators on infinite-dimensional spaces:

**Definition (spectrum).**
The set of $\lambda \in \mathbb{C}$ such that $A - \lambda \cdot \text{id}_H$ does not have an inverse in $B(H)$, is called spectrum of $A$. It is denoted by $\sigma(A)$.

**Remark 1.6.**
By the Open Mapping Theorem ([KR83], Theorem 1.8.4 and Theorem 1.8.5), having a bounded inverse and being bijective are equivalent.

**Hence, if $\lambda \in \sigma(A)$, then either $A$ has a $\lambda$-eigenvector (i.e. a vector $x \in H \setminus \{0\}$ such that $Ax = \lambda x$), or $A - \lambda x$ is not surjective.**

Some facts about the spectrum are the following:

- For every $A \in B(H)$, $\sigma(A)$ is bounded, closed and non-empty ([KR83], Theorem 3.2.3).

- The number $\max\{|x|, x \in \sigma(A)\}$ is called the **spectral radius** of $A$.
  It can be calculated explicitly ([Rud91], Theorem 10.13):
  $$\max_{x \in \sigma(A)} |x| = \lim_{n \to \infty} \|A^n\|^{\frac{1}{n}} \leq \|A\|$$

- One has $\sigma(B^*) = \overline{\sigma(B)}$ and $\sigma(A^{-1}BA) = \sigma(B)$.
  This follows directly from the facts, that $(A^{-1})^* = (A^*)^{-1}$ and $A^{-1}BA - \lambda \text{id}_H = A^{-1}(B - \lambda \text{id}_H)A$.

Normal operators play a special role as they are “dominated” by their spectrum in the following way:

For normal operators $A$, let $C(\sigma(A))$ denote the commutative $C^*$-algebra of all $\mathbb{C}$-valued continuous functions on the spectrum $\sigma(A)$, where the $*$-operation is the complex conjugation and the norm on $C(\sigma(A))$ is the supremum norm. Further, let $C^*(A)$ be the norm-closed $*$-subalgebra of $B(H)$ generated by $A$ and $\text{id}_H$. 

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Then ([KR83], Theorem 4.4.5), there is a unique $*$-isomorphism of $C^*$-algebras between $C(\sigma(A))$ and $C^*(A)$, which maps the identity function on $\sigma(A)$ to $A$. Usually, this isomorphism is denoted by $f \mapsto f(A)$ for elements $f \in C(\sigma(A))$. It is called continuous spectral calculus and has the following properties:

- constant maps and polynomials are mapped to the obvious choices (see the proof to [KR83], Theorem 4.4.5):
  \[
  f : \sigma(A) \to \mathbb{C}, \, x \mapsto \lambda \Rightarrow f(A) = \lambda \cdot \text{id}_H
  \]
  \[
  p : \sigma(A) \to \mathbb{C}, \, x \mapsto x^n \Rightarrow p(A) = A^n
  \]

- as an isomorphism of $*$-algebras, it is continuous. This implies for example for any normal $A \in B(H)$:
  \[
  \lim_{n \to \infty} f_n = f \Rightarrow \lim_{n \to \infty} f_n(A) = f(A)
  \]
  Where the limit in $C(\sigma(A))$ comes from the sup-norm (i.e. $f$ is a uniform limit of $f_n$) and the limit in $C^*(A)$ is taken with respect to the operator norm on $B(H)$.

- The spectrum transforms “nicely” with respect to the continuous spectral calculus ([KR83], Theorem 4.4.8):
  \[
  \sigma(f(A)) = f(\sigma(A))
  \]
  This implies for example
  \[
  \|f(A)\| = \max_{x \in \sigma(A)} |f(x)| = \|f\|_{C(\sigma(A))}
  \]
  and in particular for invertible $A$
  \[
  \|A^{-1}\| = \max_{x \in \sigma(A)} \left| \frac{1}{x} \right| = \left( \min_{x \in \sigma(A)} |x| \right)^{-1}
  \]  (1)

- There is an equivalence ([KR83], Theorem 4.4.5)
  \[
  \sigma(A) \subset \mathbb{R} \iff A^* = A
  \]
Moreover ([KR83], Theorem 4.4.5), one has $\sigma(A) \subset \mathbb{R}_{\geq 0} \iff A$ is positive.

Also, unitary operators are characterized through the spectrum ([KR83], Theorem 4.4.5):

$$\sigma(A) \subset S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \iff A \in U(H)$$

In particular, for (positive) real-valued functions $f$, $f(A)$ is (positive) self-adjoint and functions $f$ with values in $S^1 \subset \mathbb{C}$ yield $f(A) \in U(H)$. So, for example

$$\exp(A) \in \mathcal{P}(H) \forall A \in \mathcal{S}(H)$$

$$A^{\frac{1}{2}} \in \mathcal{P}(H) \forall A \in \mathcal{S}(H)$$

As the continuous functional calculus is multiplicative (being a map of algebras), one has

$$f(A) \cdot g(A) = (f \cdot g)(A)$$

In particular, the bounded operator $A^{\frac{1}{2}}$ solves the equation $X^2 = A$, which is why one often writes $\sqrt{A}$ instead of $A^{\frac{1}{2}}$.

For any $f \in C(\sigma(A))$, $f(A)$ is normal ([KR83], Theorem 4.4.5) and therefore subject to continuous spectral calculus itself. If now, $f \in C(\sigma(A))$ and $g \in C(f(\sigma(A)))$, one has ([KR83], Theorem 4.4.8)

$$g \circ f(A) = g(f(A))$$

In particular, the mapping $\exp : A \mapsto \exp(A)$ yields a bijection between $\mathcal{S}(H)$ and $\mathcal{P}(H)$. The inverse map is given by $\ln : A \mapsto \ln(A)$.

Since for every polynomial $p$ and every $A \in \text{Aut}(H)$, one has

$$p(A^{-1}BA) = A^{-1}p(B)A$$

and polynomials are dense in the set of all continuous functions, one has

$$f(A^{-1}BA) = A^{-1}f(B)A \quad (2)$$

whenever $f$ is a continuous function on $\sigma(A^{-1}BA) = \sigma(B)$. 

1.4 Topologies on $B(H)$

On $B(H)$, there is a range of topologies, which turn it into a topological vector space.

The interplay between those topologies has proven quite fruitful. We will state a few facts on this issue here.

Generally, the norm-topology will be the strongest of the considered topologies (i.e. it allows for more open sets than the other topologies and hence, there are fewer compact sets and convergence is harder to achieve).

The reader may be reminded, that a functional $\varphi$ on a (topological) vector space $V$ is a (continuous) linear map $\varphi : V \to K$ where $K$ is the field over which, the vector space is defined.

The dual space $V'$ of a topological vector space $V$ is the set of all functionals on $V$. Hence the notion of the dual space $B(H)'$ depends on the chosen topology on $B(H)$. Normally, when speaking about the dual space of $B(H)$, one assumes it to be defined with respect to the norm-topology. This way, $B(H)'$ is itself a Banach space (cf. [KR83], page 44).

All of the topologies introduced here, are translation invariant (as they turn $B(H)$ into a topological vector space). Hence, it suffices to say, what the neighbourhoods of $0 \in B(H)$ are.

**Definition** (weak, weak operator and strong operator topologies).

We define the following topologies on $B(H)$:

- The topology of pointwise convergence is called the strong operator topology (cf. [KR83], p. 113).

  The basic open neighbourhoods of $0$ in this topology are the sets

  $$V(x_1, \ldots, x_n, \varepsilon) := \{ A \in B(H) : \| Ax_i \| < \varepsilon \ \forall i \leq n \}$$

  and a net $(T_\alpha)_{\alpha \in I}$ converges to $0$ in the strong operator topology if and only if

  $$\lim_{n \to \infty} \| T_n x \| = 0 \ \forall x \in H.$$
1.4 Topologies on $B(H)$

- The weak operator topology is generated by the sets

$$V(x_1, \ldots, x_n, y_1, \ldots, y_n, \varepsilon) := \{ A \in B(H) : \langle Ax_i, y_i \rangle < \varepsilon \ \forall i \leq n \}$$

(cf [KR83], pages 304-305) and a net $(T_\alpha)_{\alpha \in I}$ converges to 0 in the weak operator topology, if and only if

$$\lim_{n \to \infty} \langle T_n x, y \rangle = 0 \ \forall x, y \in H$$

For an arbitrary topological vector space $V$, one defines the weak topology to be the weakest topology, such that all elements from the dual space (coming from the initial topology on $V$) are continuous.

A net converges weakly, if and only if the image under every element of the dual space is a convergent net. (cf [KR83], pages 30-31)

First of all, convergent sequences with respect to any of the topologies above will be bounded (uniform boundedness principle, [KR83], Theorem 1.8.9) and all the topologies above are stronger than the weak operator topology and weaker than the norm topology ([KR83], pages 29, 114 and 305).

All those topologies have their individual advantages we will use throughout this thesis.

By Proposition 2.7 in [Tak02], both, the weak and the strong operator topologies on $B(H)$ are metrizable on bounded sets. Therefore, continuity of maps with respect to the norm, weak operator and strong operator topologies is equivalent to sequential continuity ([Bre93], p. 6 and 26, and Lemma 1.1).

By the Riesz’ Representation Theorem (Theorem II.4 on page 43 in [RS80]), on a Hilbert space $H$, a net $(x_\alpha)_{\alpha \in I}$ converges weakly, if and only if $(\langle x_\alpha, y \rangle)_{\alpha \in I}$ converges for every $y \in H$.

As an application of the Hahn-Banach Theorem ([KR83], Theorem 1.3.4), bounded, closed and convex sets are the same for the norm and the weak topology (and, by definition, the dual spaces of with respect to both topologies coincide).
Lemma 1.7.
For any $A \in B(H)$, the maps $B(H) \to B(H)$, $X \mapsto AX$ and $X \mapsto XA$ are continuous with respect to the weak and strong operator as well as the norm topology.

Proof. Let $(X_n)_{n \in \mathbb{N}} \subset B(H)$ be a sequence. We want to show sequential continuity of both maps for the three topologies.

- if $(X_n)_{n \in \mathbb{N}}$ is norm-convergent to $X$, then $\|X_n - X\| \to 0$.
  This implies
  \[
  \|AX_n - AX\| \leq \|A\| \cdot \|X_n - X\| \to 0 \quad \forall A \in B(H)
  \]
  and
  \[
  \|X_n A - XA\| \leq \|A\| \cdot \|X_n - X\| \to 0 \quad \forall A \in B(H)
  \]

- if $(X_n)_{n \in \mathbb{N}}$ is strong operator convergent to $X$, then $\|(X_n - X)v\| \to 0$ for every $v \in H$.
  This implies
  \[
  \|(AX_n - AX)v\| \leq \|A\| \cdot \|(X_n - X)v\| \to 0 \quad \forall A \in B(H), \forall v \in H
  \]
  and
  \[
  \|(X_n A - XA)v\| = \|(X_n - X)(Av)\| \to 0 \quad \forall A \in B(H) \forall v \in H
  \]

- if $(X_n)_{n \in \mathbb{N}}$ is weak operator convergent to $X$, then $\langle (X_n - X)v, w \rangle \to 0$ for every $v, w \in H$.
  This implies
  \[
  \langle (AX_n - AX)v, w \rangle = \langle (X_n - X)v, A^*w \rangle \to 0 \quad \forall A \in B(H), \forall v, w \in H
  \]
  and
  \[
  \langle (X_n A - XA)v, w \rangle = \langle (X_n - X)(Av), w \rangle \to 0 \quad \forall A \in B(H) \forall v, w \in H
  \]

\[\square\]
Unlike the maps in the previous lemma, the map $X \mapsto X^n$ will not be continuous in general. But at least, this will still be true for the norm topology:

**Lemma 1.8.**
For any polynomial $p$, $B(H) \to B(H)$, $X \mapsto p(X)$ is continuous with respect to the norm topology.

**Proof.** Since every polynomial is a finite linear combination of monoms $X \mapsto X^n$, it suffices to show continuity for those maps only.

Again, we investigate sequential continuity and proceed by induction the case $n = 1$ being the assumption.

Hence, we assume, that $\lim_{k \to \infty} X_k = X$.

Then, the sequence is norm-bounded and we see, using the induction assumption, that

$$\left\| (X_k + X)(X_k^{n-1} - X^{n-1}) \right\| \leq C \left\| X_k^{n-1} - X^{n-1} \right\| \to 0$$

for some constant $C$ and as $k$ goes to infinity.

This just means, that

$$\lim_{k \to \infty} ((X_k + X)(X_k^{n-1} - X^n)) = 0 \quad \text{(3)}$$

But this implies,

$$\lim_{k \to \infty} (X_k^n - X^n) = \lim_{k \to \infty} \left( (X_k + X)(X_k^{n-1} - X^{n-1}) - X_k X^{n-1} + X X_k^{n-1} \right)$$

$$= \lim_{k \to \infty} \left( (X_k + X)(X_k^{n-1} - X^{n-1}) \right) + \lim_{k \to \infty} \left( X_k X^{n-1} - X X_k^{n-1} \right)$$

$$= 0 \quad \text{by equation (3)}$$

Therefore (the set of polynomials is dense in the algebra of continuous functions), the continuous spectral calculus is continuous with respect to the norm topology:
Corollary 1.9.
If \((X_n)_{n \in \mathbb{N}}\) is a sequence in the space of normal operators on \(H\) converging to \(X\) in norm, \(f(X_n)\) norm-converges to \(f(X)\) for any function \(f\), which is continuous and bounded on the union of all spectra involved.

Proof. Using the Stone-Weierstrass Theorem, we find polynomials \(p_n\) such that \(\|p_n - f\|_\infty \to 0\) as \(n\) tends to infinity and this convergence is uniform on the union of all spectra involved.

By the continuity of the continuous spectral calculus for \(Y \in \{X, X_n | n \in \mathbb{N}\}\), one has

\[
\lim_{n \to \infty} \|p_n(Y) - f(Y)\| = 0
\]  

(4)

But then

\[
\|f(X_\alpha) - f(X)\| = \|f(X_\alpha) - p_n(X_\alpha)\| + \|p_n(X_\alpha) - p_n(X)\| + \|p_n(X) - f(X)\| = 0
\]

\(\square\)

Remark 1.10.
By essentially the same arguments, one can prove the continuity of the continuous spectral calculus with respect to the strong operator topology on \(B(H)\).
But this will not be used in this thesis.

Theorem 1.11 (Theorem 5.1.3 in [KR83]).
Closed norm balls

\[
B_{\lambda} := \{A \in B(H) | \|A - B\| \leq \lambda\}
\]

are compact with respect to the weak operator topology.

This means in particular, that every norm-bounded sequence in \(B(H)\) has a subsequence, which is weak operator convergent. The limit point will be bounded by the same bound as the sequence.
Finally, closed norm balls as defined above are closed sets with respect to any of the above topologies (this follows from the fact, the weak operator topology is the weakest of the above topologies and that in Hausdorff spaces compact sets are closed).
2 Unitarisability

2.1 Unitarisable representations and unitarisable groups

This thesis is mainly concerned with the concept of unitarisability, which, intuitively speaking, means the existence of an invariant inner product to a given representation of a group on a Hilbert space $H$ (we tacitly assume that every Hilbert space is separable, Remark 1.3).

In this chapter, we shall report (by far not completely) on the history of this term. The reader may also be referred to Pisier’s article [Pis05] or, more recently, [Oza06].

**Definition** (unitarisable representation).

A group representation $\pi : G \rightarrow \text{Aut}(H)$ on a Hilbert space $H$ is said to be unitarisable, if there exists an invertible operator $S \in \text{Aut}(H)$, such that $S^{-1}\pi(g)S$ is a unitary for every $g \in G$.

Those operators $S \in \text{Aut}(H)$ shall be called a unitariser for $\pi$. The set of all such unitarisers will be denoted by $U(\pi)$.

**Remark 2.1.**

The map $A \mapsto B^{-1}AB$ is called conjugation of $A$ by $B$. So, unitarisability of a group representation says, that the image of the representation is conjugate to a subgroup of the group $U(H)$ of unitary operators on $H$.

The following lemma states, that this definition fits the intuitive idea. The reader may first be reminded of the following definition

**Definition** (invariant inner product).

Let $\pi$ be a linear representation of a group $G$ on a Hilbert space $H$. Then, we call an inner product $(\cdot,\cdot)$ on $H$ $G$-invariant, if

$$(u,v) = (\pi(g)u, \pi(g)v) \forall u,v \in H, \forall g \in G$$
Lemma 2.2.
The following are equivalent for a linear representation $\pi$ of a group $G$ on a Hilbert space $(H, \langle \cdot, \cdot \rangle)$:

- $\pi$ is unitarisable
- there exists a $G$-invariant inner product on $H$, which induces the same topology.

Proof. Let $\langle \cdot, \cdot \rangle$ be the ambient inner product on $H$ and let $(\cdot, \cdot)$ be the $G$-invariant inner product on $H$, which induces the same topology.

We choose orthonormal systems $E := \{e_i, i \in I\}$ for $(\cdot, \cdot)$ and $F := \{f_j, j \in J\}$ for $\langle \cdot, \cdot \rangle$ for both inner products and choose some bijection $\varphi : I \rightarrow J$.

Since every element in $(H, (\cdot, \cdot))$ decomposes into a unique $\ell_2$-sum over $E$, the linear extension $S$ of $\varphi$ to $H$ is a well-defined linear map

$$S : (H, (\cdot, \cdot)) \rightarrow (H, \langle \cdot, \cdot \rangle)$$

It is surjective, since every element in $H$ also has a decomposition with respect to $F$. The injectivity follows from the linear independence of $F$ and $E$.

As $S$ maps orthonormal vectors to orthonormal vectors, it is clear that

$$(v, w) = \langle Sv, Sw \rangle \forall v, w \in H \quad (5)$$

and $S$ is a bounded homeomorphism.

Moreover, as both inner products induce the same topology on $H$, the identity operator

$$\text{id} : (H, \langle \cdot, \cdot \rangle) \rightarrow (H, (\cdot, \cdot))$$

is a linear homeomorphism.
Therefore, one easily sees for \( \tilde{S} = S \circ \text{id} : (H, \langle \cdot, \cdot \rangle) \to (H, \langle \cdot, \cdot \rangle) \), that \( \langle \cdot, \cdot \rangle \) is \( G \)-invariant

\[
\Leftrightarrow (u,v) = (\pi(g)u, \pi(g)v) \quad \forall u, v \in H, \quad \forall g \in G \\
\Leftrightarrow \langle Su, Sv \rangle = \langle S\pi(g)u, S\pi(g)v \rangle \quad \forall u, v \in H, \quad \forall g \in G \\
\Leftrightarrow \langle \tilde{S}u, \tilde{S}v \rangle = \langle \tilde{S}\pi(g)u, \tilde{S}\pi(g)v \rangle \quad \forall u, v \in H, \quad \forall g \in G \\
\Leftrightarrow \langle \tilde{S}u, \tilde{S}v \rangle = \langle \left( \tilde{S}\pi(g)\tilde{S}^{-1} \right) \tilde{S}u, \left( \tilde{S}\pi(g)\tilde{S}^{-1} \right) \tilde{S}v \rangle \forall u, v \in H, \quad \forall g \in G \\
\Leftrightarrow \langle \tilde{S}u, \tilde{S}v \rangle = \langle \left( \tilde{S}\pi(g)\tilde{S}^{-1} \right)^* \left( \tilde{S}\pi(g)\tilde{S}^{-1} \right) \tilde{S}u, \tilde{S}v \rangle \forall u, v \in H, \quad \forall g \in G \\
\Leftrightarrow \langle u,v \rangle = \langle \left( \tilde{S}\pi(g)\tilde{S}^{-1} \right)^* \left( \tilde{S}\pi(g)\tilde{S}^{-1} \right) u, v \rangle \forall u, v \in H, \quad \forall g \in G \\
\Leftrightarrow \left( \tilde{S}\pi(G)\tilde{S}^{-1} \right)^* \tilde{S}\pi(G)\tilde{S}^{-1} = \text{id}_H \forall g \in G \\
\Leftrightarrow \tilde{S}\pi(G)\tilde{S}^{-1} \subset U(H, \langle \cdot, \cdot \rangle) \forall g \in G \\
\Leftrightarrow G \text{ is unitarisable with respect to } \langle \cdot, \cdot \rangle
\]

where \( A^* \) denotes the operator, which is adjoint to \( A \) with respect to \( \langle \cdot, \cdot \rangle \). \( \Box \)

**Notation.** With subgroups of \( B(H) \), we will always mean multiplicative subgroups of \( B(H) \).

Given a unitary representation \( \pi \) of some group \( G \) on a Hilbert space \( H \) with norm \( \| \cdot \| \) on \( H \), it is obvious, that for any bounded invertible operator \( S \), the subgroup \( S\pi(G)S^{-1} \subset B(H) \) is bounded with respect to the operator norm on \( B(H) \) coming from \( \| \cdot \| \).

This motivates the following definition:

**Definition** (uniformly bounded representation, size of a representation).

A group representation \( \pi \) on a normed space is uniformly bounded, if one has

\[
\sup_{g \in G} \| \pi(g) \| < \infty
\]

Then, this bound is denoted by \( |\pi| \) and called the size of \( \pi \).
2.1 Unitarisable representations and unitarisable groups

Those two definitions led to two questions both of which will be addressed in this thesis:

(1) Which uniformly bounded representations are in fact unitarisable?
   
   In other words: if a subgroup of $B(H)$ forms a bounded subset, and is therefore only boundedly far away from $U(H)$ in the operator norm, can it be conjugated into $U(H)$?

(2) Are there groups such that every uniformly bounded representation on a Hilbert space is unitarisable?

For a finite group $G$, those questions are very easy to answer: as they are finite, all representations will automatically be uniformly bounded and averaging the scalar product with respect to the $G$-action yields an invariant scalar product.

This motivates the following definition:

**Definition (unitarisable group).**

*We call a discrete group $G$ unitarisable, if every uniformly bounded representation of $G$ on a Hilbert space is unitarisable.*

So, finite groups are unitarisable, and, by the following theorem, the group $\mathbb{Z}$ of integers also is.

**Theorem 2.3 (Sz.-Nagy, [SN47]).**

*For every invertible operator $T \in B(H)$ such that $\|T^n\| < C$ for some $C$ and all $n \in \mathbb{Z}$, there is some operator $S$, such that $STS^{-1}$ is a unitary operator.*

Amenable groups are a generalization of finite groups, which often behave similarly. They are defined in various equivalent ways. We shall give three of those definitions without proving their equivalence.
2 Unitarisability

Definition (amenable group).
A discrete group $G$ is called amenable, if one of the following equivalent conditions hold

- for any finite set $F \subset G$ and any $\varepsilon > 0$, there is a finite set $U \subset G$, such that\footnote{Here $A \Delta B$ stands for the symmetric difference $A \setminus B \cup B \setminus A$ of $A$ and $B$.}
  \[
  \frac{|U \Delta gU|}{|U|} < \varepsilon \ \forall g \in F
  \]

- any action of $G$ by continuous affine transformations on a locally convex topological vector space leaving invariant a compact convex subset $C$ has a fixed point inside $C$.

- there is a $G$-invariant finitely-additive probability measure on $G$.

With the help of the third condition, one can average the scalar product of a Hilbert space even with respect to the action by an amenable group.

This was first observed independently around 1950 by M. Day, J. Dixmier and Nakamura/Takeda:

Theorem 2.4 ([DMM50],[Dix50],[NT51]).
Amenable groups are unitarisable.

As one of the main results of this thesis, we will give a geometric proof to this theorem (Corollary 8.16).
2.2 Non-unitarisable groups and Dixmier’s question

In 1955, Ehrenpreis and Mautner ([EM55]), and later Kunze and Stein ([KS60]) showed, that $\text{SL}_2(\mathbb{R})$ is not unitarisable. Here, $\text{SL}_2(\mathbb{R})$ is seen as a Lie group. In particular, it is not a discrete group. For such groups, one defines unitarisability to mean that every continuous, uniformly bounded representation is unitarisable.

Of course, a counterexample to this definition would also be a representation of the underlying discrete group and hence the non-unitarisability of $\text{SL}_2(\mathbb{R})$ seen as a Lie group, implies, that it also is non-unitarisable as a discrete group.

As with all properties for groups, one may wonder, under which constructions unitarisability is stable. On this, there is the following theorem:

**Theorem 2.5.**
The following three properties hold for discrete countable groups $G$

1. If $G$ is unitarisable and $\Gamma < G$ is a some subgroup, then $\Gamma$ is unitarisable.

2. If $\Gamma < G$ is a normal subgroup and $G/\Gamma$ is non-unitarisable, then $G$ is not unitarisable.

3. If a $\Gamma < G$ is a normal unitarisable subgroup and either $\Gamma$ or $G/\Gamma$ is amenable, $G$ is unitarisable.

4. If $\Gamma < G$ is a unitarisable subgroup of finite index, $G$ is unitarisable.

**Proof.** The first two points are Proposition 0.5 in [Pis05]. Point three is mentioned in the proof of the same proposition, where it is attributed to Nagisa and Wada ([NW99]) and the last point is Corollary 8.21.
One could use the first example of a non-unitarisable group and the preceding theorem to prove the following

**Theorem 2.6.**

*Any group* $G$ *containing a non-abelian free subgroup is not unitarisable.*

But, in order to illustrate this topic in a better way, we will present the construction of an explicit non-unitarisable, uniformly bounded representation of the free group $F_2$ on two generators, $a$ and $b$ say.

This is, by definition the set of all words in letters from $A := \{a, a^{-1}, b, b^{-1}\}$, which are reduced, i.e. there are no two consecutive letters which are opposite powers of the same generator. The empty word is the neutral element and concatenation (and, if necessary reduction) is the multiplication in $F_2$.

For $g \in F_2$, we denote by $|g|$ the **word length** of $a$, which is defined to be the unique number $n \in \mathbb{N}$ such that $g$ can be written as a reduced word of $n$ letters from $A$.

The following construction is due to Pytlik and Szwarc and was published in a series of two papers [PS86] and [Szw88].

**Definition ($\ell^2(F_2)$).**

We define $\ell^2(F_2)$ to be the complex vector space of all square summable functions on $F_2$:

$$\ell^2(F_2) = \left\{ f : F_2 \to \mathbb{C} \mid \sum_{g \in F_2} |f(g)|^2 < \infty \right\}$$

This is a Hilbert space with the inner product

$$\langle f, f' \rangle = \sum_{g \in F_2} f(g)\overline{f'(g)}$$

The set $\{\delta_g, g \in F_2\}$ of functions $\delta_g$ which equals 1 on $g$ and 0 on $h \neq g$ is an orthonormal basis of this space.

On $\ell^2(F_2)$, $F_2$ acts naturally by,

$$\lambda(g)(f) : F_2 \to \mathbb{C}, \; h \mapsto f\left(g^{-1}h\right)$$
This way \( g \) maps \( \delta_h \) to \( \delta_{gh} \) hence mapping an orthonormal basis to itself.

Therefore, every \( g \in F_2 \) defines a unitary operator on \( \ell^2(F_2) \) and

\[
\lambda : F_2 \rightarrow B(\ell^2(F_2))
\]

is a unitary representation, called the left regular representation.

Define on \( \ell^2(F_2) \) the linear operator \( P \), which is defined by \( \delta_g \mapsto \delta_{\bar{g}} \), where \( \bar{g} \in F_2 \) is derived from \( g \) (written in a reduced way) by deleting the last letter. The vector \( \delta_e \) is mapped to 0.

In [Szw88], it is now shown, that \( P \) has the following rather straightforward spectral properties:

**Lemma 2.7** ([Szw88], chapter 2).

The operator \( P : \ell^2(F_2) \rightarrow \ell^2(F_2) \) has the following properties:

- \( \|P^n\| = \sqrt{4 \cdot 3^{n-1}} \), in particular \( \|P\| = 2 \) and the spectral radius is \( \sqrt{3} \).

- If one defines \( \chi_n := \sum_{|g|=n} \delta_g \) (the characteristic function on the set of all words of length \( n \)), the function

\[
f_z = \frac{4}{3} \delta_e + \sum_{n=1}^{\infty} \left( \frac{z}{3} \right)^n \chi_n
\]

lies in \( \ell^2(F_2) \) for any \( z \in \mathbb{C} \) with \( |z| < \sqrt{3} \).

It can easily be seen to be an eigenfunction with eigenvalue \( z \). Hence, the spectrum of \( P \) coincides with the closed disc \( D_{\sqrt{3}} \) of all complex numbers with absolute value \( \leq \sqrt{3} \).

Moreover, since \( P^n f = 0 \) for every finitely supported \( f \in \ell^2(F_2) \) and \( n \in \mathbb{N} \) large enough, the sequence

\[
\sum_{n=0}^{\infty} z^n P^n f
\]

converges for every finitely supported \( f \) and hence \( 1 - zP \) is bijective as an operator on the space \( V \) of finitely supported \( f \in \ell^2(F_2) \) and for every \( z \in \mathbb{C} \).
2 Unitarisability

One now defines the following family of representations of $F_2$ one the space $V$:

**Definition** ($\pi_z$, [PS86]).

Let $\lambda$ be the left regular representation of $F_2$ on $\ell^2(F_2)$. Then $\lambda$ restricts to a representation of $F_2$ on $V$ and one defines for $|z| < 1$

$$\pi_z(g) : V \to V, f \mapsto (1 - zP)^{-1}\lambda(g)(1 - zP)(f)$$

Now, on $V$, one can write $\pi_z(g)$ as the following sum:

$$\pi_z(g) = (1 - zP)^{-1}\lambda(g)(1 - zP)$$

$$= \left( \sum_{n=0}^{\infty} (zP)^n \right) \lambda(g)(1 - zP)$$

$$= \left( 1 + \sum_{n=0}^{\infty} z^{n+1}P^{n+1} \right) \lambda(g)(1 - zP)$$

$$= \lambda(g) + \left( \sum_{n=0}^{\infty} z^{n+1}P^{n+1} \right) \lambda(g)(1 - zP) - \lambda(g)zP$$

$$= \lambda(g) + \left( \sum_{n=0}^{\infty} z^{n}P^{n} \right) zP\lambda(g) - \left( \sum_{n=0}^{\infty} z^{n}P^{n} \right) \lambda(g)zP$$

$$= \lambda(g) + \left( \sum_{n=0}^{\infty} z^{n+1}P^{n} \right) (P\lambda(g) - \lambda(g)P)$$

In particular

$$\pi_z(g)(f) = \pi_z(g) \left( \lambda \left( g^{-1} \right) \lambda(g)f \right)$$

$$= \left( \lambda(g) + \left( \sum_{n=0}^{\infty} z^{n+1}P^{n} \right) (P\lambda(g) - \lambda(g)P) \right) \lambda(g^{-1}) \lambda(g)f$$

$$= \left( 1 + \left( \sum_{n=0}^{\infty} z^{n+1}P^{n} \right) (P - \lambda(g)P\lambda(g^{-1})) \right) \lambda(g)f$$

Now one easily sees, that the image of $P - \lambda(g)P\lambda(g^{-1})$ lies in the space of functions, which is only supported on the finite set $V_g := \{P^ng, \ n \leq |g|\}$ (i.e. all words that one gets from $g$ by deleting the last $n$ letters, $n \leq |g|$):
2.2 Non-unitarisable groups and Dixmier’s question

In fact, $P\delta \equiv \lambda(g)P\lambda(g^{-1})\delta$, if the multiplication of $h$ with $g^{-1}$ on the left does not delete all letters from $h$ (because then, both $P$ and $\lambda(g)P\lambda(g^{-1})$ map $\delta$ to $\delta$). Hence, the difference can only be non-zero, if the multiplication of $h$ with $g^{-1}$ on the left “kills” $h$.

In other words $h = P^kg$ for some $k \leq |g|$ (here, we also interpret $P$ as a map from $F_2$ to $F_2$ in the obvious way).

By a simple calculation, one sees, that on $V_g$ the operators $P$ and $\lambda(g)P\lambda(g^{-1})$ act by so-called shift operators:

\[
(P(f))(P^kg) = \begin{cases} 
  f(P^{k+1}g) & k < |g| \\
  0 & k = |g|
\end{cases}
\]

\[
(\lambda(g)P\lambda(g^{-1})(f))(P^kg) = \begin{cases} 
  f(P^{k-1}g) & k > 1 \\
  0 & k = 1
\end{cases}
\]

Therefore, $P - \lambda(g)P\lambda(g^{-1})$ has norm 2 and on its image $V_g$, $P$ acts as a contraction. This implies, that

\[
\|\pi_z(g)(f)\| = \|\pi_z(g)(\lambda(g^{-1})\lambda(g)f)\| \\
\leq \left(1 + 2z\sum_{n=0}^{\infty} |z|^{n+1}\right)\|\lambda(g)f\| \\
= \left(1 + 2\frac{z}{1-z}\right)\|f\| \\
= \frac{1 + z}{1 - z}
\]

Hence, $\pi_z$ is a uniformly bounded representation of $F_2$ on $V$. Therefore, it uniquely extends to a uniformly bounded representation of $F_2$ on $\ell^2(F_2)$.

Besides, also $\pi_z^*(g) = \pi_z(g^{-1})$ holds ([PS86], Theorem 1).

Obviously, for $|z| < \frac{1}{\sqrt{3}}$, $\pi_z$ is just $\lambda$ conjugated by the invertible operator $(1 - zP)$. But it turns out, that for $1 > |z| > \frac{1}{\sqrt{3}}$, $\pi_z$ is not unitarisable:

This follows from two facts:

1. The subspace $\ker(1 - zP)$ is $\pi_z$-invariant ([Szw88], Theorem 3)
(2) Up to scalar multiples, the function \( f_{z^{-1}} \) defined in (6) is the only radial function (i.e. a function which only depends on the word length of its argument) in \( \ker(1 - zP) \):

First of all, by Lemma 2.7, \( f_{z^{-1}} \) lies in \( \ker(1 - zP) = \ker(z^{-1} - P) \) and the uniqueness up to scalar multiples follows by induction:

Having two radial functions \( f_1, f_2 \) in the kernel of \( 1 - zP \), such that \( \lambda f_1 \neq f_2 \ \forall \lambda \in \mathbb{C} \), we find some \( \mu \in \mathbb{C} \) such that \( f_1 - \mu f_2 \in \ker(1 - zP) \) vanishes on \( e \in F_2 \).

This is then easily seen to imply, that it also vanishes on all \( g \in F_2 \) such that \( |g| = 1 \).

Inductively, one shows that \( f_1 - \mu f_2 \) vanishes.

**Remark 2.8.**

Here, \(|z| > \frac{1}{\sqrt{3}}\) is used implicitly, since in the other cases, \( f_{z^{-1}} \) does not lie in \( \ell^2(F_2) \).

Those two facts together with the observation, that \( \sum_{|g|=1} \lambda(g) \), and \( P \) map radial functions to radial functions, shows, that \( f_{z^{-1}} \) is an eigenfunction of

\[
\sum_{|g|=1} \pi_z(g)
\]

In Lemma 3 in [Szw88], it is calculated, that the corresponding eigenvalue is

\[
\frac{3z^2 + 1}{z} =: \alpha
\]

Here, they use the property, that \( \pi_z^*(g) = \pi_z(g^{-1}) \) holds ([PS86], Theorem 1).

Now, for \( z \) with non-vanishing imaginary part, also \( \text{Im} \alpha \neq 0 \).
This implies, that $\pi_z$ is non-unitarisable: if $\pi_z$ were unitarisable (say, $S\pi_z(g)S^{-1}$ is unitary for every $g$), then

$$
\left( \sum_{|g|=1} S\pi_z(g)S^{-1} \right)^* = (S\pi_z(a)S^{-1} + S\pi_z(b)S^{-1} + S\pi_z(a^{-1})S^{-1} + S\pi_z(b^{-1})S^{-1})^*
$$

$$
= ((S\pi_z(a)S^{-1})^* + (S\pi_z(b)S^{-1})^* + (S\pi_z(a^{-1})S^{-1})^* + (S\pi_z(b^{-1})S^{-1})^*)
$$

$$
= (S\pi_z(a^{-1})S^{-1} + S\pi_z(b^{-1})S^{-1} + S\pi_z(a)S^{-1} + S\pi_z(b)S^{-1))^*
$$

$$
= \sum_{|g|=1} S\pi_z(g)S^{-1}
$$

Hence, $\sum_{|g|=1} S\pi_z(g)S^{-1}$ is self-adjoint and has real spectrum.

But $g_z := Sf_{z^{-1}}$ is an eigenfunction of $\sum_{|g|} S\pi_z(g)S^{-1}$ with eigenvalue $\alpha$ yielding the desired contradiction.

Thus, $F_2$ is non-unitarisable. By Theorem 2.5, it follows that every group $G$ containing a non-abelian free group, will automatically be non-unitarisable.

It was in fact a long-standing question (known as the von-Neumann conjecture), whether every non-ameanable group contains a non-abelian free subgroup.

This conjecture was proven to be wrong by Olshanskii in [Ols80].

Later, by Epstein and Monod in [EM09] (using a construction by Osin, [Osi09]) and by Monod and Ozawa in [MO10] shortly after, the existence of non-unitarisable groups without non-abelian free subgroups was proven.

All this motivates a question that was first asked by Dixmier himself:

**Does unitarisability imply amenability?**

This question has attained much attention since then. Two particular results by G. Pisier on unitarisability of groups shall be stated here, since they will be central for this thesis.
2 Unitarisability

The first theorem states, that for a given unitarisable group, the “size” of a unitariser can be chosen to be universally small with respect to the size $|\pi|$ of a uniformly bounded representation $\pi$.

**Theorem 2.9 ([Pis05]).**

For a unitarisable group $G$, there are universal constants $K$ and $\alpha \in \mathbb{R}_+$ depending only on $G$, such that for every uniformly bounded representation $\pi$ of $G$ on some Hilbert space $H$, the following holds

$$\exists S \in \mathcal{U}(\pi) : \|S\| \cdot \|S^{-1}\| \leq K \cdot |\pi|^\alpha$$

The second theorem relates those constants to the concept of amenable groups:

**Theorem 2.10 ([Pis99]).**

The following are equivalent for a discrete group $G$

(1) $G$ is amenable

(2) the universal constants $K$ and $\alpha$ in Theorem 2.9 may be chosen to be $K = 1$ and $\alpha = 2$.

For more details the reader may be referred to [Pis05], [Pis01] or, more recently [EM09].
3 A geometric approach to unitarisability

In this chapter, we will relate the unitarisability of a group representation \( G \to B(H) \) to a fixed point property for an induced action of the same group on \( \mathcal{P}(H) \). This setup will enable us to show, that there are always “smallest” unitarisers for a given representation.

3.1 Unitarisability and fixed points

An action of a group is a notion generalizing representations (which are linear actions on a vector space):

**Definition** (Group action).

An action \( \rho \) of a discrete group \( G \) on a topological space \( X \) is a homomorphism \( \rho : G \to \text{Homeo}(X) \), between the group itself and the group of homeomorphisms of \( X \).

Equivalently, it is a continuous mapping \( \rho : G \times X \to X \) such that

\[
\rho(e, x) = x \ \forall x \in X \\
\rho(gh, x) = \rho(g, \rho(h, x)) \ \forall g, h \in G, \ \forall x \in X
\]

where \( e \in G \) denotes the neutral element.

For a given representation \( \pi \) of a group \( G \) on a Hilbert space \( H \), we can define the following action of \( G \) on \( \mathcal{P}(H) \):

\[
\rho_\pi : G \times \mathcal{P}(H) \to \mathcal{P}(H), \ (g, P) \mapsto \pi(g)P\pi(g)^*\]

One easily sees, that this does in fact define an action of \( G \) on \( \mathcal{P}(H) \):

Firstly, one has

\[
\rho_\pi(e, x) = \pi(e)x\pi(e)^* = \text{id}_H x \text{id}_H^* = x
\]

and secondly

\[
\rho_\pi(gh, P) = \pi(gh)P\pi(gh)^* \\
= \pi(g)\pi(h)P(\pi(g)\pi(h))^* \\
= \pi(g)(\pi(h)P\pi(h)^*)\pi(g)^* \\
= \pi(g)\rho_\pi(h, x)\pi(g)^* \\
= \rho_\pi(g, \rho_\pi(h, x))
\]
Definition (fixed point).
For a \( G \)-action \( \rho \) on a topological space \( X \), a point \( x \in X \) is called a fixed point (or \( G \)-fixed point), if

\[
\rho(g, x) = x \ \forall g \in G
\]

The set of all \( G \)-fixed points inside \( X \) is denoted by \( X^G \).

The following lemma connects \( G \)-fixed points of \( \rho_{\pi} \) with operators in \( \mathcal{U}(\pi) \):

Lemma 3.1.
For every \( S \in \mathcal{U}(\pi) \), \( SS^* \) is a fixed point of \( \rho_{\pi} \). Conversely, for any fixed point \( T \) of \( \rho_{\pi} \) one has \( \sqrt{T} \in \mathcal{U}(\pi) \).

Proof.
This is a straightforward calculation:

\[
S \in \mathcal{U} \iff S^{-1} \pi(g) S \in U(H) \ \forall g \in G
\]

\[
\iff \text{id}_H = (S^{-1} \pi(g) S) (S^{-1} \pi(g) S)^*
\]

\[
= S^{-1} \pi(g) SS^* \pi(g)^* (S^{-1})^* \ \forall g \in G
\]

\[
\Rightarrow SS^* = \pi(g) SS^* \pi(g)^* = \rho_{\pi}(g, SS^*) \ \forall g \in G
\]

and conversely, for any \( g \in G \)

\[
T = \rho_{\pi}(g, T) \Rightarrow T = \pi(g) T \pi(g)^*
\]

\[
\Rightarrow \sqrt{T^2} = \pi(g) \sqrt{T^2} \pi(g)^*
\]

\[
\Rightarrow \text{id}_H = \sqrt{T^{-1}} \pi(g) \sqrt{T} \sqrt{T^*} \pi(g)^* \left(\sqrt{T^*}\right)^{-1} \left(\sqrt{T^*} = \sqrt{T}\right)
\]

\[
= \sqrt{T^{-1}} \pi(g) \sqrt{T} \left(\sqrt{T^{-1}} \pi(g) \sqrt{T}\right)^*
\]

\[
\Rightarrow \sqrt{T} \in \mathcal{U}(\pi)
\]

\[ \boxdot \]
3 A geometric approach to unitarisability

We have actually proven another result “en passant”:

**Lemma 3.2.**

A uniformly bounded representation $\pi$ is unitarisable, if and only if there is a positive and invertible bounded operator, which conjugates $\pi$ to a unitary representation.

**Proof.** The “if”-direction is clear from the definition of unitarisibility. Now, assuming $\pi$ to be unitarisable, there is an invertible operator $S$, which conjugates the image of $\pi$ into $U(H)$ and by the preceding lemma also $\sqrt{SS^*} \in \mathcal{U}(\pi)$ does so.

But as $SS^*$ is positive and invertible and the square root maps $\mathbb{R}^+$ to $\mathbb{R}^+$, $\sqrt{SS^*}$ is itself positive and invertible. \hfill \Box

### 3.2 Existence of smallest unitarisers

In order to prove Theorem 2.9, we need to look for some $S \in \mathcal{U}(\pi)$ that is small when considering $\|S\| \cdot \|S^{-1}\|$. Therefore, it makes sense to give a name to this quantity:

**Definition (Size of an operator).**

We will refer to $\|S\| \cdot \|S^{-1}\|$ as the size $s(S)$ of $S$ for $S \in \text{Aut}(H)$.

**Remark 3.3.**

Since

$$s(\lambda S) = \|\lambda S\| \cdot \|(\lambda S)^{-1}\| = |\lambda| \|S\| \cdot |\lambda|^{-1} \|S^{-1}\| = s(S) \forall \lambda \in \mathbb{C} \setminus \{0\}$$

we see, that the size of an operator is constant with respect to scaling $A$ with any complex number different to 0.

In the same way, scaling maps the set $\mathcal{U}(\pi)$ to itself:

$$A \in \mathcal{U}(\pi) \iff A^{-1}\pi(g)A \in U(H) \forall g \in G$$

$$\iff \lambda^{-1}A^{-1}\pi(g)\lambda A \in U(H) \forall g \in G \forall \lambda \in \mathbb{C} \setminus \{0\}$$

$$\iff \lambda A \in \mathcal{U}(\pi) \forall \lambda \in \mathbb{C} \setminus \{0\}$$
So, the size of an operator \( S \) is, for example, the norm of the scalar multiple of \( S \) such that its inverse has norm 1.

Then, for \( S \) positive and invertible, by equation (1).

\[
\| S^{-1} \| = (\min(\sigma(S)))^{-1}
\]

and

\[
\| S \| = \max \sigma(S)
\]

In other words, an operator \( S \) with an inverse of norm 1 has

\[
\sigma(S) \subset [1, \| S \|] \quad \text{and} \quad s(S) = \| S \|
\]

So, looking for a unitariser with small size means searching unitarisers with “thin” spectrum.

The following lemma shows the existence of smallest unitarisers for any unitarisable representation.

**Lemma 3.4.**

*Let \( \pi \) be a unitarisable representation, then*

\[
\inf_{S \in \mathcal{U}(\pi)} s(S) = \min_{S \in \mathcal{U}(\pi)} s(S)
\]

*Proof. Using Lemma 3.2, the mapping*

\[
\mathcal{U}(\pi) \rightarrow \mathcal{P}(H) \rightarrow \mathcal{U}(\pi) \cap \mathcal{P}(H)
\]

\[
S \mapsto SS^* \mapsto \sqrt{SS^*}
\]

*maps elements in \( \mathcal{U}(\pi) \) to positive elements in \( \mathcal{U}(\pi) \).

Moreover, since \( SS^* \) is positive, one gets

\[
s\left(\sqrt{SS^*}\right) = \left\| \sqrt{SS^*} \right\| \left\| \left(\sqrt{SS^*}\right)^{-1} \right\|
\]

\[
= \|SS^*\|^\frac{1}{2} \left\| (SS^*)^{-\frac{1}{2}} \right\|
\]

\[
= \|S\| \cdot \| (S^*)^{-1} S^{-1} \|^\frac{1}{2}
\]

\[
= \|S\| \cdot \| (S^*)^{-1} \|^2 \cdot \|S^{-1} \|
\]

\[
= s(S)
\]
which means, that for any unitariser in $\mathcal{U}(\pi)$ there is a unitariser in $\mathcal{U}(\pi)$, which is positive and of same size.

So, when looking for small unitarisers, one can assume without loss of generality, that an operator in $\mathcal{U}(\pi)$ is positive and in particular self-adjoint.

Now, by Lemma 3.1, there is a size-squaring bijection $\mathcal{U}(\pi) \cap \mathcal{P}(H) \rightarrow \mathcal{P}(H)^G$, $S \mapsto S^2$ between $\mathcal{U}(\pi) \cap \mathcal{P}(H)$ and the set of fixed points of $\rho_{\pi}$.

The monotonicity of the map $\mathbb{R}_+ \rightarrow \mathbb{R}_+, x \mapsto x^2$ implies now, that the claim is equivalent to predicting the existence of some operator $T$ in the convex and norm-closed set $\mathcal{P}(H)^G$, which minimizes the size.

Finally, for any $T \in \mathcal{P}(H)^G$ and $g \in G$

$$\rho_{\pi}(g, \lambda T) = \pi(g) (\lambda T) \pi(g)^* = \lambda (\pi(g) T \pi(g)^*) = \lambda T \forall \lambda \in \mathbb{R}_+$$

Hence, $\mathcal{P}(H)^G$ is closed under multiplication with positive scalars, which preserves both, size and positivity.

Define $\mathcal{P}(H)^G_1$ to be the set $\mathcal{P}(H)^G \cap \{A : \|A\| = 1\} \subset \mathcal{P}(H)^G$ of those fixed points of $G$, which have norm 1.

Taking together all the facts from above, we have reduced the claim to

$$\inf_{T \in \mathcal{P}(H)^G_1} s(T) = \min_{T \in \mathcal{P}(H)^G_1} s(T)$$

Using continuous spectral calculus and the fact, that for positive invertible operators of norm 1, the spectrum $\sigma(T)$ lies in $(0, 1]$, one easily gets

$$s(T) = \|T\| \cdot \|T^{-1}\| = \|T^{-1}\| = \frac{1}{\min \sigma(T)}$$

In other words, we want to show, that there is an operator $T_{\pi}$ such that

$$\min \sigma(T_{\pi}) = \sup_{T \in \mathcal{P}(H)^G_1} \min \sigma(T)$$

Now, for $\|T\| = 1$, one has $\sigma(T) \subset [\min(\sigma(T)), \|T\|] \subset (0, 1]$ and therefore

$$\min(\sigma(T)) = 1 - \max(1 - \sigma(T))$$

$$= 1 - \max(\sigma(\text{id}_H - T))$$

$$= 1 - \| \text{id}_H - T\|$$
So, searching an operator $T$ with $\min \sigma(T)$ being supremal within $\mathcal{P}(H)^G_1$ is the same as looking for an operator $T \in \mathcal{P}(H)^G_1$ that minimizes the linear distance $\| \text{id}_H - T \|$ from $T$ to $\text{id}_H \in \mathcal{P}(H)$ and therefore realizes the distance from $\text{id}_H$ to $\mathcal{P}(H)^G_1$, which we will denote by $\kappa \in (0,1)$.

Let $(T_i)_{i \in \mathbb{N}}$ be a sequence in $\mathcal{P}(H)^G_1$, such that

$$\lim_{n \to \infty} \| 1 - T_n \| = \kappa$$

We have $\mathcal{P}(H)^G_1 \subset \{ A \in B(H) : \| A \| \leq 1 \}$ and the latter set is compact with respect to the weak operator topology (Theorem 1.11).

So, after going over to a subsequence (which, for simplicity, we will also denote by $(T_n)_{n \in \mathbb{N}}$), one can assume that

$$\lim_{i \to \infty} T_i = T_\pi \in \{ A : \| A \| \leq 1 \}$$

where the limit is taken with respect to the weak operator topology.

We claim, that $T_\pi$ has the demanded properties.

Firstly

$$\| T_\pi \| = \sup_{\| x \| = \| y \| = 1} \langle T_\pi x, y \rangle = \sup_{\| x \| = \| y \| = 1} \lim_{i \to \infty} \langle T_i x, y \rangle \leq \sup_{\| x \| = \| y \| = 1} \lim_{i \to \infty} \| T_i \| \cdot \| x \| \cdot \| y \| = 1$$

and

$$\langle T_\pi x, x \rangle = \lim_{i \to \infty} \langle T_i x, x \rangle \geq 0$$

show that $T_\pi$ is a positive operator of norm $\leq 1$. 

---

3.2 Existence of smallest unitarisers
Due to
\[
\langle T_\pi x, y \rangle = \lim_{i \to \infty} \langle T_i x, y \rangle = \lim_{i \to \infty} \langle \rho_\pi(g, T_i)x, y \rangle = \langle T_\pi \pi(g)^*x, \pi(g)^*y \rangle = \langle \rho_\pi(g, T_\pi)x, y \rangle \quad \forall g \in G, \forall x, y \in H
\]
it is $G$-invariant and with
\[
\|T_\pi - \text{id}_H\| = \sup_{\|x\|=\|y\|=1} |\langle T_\pi x, y \rangle - \langle x, y \rangle| = \sup_{\|x\|=\|y\|=1} \lim_{i \to \infty} |\langle T_i x, y \rangle - \langle x, y \rangle| \leq \sup_{\|x\|=\|y\|=1} \lim_{i \to \infty} \|T_i - \text{id}_H\| \cdot \|x\| \cdot \|y\| = \sup_{\|x\|=\|y\|=1} \kappa \cdot \|x\| \cdot \|y\| = \kappa
\]
the distance of $T_\pi$ to 1 is smaller than or equal to $\kappa$.

In particular, this shows, that
\[
\sigma(T_\pi - 1) \subset [-\kappa, \kappa] \Rightarrow \sigma(T_\pi) \subset [1 - \kappa, 1] \subset (0, 1]
\]
Here we used the facts, that $\|T_\pi\| \leq 1$ and $\|\text{id}_H - T\| \leq \kappa$.

Therefore, $T_\pi$ is invertible and it remains to show, that
\[
T_\pi \in \mathcal{P}(H)_1^G
\]
Assume for contradiction, that $\|T_\pi\| < 1$, which means $\sigma(T_\pi) \subset [a, b]$ with
\[
0 < 1 - \kappa \leq a < b < 1
\]
If we now normalize $T_\pi$ by multiplying with $b^{-1} > 1$, this will yield an operator $T$ of norm 1, which is still positive, invertible and $G$-invariant, hence
\[
T \in \mathcal{P}(H)_1^G
\]
3.2 Existence of smallest unitarisers

Then, $T$ will have

$$\sigma(T) \subset [ab^{-1}, 1]$$

But this implies

$$\|1 - T\| = 1 - ab^{-1} < 1 - a = \|1 - T_\pi\| \leq \kappa$$

which is the desired contradiction. \qed

**Definition.**

For a unitarisable representation $\pi$ of $G$, some $t \in [0,1]$ and a chosen smallest and positive unitariser $S$ of $\pi$, we define $\pi_t$ to be the representation defined by

$$\pi_t : g \mapsto S^{-t} \pi(g) S^t$$

**Remark 3.5.**

This is in fact a representation:

$$\pi_t(gh) = S^{-t} \pi(gh) S^t = S^{-t} \pi(g) S^t S^{-t} \pi(h) S^t = \pi_t(g) \pi_t(h)$$

$$\forall g, h \in G, \forall t \in [0,1]$$

and

$$\pi_t(e) = S^{-t} \pi(e) S^t = S^{-t} \text{id}_H S^t = \text{id}_H$$

where $e \in G$ is the neutral element.

**Lemma 3.6.**

Let $\pi$ be unitarisable and $S$ a smallest and positive unitariser. Then $S^{1-t}$ is a smallest unitariser for $\pi_t$.

**Proof.** Assume for contradiction the existence of some “better” unitariser $Q \in W(\pi_t)$ with $\|Q\| = 1$, such that $\|Q^{-1}\| = s(Q) \leq s(S^{1-t}) = \|S^{1-t}\|$.

Then $S^t Q$ obviously unitarises $\pi_t$ and

$$s(S^t Q) = \|S^t Q\| \cdot \|(S^t Q)^{-1}\|$$

$$\leq \underbrace{\|S^t\| \cdot \|S^{-t}\|}_{=1} \cdot \|Q\| \cdot \|Q^{-1}\|$$

$$\leq \|S^{-1}\|^{1-t} \cdot \|S^{-(1-t)}\|^{1-t}$$

$$= \|S^{-1}\| = s(S)$$

which contradicts $S$ to be the smallest unitariser of $\pi$. \qed
Corollary 3.7.
If $\alpha$ is the size of the smallest unitariser of $\pi$, then the smallest unitariser of $\pi_t$ has size $\alpha^{1-t}$.

Proof. If $S$ is the smallest unitariser of $\pi$, then, by the lemma above, $S^{1-t}$ is the smallest unitariser for $\pi_t$, and we calculate

$$s(S^{1-t}) = \|S^{1-t}\| \cdot \|S^{-1}\|^{1-t}$$
$$= \|S\|^{1-t} \cdot \|S^{-1}\|^{1-t}$$
$$= \|S\|^{1-t} \cdot \|S^{-1}\|^{1-t}$$
$$= s(S)^{1-t}$$

\qed
4 The cone of positive invertible operators

In this chapter, we will look at the cone $\mathcal{P}(H)$ of positive invertible operators from a different, more geometric, point of view using a metric topology. This will then lead to a translation of Theorems 2.9 and 2.10 into more geometric versions.

4.1 A differential geometric view on the space of positive invertible operators

In [CPR94], G. Corach, H. Porta and L. Recht allowed for a different, rather geometric view on the space $\mathcal{P}(H)$ by introducing a different, metric topology. We will introduce their results here without stating the proofs. The methods used in [CPR94] are differential geometric. Since we are only interested in the metric aspects of the results, we will only outline some parts. The reader may be referred to [CPR94] for more details.

In this section, we give a short and sketchy overview of the constructions given in [CPR94]. After this, we will put together in a more precise way the details we need for this thesis.

Definition (transitive action, continuous action).

An action of a discrete group $G$ on a topological space $X$ is called transitive, if for any $x, y \in X$ there is some $g \in G$ such that $gx = y$.

It is called continuous, if for every $g$ the induced map $g : X \to X, x \mapsto gx$ is a homeomorphism.

Definition (homogeneous space).

A topological space is called homogeneous with respect to a group $G$, if $X$ is acted upon by $G$ in a transitive and continuous way.

Example 2. Aut$(H)$ acts naturally on $\mathcal{P}(H)$ by

$$\rho : \text{Aut}(H) \times \mathcal{P}(H) \to \mathcal{P}(H), \quad \rho(S, T) = STS^*$$

(7)

This action makes $\mathcal{P}(H)$ a homogeneous space with respect to Aut$(H)$:
4.1 A differential geometric view on the space of positive invertible operators

In fact, transitivity is seen in the following way: let $S, T \in \mathcal{P}(H)$, then

$$\rho(S^{\frac{1}{2}}T^{-\frac{1}{2}}, T) = S^{\frac{1}{2}}T^{-\frac{1}{2}}TT^{-\frac{1}{2}}S^{\frac{1}{2}} = S$$

The continuity of the map $T \mapsto ATA^*$ for some $A \in \text{Aut}(H)$ is obvious.

Moreover, as a topological space, $\mathcal{P}(H)$ comes as an open subspace of the real vector space $\mathcal{S}(H)$ of selfadjoint operators on $H$. Therefore, it can be given the structure of a smooth (infinite-dimensional) manifold.

The tangential space $T_X(\mathcal{P}(H))$ at any point $X$ in $\mathcal{P}(H)$ can then naturally be identified with $\mathcal{S}(H)$. Using the action of $\text{Aut}(H)$, Corach, Porta and Recht defined a connection on the tangential bundle of $\mathcal{P}(H)$ that is preserved by the group action.

The corresponding exponential map at $T_{\text{id}_H}(\mathcal{P}(H))$ is the ordinary exponential map $\exp : \mathcal{S}(H) \to \mathcal{P}(H)$, which one gets from the continuous spectral calculus.

By the group action, this is transported to all tangential fibres:

$$\exp_X : T_X(\mathcal{P}(H)) \to \mathcal{P}(H) : S \mapsto X^{\frac{1}{2}}\exp\left(X^{-\frac{1}{2}}SX^{-\frac{1}{2}}\right)X^{\frac{1}{2}} \forall X \in \mathcal{P}(H)$$

Moreover, the authors introduce a Finsler structure on $\mathcal{P}(H)$ (i.e. a choice of norms on each tangential space which varies smoothly over $\mathcal{P}(H)$):

$$d_A : T_A(\mathcal{P}(H)) \to \mathbb{R}_+, d_A(B) = \left\|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right\| \forall A \in \mathcal{P}(H)$$

The group $\text{Aut}(H)$ acts by isometries for this Finsler metric.

Using the covariant derivative corresponding to the connection, one finds unique geodesics between any two points in $\mathcal{P}(H)$. Explicitly, they are given by

$$\gamma(A, B, \cdot) : I \to \mathcal{P}(H)$$

$$\gamma(A, B, t) = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^tA^{\frac{1}{2}} \forall A, B \in \mathcal{P}(H)$$

where $I$ denotes the unit interval $[0, 1] \subset \mathbb{R}$.

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Here, a geodesic is a self-parallel curve (i.e. a smooth curve $\gamma$ such that $\nabla_{\gamma'}(t) = 0$ for the covariant derivative $\nabla$). Later, in the language of metric spaces, the term “geodesic” will denote a “shortest path” between two points.
4 The cone of positive invertible operators

Corresponding to this Finsler structure, there is a metric $d$ on $\mathcal{P}(H)$:

$$d(A, B) = \text{length}(\gamma(A, B)) = \left\| \ln(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \right\| \forall A, B \in \mathcal{P}(H)$$

This metric is convex and the action of Aut($H$) is isometric with respect to the Finsler metric.

4.2 A metric view on the space of positive invertible operators

In this section, we will collect properties of the metric space $\mathcal{P}(H)$ as introduced by Corach, Porta and Recht in a more precise way.

For this, let us introduce the basic definitions

**Definition** (metric space, metric balls).

A metric on a topological space $X$ is a map $d : X \times X \to \mathbb{R}_{\geq 0}$, which fulfills

- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x) \forall x, y \in X$, i.e. $d$ is symmetric
- $d(x, y) \leq d(x, z) + d(z, y) \forall x, y, z \in X$ (triangle inequality)

For a metric on a topological space $X$, one defines the metric ball of radius $r$ around $x \in X$ to be the set $B(x, r)$ of all points $y \in X$, which are of distance less than $r$ from $x$:

$$B(x, r) := \{y \in X : d(x, y) < r\}$$

Closed balls are defined by

$$\overline{B(x, r)} \{y \in \mathcal{P}(H) : d(x, y) \leq r\}$$

The space $X$ is then called metric, iff the topology on $X$ is generated by the balls $B(x, \varepsilon)$ for $x \in X$ and $\varepsilon > 0$.

Then, the topological closures of the open balls are the corresponding closed balls.
4.2 A metric view on the space of positive invertible operators

**Notation.** We shall denote open and closed metric balls by $B(r, x)$ and $\overline{B}(r, x)$ respectively. If the difference to other metrics needs to be underlined, we shall write $B^d(r, x)$ and $\overline{B^d}(x, r)$.

Moreover, $\overline{A}$ shall always denote the metric closure of $A$.\(^3\)

**Definition** (bounded subset, diameter of a set).
A subset $A$ of a metric space $X$ is called bounded, if

$$\sup_{x, y \in A} d(x, y) < \infty$$

this is equivalent to requiring that $A$ is contained in a ball.

For bounded sets, one defines $\text{diam}(A) := \sup_{x, y \in A} d(x, y)$ to be the diameter of $A$.

**Definition** (distances between sets).
For subsets $A$ and $B$ of a metric space $X$, the distance $d(A, B)$ between $A$ and $B$ is defined to be

$$d(A, B) = \inf_{a \in A, b \in B} d(a, b)$$

**Definition** (rectifiable curve, length of a curve, geodesic).
A curve in a metric space $X$ is a continuous map $\gamma : I \to X$.

We say, $\gamma$ starts at $\gamma(0)$ and ends at $\gamma(1)$. It is called rectifiable, if the following supremum exists

$$\sup_{0 = t_0 < t_1 < \ldots < t_n = 1} \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i))$$

In this case, the above quantity is defined to be the length $l(\gamma)$ of $\gamma$.

A geodesic between $x, y \in X$ is a rectifiable curve between $x$ and $y$ such that

$$l(\gamma) = d(\gamma(0), \gamma(1))$$

In particular, since by the triangle inequality, always $l(\gamma) \geq d(\gamma(0), \gamma(1))$, a geodesic is a shortest curve between its start and end point.

---

\(^3\)we shall see later that $d$-closed balls in $\mathcal{P}(H)$ are weakly compact and hence weakly closed anyway. This implies closedness for all topologies that are discussed here.
Remark 4.1.
It is not true, that geodesics always exist in an arbitrary metric space. For example, the space $\mathbb{R}^2 \setminus \{(0,0)\}$ with the usual euclidean metric does not contain a geodesic between $(0,-1)$ and $(0,1)$.

Even if geodesics exist, generally there will be no hope, that they are unique: if we equip $\mathbb{R}^2$ with the $\infty$-distance $d_\infty((a,b),(c,d)) = \max\{|c-a|,|d-b|\}$ the following two curves are geodesics between $(0,0)$ and $(1,2)$ of length 2:

$$
\gamma_1 : I \to \mathbb{R}^2 : t \mapsto \begin{cases} (2t,2t) & t \in [0;0.5) \\ (1,2t) & t \in [0.5;1] \end{cases}
$$

$$
\gamma_2 : I \to \mathbb{R}^2 : t \mapsto (t,2t)
$$

In differential geometry, the restrictions for a curve to be a geodesic are stricter. So, there might be many metric geodesics between a pair of points in a manifold that has a unique differential geometric geodesic. This is the case for $\mathcal{P}(H)$.

Definition (geodesic bicombing).
On a metric space $X$, such that there exist geodesics between any two points, a geodesic bicombing is a map $\gamma : X \times X \times [0,1] \to X$, such that

- the map $\gamma(x,y,\cdot) : [0,1] \to X$ is a geodesic for any pair $(x,y) \in X \times X$
- $\gamma(y,x,t) = \gamma(x,y,1-t) \ \forall t \in I, \forall x,y \in X$
- $\gamma(x,\gamma(x,y,t),s) = \gamma(x,y,ts) \ \forall s,t \in I, \forall x,y \in X$
- the geodesics depend continuously on $x$ and $y$, i.e.

$$
\lim_{n \to \infty} x_n = x, \lim_{n \to \infty} y_n = y \implies \lim_{n \to \infty} \gamma(x_n,y_n,t) = \gamma(x,y,t) \ \forall t \in I
$$

Let $(X,d)$ and $(Y,d')$ be two spaces with a distinguished geodesic bicombing. Then, a map $f : (X,d) \to (Y,d')$ is said to respect the geodesic bicombing, iff

$$
f \circ \gamma(x,y,t) = \gamma(f(x),g(y),t) \ \forall x,y \in X, \forall t \in [0,1]
$$
4.2 A metric view on the space of positive invertible operators

**Definition** (isometric action).
An action of some group $G$ on a metric space $X$ is called isometric, iff

$$d(gx, gy) = d(x, y) \forall x, y \in X, \forall g \in G$$

**Definition** (convex set).
Let $X$ be a metric space with a geodesic bicombing. Then, a subset $C \subset X$ is called convex, if for any $x, y \in C$ one has $\gamma(x, y, t) \in C \forall t \in I$.

Now, we can sum up the results by Corach, Porta and Recht in a more precise way.

**Lemma 4.2.**
On the cone $\mathcal{P}(H)$, the assignment

$$d : \mathcal{P}(H) \times \mathcal{P}(H) \to \mathbb{R}_{\geq 0}, \ d(A, B) = \left\| \ln \left( A^{-\frac{1}{2}} BA^{-\frac{1}{2}} \right) \right\|$$

defines a metric.

The action of $\text{Aut}(H)$ on $\mathcal{P}(H)$ as defined in (7) is isometric with respect to this metric.

In order to prove the lemma above, we need the following lemma, which will be used throughout this thesis:

**Lemma 4.3.**
For any $A \in \mathcal{P}(H)$, one has

$$d(\text{id}_H, A) = \| \ln A \| = \max \left\{ \ln \| A \|, \ln \| A^{-1} \| \right\}$$

**Proof.** By continuous spectral calculus, one gets:

$$d(\text{id}_H, A) = \| \ln(A) \|
\quad = \max \left\{ \ln(\max(\sigma(A))), -\ln(\min(\sigma(A))) \right\}
\quad = \max \left\{ \ln(\| A \|), -\ln \left( \frac{1}{\| A^{-1} \|} \right) \right\}
\quad = \max \left\{ \ln(\| A \|), \ln (\| A^{-1} \|) \right\}$$

$\Box$
Now, we can prove Lemma 4.2:

**Proof.** First, we show, that the action of $\text{Aut}(H)$ is isometric.

For this, let $A \in B(H)$ be invertible and let $A = PU$ be its polar decomposition. Then, one observes

$$d(\rho(A, \text{id}_H), \rho(A, B)) = d(AA^*, ABA^*)$$
$$= d(PUU^*P^*, PUBU^*P^*)$$
$$= d(P^2, PUBU^{-1}P)$$
$$= \| \ln (P^{-1}PUBU^{-1}PP^{-1}) \|$$
$$\stackrel{(2)}{=} \| U \ln(B)U^* \|$$
$$= \| \ln B \|$$
$$= d(1, B) \quad (8)$$

Now, for any $A \in \text{Aut}(H)$ and $B, C \in \mathcal{P}(H)$ define

$$K := \sqrt{B} \quad \text{and} \quad L := \rho(K^{-1}, C)$$

Then

$$ABA^* = AKK^*A^* = \rho(AK, \text{id}_H)$$

and one sees using (8)

$$d(\rho(A, B), \rho(A, C)) = d(\rho(A, \rho(K, \text{id}_H)), \rho(A, \rho(K, L)))$$
$$= d(\rho(AK, \text{id}_H), \rho(AK, L))$$
$$\stackrel{(8)}{=} d(\text{id}_H, L)$$
$$\stackrel{(8)}{=} d(\rho(K, \text{id}_H), \rho(K, \rho(K^{-1}, C)))$$
$$= d(B, C)$$

We have shown, that $\text{Aut}(H)$ acts by isometries.
4.2 A metric view on the space of positive invertible operators

We still need to show the properties of a metric:

(1) since the norm is a positive definite function, we have $d$ mapping to $\mathbb{R}_{\geq 0}$ and

$$0 = d(A, B) \Rightarrow 0 = \left\| \ln \left( A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \right) \right\|$$
$$\Rightarrow 0 = \ln \left( A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \right)$$
$$\Rightarrow 1 = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$$
$$\Rightarrow A = B$$

the converse being obvious, this shows

$$d(A, B) = 0 \iff A = B$$

(2) The symmetry of $d$ follows from the fact, that $\mathcal{P}(H) \subset \text{Aut}(H)$ acts by isometries:

Firstly, since by Lemma 4.3 $\| \ln A \| = \| \ln(A^{-1}) \|$, one has for every $A \in \mathcal{P}(H)$

$$d(\text{id}_H, A) = \| \ln A \|$$
$$= \| \ln (A^{-1}) \|$$
$$= \| \ln \left( (A^{-\frac{1}{2}})^2 \right) \|$$
$$= d(A, \text{id}_H)$$

therefore, for any $A, B \in \mathcal{P}(H)$ (here, we use, that $\mathcal{P}(H) \subset \text{Aut}(H)$ acts isometrically):

$$d(A, B) = d \left( \text{id}_H, \rho \left( A^{-\frac{1}{2}}, B \right) \right)$$
$$= d \left( \rho \left( A^{-\frac{1}{2}}, B \right), \text{id}_H \right)$$
$$= d \left( B, \rho \left( A^{\frac{1}{2}}, \text{id}_H \right) \right)$$
$$= d(B, A)$$
The cone of positive invertible operators

(3) Again, with Lemma 4.3, we have for any \( A, B \in \mathcal{P}(H) \)

\[
d(A, \text{id}_H) + d(B, \text{id}_H) = \| \ln A \| + \| \ln B \|
\]

\[
= \max \left\{ \ln \| A \|, \ln \| A^{-1} \| \right\} + \max \left\{ \ln \| B \|, \ln \| B^{-1} \| \right\}
\]

\[
\geq \max \left\{ \ln \| A^{-1} \| + \ln \| B \| \right\}
\]

\[
= \max \left\{ \ln \left( A^{-\frac{1}{2}} \|B\| \cdot \left| A^{-\frac{1}{2}} \right| \right) \right\}
\]

\[
\geq \max \left\{ \ln \| A^{-\frac{1}{2}} \cdot B^{-\frac{1}{2}} \| \right\}
\]

\[
= \ln \left( A^{-\frac{1}{2}} B^{-\frac{1}{2}} \right) = d(A, B)
\]

This proves the triangle inequality for any \( A, B, C \in \mathcal{P}(H) \), since

\[
d(A, B) + d(B, C) = d \left( \rho \left( B^{-\frac{1}{2}}, A \right), \text{id}_H \right) + d \left( \text{id}_H, \rho \left( B^{-\frac{1}{2}}, C \right) \right)
\]

\[
= d \left( \rho \left( B^{-\frac{1}{2}}, A \right), \text{id}_H \right) + d \left( \rho \left( B^{-\frac{1}{2}}, C \right), \text{id}_H \right)
\]

\[
\geq d \left( \rho \left( B^{-\frac{1}{2}}, A \right), \rho \left( B^{-\frac{1}{2}}, C \right) \right)
\]

\[
= d(A, C)
\]

Lemma 4.4.

The maps

\[
\gamma(A, B) : [0, 1] \to \mathcal{P}(H), \ \gamma(A, B, t) = A^{\frac{t}{4}} \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{t} A^{\frac{t}{4}}
\]

yield a geodesic bicombing on \( \mathcal{P}(H) \), which is respected by the action of \( \text{Aut}(H) \):

\[
\rho(\alpha, \gamma(B, C)) = \gamma(\rho(\alpha, B), \rho(\alpha, C)) \ \forall \alpha \in \text{Aut}(H), \ \forall B, C \in \mathcal{P}(H)
\]

Proof. We are going to show, that

\[
\gamma(\rho(A, \text{id}_H), \rho(A, B)) = \rho(A, \gamma(\text{id}_H, B)) \ \forall A \in \text{Aut}(H), \ \forall B \in \mathcal{P}(H)
\]

\[
\]
and, that the curves $\gamma(id_H, A)$ are indeed geodesics. The continuity of the geodesic bicombing will be shown explicitly as Lemma 6.10 in Chapter 6.

By the fact, that $\text{Aut}(H)$ acts isometrically and transitively on $\mathcal{P}(H)$, this will then yield the claim.

To show the above equality, let $A \in \text{Aut}(H)$ be arbitrary and let $A = PU$ be its polar decomposition.

Then, as in the proof of Lemma 4.3, $A^* = U^*P$ and $AA^* = P^2$ and we see

$$
\rho(A, \gamma(id_H, B, t)) = AB^tA^*
= PUB^tU^*P
= P(UBU^{-1})^tP
= P(P^{-1}ABA^*P^{-1})^tP
= (AA^*)^{\frac{1}{2}} \left((AA^*)^{-\frac{1}{2}}ABA^*(AA^*)^{-\frac{1}{2}}\right)^t(\AA^*)^{\frac{1}{2}}
= \gamma(\rho(A, id_H), \rho(A, B), t)
$$

Now, we generalize this to the action on any geodesic:

For this, define, as in the previous proof, $K := \sqrt{B}$ and $L = \rho(K^{-1}, C)$ for arbitrary $B, C \in \mathcal{P}(H)$.

Then, for any $A \in \text{Aut}(H)$ and arbitrary $t \in [0, 1]$, one has

$$
\rho(A, \gamma(B, C, t)) = \rho(A, \gamma(\rho(K, id_H), \rho(K, L), t))
= \rho(A, \rho(K, \gamma(id_H, L, t)))
= \rho(AK, \gamma(id_H, L, t))
= \gamma(\rho(AK, id_H), \rho(AK, L), t)
= \gamma(\rho(A, B), \rho(A, C), t)
$$

To show, that $\gamma(id_H, A)$ is a geodesic for any $A \in \mathcal{P}(H)$, we need to calculate its length.

For this, let $0 = t_0 < \cdots < t_n = 1$ be a partition of the unit interval.
Then

\[ \sum_{i=0}^{n-1} d(\gamma(id_H, A, t_i), \gamma(id_H, A, t_{i+1})) = \sum_{i=0}^{n-1} d\left(A^{t_i}, A^{t_{i+1}}\right) \]

\[ = \sum_{i=0}^{n-1} \left\| \ln\left(A^{-\frac{1}{2}} A^{t_{i+1}} A^{-\frac{1}{2}}\right) \right\| \]

\[ = \sum_{i=0}^{n-1} \left\| \ln\left(A^{t_{i+1} - t_i}\right) \right\| \]

\[ = \sum_{i=0}^{n-1} \left\| (t_{i+1} - t_i) \ln A \right\| \]

\[ = \sum_{i=0}^{n-1} (t_{i+1} - t_i) d(id_H, A) \]

\[ = (t_n - t_0) d(id_H, A) \]

\[ = d(id_H, A) \]

Since the length of \(\gamma(id_H, A)\) equals the supremum of this taken over all possible partitions of the unit interval, we see, that

\[ l(\gamma(id_H, A)) = d(id_H, A) \]

and hence \(\gamma(id_H, A)\) is a geodesic. Then, also \(\gamma(A, B)\) is a geodesic for any \(A, B \in \mathcal{P}(H)\) because Aut\((H)\) acts bicombing respectingly and transitively.

We still need to show the defining properties of a geodesic bicombing. Namely:

\[ \gamma(A, \gamma(A, B, t), s) = \gamma(A, B, st) \quad \forall A, B \in \mathcal{P}(H), \ \forall s, t \in [0, 1] \]

and

\[ \gamma(X, Y, t) = \gamma(Y, X, 1 - t) \]

Both cases will be reduced to the case \(A = id_H\) by the fact, that Aut\((H)\) acts respecting the bicombing.
4.2 A metric view on the space of positive invertible operators

So, let $A, B \in \mathcal{P}(\mathcal{H})$ and $s, t \in [0, 1]$ be arbitrary. Then, with the help of the continuous spectral calculus

$$
\gamma(A, \gamma(A, B, t), s) = \gamma\left(\text{id}_H, A^{\frac{1}{2}} \gamma(A, B, t) A^{-\frac{1}{2}}, s\right)
= \gamma\left(\text{id}_H, \gamma\left(\text{id}_H, A^{-\frac{1}{2}} BA^{-\frac{1}{2}}, t\right), s\right)
= \left(\gamma\left(\text{id}_H, A^{-\frac{1}{2}} BA^{-\frac{1}{2}}, t\right)\right)^s
= \left(A^{-\frac{1}{2}} BA^{-\frac{1}{2}}\right)^{st}
= \gamma\left(\text{id}_H, A^{-\frac{1}{2}} BA^{-\frac{1}{2}}, st\right)
= \gamma(A, B, st)
$$

and similarly

$$
\gamma(A, B, t) = \gamma\left(\text{id}_H, A^{-\frac{1}{2}} BA^{-\frac{1}{2}}, t\right)
= \left(A^{-\frac{1}{2}} BA^{-\frac{1}{2}}\right)^{t}
= \left(A^{-\frac{1}{2}} BA^{-\frac{1}{2}}\right)^{\frac{1}{2}} \left(A^{-\frac{1}{2}} BA^{-\frac{1}{2}}\right)^{\frac{1}{2}} \left(A^{-\frac{1}{2}} BA^{-\frac{1}{2}}\right)^{1-t}
= \left(A^{-\frac{1}{2}} BA^{-\frac{1}{2}}\right)^{\frac{1}{2}} \left(A^{-\frac{1}{2}} BA^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \text{id}_H \left(A^{-\frac{1}{2}} BA^{-\frac{1}{2}}\right)^{1-t}
= \gamma\left(A^{-\frac{1}{2}} BA^{-\frac{1}{2}}, \text{id}_H, 1 - t\right)
= \gamma(B, A, 1 - t)
$$

\[\square]\]

**Theorem 4.5** (Corach, Porta, Recht).

The metric $d$ on $\mathcal{P}(\mathcal{H})$ is convex with respect to the geodesic bicombing defined above. This means

$$
d(\gamma(A, B, t), \gamma(C, D, t)) \leq (1 - t)d(A, C) + td(B, D) \quad \forall t \in I \tag{9}
$$

for any four points $A, B, C$ and $D \in \mathcal{P}(\mathcal{H})$.

In particular, metric balls (closed and open) are convex.

**Remark 4.6.**

This theorem is Theorem 2 in [CPR94]. The proof uses the fact
(Theorem 1 in the same article), that the norm of the Jacobi field along a geodesic is a convex function.

In fact, in [CPR94] the action $L : (S, T) \mapsto LST := (S^*)^{-1}TS^{-1}$ is used.

But the results are the same for $\rho$, as $LST = \rho((S^*)^{-1}, T)$ and the map $S \mapsto (S^*)^{-1} : \text{Aut}(H) \to \text{Aut}(H)$ is an automorphism.

**Notation.** If it is clear from the context, we will often write $gx$ instead of $\rho(g, x)$. In particular, $gx$ will denote $\pi(g)(x)$ for $x \in H$ and $gT$ will denote $\rho_n(g, T)$ for $T \in \mathcal{P}(H)$ and a representation $\pi$.

This way, the equations in Theorem 4.5 and Lemma 4.4 turn into

$$x\gamma(a, b, t) = \gamma(xa, xb, t) \quad \forall x \in \text{Aut}(H), \forall a, b \in \mathcal{P}(H), \forall t \in I$$

$$d(xa, xb) = d(a, b) \quad \forall x \in \text{Aut}(H), \forall a, b \in \mathcal{P}(H)$$

**Definition** (closed convex hull).

For a metric space $X$ with a geodesic bicombing, the closed convex hull $\text{conv}(A)$ of a bounded set $A$ is the smallest closed and convex set containing $A$.

**Lemma 4.7.**

Let $X$ be a metric space together with a geodesic bicombing. Then, if the metric of $X$ is convex with respect to the geodesic bicombing in the sense of (9),

$$\text{diam}(\text{conv}A) = \text{diam}(A)$$

holds for any bounded subset $A$ of $X$.

**Proof.** Let $A$ be an arbitrary subset. Then it is obvious, that

$$A_1 := \{\gamma(a, a')|a, a' \in A\} \subset \text{conv}(A)$$

and therefore

$$\text{conv}(A_1) = \text{conv}(A)$$

Moreover, one has (by the convexity of the metric)

$$A \subset A_1 \text{ and } \text{diam}(A_1) = \text{diam}(A)$$

Inductively, we may now construct $A_n$ out of $A_{n-1}$ ($n \in \mathbb{N}$), in the same way as we constructed $A_1$ from $A$. 
For the corresponding sequence of subsets $A_n$ of $X$, one then has:

1. $A \subset A_j$, $A_i \subset A_j \forall i \leq j \in \mathbb{N}$
2. $\text{diam}(A) = \text{diam}(A_j) \forall j \in \mathbb{N}$
3. $\text{conv}(A) = \text{conv}(A_j) \forall j \in \mathbb{N}$

We now claim, that the closure of the corresponding ascending union will be $\text{conv}(A)$:

$$\text{conv}(A) = \bigcup_{j \in \mathbb{N}} A_j$$

In fact, since $\text{conv}(A)$ is closed and convex and property (3) above holds, it is obvious, that at least

$$\text{conv}(A) \supset \bigcup_{j \in \mathbb{N}} A_j$$

and the claim is proven, if we show, that the right hand side (which we will denote by $\overline{W}$) is convex.

In fact, let $x, y \in \overline{W}$, then there are sequences $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subset W$ such that

1. $x_n, y_n \in A_n \forall n \in \mathbb{N}$
2. $\lim_{n \to \infty} x_n = x$
3. $\lim_{n \to \infty} y_n = y$

Hence, by the definition of the $A_n$, one has

$$\gamma(x_n, y_n) \subset A_{n+1} \forall n \in \mathbb{N}$$

And consequently, by the continuity of the geodesic bicombing

$$\gamma(x, y, t) = \gamma\left(\lim_{n \to \infty} x_n, \lim_{n \to \infty} y_n, t\right)$$

$$= \lim_{n \to \infty} \gamma(x_n, y_n, t)$$

$$\in \bigcup_{n \in \mathbb{N}} A_n \forall t \in [0, 1]$$

And $\overline{W}$ is convex (i.e., it equals $\text{conv}(A)$ by the arguments from above).
But this implies, that (using the notation from above)

\[ \text{diam}(\text{conv}(A)) = \sup_{x,y \in \text{conv}(A)} d(x, y) \]
\[ = \sup_{x,y \in \text{conv}(A)} d\left( \lim_{n \to \infty} x_n, \lim_{n \to \infty} y_n \right) \]
\[ \leq \sup_{x,y \in \text{conv}(A)} \text{diam}(A_n) \]
\[ = \sup_{x,y \in \text{conv}(A)} \text{diam}(A) \]
\[ = \text{diam}(A) \]  

and the converse inequality is obvious. \qed
In this chapter, we want to re-prove Theorem 2.9 and give equivalent and more geometric statements of the Theorems 2.9 and 2.10.

In order to do this, we will interpret the universal constants that appear in the statement of the theorems geometrically. After this, we will give a geometric proof for Theorem 2.9 and eventually translate both theorems into the metric language on $\mathcal{P}(H)$.

### 5.1 Re-proving Pisier’s theorem

In order to understand Theorem 2.9 more geometrically, we will establish some metric properties for the $G$-action on $\mathcal{P}(H)$ that is induced by a unitarisable representation of $G$ on $H$.

First of all, let us remark, that there are two notions of convexity on $\mathcal{P}(H)$ coming from the underlying linear structure of $B(H)$ and the metric structure on $\mathcal{P}(H)$. From now on, convex, shall always denote convex with respect to $d$. (we will say linearly convex and $d$-convex, in case confusion might occur)

**Lemma 5.1.**

For any action of a group $G$ on $\mathcal{P}(H)$, that respects the geodesic bicombing on $\mathcal{P}(H)$ in the sense of Lemma 4.4, $\mathcal{P}(H)^G$ is convex.

**Proof.** For $X, Y \in \mathcal{P}(H)^G$ we have

$$g\gamma(X, Y, t) = \gamma(gX, gY, t) = \gamma(X, Y, t) \forall g \in G$$

$\square$

For a uniformly bounded representation $\pi$ of a group $G$ and arbitrary $g \in G$, one has

$$|\pi|^2 \geq \|\pi(g)\|^2 = \|\pi(g)\pi(g)^*\| \forall g \in G$$

and analogously for the inverse of $g$

$$|\pi|^2 \geq \|\pi(g^{-1})^* \pi(g^{-1})\| = \|(\pi(g)\pi(g)^*)^{-1}\|$$

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Hence, the $G$-orbit $\rho_\pi(G, \text{id}_H)$ of $\text{id}_H \in \mathcal{P}(H)$ under $\rho_\pi$ (and thus its closed convex hull) is bounded, since by Lemma 4.3

\[
d(\rho_\pi(g, \text{id}_H), \text{id}_H) = \| \ln(\pi(g)\pi(g^*)) \|
= \max \left\{ \ln \left( \| \pi(g)\pi(g^*) \| \right), \ln \left( \| \pi(g)\pi(g^*) \|^{-1} \right) \right\}
\geq \ln \left( |\pi|^2 \right)
\]

**Definition** (diameter of a uniformly bounded representation).

For a uniformly bounded representation $\pi$ of a group $G$ on a Hilbert space $H$, one defines

\[
diam(\pi) := \sup_{g,h \in G} d(\rho_\pi(h, \text{id}_H), \rho_\pi(g, \text{id}_H)) = diam(\rho_\pi(G, \text{id}_H))
\]

to be the diameter of $\pi$.

Since $\rho_\pi$ is an action of isometries on $\mathcal{P}(H)$, this is the same as

\[
\sup_{g \in G} d(\text{id}_H, \rho_\pi(g, \text{id}_H))
\]

The next lemmas connect $|\pi|$ with the diameter of $\pi$. It turns out, that for $\pi_t$ as defined in Lemma 3.7, $|\pi_t|$ behaves in a similar way as the size of its smallest unitariser (in the sense of Corollary 3.7)

**Lemma 5.2.**

For a uniformly bounded representation $\pi$, the following equality holds

\[
2 \ln |\pi| = diam(\pi)
\]

**Proof.** One calculates

\[
diam(\pi) = \sup_{h \in G} d(\text{id}_H, \rho_\pi(h, \text{id}_H))
= \sup_{h \in G} \| \ln (\pi(h)\pi(h^*)) \|
= \sup_{g \in G} \max \left\{ \ln \left( \| \pi(g)\pi(g^*) \| \right), \ln \left( \| (\pi(g)\pi(g^*))^{-1} \| \right) \right\}
= \sup_{g \in G} \max \left\{ \ln \left( \| \pi(g) \| \right)^2, \ln \left( \| \pi(g^{-1}) \| \right)^2 \right\}
= 2 \sup_{g \in G} \ln (\| \pi(g) \|)
= 2 \ln |\pi|
\]

\[
\]

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Lemma 5.3.
For $\pi_t$ as in Lemma 3.6, the following holds

$$|\pi_t| \leq |\pi|^{1-t} \quad \forall t \in I$$

Proof. Using the facts from Theorem 4.5 for the metric $d$, one calculates (we use the notation that we introduced in the previous chapter for the action $\rho_\pi$)

$$2 \ln |\pi_t| = \text{diam}(\pi_t)$$

$$= \sup_{g \in G} d(\text{id}_H, \pi_t(g)\pi_t(g)^*)$$

$$= \sup_{g \in G} d\left(\text{id}_H, S^{-t}\pi(g)S^{2t}\pi(g)^*S^{-t}\right)$$

$$= \sup_{g \in G} d\left(S^{2t}, gS^{2t}\right)$$

$$= \sup_{g \in G} d\left(\gamma(\text{id}_H, S^2, t), \gamma(g\text{id}_H, gS^2, t)\right)$$

$$= \sup_{g \in G} \left((1-t)d(\text{id}_H, g\text{id}_H) + td(S^2, gS^2)\right)$$

(Lemma 3.1)

$$= \sup_{g \in G} (1-t)d(\text{id}_H, g\text{id}_H)$$

$$= (1-t) \text{diam}(\pi)$$

$$= 2(1-t) \ln (|\pi|)$$

$$= 2 \ln (|\pi|^{1-t})$$

This obviously implies that

$$\ln |\pi_t| \leq \ln (|\pi|^{1-t})$$

Exponentiating both sides yields the claim. \qed

Lemma 5.4.
For $\pi_t$ as in Lemma 3.6, the function $t \mapsto |\pi_t|$ is continuous

Proof. In the proof of the previous lemma, we have seen, that

$$2 \ln |\pi_t| = \sup_{g \in G} d\left(\gamma(\text{id}_H, S^2, t), \gamma(g\text{id}_H, S^2, t)\right)$$

$$=: \sup_{g \in G} f_g(t)$$

Now, the claim is proven, if the right hand side depends continuously on $t$. 

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This follows easily from the fact, that the family \( \{f_g, g \in G\} \), over which we take the supremum, is uniformly equicontinuous. That means, by definition, that the following holds (cf [RS80], Chapter I.6)

\[
\forall \varepsilon > 0 \exists \delta > 0, \forall t \in [0, 1] : |t - t'| < \delta \Rightarrow |f_g(t) - f_g(t')| < \varepsilon \quad \forall g \in G
\]

In order to prove this, we use the following consequence of the triangle inequality: for arbitrary 4 points \( a, b, c, d \) in a metric space, the following holds

\[
d(a, d) \leq d(a, b) + d(b, c) + d(c, d) \Rightarrow d(a, d) - d(b, c) \leq d(a, b) + d(c, d)
\]

hence, by symmetry (doing the same with \( d(a, d) \) and \( d(b, c) \) interchanged)

\[
|d(a, d) - d(b, c)| \leq d(a, b) + d(c, d) \tag{11}
\]

Now, for \( \varepsilon > 0 \), let \( \delta = \frac{\varepsilon}{4\|\ln S\|} \) and choose \( t, t' \in [0, 1] \) such that \( |t - t'| < \delta \).

Then, for arbitrary \( g \in G \), one has (as \( G \) acts by isometries)

\[
|f_g(t) - f_g(t')| \\
= \left| d \left( \gamma(id_H, S^2, t), g\gamma(id_H, S^2, t) \right) - d \left( \gamma(id_H, S^2, t'), g\gamma(id_H, S^2, t') \right) \right| \\
\overset{(11)}{\leq} d(\gamma(id_H, S^2, t), \gamma(id_H, S^2, t')) + d(g\gamma(id_H, S^2, t), g\gamma(id_H, S^2, t')) \\
= 2d(\gamma(id_H, S^2, t), \gamma(id_H, S^2, t')) \\
= 2\|\ln \left( S^{2(t'-t)} \right)\| \\
= 4|t' - t|\|\ln S\| \\
< \varepsilon
\]

\[\square\]

Remark 5.5.

The fact, that uniform equicontinuity of a family \( \{f_i : \mathbb{R} \to \mathbb{R}\} \) implies, that the pointwise supremum is continuous again is a straightforward proof by contradiction and we assume the reader to know this fact.

Now, we can re-prove Theorem 2.9.
For the convenience, it shall be stated again:

**Theorem.**
For a unitarisable group $G$, there are universal constants $K$ and $\alpha \in \mathbb{R}_+$ depending only on $G$, such that for every uniformly bounded representation $\pi$ of $G$ on some Hilbert space $H$ the following holds
\[
\exists S \in \mathcal{U}(\pi) : s(S) \leq K \cdot |\pi|^\alpha
\]

**Proof.** We assume for contradiction, that this is not the case.

So, let $G$ be an arbitrary group such that for every choice $K, \alpha \in \mathbb{R}_+$, there is a uniformly bounded representation $\pi_{K,\alpha}$, such that its smallest unitariser $S_{K,\alpha}$ will fullfill
\[
s(S_{K,\alpha}) > K \cdot |\pi_{K,\alpha}|^\alpha
\]

Hence, choosing $K = \alpha = n \in \mathbb{N}$ yields uniformly bounded representations $\pi_n := \pi_{n,n}$ of $G$ on Hilbert spaces $H_n$ with smallest unitarisers $S_n := S_{n,n}$, such that the following holds
\[
s(S_n) > n|\pi_n|^n
\]

In order to find a contradiction, we want to consider the direct sum of those representations.

Unfortunately, this does not have to be uniformly bounded, as the sequence $(|\pi_n|)_n \in \mathbb{N}$ has no reason to be bounded from above.

For a given $\pi_n$ such that $|\pi_n| > 2$ and in the flavour of Lemma 5.3, we define
\[
\pi_{n,t}(g) := S_n^{-t} \pi_n(g) S_n^t
\]

By Lemma 5.3 and Lemma 5.4, we can then find a $0 < t < 1$ yielding
\[
2 = |\pi_{n,t}| \leq |\pi_n|^{1-t}
\]
and the corresponding smallest unitariser $S_{n,t} = S_n^{1-t}$ of $\pi_{n,t}$ fullfills
\[
s(S_{n,t}) = s(S_n)^{1-t} \geq n^{1-t} |\pi_{n,t}|^n > 2^n > n
\]
5.1 Re-proving Pisier’s theorem

As the size of every representation is at least 1, we also have for all those $\pi_n$ with $|\pi_n| \leq 2$

$$s(S_n) > n|\pi_n|^n \geq n$$

This way, we get a sequence $(\pi_n : G \to \text{Aut}(H_n))_{n \in \mathbb{N}}$ of uniformly bounded representations of $G$, such that the following two inequalities hold for any $n \in \mathbb{N}$

$(1)$ $|\pi_n| \leq 2$

$(2)$ $s(S_n) > n$

We define the following representation $\pi$ of $G$ on $\bigoplus_{n \in \mathbb{N}} H_n$

$$\pi : g \mapsto \bigoplus_{n \in \mathbb{N}} \pi_n(g)$$

By taking suprema over all $g \in G$ and using Lemma 1.4, we get

$$|\pi| = \sup_{n \in \mathbb{N}} |\pi_n| \leq 2$$

and hence, $\pi$ is itself a uniformly bounded representation of $G$.

Now, since $G$ is unitarisable, we find a bounded $S$ unitarising $\pi$.

But then $S\pi_n(g)S_{S_{n}}^{-1}$ is unitary for every $n \in \mathbb{N}$, $g \in G$. Here, we used that $S_n : H_n \to SH_n$ is a homeomorphism of Hilbert spaces and identified $H_n$ with the corresponding subspace in the orthogonal sum $\bigoplus_{n \in \mathbb{N}} H_n$

(cf. [KR83], p. 120-121).

Now (Remark 1.3), any two separable Hilbert spaces of same dimension are unitarily equivalent. Let $U : SH_n \to H_n$ be a unitary equivalence.

Then, $US_n|_{H_n} : H_n \to H_n$ unitarises $\pi_n$ and we see that

$$s \left(US_n|_{H_n}\right) \geq s(S_n) > n \ \forall n \in \mathbb{N}$$

But this contradicts the boundedness of $S$ as $\|US_n|_{H_n}\| \leq \|S\|$. □
5 A proof and a geometric version of Pisier’s theorem

5.2 Translating Pisier’s theorems

In this section, we want to translate Theorems 2.9 and 2.10 into the language of metric spaces developed in the last chapter.

As already mentioned before, the size of an operator, the set of unitarisers as well as the fixed-point-set \( \mathcal{P}(H)^G \) are closed under scaling with positive real numbers.

But unlike the size \( s(S) \) of a unitariser, which does not notice scaling, its “metric counterpart” \( d(SS^*, \text{id}_H) \) does.

The claim of the following lemma is now, that one can construct a fixed point coming from a smallest unitariser, which minimizes the distance to \( \text{id}_H \) and hence to the orbit \( \rho(G, \text{id}_H) \).

**Lemma 5.6** (symmetric spectrum).

Let \( \pi \) be a unitarisable representation of \( G \) and \( \rho_\pi \) the induced action of \( G \) on \( \mathcal{P}(H) \).

Then, there is a fixed point \( T \) associated to a smallest unitariser of \( \pi \) in the sense of Lemma 3.1, such that

\[
d(T, \text{id}_H) = d(\mathcal{P}(H)^G, \text{id}_H) = \ln(s(S))
\]

**Proof.** By multiplying the fixed point \( SS^* = T \) corresponding to a smallest unitariser \( S \) with

\[
(\min(\sigma(T)) \cdot \max(\sigma(T)))^{-\frac{1}{2}} = \sqrt{\|T\|_{-1} \cdot \|T^{-1}\|}
\]

one gets a fixed point \( T' \) such that

\[
(\min(\sigma(T')))^{-1} = \left( (\min(\sigma(T) \cdot \max(\sigma(T)))^{-\frac{1}{2}} \min(\sigma(T)) \right)^{-1}
\]

\[
= \left( \frac{\min(\sigma(T))}{\max(\sigma(T))} \right)^{\frac{1}{2}(1)}
\]

\[
= \left( \frac{\max(\sigma(T))}{\min(\sigma(T))} \right)^{\frac{1}{2}}
\]

\[
= \left( (\min(\sigma(T) \cdot \max(\sigma(T)))^{-\frac{1}{2}} \max(\sigma(T)) \right)
\]

\[
= \max(\sigma(T'))
\]
5.2 Translating Pisier’s theorems

And therefore
\[
\|T’\| = \|T^{-1}\| = \left(\frac{\max \sigma(T)}{\min \sigma(T)}\right)^{\frac{1}{2}} = \sqrt{\|T\| \cdot \|T^{-1}\|}
\]

which in turn implies
\[
s(S) = \sqrt{s(T)} = \sqrt{s(T')}
\]
\[
= \exp(\ln \sqrt{\|T’\|^2})
\]
\[
= \exp(\ln \|T’\|)
\]
\[
= \exp(d(\text{id}_H, T'))
\]

Besides, operators with such spectral symmetry are precisely those, that minimize
\[
\min \{ \| \ln(\alpha S) \| \mid \alpha \in \mathbb{R}_+ \} = \min_{\alpha \in \mathbb{R}_+} \left\{ \max \left\{ \ln \| \alpha S \|, \ln \left( \| (\alpha S)^{-1} \|^{-1} \right) \right\} \right\}
\]
\[
= \min_{\alpha \in \mathbb{R}_+} \left\{ \max \left\{ \ln \min(\sigma(\alpha S)) \right\}, \ln \max(\sigma(\alpha S)) \right\}
\]
\[
= \min_{\alpha \in \mathbb{R}_+} \left\{ \max \left\{ \ln \alpha + \ln \min(\sigma(S)) \right\}, \ln \alpha + \ln \max(\sigma(S)) \right\}
\]

Hence we can argue conversely, that a fixed point \( T \) of the \( G \)-action \( \rho_\pi \), which minimizes the distance \( d(T, \text{id}_H) \), will have \( \|T\| = \|T^{-1}\| \) and hence:
\[
d(\text{id}_H, T) = \| \ln T \| = \ln \|T\| = \ln \left( s(T)^{\frac{1}{2}} \right) = \ln s(S)
\]

Thus we have seen, that smallest unitarisers with symmetric spectrum stand in 1:1-correspondance with points in \( \mathcal{P}(H)^G \) having minimal distance to \( \text{id}_H \) and (by the \( G \)-invariance of \( d \)) to the \( G \)-orbit of \( \text{id}_H \). \( \square \)
We can now rephrase Pisier’s theorem:

**Corollary 5.7.**

Let $G$ be a unitarisable group. Then, there are universal constants $C$ and $\alpha$ depending only on $G$ such that, for any action $\rho_{\pi}$ of $G$ on $\mathcal{P}(H)$ induced by a uniformly bounded representation $\pi$ on $H$,

$$
d (\text{id}_H, \mathcal{P}(H)^G) = d (\rho_{\pi}(G, \text{id}_H), \mathcal{P}(H)^G) = C + \frac{\alpha}{2} \text{diam}(\pi)\n$$

**Proof.** First of all, by Lemmas 3.1 and 3.4, $\mathcal{P}(H)^G$ is non-empty and the distance

$$
d (\rho_{\pi}(G, \text{id}_H), \mathcal{P}(H)^G) = \inf_{g \in G, T \in \mathcal{P}(H)^G} d (\rho_{\pi}(g, \text{id}_H), T) = \inf_{g \in G, T \in \mathcal{P}(H)^G} d (\text{id}_H, \rho_{\pi}(g^{-1}, T)) \text{ Lemma 4.4} = \inf_{T \in \mathcal{P}(H)^G} d (\text{id}_H, T) = d (\text{id}_H, \mathcal{P}(H)^G)\n$$

is realized by some particular $T$ such that (symmetric spectrum, Lemma 5.6)

$$
d (\text{id}_H, T) = \ln s(S)\n$$

for the smallest unitariser $S = \sqrt{T}$ corresponding to $T$.

Now, by Theorem 2.9, there are constants $K$ and $\alpha$ such that

$$
s (S) \leq K|\pi|^\alpha\n$$

Therefore, taking together both results

$$
d (\rho_{\pi}(G, \text{id}_H), \mathcal{P}(H)^G) = \ln s(S) \leq \ln (K|\pi|^{\alpha}) = \ln K + \alpha \ln |\pi| = \ln K + \frac{\alpha}{2} \text{diam}(\pi) \text{ (Lemma 5.2)}\n$$

Which proves the claim for $C = \ln K$. \qed
We can also “translate” Theorem 2.10 into our geometrical setup:

**Corollary 5.8.**

The following are equivalent for a discrete group $G$

1. $G$ is amenable

2. For any uniformly bounded representation of $G$ on a Hilbert space $H$, the induced action on $\mathcal{P}(H)$ allows for fixed points and

   $$d \left( \rho_\pi(G, \text{id}_H), \mathcal{P}(H)^G \right) \leq \text{diam}(\pi)$$

**Proof.** By Theorem 2.10, amenability of $G$ is equivalent to being able to choose (using the same notation as in the proof above)

$$K = \exp(C) = 1$$

and

$$\alpha = 2$$

for the universal coefficients $C$ and $\alpha$ in the preceding theorem.

But then, in the realm of Theorem 5.7, $C = \ln K = 0$ and one gets

$$G \text{ is amenable} \iff d \left( \rho_\pi(G, \text{id}_H), \mathcal{P}(H)^G \right) \leq \text{diam}(\pi) \ \forall \pi \text{ uniformly bounded rep.}$$

This theorem motivates the following definition:

$$X_\pi := \{ x \in \mathcal{P}(H) : d(x, \rho_\pi(g, \text{id}_H)) \leq \text{diam } \rho_\pi(G, \text{id}_H) \ \forall g \in G \} = \{ x \in \mathcal{P}(H) : d(x, y) \leq \text{diam } \rho_\pi(G, \text{id}_H) \ \forall y \in \text{conv } \rho_\pi(G, \text{id}_H) \}$$

In Theorem 6.9 we will see, that this space is some nice compromise between the weak operator and the metric topologies on $\mathcal{P}(H)$
The Corollary above now reads

**Corollary 5.9.**
A group $G$ is amenable, if and only if

$$X_{\pi} \cap \mathcal{P}(H)^G \neq \emptyset$$

for every uniformly bounded representation $\pi$.

So, if one could answer the following question affirmatively, unitarisability and amenability were equivalent:

**Does the existence of a fixed point for a group action $\rho$ on the space $\mathcal{P}(H)$ by isometries respecting the geodesic bicombing imply the existence of a fixed point inside $X_{\rho}$ or even inside the closed convex hull of the orbit of $\text{id}_H$ under $G$?**
In this chapter, we want to take a closer look at the interplay of the different topologies introduced in Chapter 1.

In the first section we will address issues of compactness and show, that the restriction of the norm topology in $B(H)$ to $\mathcal{P}(H)$ and the metric topology agree.

Afterwards, we will investigate properties of the geodesic structure of $\mathcal{P}(H)$. In the end of this chapter, we will construct midpoints and circumradii of bounded subsets of $\mathcal{P}(H)$. The midpoint set of the convex hull of $\rho_n(G, \text{id}_H)$ will be shown to be a $G$-set inside $X_\pi$ (i.e. the action $\mathcal{P}(H)$ maps the set to itself).

We will then construct more $G$-invariant subsets.

### 6.1 Compactness

*Notation.* We shall denote the topology coming from the metric $d$ on $\mathcal{P}(H)$ by $\tau_d$, the ordinary norm-topology, by $\tau_{\|\|}$ and weak and strong operator topologies by $\tau_w$ and $\tau_s$ respectively.

We will not make a difference in notation between $\tau_w$ or $\tau_s$ and their restrictions to the space of positive operators.

Furthermore, we will denote by $\overline{A}$ the closure of $A$ with respect to the ambient topology. If it is needed, the topology with respect to which we mean $\overline{A}$ to be closed, will be noted $\overline{A}^\tau$.

*Remark 6.1.*

*We remark first of all, that all topologies discussed in this chapter are invariant under the $d$-isometries $A \mapsto B^\frac{1}{2}AB^\frac{1}{2}$ with positive and invertible operators $B$.*

This is true, since both maps $A \mapsto AB\frac{1}{2}$ and $A \mapsto B\frac{1}{2}A$ are continuous with respect to $\tau_d, \tau_{\|\|}, \tau_w, \tau_s$ for every self-adjoint invertible $B$ (by Lemma 1.7), and their inverses are given by multiplying on the left or on the right with $B^{-\frac{1}{2}}$ which are continuous maps for the same reasons.

*As the space of positive invertible operators is not closed (with respect to any of the topologies discussed here apart from $\tau_d$), one has to keep in mind, that generally speaking $\tau$-convergent nets do not have to have their limit in $\mathcal{P}(H)$.***
Lemma 6.2.
Open (closed) \( d \)-balls of radius \( r \) around \( \text{id}_H \) correspond to open (closed) norm-balls (intersected with the space of positive operators).

Therefore, closed \( d \)-balls of finite radius around \( \text{id}_H \) are compact with respect to \( \tau_w \).

Proof. One sees that

\[
B^d(\text{id}_H, r) = \{ S \in \mathcal{P}(H) \mid \| \ln S \| < r \} = \{ S \in \mathcal{P}(H) \mid \max \{ |\ln \min(\sigma(S))|, |\ln \max(\sigma(S))| \} < r \} = \{ S \in \mathcal{P}(H) \mid \sigma(S) \subseteq (\exp(-r), \exp(r)) \}
\]

which gives a spectral definition of \( d \)-balls around \( \text{id}_H \in \mathcal{P}(H) \).

Furthermore, this yields

\[
B^d(\text{id}_H, r) = \frac{\exp(-r) + \exp(r)}{2} \text{id}_H + \left\{ S = S^* \mid \sigma(S) \subseteq \left( -\frac{\exp(r) - \exp(-r)}{2}, \frac{\exp(r) - \exp(-r)}{2} \right) \right\} = \frac{\exp(-r) + \exp(r)}{2} \text{id}_H + B^{\| \cdot \|}(0, \frac{\exp(r) - \exp(-r)}{2}) \cap \mathcal{P}(H) = B^{\| \cdot \|}\left( \frac{\exp(-r) + \exp(r)}{2} \text{id}_H, \frac{\exp(r) - \exp(-r)}{2} \right) \cap \mathcal{P}(H)
\]

The same is obviously true, if \( < \) is replaced by \( \leq \) and open intervals by closed intervals in the calculation above.

To prove compactness of closed \( d \)-balls of radius \( r \), one has to see, that operators in \( B := B^{\| \cdot \|}\left( \frac{\exp(-r) + \exp(r)}{2} \text{id}_H, \frac{\exp(r) - \exp(-r)}{2} \right) \) have spectrum away from 0 and are therefore invertible.

This implies, that the intersection of \( B \) with \( \mathcal{P}(H) \) is the same as its intersection with the \( \tau_w \)-closed space of positive operators. Hence, as an intersection of a \( \tau_w \)-compact set with a \( \tau_w \)-closed set, \( B^d(id_H, r) \) is itself \( \tau_w \)-compact. \( \Box \)
The topologies \( \tau_d \) and \( \tau_{\parallel \cdot \parallel} \) agree on \( \mathcal{P}(H) \).

**Proof.** Above, we have seen that \( d \)-balls around \( \text{id}_H \in \mathcal{P}(H) \) are also balls (of different radius and around different midpoints) with respect to the norm.

Conversely, given a radius \( \alpha \in (0,1) \) the norm-ball \( B_{\parallel \cdot \parallel}(\text{id}_H, \alpha) \) of radius \( \alpha \) around \( \text{id}_H \) (intersected with \( \mathcal{P}(H) \)) consists of all positive operators with spectrum in

\[
(1 - \alpha, 1 + \alpha)
\]

Let \( r < \ln(1 + \alpha) \), then

\[
1 > 1 - \alpha^2 = (1 - \alpha)(1 + \alpha) \Rightarrow 1 - \alpha < \frac{1}{1 + \alpha} = \exp \left( \ln \frac{1}{1 + \alpha} \right) < \exp(-r)
\]

\[
\Rightarrow (\exp(-r), \exp(r)) \subset (1 - \alpha, 1 + \alpha)
\]

Now, by the spectral characterization of \( d \)-balls from the proof to Lemma 6.2, it is obvious, that

\[
B_d(\text{id}_H, r) \subset B_{\parallel \cdot \parallel}(\text{id}_H, \alpha)
\]

Moreover, the mapping \( X \mapsto AXA \) is a homeomorphism for any \( A \in \mathcal{P}(H) \) with respect to both, \( \tau_d \) and \( \tau_{\parallel \cdot \parallel} \).

We have shown that the local bases at \( \text{id}_H \) for the topologies \( \tau_d \) and \( \tau_{\parallel \cdot \parallel} \) are equivalent in the way that every element of one of the local bases contains a neighbourhood of \( \text{id}_H \) from the other topology. Moreover, both topologies share a transitive subgroup of their homeomorphisms. Hence both topologies are the same.

**Remark 6.4.**

The fact, that the metric topologies \( \tau_{\parallel \cdot \parallel} \) and \( \tau_d \) coincide on \( \mathcal{P}(H) \) does not imply, that the corresponding metrics are equivalent in the following sense

\[
\exists c, C : \forall x, y \in \mathcal{P}(H) : c \leq \|x - y\| \leq d(x, y) \leq C \|x - y\|
\]
6.1 Compactness

This can be seen easily by the sequence

$$(x_n)_{n \in \mathbb{N}}, \ x_n := \frac{1}{n} \text{id}_H$$

which is bounded in norm but not in $\tau_d$.

This does not contradict the equality of $\tau_d$ and $\tau_{\|\|}$, since $\tau_d$ does not come from a Banach topology on $B(H)$ or the space of self-adjoint operators, for which equality of topologies implies equivalence of the corresponding norms (and hence metrics). But we do have the following

Lemma 6.5.

d-bounded subsets of $\mathcal{P}(H)$ are norm-bounded.

Proof. This is an easy consequence of Lemma 6.2:
If $A \subset \mathcal{P}(H)$ is bounded, then so is $A \cup \{\text{id}_H\}$ and hence

$A \subset A \cup \{\text{id}_H\}$

$\subset B^d(\text{id}_H, r)$

$\overset{\text{Lemma 6.2}}{=} B_{\|\|} \left( \frac{\exp(-r) + \exp(r)}{2} \text{id}_H, \frac{\exp(r) - \exp(-r)}{2} \right) \cap \mathcal{P}(H)$

for $r = \text{diam}(A \cup \{\text{id}_H\})$. This implies norm-boundedness. \qed

Corollary 6.6.

Closed d-balls are $\tau_w$ compact.

Proof. Let $B = \overline{B^d(A, r)}$ be a closed d-ball. Then, (remember, that Aut($H$) acts on the metric space $\mathcal{P}(H)$ by isometries)

$B = A^{\frac{1}{2}}B^d(\text{id}_H, r)A^{\frac{1}{2}}$

Now, by Lemma 1.7 the map $\varphi : X \mapsto A^{\frac{1}{2}}XA^{\frac{1}{2}}$ is a $\tau_w$-homeomorphism and by Lemma 6.2, $B$ is $\tau_w$-compact as an image of the compact set $\overline{B^d(\text{id}_H, r)}$ under the continuous map $\varphi$. \qed
Corollary 6.7.
Every $d$-bounded subset of $\mathcal{P}(H)$ is $\tau_w$-precompact

Proof. If a set $U$ is $d$-bounded, it is contained in a $\tau_w$-compact closed $d$-ball, which is weak operator closed (this follows from $\tau_w$-compactness and the fact, that $\tau_w$ is a Hausdorff topology).
Hence, it contains the $\tau_w$-closure $\overline{U}$, which, as a closed subset of a $\tau_w$-compact set is itself $\tau_w$-compact.

It is not clear, whether or not $d$-closed, $d$-convex and $d$-bounded sets are $\tau_w$ closed. What we do know, though, is the following

Theorem 6.8.
Let $A \subset \mathcal{P}(H)$ be $d$-closed, $d$-convex and $d$-bounded. Then any $\tau_w$-limitpoint $x$ of some sequence $(x_n)_{n \in \mathbb{N}} \subset A$ fullfills

$$d(x, y) \leq \text{diam}(A) \quad \forall y \in A$$

Proof. The sequence $(x_n)_{n \in \mathbb{N}}$ lies in the $\overline{B^d(y, \text{diam}(A))}$ for any $y \in A$. This ball is $\tau_w$-compact by Lemma 6.6.

So, generally speaking, $\tau_w$-limit points of sequences inside $d$-convex and $d$-closed sets are “not too far away” from the sequence. We give a name to such positions:

Definition (convex close position).
We say that a point $x$ is in convex close position (or convex close) to a subset $A$ of a metric space $X$, if

$$d(x, a) \leq \text{diam}(A) \quad \forall a \in A$$

In the last chapter, we introduced the space $X_\pi$, which, with the definition above, is the set of points convex close to $\rho_\pi(G, \text{id}_H)$. 

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With the help of Theorem 6.8, we may now collect some facts about this space:

**Theorem 6.9.**
The space $X_\pi \subset \mathcal{P}(H)$ has the following properties for uniformly bounded representations $\pi : G \to B(H)$:

- it is $\tau_w$-compact
- it is a $G$-space
- it is $d$-convex, $d$-bounded and $d$-closed

**Proof.**

- By Theorem 6.8, $X_\pi$ is $\tau_w$-closed and by definition, it is bounded. Therefore, it is $\tau_w$ compact.

- Let $x \in X_\pi$. Then for any $g \in G$ one has, due to the invariance of $d$ under $\rho$,

\[
d(gx, \rho(h, \text{id}_H)) = d(x, \rho(g^{-1} h, \text{id}_H))
\]
\[
\leq \text{diam} \rho_\pi(G, \text{id}_H) \forall h \in G
\]
\[
\Rightarrow gx \in X_\pi
\]
proving that $X_\pi$ is a $G$-space.

- $d$-boundedness and $d$-closedness are obvious.

For the $d$-convexity, one sees, that for any $g \in G$ and $t \in [0,1]$

\[
d(\gamma(x, y, t), \rho_\pi(g, \text{id}_H)) \leq (1 - t)d(x, \rho_\pi(g, \text{id}_H)) + td(x, \rho_\pi(g, \text{id}_H))
\]
\[
\leq (1 - t) \text{diam} \rho_\pi(G, \text{id}_H) + t \text{diam} \rho_\pi(G, \text{id}_H)
\]
\[
= \text{diam} \rho_\pi(G, \text{id}_H)
\]
6 Topological lemmas about $\mathcal{P}(H)$

6.2 Geodesics

In this section, we show the continuity of the geodesic bicombing on $\mathcal{P}(H)$ given by $\gamma(a, b, \cdot) : [0, 1] \to \mathcal{P}(H), t \mapsto a^\frac{1}{2} \left( a^{-\frac{1}{2}} b a^{-\frac{1}{2}} \right)^t a^\frac{1}{2}$.

Lemma 6.10.

If $a_i$ and $b_i$ are sequences in $\mathcal{P}(H)$ converging to $a$ and $b$ respectively with respect to $d$, one has pointwise convergence of $\gamma(a_i, b_i, \cdot)$ to $\gamma(a, b, \cdot)$.

Proof. We have to show, that $\lim_{i \to \infty} d(a_i, a) = \lim_{i \to \infty} d(b_i, b) = 0$ implies

$$\lim_{i \to \infty} d(\gamma(a_i, b_i, t), \gamma(a, b, t)) = 0 \quad \forall t \in [0, 1]$$

First of all, since the metric topology coming from $d$ and the norm topology agree on $\mathcal{P}(H)$ (Theorem 6.3), the assumed convergence $a_i \to a$ and $b_i \to b$ with respect to $d$ implies the same convergence with respect to the norm.

Now, calculating

$$\|x_i y_i - x y\| = \|x_i y_i - x_i y + x_i y - x y\|$$
$$= \|x_i (y_i - y) + (x_i - x) y\|$$
$$\leq \|x_i\| \|y_i - y\| + \|x_i - x\| \|y\|$$

one sees, that $x_i y_i$ converges to $x y$ for sequences $x_i$ and $y_i$ converging to $x$ and $y$ respectively.

Using this argument twice, one sees that the assumed convergence of $a_i$ ($b_i$) to $a$ ($b$, resp.) in $d$ (and hence in norm) implies

$$a_i^{-\frac{1}{2}} b_i a_i^{-\frac{1}{2}} \to a^{-\frac{1}{2}} b a^{-\frac{1}{2}} \Rightarrow \left(a_i^{-\frac{1}{2}} b_i a_i^{-\frac{1}{2}}\right)^t \to \left(a^{-\frac{1}{2}} b a^{-\frac{1}{2}}\right)^t \quad \text{Cor. 1.9}$$
$$\Rightarrow a_i^\frac{1}{2} \left(a_i^{-\frac{1}{2}} b_i a_i^{-\frac{1}{2}}\right)^t a_i^\frac{1}{2} \to a^\frac{1}{2} \left(a^{-\frac{1}{2}} b a^{-\frac{1}{2}}\right)^t a^\frac{1}{2}$$
$$\Rightarrow \gamma(a_i, b_i, t) \to \gamma(a, b, t)$$

in norm and for arbitrary $t \in [0, 1]$.

Again, by Theorem 6.3, this implies the desired convergence with respect to $d$. □
6.3 Midpoints and circumradii

For a bounded subset $A$ of a Banach space, there exists a unique $r$, such that $A$ is contained in a ball of radius $r$. We show in this section, that this is also a property of $\mathcal{P}(H)$ with its metric topology coming from $d$.

**Definition** (midpoint, circumradius).

Let $U \subset \mathcal{P}(H)$ be a $d$-bounded and $d$-convex set. We define the circumradius of $U$ to be

$$\inf_{r \in \mathbb{R}} \{ \exists x_r \in \mathcal{P}(H) : U \subset \overline{B(x_r, r)} \}$$

If for the circumradius $r^*$ of $U$ there is some $x^*$ such that $U \subset \overline{B(x^*, r^*)}$, we call $x^*$ a midpoint of $U$.

**Remark 6.11.**

Let $U$ be bounded with diameter $l$. Then obviously $\frac{l}{2} \leq r \leq l$ for the circumradius $r$. Also, the midpoints of $U$ are obviously convex close to $U$.

**Lemma 6.12.**

Midpoints exist for every bounded set $U$.

**Proof.** Let $r$ be the circumradius of $U$. For $n \in \mathbb{N}$ define $r_n$ by $r_n = r + \frac{1}{n}$ and let $(x_n)_{n \in \mathbb{N}}$ be a corresponding sequence of points $x_n \in \mathcal{P}(H)$, such that $B(x_n, r_n) \supset U$

Then

$$d(x_n, x_m) \leq d(x_n, y) + d(y, x_m) \leq 2r_{\min\{n,m\}} \leq 2(r + 1)$$

for some $y \in U$ and hence $(x_n)_{n \in \mathbb{N}}$ is a bounded sequence.

By Lemma 6.5, this sequence is also norm-bounded and has a weak limit point $x$.

To prove that $x$ is a midpoint realizing the circumradius $r$, let $u \in U$ be arbitrary. Then, by the definition of $x_m$,

$$\forall n \in \mathbb{N}: d(x_m, u) \leq r_m \leq r_n \forall m > n \Rightarrow (x_m)_{m \geq n} \subset \overline{B(u, r_n)} \forall n \in \mathbb{N}$$
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Due to weak compactness of this ball and the uniqueness of limit points of $\tau_w$-converging sequences, $x$ has to be in $B^d(u,r_n)$, too ($n \in \mathbb{N}$ arbitrarily large). But this just means

$$d(x,u) \leq r_n = r + \frac{1}{n} \forall n \in \mathbb{N} \Rightarrow d(x,u) \leq r$$

Since $u \in U$ was arbitrary, this finishes the proof.

A priori, the set of midpoints does not have to be a one-point-set. But the following holds:

**Lemma 6.13.**
For any bounded set $U$ with circumradius $r$, the set $M(U)$ of midpoints is $d$-convex, $d$-closed and bounded.

*Proof.* Let $x_1$ and $x_2$ be two midpoints of $U$, then by the convexity of $d$ for any $t \in [0,1]$

$$d(\gamma(x_1,x_2,t),u) \leq td(x_2,u) + (1-t)d(x_1,u) \leq t \cdot r + (1-t)r = r \forall u \in U$$

hence $\gamma(x_1,x_2,t) \in M(U)$, which shows $d$-convexity of $M(U)$.

The boundedness of $M(U)$ is obvious:

$$d(x_1,x_2) \leq d(x_1,y) + d(x_2,y) \leq 2r \forall x_1,x_2 \in M(U), \ (\forall y \in U)$$

Now, let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $M(U)$, which converges to $x$ with respect to $d$. Then

$$d(x,u) \leq d(x,x_n) + d(x_n,u) \leq d(x,x_n) + r \xrightarrow{n \to \infty} r$$

which shows that $x \in M(U)$. Hence $M(U)$ is a $d$-convex, $d$-bounded and $d$-closed set.

**Remark 6.14.**
One cannot assume, that there is only one midpoint for arbitrary bounded sets as the following example shows:

**Example 2.**
Let $G$ be a non-unitarisable group and $\pi : G \to \text{Aut}(H)$ be a uniformly bounded, non-unitarisable representation.
Consider now the orbit $X := \rho_{\pi}(G, \text{id}_H)$ of the identity with respect to the action of $G$ on $\mathcal{P}(H)$ induced by this representation.

Since $X$ is bounded (by the uniform boundedness of $\pi$), it has a circumradius, which we will denote by $r$.

We want to show, that the $G$-action on $\mathcal{P}(H)$ restricts to a $G$-action on the set of midpoints $M(X)$ of $X$.

In fact, one sees

\[
x \in M(X) \Rightarrow d(x, gg^*) \leq r \quad \forall g \in G
\]

\[
\Leftrightarrow d\left(x, h^{-1}gg^*(h^*)^{-1}\right) \leq r \quad \forall g, h \in G
\]

\[
\Leftrightarrow d(hxh^*, gg^*) \leq r \quad \forall g, h \in G
\]

\[
\Rightarrow hxh^* \in M(X) \quad \forall h \in G
\]

So, if the set of midpoints consisted only of one point, this point would be fixed by the action of $G$ and hence by Lemma 3.1, this would imply unitarisability of $\pi$.

The following example shows, that even in the linear case, the midpoints discussed above are counter-intuitive:

**Example 3.**

Consider $\ell^\infty(\mathbb{N})$ and in it 0 and the characteristic functions $f_n$ of $n \in \mathbb{N}$.

Now, the (algebraic) convex hull of those functions consists of all finitely supported functions with values in $[0, 1]$ such that the $\ell^1$-norm is 1.

Closing this in the $\ell^\infty$-norm means adding those functions of $\ell^1$-norm 1 taking values in $[0, 1]$ and vanishing at infinity.

This set $U$ is obviously convex, $\ell^\infty$-norm-closed and has “inner” circumradius 1:

\[
f \in U \Rightarrow \lim_{x \to \infty} f(x) = 0 \Rightarrow \|f - f_n\|_{\infty} \to 1
\]

so that every point in $U$ has an “opposite” point within $U$. In other words: midpoints in $U$ would imply the circumradius to be 1.

The circumradius “from the outside” is less: let $g$ be the constant function
with value $\frac{1}{2}$. Then
\[ f \in U \Rightarrow \frac{1}{2} \geq |f(x) - g(x)| \xrightarrow{x \to \infty} \frac{1}{2} \Rightarrow \|f - g\|_\infty = \frac{1}{2} \]

In other words, the “true midpoints” (those realizing the smallest possible radius of a ball containing $U$) do not have to be inside $U$, even if $U$ is convex!

### 6.4 $G$-subsets of $\mathcal{P}(H)$

Since, by Lemma 3.1, fixed point sets of the induced action $\rho_\pi$ coming from a uniformly bounded representation of $G$ correspond to the unitarisability of $\pi$, it is interesting to find (small) $G$-invariant subsets of $\mathcal{P}(H)$.

We will construct some of those sets in this section.

**Remark 6.15.**

In the sequel, compact will always refer to $\tau_w$-compact and convexity and boundedness are meant be $d$-convexity and $d$-boundedness respectively.

The following lemma is not only of general interest itself, but will also be used in the sequel:

**Lemma 6.16.**

For a compact set $A \subset \mathcal{P}(H)$ and a closed set $B \subset \mathcal{P}(H)$, there exist points $a \in A$, such that
\[ d(a, B) = d(A, B) \]

If, moreover, $B$ is compact, there is points $a \in A$ and $b \in B$ such that
\[ d(a, b) = d(a, B) = d(A, B) = d(A, b) \]

**Proof.** Let $(a_i)_{i \in \mathbb{N}} \subset A$ be a sequence realizing $d(A, B)$:
\[ d(a_i, B) = \inf_{b \in B} d(a_i, b) \xrightarrow{i \to \infty} \inf_{(a, b) \in A \times B} d(a, b) = d(A, B) \]
After going over to a subsequence ($A$ is $\tau_w$-compact), one can assume that $(a_i)_{i \in \mathbb{N}}$ $\tau_w$-converges with limitpoint $a \in A$ and
\[
d(a_i, B) \leq d(A, B) + \frac{1}{N} \quad \forall i > N
\]
But then, the subsequence $(a_i)_{i > N}$ lies in
\[
\left\{ x \in \mathcal{P}(H) : d(x, B) \leq d(A, B) + \frac{1}{N} \right\} = \bigcup_{b \in B} \left( b, d(A, B) + \frac{1}{N} \right) \cap A
\]
which, as an intersection of a closed and a compact set, is again compact.

Hence, its limit point $a$ is in this set as well. But as $N$ was arbitrary, this means that
\[
a \in \left\{ x \in \mathcal{P}(H) : d(x, B) \leq d(A, B) \right\} \Rightarrow d(a, B) = d(A, B)
\]
This proves the first claim.

Doing the same process with $A$ replaced by $B$ and $B$ replaced by $\{a\}$ yields a point $b$ such that
\[
d(a, b) = d(\{a\}, b) = d(\{a\}, B) = d(A, B)
\]
\[
\square
\]

\textit{Notation.} For a compact subset $A$ of $\mathcal{P}(H)$ and a closed subset $B \subset \mathcal{P}(H)$, we denote by $S(A, B) \subset A$, the set of points $a \in A$, for which
\[
d(a, B) = d(A, B)
\]
Those points exist by Lemma 6.16.

Now, we shall be interested in compact, convex sets that are themselves $G$-spaces with the $G$-action being the restriction of some action on $\mathcal{P}(H)$, which respects the geodesic bicombing.
Lemma 6.17.
Let $A$ be a bounded $G$-subset of $\mathcal{P}(H)$. Then the following sets are bounded, closed and convex $G$-subsets of $\mathcal{P}(H)$:

1. The set $M(A)$ of midpoints of $A$.

2. $S(A, B)$, for a compact and convex set $A \subset \mathcal{P}(H)$ and a closed set $B \subset \mathcal{P}(H)$.

Proof. We have to prove closedness, convexity and the fact that the sets are $G$-sets.

Convexity will always follow from the convexity of the function $x \mapsto d(x, y)$, which implies that for two points $a$ and $b$ equally far away from a third point $c$, elements $\gamma(a, b, t)$, $t \in [0, 1]$ are at most as far away from $c$ as $a$ and $b$.

1. This has been proven before as Lemma 6.13.

2. For a point $a$ in $S(A, B)$, we have $d(a, B) = d(A, B)$ and therefore

$$d(g^*ag, B) = \inf_{b \in B} d(g^*ag, b) = \inf_{b \in B} d(a, g^{-1}bg^{-1}) = d(a, B) = d(A, B)$$

We have proven, that $S(A, B)$ is a $G$-space, which as a subset of $A$ is obviously bounded.

The convexity of $d$ and $A$ imply the convexity of $S(A, B)$. Moreover, if $(x_n)_{n \in \mathbb{N}}$ is a $d$-convergent sequence in $S(A, B)$, the limitpoint $x$ lies in $A$ ($A$ being closed) and obviously,

$$d(x, B) = \inf_{b \in B} d(x, b) = \inf_{b \in B \ x \to \infty} d(x_n, b) = \inf_{b \in B \ x \to \infty} d(A, B) = d(A, B)$$

and hence $x \in S(A, B)$ implying closedness.
7 More general metric spaces

In this chapter, we introduce a concept of metric spaces generalizing the metric setup we have discussed on $\mathcal{P}(H)$. It also generalizes the concept of complete CAT(0)-spaces (broadly discussed by Martin Bridson in [MBAH99]) and will be a special case of “continuous midpoint spaces” as discussed in [Hor09].

**Definition (GCB-space).**
A complete metric space $(X, d)$ together with a fixed geodesic bicombing

$$\gamma: X \times X \times [0, 1] \to X$$

such that the metric is convex with respect to this bicombing (i.e., equation (9) holds) shall be called a GCB-space.

On an arbitrary GCB-space, there is no such thing as a “natural” weak topology $\tau_w$, which has shown to be very fruitful in the case of $\mathcal{P}(H)$.

The following property will make up for this at some points

**Definition (Property (C)).**
We say, a GCB-space $X$ has property (C), iff

Given a bounded sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ and a family $\{f_\alpha | \alpha \in I\}$ of isometries of $X$ respecting the geodesic bicombing (where $I$ is any index set) such that $d(x_n, f_\alpha(x_n)) \to 0$ for any $\alpha \in I$, there is some $x \in X$ convex close to the sequence $(x_n)_{n \in \mathbb{N}}$ such that

$$x = f_\alpha(x) \ \forall \alpha \in I$$

holds.

**Remark 7.1.**
The point $x$ in property (C) does not necessarily have to be a $d$-limit point (for which the latter property is obvious) as the following example shows.
Lemma 7.2.
For a Hilbert space $H$ and with the definitions from above one has that $\mathcal{P}(H)$ is a GCB-space with property (C) (considering only isometries $f_A$, where $f_A(x) = A^*xA$ for $A \in B(H)$).

Proof. We only need to show property (C).

Given a bounded sequence $(x_n)_{n \in \mathbb{N}}$, $X := \text{conv}(\{x_n | n \in \mathbb{N}\})$ is convex, bounded and closed. By Theorem 6.8, we find a $\tau_w$-limit point $x$ convex close to $X$. Generally speaking, this point does not at all have to be a $d$-limit point.

Now, given a family $\{f_{A_i} | A_i \in B(H), i \in I\}$ and the assumption in property (C) by definition of $d$, we have for any $A \in \{A_i, i \in I\}$

$$d(x_n, f_A(x_n)) \xrightarrow{n \to \infty} 0$$

and by Theorem 6.3

$$\|A^*x_nA - x_n\| \xrightarrow{n \to \infty} 0$$

Hence

$$|\langle (A^*x_nA - x_n) u, v \rangle| \leq \|A^*x_nA - x_n\| \|u\| \|v\| \xrightarrow{n \to \infty} 0 \quad \forall u, v \in H$$

But this then tells us

$$0 = \lim_{n \to \infty} |\langle (A^*x_nA - x_n) u, v \rangle|$$

$$= \lim_{n \to \infty} |\langle x_nAu, Av \rangle - \langle x_nu, v \rangle|$$

$$= |\langle xAu, Av \rangle - \langle xu, v \rangle|$$

$$= |\langle (A^*xA - x)u, v \rangle| \quad \forall u, v \in H$$

By the fact, that $\lim_{n \to \infty} x_n = x$ with respect to $\tau_w$.

This was true for any $u, v \in H$ and arbitrary $i \in I$ so that

$$A_i^*xA_i = x \quad \forall i \in I$$

$\square$
Example 4.
For a reflexive Banachspace \((X, \| \cdot \|)\), a geodesic bicombing can be defined by

\[ \gamma(x, y, t) = tx + (1 - t)y, \quad t \in I, \quad x, y \in X \]

The triangle inequality yields convexity of this bicombing and weak limit points comply with property (C). Hence, \(X\) is a GCB-space with property (C).

Example 5.

**Definition** (triangle in a metric space, comparison triangle).
For a metric space \(X\) with geodesic bicombing, a triangle between any three points \(x_1, x_2, x_3 \in X\) is the union of the images of \(\gamma(x_1, x_2, \cdot)\), \(\gamma(x_2, x_3, \cdot)\) and \(\gamma(x_1, x_3, \cdot)\).

To this triangle, there is a unique triangle in euclidean geometry that has the same side lengths \(d(x_i, x_j)\), \(i, j \in \{1, 2, 3\}\). This triangle is called comparison triangle.

**Definition** (\(\text{CAT}(0)\)-space).
A geodesic metric space \(X\), such that for any three points \(x_1, x_2, x_3 \in X\), the distances between any points in the geodesic triangle are smaller than distances between the corresponding points in the comparison triangle, is called a \(\text{CAT}(0)\)-space.

Complete \(\text{CAT}(0)\) spaces are called Hadamard spaces, they form another class of GCB-spaces (they are easily seen to be uniquely geodesic. Hence they carry a natural geodesic bicombing). Compact, closed subspaces will also have property (C).

**Remark 7.3.**
Obviously, points in \(\text{conv}(A)\) are convex close to \(A\) and in all the examples above apart from \(\mathcal{P}(H)\), we may find the point \(x\) from property (C) to lie inside the closed convex hull \(\text{conv}\{x_n, n \in \mathbb{N}\}\).
8 Barycenters for finite sets in GCB-spaces

In this chapter, we will introduce the concept of a barycenter of a finite set in an arbitrary GCB-space. Informally speaking, those are points in the closed convex hull of the given finite set, which are “equally far away” from all the points in the set, and which can be defined in a way that is invariant under bicombing-respecting maps.

We will start by proving the existence of barycenters for small sets and then generalize this to arbitrary finite sets.

Notation. We will denote by \([n]\) the set \(\{1, \ldots, n\} \subset \mathbb{N}\).

Definition (n-tuple space).
We define the \(n\)-tuple space of a topological space \(X\) to be
\[
X_{(n)} = \prod_{i \in \{1, \ldots, n\}} X / S_n
\]
the space of unordered \(n\)-tuples.
(Here \(S_n\) stands for the symmetric group on \(n\) elements).

Elements in the \(n\)-tuple-space are denoted by \((x_1, \ldots, x_n)\) or by \((x_i, i \in [n])\).

Lemma 8.1.
If \(X\) is a metric space,
\[
d_{(n)} : X_{(n)} \times X_{(n)} \to \mathbb{R}, \qquad ((x_i, i \in [n]), (y_i, i \in [n])) \mapsto \min_{\sigma \in S_n} \frac{1}{n} \sum_{i \in [n]} d(x_i, y_{\sigma(i)})
\]
defines a metric on \(X_{(n)}\).

Proof. We need to show the defining properties:

1. Since \(d\) is a metric, \(d_{(n)}\) will be a non-negative map, and
\[
d_{(n)} ((x_i, i \in [n]), (y_i, i \in [n])) = 0 \Rightarrow \exists \sigma \in S_n : \frac{1}{n} \sum_{i \in [n]} d(x_i, y_{\sigma(i)}) = 0
\]
\[
\Rightarrow \exists \sigma \in S_n : d(x_i, y_{\sigma(i)}) = 0 \quad \forall i \in [n]
\]
\[
\Rightarrow \exists \sigma \in S_n : x_i = y_{\sigma(i)} \quad \forall i \in [n]
\]
\[
\Rightarrow (x_i, i \in [n]) = (y_i, i \in [n]) \in X_{(n)}
\]
(2) Symmetry is straightforward:

\[ d_{(n)}((x_i, i \in [n]), (y_i, i \in [n])) = \min_{\sigma \in S_n} \frac{1}{n} \sum_{i \in [n]} d(x_i, y_{\sigma(i)}) \]

\[ = \min_{\sigma \in S_n} \frac{1}{n} \sum_{i \in [n]} d(y_{\sigma(i)}, x_i) \]

\[ = \min_{\sigma \in S_n} \frac{1}{n} \sum_{i \in [n]} d(y_i, x_{\sigma(i)}) \]

\[ = d_{(n)}((y_i, i \in [n]), (x_i, i \in [n])) \]

This holds for all \((x_i, i \in [n]), (y_i, i \in [n]) \in X_{(n)}, \forall n \in \mathbb{N}\)

(3) also, the triangle inequality follows directly:

\[ d_{(n)}((x_i, i \in [n]), (z_i, i \in [n])) = \min_{\sigma \in S_n} \frac{1}{n} \sum_{i \in [n]} d(x_i, z_{\sigma(i)}) \]

\[ \leq \min_{\sigma, \mu \in S_n} \frac{1}{n} \sum_{i \in [n]} \left( d(x_i, y_{\mu(i)}) + d(y_{\mu(i)}, z_{\sigma(i)}) \right) \]

\[ = \min_{\sigma, \mu \in S_n} \frac{1}{n} \sum_{i \in [n]} \left( d(x_i, y_{\sigma(i)}) + d(y_{\sigma(i)}, z_{\mu(i)}) \right) \]

\[ = d_{(n)}((x_i, i \in [n]), (y_i, i \in [n])) + d_{(n)}((y_i, i \in [n]), (z_i, i \in [n])) \]

This holds for all \((x_i, i \in [n]), (y_i, i \in [n]), (z_i, i \in [n]) \in X_{(n)}, \forall n \in \mathbb{N}\)

\[ \square \]

**Lemma 8.2.**

The metric \(d_{(n)}\) turns \(X_{(n)}\) into a complete metric space for complete metric spaces \(X\).

**Proof.** We need to show, that every Cauchy sequence in \(X_{(n)}\) has a limit point in \(X_{(n)}\). For this, let \((A_j)_{j \in \mathbb{N}} \subset X_{(n)}\) be a Cauchy sequence and choose \((i_k)_{k \in \mathbb{N}} \subset \mathbb{N}\) in such a way, that

\[ d_{(n)}(A_m, A_n) \leq 2^{-k} \forall m, n \geq i_k \] (12)

We may now permute the tuples \(A_{i_k} = (a_{j_k}^i, j \in [n])\) in a way, that

\[ d_{(n)}(A_{i_k}, A_{i_{k+1}}) = \frac{1}{n} \sum_{j \in [n]} d(a_{j_k}^{i_k}, a_{j_k}^{i_{k+1}}) \] (13)
We claim, that the sequences \((a^i_j)_{k \in \mathbb{N}} \subset X\) are Cauchy sequences for every \(j \in [n]\).

To show this, fix \(k \in \mathbb{N}\) and let \(l > k\). We are going to estimate \(d(a^i_k, a^i_l)\) independently of \(l > k\) and show, that

\[ d(a^i_k, a^i_l) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty \]

For this, we use the above equations:

\[
\begin{align*}
    d(a^i_k, a^i_l) &\leq \sum_{\alpha = k}^{l-1} d(a^i_{\alpha}, a^i_{\alpha+1}) \\
    &\leq \sum_{\alpha = k}^{l-1} n d(A_{i_{\alpha}}, A_{i_{\alpha+1}}) \\
    &\leq n \sum_{\alpha = k}^{l-1} 2^{-\alpha} \\
    &= n \left( 1 - \frac{2^{-l}}{1 - 2^{-1}} - \frac{1}{1 - 2^{-1}} \right) \\
    &= 2n \left( 1 - 2^{-l} - 1 + 2^{-k} \right) \\
    &= \frac{n}{2^k} \left( 1 - 2^{k-l} \right) \\
    &\leq \frac{n}{2^k} \left( 1 - 2^{k-l} \right) \\
    &\rightarrow 0 \quad \text{as} \quad k \rightarrow \infty
\end{align*}
\]

Using the completeness of \(X\), we find

\[ a^i_k \rightarrow a_j \in X \]

and define

\[ A := (a_j, j \in [n]) \]
We claim, that $A$ is a limit point for $(A_i)_{i \in \mathbb{N}} \subset X_{(n)}$.

For this, let $\varepsilon > 0$ be arbitrary and choose $k \in \mathbb{N}$ in a way, that
\[
2^{-k} < \varepsilon
\]
and
\[
d(a_{jk}^i, a_j) < \varepsilon \quad \forall j \in [n]
\]
Now, for $\alpha > i_k$
\[
d_{(n)}(A_\alpha, A) \leq d_{(n)}(A_\alpha, A_{i_k}) + d_{(n)}(A_{i_k}, A)
\]
\[
\overset{(12)}{\leq} 2^{-k} + \frac{1}{n} \sum_{j \in [n]} d(a_{jk}^i, a_j)
\]
\[
\leq 2^{-k} + \varepsilon
\]
\[
< 2\varepsilon
\]

Definition (diameter of an $n$-tuple).
For an $n$-tuple $(x_i, i \in [n]) \in X_{(n)}$, we define the diameter $\text{diam}((x_i, i \in [n]))$ to be
\[
\text{diam}((x_i, i \in [n])) := \max_{i,j \in [n]} d(x_i, x_j)
\]

Remark 8.3.
Any map $\varphi : X \to X$ naturally induces an map $\tilde{\varphi}$ on $X_{(n)}$ by
\[
\tilde{\varphi} : (x_i, i \in [n]) \mapsto (\varphi(x_i), i \in [n])
\]

Definition (barycenter map).
A map $b_n : X_{(n)} \to X$ is called a barycenter map, if

(1) $b_n((x_1, \ldots, x_n)) \in \text{conv}([x_1, \ldots, x_n])$

(2) $b$ is equivariant with respect to the group of bicombing-respecting maps $\varphi : X \to X$:

\[
i.e. \quad X_{(n)} \xrightarrow{\tilde{\varphi}} X_{(n)} \quad \text{commutes}
\]
\[
\xrightarrow{b}
\]
\[
X \xrightarrow{\varphi} X
\]

\[100\]
Definition (barycenter).
The image of an $n$-tuple by a barycenter map is called a barycenter of this tuple.

Remark 8.4.
Even though, by definition, the barycenter map maps tuples of elements in $X$ to points in $X$, we will frequently speak of “barycenters of a subset of $X$”.

A set $\{x_1, \ldots, x_n\}$ is then naturally identified with the obvious corresponding tuple $(x_i, i \in [n])$.

We define:

Definition (Barycenter of a set).
Let $A = \{a_1, \ldots, a_n\} \subset X$ be an $n$-point subset of a GCB space. Then, for a barycenter map $b_n : X^{(n)} \to X$, we define

$$b_n(A) := b_n((a_i, i \in [n]))$$

The set of all $n$-tuples still is bigger than the set of $n$-subsets of $X$, because in a tuple there might be points from $X$ occuring more than once.

Vice versa, one can associate to an $n$-tuple over $X$ the subset of $X$ that contains all points from the $n$-tuple. Therefore, it is possible, to associate to an $n$-tuple $A$ the closed convex hull $\text{conv}(A) \subset X$ or - as above - the diameter $\text{diam}(A)$.

In particular, by saying $x \in (x_1, \ldots, x_n)$ we mean that there is some $i \in [n]$ such that $x = x_i$.

Remark 8.5.
For $n \in \{1, 2\}$, there are obvious choices for barycenter maps:

$$b_1 : X^{(1)} \to X, (x) \mapsto x$$

$$b_2 : X^{(2)} \to X, (x, y) \mapsto \gamma \left( x, y, \frac{1}{2} \right)$$

In fact, we were forced to come up with precisely those definitions:

The fact, that $x$ is the only point in the closed convex hull of $x$ fixes $b_1$ and exchanging $x_1$ and $x_2$ in the definition of $b_2$ had to leave the result invariant. The midpoint $\gamma \left( x_1, x_2, \frac{1}{2} \right)$ is the only point in $\text{conv}\{x_1, x_2\}$ that has this property.
8.1 Barycenters for 3- and 4-tuples

As an introductory example, we will construct barycenters for three- and four-tuples over GCB-spaces.

Example 6.

Let \( D_0 = (x, y, z) \in X^{(3)} \) be a three-tuple. Due to the metric being convex, the diameter of its convex hull is the maximal distance between the points \( x \), \( y \) and \( z \).

Consider now the new tuple

\[
D_1 := \left( \gamma \left( x, y, \frac{1}{2} \right), \gamma \left( x, z, \frac{1}{2} \right), \gamma \left( y, z, \frac{1}{2} \right) \right) = (b_2((x, y)), b_2((z, y)), b_2((x, z))) \in X^{(3)}
\]

Now, the distance between two of those points is (using the convexity of the metric) at most half the diameter of \( D_0 \).

Thus, by inductively constructing \( D_n \) to be the tuple of midpoints between points in \( D_{n-1} \), one gets a sequence \( (D_n)_{n \in \mathbb{N}} \) of three-tuples, such that the sequence \( (\text{conv}(D_n))_{n \in \mathbb{N}} \subset X \) forms a nested sequence of closed, convex sets with diameter

\[
\text{diam conv}(D_n) \leq \left( \frac{1}{2} \right)^n \text{diam}(D_0)
\]

But this implies, that any sequence \( (x_n)_{n \in \mathbb{N}} \) of points \( x_n \in \text{conv}(D_n) \) is a Cauchy sequence:

\[
d(x_n, x_m) \leq \text{diam} \left( D_{\min\{m,n\}} \right) \xrightarrow{n,m \to \infty} 0
\]

Since \( X \) is complete, the sequence \( (x_n)_{n \in \mathbb{N}} \) will converge in \( X \) to some limit point.

Obviously, the limit point does not depend on the choice of \( x_n \in D_n \):

A different choice would result in a sequence, which is asymptotically close to
(x_n)_{n \in \mathbb{N}} and hence has the same limit point.

Therefore, the intersection of all the convex and closed sets \( \text{conv}(D_n) \) contains exactly one point.

This point is defined to be the barycenter of \((x, y, z)\):

\[
b_3((x, y, z)) := \bigcap_{n \in \mathbb{N}} \text{conv} D_n = \lim_{n \to \infty} x_n \text{ where } x_n \in D_n \text{ arbitrary}
\]

**Remark 8.6.**

*Observe, that in the construction above, we did not assume any ordering on the tuple and therefore \( b_3 \) is a well-defined map \( X_{(3)} \to X \). Moreover, as midpoints of geodesics are mapped to one another by bicombing respecting maps, \( b_3 \), as the limit of equivariant maps, is equivariant (with respect to bicombing respecting maps). Hence, \( b_3 \) is a well-defined barycenter map.*

Now, this construction shall be generalized to four-tuples \((a, b, c, d) \in X_{(4)}\).

Again, starting from \( A_1 \), one wants to construct out of a \( A_n \in X_{(4)} \) the four-tuple \( A_{n+1} \) consisting of the four barycenters of all sup-tuples consisting of three elements from \( A_n \).

We shall call those subtuples “triangles” or “faces” of the closed convex hull of \( A_n \).

In order to show, that the diameter of \( A_{n+1} \) is uniformly smaller than the diameter of \( A_n \), one uses the fact, that the barycenter of a triangle \( D_0^i \) (one of the faces \( D_0^i = \text{conv}(x_1, \ldots, \hat{x}_i, \ldots, x_4) \) of \( A_n = (x_1, \ldots, x_4)^4 \)) lies in all the “intermediate” triangles \( D_m^i \) from above.

Thus, the diameter of \( A_{n+1} \) is at most as big as the diameter of the convex hull of the union of all faces \( D_m^i \), \( i \in [4], \ m \in \mathbb{N} \), which in turn is the maximal distance between vertices of those triangles.

\^{4}(x_1, \ldots, \hat{x}_i, \ldots, x_4) \text{ shall denote } (x_j, \ j \in [4] \setminus \{i\})
Now, let \( m = 2 \) and choose \( x \in D^2_i \) and \( y \in D^2_j \), two of those vertices.

We claim, that

**Lemma 8.7.**

\[
d(x, y) \leq \frac{3}{4} \text{diam}(A_n)
\]

**Proof.** If \( i = j \), for start, we have

\[
d(x, y) \leq \text{diam}(D^i_2) \leq \frac{1}{4} \text{diam}(D^0_0) \leq \frac{1}{4} \text{diam}(A_n)
\]

and the claim is proven.

For \( i \neq j \), we have \( x \) and \( y \) living on different faces (say \( D^0_0 \) and \( D^1_1 \)), which share (at least) a geodesic (in our case \( \gamma(x_2, x_3, \cdot) \)) as a “common edge”.

In the sketch below, we see those two faces “clapped” open: the 4 outer points define our tuple \( A_n \), the \( y_i, i \in [6] \) are the “corners” of \( D^4_1 \) and \( D^1_1 \) and the points \( z_i, i \in [6] \) then stand for the corners of \( D^4_2 \) and \( D^1_2 \).

One then has

\[
x \in (z_1, z_2, z_3) \text{ and } y \in (z_4, z_5, z_6)
\]
To estimate the distance of $x$ and $y$, we therefore have to estimate the maximal distance between the $z_i, i \in [6]$.

Due to symmetry reasons, it suffices to look for

$$\max\{d(z_1, z_4), d(z_2, z_4), d(z_2, z_6)\}$$

For this, we observe (using the convexity of $d$), that the following identities hold:

\begin{align*}
    d(z_4, z_6) &\leq \frac{1}{2}d(y_3, y_6) \leq \frac{1}{4}d(x_2, x_4) \leq \frac{1}{4}\text{diam}(A_n) \\
    d(z_1, z_2) &\leq \frac{1}{2}d(y_3, y_2) \leq \frac{1}{4}d(x_1, x_2) \leq \frac{1}{4}\text{diam}(A_n) \\
    d(y_1, y_5) &\leq \frac{1}{2}(x_1, x_4) \leq \frac{1}{2}\text{diam}(A_n)
\end{align*}

Hence,

- Case $(z_2, z_4)$:
  \[d(z_2, z_4) \leq \frac{1}{2}d(y_1, y_5) \leq \frac{1}{4}\text{diam}(A_n)\]

- Case $(z_1, z_4)$:
  \[d(z_1, z_4) \leq d(z_1, z_2) + d(z_2, z_4) \leq \frac{1}{2}\text{diam}(A_n)\]

- Case $(z_1, z_6)$:
  \[d(z_1, z_6) \leq d(z_1, z_2) + d(z_2, z_4) + d(z_4, z_6) \leq \frac{3}{4}\text{diam}(A_n)\]

- Case $(z_2, z_6)$:
  \[d(z_2, z_6) \leq d(z_2, z_4) + d(z_4, z_6) \leq \frac{1}{2}\text{diam}(A_n)\]

But this forces the diameter of $A_{n+1}$ to be no more than $\frac{3}{4}\text{diam}(A_n)$ and again, as $\text{conv}(A_{n+1}) \subset \text{conv}(A_n)$, this implies $\bigcap_{n\in N}\text{conv}(A_n)$ to contain exactly one point, which shall be called the barycenter $b_{4}(A)$ of $A$. 
Remark 8.8.

Since the construction described above only involved geodesics from the convex bicombing that are mapped to one another by bicombing-respecting maps, we see, that $b_n : X(n) \to X$ is equivariant with respect to the action of the group of bicombing-respecting maps (for $n = 3$ and $n = 4$).

Since, on top of that, $\gamma(x, y, \frac{1}{2}) = \gamma(y, x, 1 - \frac{1}{2}) = \gamma(y, x, \frac{1}{2})$ for any $x, y \in X$, we see that the action on $S_d$ on $A$ does not change $b_d(A)$ (here, $d \in \{2, 3, 4\}$).

Hence $b_n(A)$ is a well-defined barycenter of $A \in X(n)$ in those cases.

8.2 Barycenters for finite sets

Now, we want to generalize the above construction: iteratively and starting from $A = A_1$, out of the $(n + 1)$-tuple $A_k \in X(n+1)$, we will construct $A_{k+1}$ as the $(n + 1)$-tuple of $n$-barycenters of the $n + 1$ sub-tuples of length $n$.

We will assume by induction, that those $n$-barycenters are already constructable.

By showing, that the diameter of $A_{k+1}$ is then uniformly smaller than the diameter of $A_k$, it will again and by the same arguments as above be obvious, that

$$\left| \bigcap_{k \in \mathbb{N}} \text{conv}(A_k) \right| = 1$$

and one defines

$$b_n(A_1) := \bigcap_{k \in \mathbb{N}} \text{conv}(A_k)$$

This construction shall be called the barycenter construction.

Moreover, we will prove the barycenter map to be non-expansive with respect to the distance on $X(n)$:

**Definition** (non-expansive map).

A map $f : X \to Y$ between metric spaces $(X, d_X)$ and $(Y, d_Y)$ is called non-expansive, iff

$$d_Y(f(x), f(x')) \leq d_X(x, x')$$
Remark 8.9.
For a convergent sequence \( (x_n)_{n \in \mathbb{N}} \subset X \) converging to \( x \in X \) and a non-expansive map \( f : X \to Y \), one has

\[
\lim_{n \to \infty} x_n = x \in X \\
\Rightarrow 0 = \lim_{n \to \infty} d_X(x_n, x) \geq \lim_{n \to \infty} d_Y(f(x_n), f(x)) \geq 0 \\
\Rightarrow \lim_{n \to \infty} f(x_n) = f(x)
\]

and we see, that non-expansiveness implies continuity.

In order to prove, that the barycenter construction works, we define the following auxiliary map

\[
\tilde{b}_{n+1} : X_{(n+1)} \to X_{(n+1)} \\
(x_i, i \in [n+1]) \mapsto (b_n(x_j, i \neq j \in [n+1]), i \in [n+1])
\]

We want to prove the following theorem

**Theorem 8.10.**
The barycenter construction works for every \( n \in \mathbb{N} \) and any \( n \)-tuple \( (x_i, i \in [n]) \).

It yields a non-expansive map \( b_n : X_{(n)} \to X \).

**Proof.** As announced above, we will proceed by induction.

We already know, that the barycenter construction works for \( n \leq 4 \) and is non-expansive for \( n \leq 2 \) (this is the convexity of \( d \)).

We assume by induction, that the barycenter construction yields non-expansive barycenter maps \( b_k : X_{(k)} \to X \) for \( k \) up to \( n \), so that the map \( \tilde{b}_{n+1} \) defined above is well-defined.

Moreover, by construction, \( \tilde{b}_{n+1} \) is equivariant with respect to bicombing-respecting maps.

If we can show, that

\[
\text{diam} \left( \tilde{b}_{n+1}(A) \right) \leq \kappa \text{diam}(A) \forall A \in X_{(n+1)}
\]

for some \( \kappa < 1 \), the sequence

\[
\left( \tilde{b}_{n+1}^k(A) \right)_{k \in \mathbb{N}} \subset X_{(n+1)}
\]
forms a Cauchy sequence for every $A \in X_{(n+1)}$.

Since, by construction $\text{conv} \left( \tilde{b}_{n+1}(A) \right) \subset \text{conv}(A) \ \forall A \in X_{(n+1)}$, we have

$$d_{(n+1)} \left( \tilde{b}_{n+1}^k(A), \tilde{b}_{n+1}^l(A) \right) \leq \text{diam} \left( \tilde{b}_{n+1}(A) \right) \leq \kappa^k \text{diam}(A) \ \forall l > k$$

Hence, the limit map

$$\hat{b}_{n+1} : X_{(n+1)} \to X_{(n+1)}, \ (x_i, i \in [n+1]) \mapsto \lim_{k \to \infty} \tilde{b}_{n+1}^k((x_i, i \in [n+1]))$$

is well-defined and still equivariant with respect to bicombing-respecting maps as we assumed the bicombing to be continuous.

But then, by construction

$$\hat{b}_{n+1}(A) = (f(A), \ldots, f(A))$$

for some $f(A) \in \text{conv}(A) \subset X$ and every $A \in X_{(n+1)}$ (i.e., the tuple is constant), since, as in the last section, every sequence $(x_k)_{k \in \mathbb{N}}$ with points $x_k \in \tilde{b}_n^k(A)$ is a Cauchy sequence and all those Cauchy sequences have the same limit point.

We define this limit point $f(A)$ as $b_{n+1}(A)$, the $(n+1)$-barycenter of $A \in X_{(n+1)}$.

The resulting barycenter map is now equivariant, since the barycenter construction only used maps which are equivariant. It obviously maps tuples to points in the corresponding convex hull and is invariant under permuting the tuples, since in every step of the barycenter construction was already invariant. Hence, the resulting map is a barycenter map according to our definition.

The theorem now follows from the following

**Lemma 8.11.**

Let $b_n : X_{(n)} \to X$ be a non-expansive barycenter map. Then the map $\hat{b}_{n+1}$ defined above fulfills the following properties:

1. $\text{diam} \left( \hat{b}_{n+1}(A) \right) \leq \frac{1}{n} \text{diam } A \ \forall A \in X_{(n+1)}$

2. $\hat{b}_{n+1} : X_{(n+1)} \to X_{(n+1)}$ is non-expansive.
Let us first finish the proof of Theorem 8.10:

As discussed above, property (1) implies the existence of a barycenter map \( b_{n+1} \).

Now, we prove, that property (2) of Lemma 8.11 implies the non-expansiveness of \( b_{n+1} \):

\[
\begin{align*}
\| b_{n+1}(x_i, i \in [n+1]) - b_{n+1}(y_i, i \in [n+1]) \| & = \lim_{k \to \infty} d_{n+1}(b_{n+1}(x_i, i \in [n+1]), b_{n+1}(y_i, i \in [n+1])) \\
& \leq \lim_{k \to \infty} d_{n+1}(x_i, i \in [n+1], y_i, i \in [n+1]) \\
& \leq \text{diam } A
\end{align*}
\]

Proof. (of Lemma 8.11)

We show both properties individually:

(1) Let \( A = (x_i, i \in [n+1]) \in X_{(n+1)} \) and \( y_1 \neq y_2 \in \hat{b}_{n+1}(A) \) be arbitrary.

Then, there are \( j \neq k \in [n+1] \) such that

\[
y_1 = b_n((x_1, \ldots, \hat{x}_j, \ldots, x_{n+1})), \quad y_2 = b_n((x_1, \ldots, \hat{x}_k, \ldots, x_{n+1}))
\]

and one easily sees by using the non-expansiveness of \( b_n \) and the definition of \( d_{(n)} \), that

\[
d(y_1, y_2) = d(b_n((x_1, \ldots, \hat{x}_j, \ldots, x_{n+1})), b_n((x_1, \ldots, \hat{x}_k, \ldots, x_{n+1}))) \\
\leq \frac{1}{n} \left( d(x_k, x_j) + \sum_{i \in [n+1] \setminus \{j,k\}} d(x_i, x_i) \right) \\
= \frac{1}{n} d(x_k, x_j) \\
\leq \frac{\text{diam } A}{n}
\]
Since this was true for arbitrary \( y_1 \) and \( y_2 \in \hat{b}_{n+1}(A) \), we see, that

\[
\text{diam} \left( \hat{b}_{n+1}(A) \right) \leq \frac{1}{n} \text{diam } A
\]

(2) Let \((x_i, i \in [n+1])\) and \((y_i, i \in [n+1])\) be arbitrary \((n+1)\)-tuples. Choose the labelling in such a way that

\[
d_{(n+1)} \left( (x_i, i \in [n+1]), (y_i, i \in [n+1]) \right) = \min_{\sigma \in S_n} \frac{1}{n+1} \sum_{i \in [n+1]} d(x_i, y_{\sigma(i)})
\]

\[
= \frac{1}{n+1} \sum_{i \in [n+1]} d(x_i, y_i)
\]

To abbreviate the equations below, we introduce the notation

\[
U_k := [n+1] \setminus \{k\} \text{ and } \bar{x}_k = (x_i, i \in U_k) \in X_{(n)}
\]

Let \(\text{Perm}(U_k)\) denote the group of all permutations of \(U_k\).

Then, by using the non-expansiveness of \(b_n\) (by induction), we see

\[
d_{(n+1)} \left( \hat{b}_{n+1}((x_i, i \in [n+1])), \hat{b}_{n+1}((y_i, i \in [n+1])) \right)
\]

\[
= d_{(n+1)} \left( (b_n(\bar{x}_k), k \in [n+1]), (b_n(\bar{y}_k), k \in [n+1])) \right)
\]

\[
= \min_{\sigma \in S_n} \frac{1}{n+1} \sum_{k \in [n+1]} d \left( b_n(\bar{x}_k), b_n(\bar{y}_{\sigma(k)}) \right)
\]

\[
\leq \frac{1}{n+1} \sum_{k \in [n+1]} d \left( b_n(\bar{x}_k), b_n(\bar{y}_k) \right)
\]

\[
= \frac{1}{n+1} \sum_{k \in [n+1]} d_{(n)}(\bar{x}_k, \bar{y}_k)
\]

\[
= \frac{1}{n+1} \sum_{k \in [n+1]} \min_{\tau \in \text{Perm}(U_k)} \frac{1}{n} \sum_{j \in U_k} d \left( x_j, y_{\tau(j)} \right)
\]

\[
\leq \frac{1}{n+1} \sum_{k \in [n+1]} \frac{1}{n} \sum_{j \in U_k} d \left( x_j, y_j \right)
\]

\[
= \frac{1}{n+1} \sum_{k \in [n+1]} d(x_k, y_k)
\]

\[
= d_{(n+1)} \left( (x_i, i \in [n+1]), (y_i, i \in [n+1]) \right)
\]
Corollary 8.12.

One has

$$d(b(A \cup B), b(A \cup C)) \leq \frac{|B|}{|A \cup B|} \text{diam}(B \cup C)$$

for finite and disjoint subsets $A, B, C \subset X$ of a GCB-space $(X, d)$ with $B$ and $C$ having the same cardinality.

Proof. Choose some bijection $\sigma' : B \to C$ and define

$$\sigma : A \cup B \to A \cup C, \quad \sigma(x) = \begin{cases} x & x \in A \\ \sigma'(x) & x \in B \end{cases}$$

Then, since the barycenter map is non-expansive, one sees (here $\text{Bij}(X, Y)$ shall denote the group of all bijections between sets $X$ and $Y$)

$$d(b(A \cup B), b(A \cup C)) \leq d_{|A|+|B|}((x, x \in A \cup B), (y, y \in A \cup C))$$

$$= \min_{\varphi \in \text{Bij}(A \cup B, A \cup C)} \frac{1}{|A| + |B|} \sum_{x \in A \cup B} d(x, \varphi(x))$$

$$\leq \frac{1}{|A| + |B|} \sum_{x \in A \cup B} d(x, \sigma(x))$$

$$= \frac{1}{|A| + |B|} \left( \sum_{x_i \in A} d(x_i, x_i) + \sum_{y_i \in B} d(y_i, \sigma'(y_i)) \right)$$

$$= \frac{1}{|A| + |B|} \sum_{y \in B} d(y, \sigma'(y))$$

$$\leq \frac{|B|}{|A \cup B|} \text{diam}(B \cup C)$$
8.3 Dependence of barycenters on the finite set

One could wonder, whether the barycenter maps defined above respect the GCB-structure in the sense that they send tuples of geodesics to a geodesic.

The following theorem shows, that this is true for any \( n \in \mathbb{N} \), if it holds in the case \( n = 2 \).

**Theorem 8.13.**

Let \( X \) be a GCB space such that the set of midpoints between any two geodesics is itself again a geodesic.

Then, for given \( A = (x_i, \ i \in [n]) \) and \( B = (y_i, \ i \in [n]) \in X_{(n)} \), one has

\[
b_n \left( (\gamma(x_i, y_i, t), \ i \in [n]) \right) = \gamma(b_n(A), b_n(B), t) \quad \forall n \in \mathbb{N}
\]

**Proof.** We prove this by induction the first step \( n = 2 \) being assumed.

Assume the theorem to hold for \( n - 1 \).

Then, by construction, the barycenter \( b_n \left( (\gamma(x_i, y_i, t), \ i \in [n]) \right) \) is the \( d \)-limit of \( (z_i(t))_{i \in \mathbb{N}} = z_i^{(1)}(t) \), where

\[
z_i^{(k)}(t) = \begin{cases} 
\gamma(x_k, y_k, t) & i = 1 \\
b_{n-1} \left( \left( z_i^{(l)}(t), \ k \neq l \in [n] \right) \right) & n \neq 1
\end{cases}
\]

By induction, we know, that \( z_i(t) = \gamma(z_i(0), z_i(1), t) \).

Using the fact, that \( z_i(0) \) and \( z_i(1) \) converge to the barycenters \( b_n((x_i, i \in [n])) \) and \( b_n((y_i, i \in [n])) \) respectively and the continuity of the geodesic bicombing, we see, that

\[
b_n \left( (\gamma(x_i, y_i, t), \ i \in [n]) \right) = \lim_{i \to \infty} z_i(t)
\]

\[
= \lim_{i \to \infty} \gamma(z_i(0), z_i(1), t)
\]

\[
= \gamma \left( \lim_{i \to \infty} z_i(0), \lim_{i \to \infty} z_i(1), t \right)
\]

\[
= \gamma \left( b_n([x_i, i \in [n]]), b_n([y_i, i \in [n]]), t \right)
\]

which proves the claim. \( \square \)
8 Barycenters for finite sets in GCB-spaces

8.4 Amenable groups and GCB-spaces

In this section, we will prove that bicombing-respecting actions by discrete countable groups on a GCB space with property (C) have fixed points convex close to any bounded orbit, if the action restricted to the orbit is amenable.

Amenable actions by a group $G$ on a space $X$ are normally defined as actions allowing for $G$-invariant means (see [YGM07] for example). As proven for example by Rosenblatt in [Ros73], this is equivalent to the following definition:

**Definition (amenable action).**
An action of a countable discrete group $G$ on a set $X$ is called amenable, if for any finite $S \subset G$ and any $\varepsilon > 0$, one can find a finite set $A \subset X$, such that
\[
\frac{|A \Delta gA|}{|A|} < \varepsilon \quad \forall g \in S
\]

**Remark 8.14.**
Amenable actions always act amenably (one uses for example the first of the 3 equivalent ways of defining amenability as given in Chapter 2).

**Definition (Følner sequence for a group action).**
Let $G$ act amenably on $X$. Since $G$ is countable, it is an ascending union of finite sets $U_n$. Let $\varepsilon_n = \frac{1}{n}$, then the corresponding sequence $(F_n)_{n \in \mathbb{N}}$ such that
\[
\frac{|F_n \Delta gF_n|}{|F_n|} < \varepsilon \quad \forall g \in U_n
\]
is called Følner sequence for this action.

**Theorem 8.15.**
Let $X$ be a GCB-Space with Property (C) and $G$ be a group acting on $X$ bicombing respectively, such that the action

1. allows for at least one bounded orbit $Gx$
2. restricts to an amenable action on $Gx$

Then, there is a fixed point $x$, such that $d(x, Gx) \leq \text{diam}(\text{conv}(Gx))$. 

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8.4 Amenable groups and GCB-spaces

Proof. Let $F_n \subset Gx$ be a Følner-sequence for the restricted action of $G$ on $Gx$. Consider the sequence

$$ (x_n)_{n \in \mathbb{N}} := (b_{|F_n|} (F_n))_{n \in \mathbb{N}} \subset \text{conv}(Gx) $$

By construction, any $g$ eventually lies in $U_n \forall n > N$ (with $N$ big enough) and by the definition of $F_n$,

$$ \frac{|F_n \Delta gF_n|}{|F_n|} \rightarrow 0 \forall g \in G $$

But this means by Corollary 8.12 that for any $g \in G$

$$ d(x_n, gx_n) = d\left(b_{|F_n|} (F_n), gb_{|F_n|} (F_n)\right) $$

$$ = d\left(b_{|F_n|} (F_n), b_{|F_n|} (gF_n)\right) $$

$$ = d\left(b_{|F_n|} ((F_n \cap gF_n) \cup (F_n \setminus gF_n)), b_{|F_n|} ((F_n \cap gF_n) \cup (F_n \setminus F_n))\right) $$

$$ \leq \frac{|F_n \setminus gF_n|}{|F_n|} \text{diam } (F_n \Delta gF_n) $$

$$ \leq \frac{|F_n \Delta gF_n|}{2|F_n|} \text{diam } (gF_n \cup F_n) $$

$$ \leq \frac{|F_n \Delta gF_n|}{2|F_n|} \text{diam } Gx $$

$$ \rightarrow 0 $$

By definition, Property (C) implies the existence of some $\hat{x}$ being convex close to $\text{conv}(\{x_n, n \in \mathbb{N}\})$ such that $g\hat{x} = \hat{x}$ for any $g \in G$ and

$$ d(\hat{x}, \text{conv}(Gx)) \leq d(\hat{x}, \text{conv}(\{x_n, n \in \mathbb{N}\})) $$

$$ \leq \text{diam } \text{conv}(\{x_n, n \in \mathbb{N}\}) $$

$$ \leq \text{diam } \text{conv}(Gx) $$

\[ \square \]

Corollary 8.16.

Let $G$ be an amenable group acting on $\mathcal{P}(H)$ by bicothing-respecting isometries. Then there is a fixed point in $X_\pi$. In particular, this implies Theorem 2.4 and one direction in Corollary 5.9 as well as Theorem 2.10.

Proof. Apply the above result to $X = X_\pi$. \[ \square \]
One could wonder, if for non-unitarisable groups (or possibly for unitarisable and non-amenable groups, where the fixed point to some group action on $\mathcal{P}(H)$ is far away from the $G$-orbit of $\text{id}_H$), one may find a model for the classifying space (defined in [Luc02], for example) as a bounded subspace of $\mathcal{P}(H)$.

The following corollary gives a partial answer to this. The reader may be reminded that an action of a group on a space $X$ is **free**, if $gx = hx$ for some $g, h \in G$, $x \in X$ implies $g = h$.

**Corollary 8.17.**

Let $G$ act on $\mathcal{P}(H)$ in a way that is induced by a representation of $G$ on $H$. Then $G$ never acts freely on the set $X_\pi$.

Moreover, every element in $G$ fixes some point inside $X_\pi$.

**Proof.** Let $g \in G$ be arbitrary. Then $g$ generates a subgroup which is amenable (finite or $\mathbb{Z}$). Thus, there is a fixed point for this subgroup in $X_\pi$ (we apply Theorem 8.15 to $X_\pi$). \hfill $\Box$

Above, we have seen, that actions on $\mathcal{P}(H)$ coming from a representation of $G$ will never be free on $\text{conv}(\rho_\pi(G, \text{id}_H))$. Naturally, one could therefore ask for possible stabilizers.

**Remark 8.18.**

The following theorem shows, that a group $G$ acting on a GCB space $X$ by bicombing respecting maps will either have a fixed point or all stabilizers of elements $x \in X$ will be of infinite index.

**Theorem 8.19.**

Let $G$ act by bicombing-respecting maps on a GCB-space $X$ such that some finite index subgroup $H \triangleleft G$ fixes a point in $X$. Then $G$ has a fixed point.

**Proof.** Let $H \triangleleft G$ be a subgroup of index $n$ having a fixed point $x$ in $X$. Furthermore, let $\{e = g_1, \ldots, g_n\} \subset G$ be a choice of representatives of the cosets $G/H$.

Then $G = \bigsqcup_{i \in [n]} g_i H$, where $\bigsqcup$ denotes the disjoint union. Multiplying from the left with elements from the set $\{g_i, i \in [n]\}$ or $H$ permutes the cosets $g_i H$.

In other words, multiplying with $g \in G$ yields a bijection $\varphi_g : [n] \to [n]$ in such a way, that $gg_i \in g\varphi_g(i) H$. 

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Define \( y := b(\{g_i \cdot x|i \in [n]\}) \). Then, for arbitrary \( g \in G \), we see, that for some \( h \in H \)

\[
g \cdot y = \begin{align*}
g \cdot b(\{g_i \cdot x|i \in [n]\}) \\
= b(\{gg_i \cdot x|i \in [n]\}) \\
= b(\{g_{\phi(i)}h \cdot x|i \in [n]\}) \\
= b(\{g_{\psi(i)} \cdot x|i \in [n]\}) \\
= y
\end{align*}
\]

and \( y \) is a fixed point for the \( G \)-action. \( \square \)

**Remark 8.20.**

*In the theorem above, we did neither assume Property (C) nor any boundedness.*

**Definition** (virtual property).

A group \( G \) is said to have a property \( P \) virtually (we say, \( G \) is "virtually \( P \")), if there is some finite-index subgroup of \( G \), which has property \( P \).

We can immediately conclude the following corollary

**Corollary 8.21.**

Virtually unitarisable groups are unitarisable. Moreover, the constants in Theorems 2.9 and 5.7 are at most the infimum over the corresponding constants coming from finite index subgroups of \( G \).

**Proof.** Let \( \pi : G \to \text{Aut}(H) \) be a uniformly bounded representation of \( G \) on some Hilbert space \( H \), \( \Gamma < G \) be a finite-index unitarisable subgroup and \( \rho_\pi \) the induced action of \( G \) on \( \mathcal{P}(H) \).

Then (Corollary 5.7), \( \Gamma \) fixes a point \( x_\Gamma \) in the \( C_\Gamma + \frac{\alpha_\Gamma}{2} \text{diam}(\pi) \)-neighbourhood of the \( \Gamma \)-orbit of \( \text{id}_H \in \mathcal{P}(H) \), which is a subset of \( C_\Gamma + \frac{\alpha_\Gamma}{2} \text{diam}(\pi) \)-neighbourhood of the \( G \)-orbit of \( \text{id}_H \).

Now, Theorem 8.19 yields a \( G \)-fixed point (proving, that \( G \) is unitarisable), which (by construction) is the barycenter of the finite set of \( G \)-translates of \( x_\Gamma \). Hence, it is at most \( C_\Gamma + \frac{\alpha_\Gamma}{2} \text{diam}(\pi) \) away from the \( G \)-orbit of \( \text{id}_H \) (and therefore, as it is a \( G \)-fixed point, from \( \text{id}_H \) itself). Hence, we have \( \alpha_G \leq \alpha_\Gamma \) as well as \( C_G \leq C_\Gamma \) (and therefore, \( K_G \leq K_\Gamma \) for the universal constant \( K \) in Theorem 2.9).

\( \square \)
An important property of group actions is properness:

**Definition** (proper action).

An action of a group $G$ on a topological space $X$ is called proper, if preimages of compact subsets of $X \times X$ under the map $\rho : G \times X \to X \times X$, $(g, x) \mapsto (gx, x)$ are compact.

We can now derive the following result about proper actions of arbitrary groups on GCB spaces:

**Corollary 8.22.**

Let $G$ be a discrete group acting properly on a Property (C) GCB-space $X$ by bicombing-respecting isometries and with at least one bounded orbit.

Then, every amenable subgroup of $G$ is finite.

In particular, $G$ is a torsion group, i.e. every element has finite order.

**Proof.** $\Gamma < G$ be an amenable subgroup and $x \in X$ be a fixed point of $\Gamma$ (by Theorem 8.15).

Then, if $\Gamma$ is of infinite order, there is an infinite stabilizer for some $x \in X$ and the action cannot be proper:

The projection of $\rho^{-1} (\{(x, x)\})$ to $G$ is infinite, where $\rho$ is the map from the definition of a proper action. But infinite subsets of a discrete group are non-compact (the covering by the open 1-point sets containing the elements from the set itself does not have a finite subcover), and (the projection is continuous) since continuous images of compact sets are compact, this implies non-compactness of $A$.

Finally, since the group generated by some element $g \in G$ is amenable (finite or $\mathbb{Z}$), $G$ is torsion by the above argument. \hfill $\Box$
9 Distal actions and fixed points

In this last chapter, we will prove a generalization of the well-known Ryll-Nardzewski fixed-point-Theorem for distal actions of groups on particular GCB-spaces. The proof is very much along the lines of the geometric proof by Asplund and Namioka in [EAIN67]. It is also referred to in [Pet89], where it is attributed to Glasner ([Gla76]).

In order to state the original Ryll-Nardzewski Theorem, we first need to define distality for (semi-)group actions:

**Definition** (distal action).
A group action on a metric space is called distal, if for any \( x \neq y \in X \) there is an \( \varepsilon > 0 \) such that
\[
d(g(x), g(y)) > \varepsilon \quad \forall g \in G
\]

**Theorem 9.1** (Ryll-Nardzewski fixed point Theorem, [RN67]).
Let \( E \) be a semigroup acting distally and affinely on a non-empty, weakly compact convex subset \( X \) of a locally convex Hausdorff topological vector space \( A \). Then \( E \) has a fixed point in \( X \).

Conceptually, the proof uses the fact that one knows of the existence of fixed points for a single affine map \( X \rightarrow X \) and uses this to construct fixed points for finite sets by a midpoint-construction.

For the proof to work, one needs the following property, which holds for any convex and weakly compact set in \( A \) with respect to the weak topology:

**Definition** (small caps).
Let \( A \) be a locally convex topological vector space and \( \tau \) a topology on \( A \). Then we say, a \( \tau \)-closed subset \( K \) of \( A \) has small caps, if for any continuous seminorm \( \rho \) and any \( \varepsilon > 0 \) there is a non-empty set \( K_{\varepsilon} \subset K \) such that
\[
(1) \quad \sup_{x,y \in K_{\varepsilon}} \rho(x - y) < \varepsilon
\]
\[
(2) \quad K \setminus K_{\varepsilon} \text{ is convex and } \tau\text{-closed.}
\]

\(^5\) a topological vector space is called locally convex, if every point has a neighbourhood basis of convex sets
Remark 9.2.
In the articles mentioned above, this property is not defined as a property of the topological vector spaces. It is proven for convex closed sets, which in addition are compact with respect to the weak topology. The proof uses the Baire Category and Krein-Millman Theorems.

We translate this property to the language of GCB-spaces:

**Definition** (small caps).
Let $X$ be a GCB space together with a topology $\tau$. Then, a convex closed subset $C \subset X$ is said to have small caps, for any $\varepsilon > 0$ there is a non-empty set $K_\varepsilon \subset C$ such that

1. $\text{diam}(K_\varepsilon) < \varepsilon$
2. $K \setminus K_\varepsilon$ is convex and $\tau$-closed.

Additionally, we introduce the following geometric property

**Definition** (barycenter stability).
A GCB-space $X$ is called $n$-barycenter-stable, if for any finite set $A \subset X$ such that $|A| \leq n - 1$

$$b(A \sqcup \{x\}) = x \iff b(A) = x$$

**Lemma 9.3.**
Every GCB-space is at least 3-barycenter stable.

*Proof.* As 2-barycenter-stability is obvious, we need to show the following equivalence to hold for every $x, y, z$ from a GCB-space $X$.

$$b(x, y, z) = z \iff z = b(x, y)$$

First, we show, that for $b(x, y, z) = z$ to hold, $z$ needs to be a point on the geodesic between $x$ and $y$. 

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Assume this not to be the case and choose \( s \in [0, 1] \) such that
\[
d(z, \gamma(x, y)) := \min_{t \in [0, 1]} d(z, \gamma(x, y, t)) = d(z, \gamma(x, y, s))
\]
This minimum has to exist, as \( \gamma(x, y, \cdot) : [0, 1] \to X \) is a continous map and [0,1] is compact.

Observe, that by assumption this number is strictly positive.

Then by the convexity of \( d \)
\[
d(b(x, z), \gamma(x, y)) \leq d \left( \gamma \left( x, z, \frac{1}{2} \right), \gamma \left( x, y, \frac{1}{2} s \right) \right)
= d \left( \gamma \left( x, z, \frac{1}{2} \right), \gamma \left( x, \gamma(x, y, s), \frac{1}{2} \right) \right)
\leq \frac{1}{2} d(z, \gamma(x, y, s))
= \frac{1}{2} d(z, \gamma(x, y))
\]
and with the same argument, one gets
\[
d(b(y, z), \gamma(x, y)) \leq \frac{1}{2} d(z, \gamma(x, y))
\]
Now, the barycenter \( b(x, y, z) \) lies in the closed convex hull of \( b(x, y), b(y, z) \) and \( b(x, y) = \gamma(x, y, \frac{1}{2}) \).

Hence,
\[
d(b(x, y, z), \gamma(x, y)) \leq \frac{1}{2} d(z, \gamma(x, y)) < d(z, \gamma(x, y))
\]
which obviously implies that \( b(x, y, z) \neq z \).

This reduces the claim to the following:
\[
b(x, y, \gamma(x, y, t)) = b(x, y) \iff t = \frac{1}{2}
\]
We will prove that
\[
b(x, y, \gamma(x, y, t)) = \gamma \left( x, y, \frac{1}{3} (1 + t) \right)
\]
which coincides with \( b(x, y) \) if and only if \( t \neq \frac{1}{2} \).
One has
\[ b(\gamma(x, y, a), \gamma(x, y, b)) = \gamma \left(x, y, \frac{a + b}{2}\right) \]  
(14)

(here, we use the third property from the definition of a geodesic bicombing).

Therefore, the intermediate 3-tuples \( D_n = (x_n, y_n, z_n) \) in the barycenter construction to construct \( b(x, y, \gamma(x, y, t)) \), all consist of points on the geodesic between \( x \) and \( y \) and are uniquely defined through its “time”:

\[ D_n = (\gamma(x, y, s_n), \gamma(x, y, t_n), \gamma(x, y, u_n)) \]

Then, by the barycenter construction and equation (14), one gets the following recursions:

\[ s_{n+1} = \frac{t_n + u_n}{2}, \quad t_{n+1} = \frac{s_n + u_n}{2}, \quad u_{n+1} = \frac{s_n + t_n}{2}, \quad s_0 = 0, \quad t_0 = t, \quad u_0 = 1 \]

We claim, that

\[ s_{2n} = \frac{1}{4^n} \left(\frac{4^n - 1}{3} (1 + t)\right), \quad t_{2n} = \frac{s_{2n} + t}{4^n}, \quad u_{2n} = s_{2n} + \frac{1}{4^n} \]

which we prove by induction.

The case \( n = 0 \) is obvious.

For the induction step, one calculates

\[ s_{2(n+1)} = \frac{1}{2} \left(t_{2n+1} + u_{2n+1}\right) \]
\[ = \frac{1}{4} \left(2s_{2n} + t_{2n} + u_{2n}\right) \]
\[ = \frac{1}{4} \left(4s_{2n} + \frac{1 + t}{4^n}\right) \]
\[ = \frac{1}{4} \left(\frac{1}{4^n} \left(4^{4^n - 1} \left(\frac{4^n - 1}{3} (1 + t) + (1 + t)\right)\right)\right) \]
\[ = \frac{1}{4^{n+1}} \left(\frac{4^{n+1} - 4 + 3}{3} (1 + t)\right) \]
\[ = \frac{1}{4^{n+1}} \left(\frac{4^{n+1} - 1}{3} (1 + t)\right) \]

and the other cases are completely analogous.
Finally
\[
b(x, y, \gamma(x, y, t)) = \lim_{n \to \infty} x_n
\]
\[
= \lim_{n \to \infty} \gamma(x, y, s_n)
\]
\[
= \gamma \left( x, y, \lim_{n \to \infty} \frac{1}{4^n} \frac{1}{4^n} - \frac{1}{3} (1 + t) \right)
\]
\[
= \gamma \left( x, y, \frac{1}{3} (1 + t) \right)
\]
\[\Box\]

In many situations, one knows of the existence of fixed points for a single continuous self-map. (For example, every affine self-map of a Banach space leaving invariant a weakly compact convex subset has a fixed point, every continuous map of the \(n\)-dimensional ball \(D^n \subset \mathbb{R}^n\) has a fixed point).

This motivates the following

**Definition** (fixed point property).

We say that a topological space \(X\) has the fixed point property with respect to the space \(\mathcal{F}\) of functions \(f: X \to X\), if every element \(f \in \mathcal{F}\) has a fixed point in \(X\).

We will now prove the following generalization to the Ryll-Nardzewski Theorem:

**Theorem 9.4.**

Let \(X\) be an \(n\)-barycenter stable GCB space with Property (C) and let \(G\) be a group, which is generated by \(n\) elements and acts distally on \(X\) respecting the bicombing, such that \(X\) fixes a bounded, convex \(d\)-closed set \(C\).

Further, let \(\tau\) be a topology on \(C\) such that

- every bounded, \(\tau\)-closed and convex subset \(K \subset X\) has small caps
- \(X\) has the fixed point property with respect to continuous maps leaving invariant bounded convex sets

Then \(G\) fixes a point in \(C\).

If moreover \(C\) is \(\tau\)-compact and the group action is \(\tau\)-continuous, the assumption on \(G\) being finitely generated can be dropped.
Proof. First, one may drop the finitely generated assumption, if $C$ is $\tau$-compact and the action $\tau$-continuous, because then, the set of fixed points for finitely generated subgroups in $C$ is closed and hence $\tau$-compact, and convex.

Therefore, by the finite intersection property, there will be a fixed point for $G$, if all finite sets of $G$ have a fixed point.

Let $g_1, \ldots, g_n$ be a finite set of generators for $G$. Then, define the following map

$$g_0 : X \to X, \ x \mapsto b((g_i(x), \ i \in [n]))$$

By Theorem 8.13 we know, that as all $g_i$ fix the convex and $d$-closed set $C$, so does $g_0$. Moreover, as all $g_i$ and the barycenter map are continuous, $g_0$ also is.

This means, that from the fixed point property, we may conclude, that there is a fixed point $x_0$ of $g_0$.

Assume for contradiction that $g_i(x_0) \neq x_0$ $\forall i \in [n]$: otherwise, we may change the finite set of generators to the set of those generators not fixing $x_0$ and redefine $g_0$ as the barycenter to the smaller set. Due to the barycenter stability, this map will still fix $x_0$.

Then, by distality, there is some $\varepsilon > 0$ such that

$$d(gx_0, gg_i x_0) > \varepsilon \ \forall i \in [n], \ \forall g \in G \tag{15}$$

Now, the $\tau$-closure of the convex hull of the orbit $Gx_0$ is a $\tau$-closed, convex subset of $C$. Since $\tau$ allows for small caps, there is some non-empty $K \subset C$ such that diam $K < \varepsilon$ and $K \setminus C$ is $\tau$-closed and convex. As $K \neq \emptyset$ there must be some $g \in G$ such that $gx_0 \in K$.

But then

$$K \ni gx_0 = gg_0 x_0 = g(b(\{g_i x_0, \ i \in [n]\})) = b(\{gg_i x_0, \ i \in [n]\})$$

and therefore some $gg_i x_0 \in K$. But this means

$$\varepsilon > d(gg_i(x_0), gx_0)$$

contradicting (15).
Remark 9.5.
If the group $G$ in the preceding theorem acts by geodesic bicombing respecting isometries, the map $g_0$, as the barycenter of isometries, is non-expansive. Hence, one only needs $X$ to have the fixed point property with respect to non-expansive maps.

Putting this remark together with Theorem 8.19, we get the following

Corollary 9.6.
Let $G$ be a group acting by bicombing respecting isometries on a GCB space $X$. If $G$ has a finite index subgroup $H$, such that

1. $H$ is generated by at most 3 elements
2. $H$ fixes a bounded $d$-closed set $C$ in $X$
3. $C$ has the fixed point property with respect to non-expansive maps
4. there is some topology $\tau$ on $C$ such that $C$ allows for small caps

Then $G$ has a fixed point in $\text{conv}(C)$.

Proof. By Theorem 8.19, the existence of a fixed point in $C$ follows from the existence of an $H$-fixed point. As every GCB-space is 2-barycenter stable, the existence of fixed point follows from Theorem 9.4.

\qed
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