Fuzzy Description Logics with General Concept Inclusions

Dissertation

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Doktoringenieur (Dr.-Ing.)

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## Lattices

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<th>Symbol</th>
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<tr>
<td>$L$</td>
<td>A lattice (or its carrier set)</td>
<td>12</td>
</tr>
<tr>
<td>$\wedge$</td>
<td>A lattice infimum (meet)</td>
<td>12</td>
</tr>
<tr>
<td>$\vee$</td>
<td>A lattice supremum (join)</td>
<td>12</td>
</tr>
<tr>
<td>$\leq$</td>
<td>A partial order, e.g. on a lattice</td>
<td>12</td>
</tr>
<tr>
<td>$0$</td>
<td>The smallest element of a bounded lattice</td>
<td>12</td>
</tr>
<tr>
<td>$1$</td>
<td>The largest element of a bounded lattice</td>
<td>12</td>
</tr>
<tr>
<td>$2$</td>
<td>The $2$-element lattice</td>
<td>14</td>
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<tr>
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<td>The $4$-element lattice of Figure 2.4</td>
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## Mathematical Fuzzy Logic

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<td>$\otimes$</td>
<td>A t-norm</td>
<td>7, 12</td>
</tr>
<tr>
<td>$\oplus$</td>
<td>A t-conorm</td>
<td>11, 13</td>
</tr>
<tr>
<td>$\Rightarrow$</td>
<td>A residuum</td>
<td>10, 12</td>
</tr>
<tr>
<td>$\ominus$</td>
<td>A residual negation</td>
<td>10, 13</td>
</tr>
<tr>
<td>$\sim$</td>
<td>A De Morgan negation</td>
<td>11, 12</td>
</tr>
<tr>
<td>$G$</td>
<td>The Gödel t-norm</td>
<td>8</td>
</tr>
<tr>
<td>$\Pi$</td>
<td>The Product t-norm</td>
<td>8</td>
</tr>
<tr>
<td>$\mathfrak{t}$</td>
<td>The Łukasiewicz t-norm</td>
<td>8</td>
</tr>
<tr>
<td>$a, b$</td>
<td>Component bounds of a continuous t-norm over $[0, 1]$</td>
<td>8</td>
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## Description Logics

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<tr>
<td>$A, B$</td>
<td>Concept names</td>
<td>14</td>
</tr>
<tr>
<td>$r, s$</td>
<td>Role names or complex roles</td>
<td>14</td>
</tr>
<tr>
<td>$c, d$</td>
<td>Individual names</td>
<td>14</td>
</tr>
<tr>
<td>$C, D$</td>
<td>Concepts</td>
<td>14</td>
</tr>
<tr>
<td>$\mathcal{I}$</td>
<td>An interpretation</td>
<td>14</td>
</tr>
<tr>
<td>$\Delta^\mathcal{I}$</td>
<td>An interpretation domain</td>
<td>14</td>
</tr>
<tr>
<td>$\mathcal{I}^r$</td>
<td>An interpretation function</td>
<td>14</td>
</tr>
<tr>
<td>$r^{-}$</td>
<td>The inverse of a role name</td>
<td>14</td>
</tr>
<tr>
<td>$\pi$</td>
<td>The inverse of a complex role</td>
<td>14</td>
</tr>
<tr>
<td>$\top$</td>
<td>The top concept</td>
<td>15</td>
</tr>
<tr>
<td>$\bot$</td>
<td>The bottom concept</td>
<td>15</td>
</tr>
<tr>
<td>$\sqcap$</td>
<td>Concept conjunction</td>
<td>15</td>
</tr>
<tr>
<td>$\sqcup$</td>
<td>Concept disjunction</td>
<td>15</td>
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<tr>
<td>$\rightarrow$</td>
<td>Concept implication</td>
<td>15</td>
</tr>
<tr>
<td>$\neg$</td>
<td>Involutive concept negation</td>
<td>15</td>
</tr>
<tr>
<td>$\Box$</td>
<td>Residual concept negation</td>
<td>15</td>
</tr>
<tr>
<td>${c}$</td>
<td>Nominal concept</td>
<td>15</td>
</tr>
<tr>
<td>$\exists r.C$</td>
<td>Existential restriction</td>
<td>15</td>
</tr>
<tr>
<td>$\forall r.C$</td>
<td>Value restriction</td>
<td>15</td>
</tr>
<tr>
<td>$\mathcal{L}$</td>
<td>A classical DL</td>
<td>14</td>
</tr>
<tr>
<td>$L$-$\mathcal{L}$</td>
<td>A fuzzy DL over $L$</td>
<td>14</td>
</tr>
<tr>
<td>$\otimes$-$\mathcal{L}$</td>
<td>A fuzzy DL over $[0, 1]$ with t-norm $\otimes$</td>
<td>14</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td></td>
</tr>
<tr>
<td>c</td>
<td>Subscript indicating the restriction to crisp roles</td>
<td>41</td>
</tr>
<tr>
<td>A</td>
<td>A (local) (ordered) ABox</td>
<td>20, 70</td>
</tr>
<tr>
<td>&gt;</td>
<td>A place-holder for = or ≥</td>
<td>20</td>
</tr>
<tr>
<td>T</td>
<td>A (general or acyclic) TBox</td>
<td>20</td>
</tr>
<tr>
<td>R</td>
<td>An RBox</td>
<td>20</td>
</tr>
<tr>
<td>O</td>
<td>An ontology</td>
<td>20</td>
</tr>
<tr>
<td>Ind</td>
<td>The set of individual names occurring in an ABox or ontology</td>
<td>56, 141</td>
</tr>
<tr>
<td>sub</td>
<td>The set of subconcepts of a concept or an ontology</td>
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**Chapter 3. Finite Lattices**

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<thead>
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<tr>
<td>H</td>
<td>A Hintikka function or Hintikka set</td>
</tr>
<tr>
<td>supp</td>
<td>The support of a Hintikka function</td>
</tr>
<tr>
<td>K</td>
<td>The set {1, \ldots, k} of indices of successors in a Hintikka tree</td>
</tr>
<tr>
<td>(\varphi)</td>
<td>A bijection between the set of relevant quantified concepts and (K)</td>
</tr>
<tr>
<td>(\varphi_r)</td>
<td>The set of all indices of quantified concepts with role (r)</td>
</tr>
<tr>
<td>T</td>
<td>A Hintikka tree or tableau</td>
</tr>
<tr>
<td>g</td>
<td>A place-holder for the role degrees in a Hintikka tree</td>
</tr>
<tr>
<td>A</td>
<td>A looping (tree) automaton</td>
</tr>
<tr>
<td>r</td>
<td>A run of a looping automaton</td>
</tr>
<tr>
<td>A(_O)</td>
<td>A Hintikka automaton</td>
</tr>
<tr>
<td>(\leftarrow)</td>
<td>A blocking relation for a looping automaton</td>
</tr>
<tr>
<td>A(_S)</td>
<td>The subautomaton induced by a faithful family of functions</td>
</tr>
<tr>
<td>rd</td>
<td>The role depth of a concept or a Hintikka function</td>
</tr>
<tr>
<td>sub(_\leq n)</td>
<td>The set of subconcepts of role depth (\leq n)</td>
</tr>
<tr>
<td>(\mathcal{A}^*)</td>
<td>A pre-completion or a finite extension of a (local) ABox (\mathcal{A})</td>
</tr>
<tr>
<td>(\mathcal{A}_c)</td>
<td>The restriction of an (ordered) ABox or pre-completion to (c)</td>
</tr>
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**Chapter 4. Decidability**

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<tr>
<td>2</td>
<td>The reduction function (2: L \rightarrow 2)</td>
</tr>
<tr>
<td>crisp</td>
<td>A modified ontology resulting from replacing all truth degrees by 1</td>
</tr>
<tr>
<td>(\infty)</td>
<td>An arbitrary element of ({=, \geq, &gt;, \leq, &lt;})</td>
</tr>
<tr>
<td>S</td>
<td>An order structure</td>
</tr>
<tr>
<td>inv(_S)</td>
<td>The involution of an order structure (S)</td>
</tr>
<tr>
<td>order</td>
<td>The set of well-behaved total preorders over an order structure</td>
</tr>
<tr>
<td>(\preceq_{\ast})</td>
<td>A total preorder, usually an element of (\text{order}(S))</td>
</tr>
<tr>
<td>min(_{\ast})</td>
<td>The minimum function induced by (\preceq_{\ast} \in \text{order}(S))</td>
</tr>
<tr>
<td>res(_{\ast})</td>
<td>The residuum induced by (\preceq_{\ast} \in \text{order}(S))</td>
</tr>
<tr>
<td>cl</td>
<td>The closure under negation of the subconcepts in an ontology</td>
</tr>
<tr>
<td>cl(_V)</td>
<td>The closure under negation of the values in an ontology</td>
</tr>
<tr>
<td>(C_T)</td>
<td>An expression referring to the value of (C) at the parent node</td>
</tr>
<tr>
<td>cl(_T)</td>
<td>The set of relevant concepts of the parent node</td>
</tr>
<tr>
<td>(\mathcal{U})</td>
<td>The order structure for a Hintikka ordering</td>
</tr>
<tr>
<td>rcl</td>
<td>The closure under negation of the roles in an ontology</td>
</tr>
<tr>
<td>(\mathcal{W})</td>
<td>The order structure for an ordered pre-completion</td>
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**Chapter 5. Undecidability**

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<tbody>
<tr>
<td>(\mathcal{P})</td>
<td>An instance of the Post correspondence problem</td>
</tr>
<tr>
<td>(\Sigma)</td>
<td>The alphabet ({1, \ldots, s}) of (\mathcal{P})</td>
</tr>
<tr>
<td>(\Sigma_0)</td>
<td>(\Sigma \cup {0})</td>
</tr>
<tr>
<td>(u, v, w)</td>
<td>Words over (\Sigma) or (\Sigma_0), sometimes viewed as integers in base (s + 1)</td>
</tr>
<tr>
<td>(\leftarrow u)</td>
<td>The word resulting from (u) by reading it backwards</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>$N$</td>
<td>The set ${1, \ldots, n}$ of indices of the pairs in $\mathcal{P}$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>A word over $N$</td>
</tr>
<tr>
<td>$\text{Enc}$</td>
<td>A (valid) encoding function</td>
</tr>
<tr>
<td>$\mathcal{I}_\mathcal{P}$</td>
<td>The canonical model for $\mathcal{P}$</td>
</tr>
<tr>
<td>$\mathcal{P}_\triangle$</td>
<td>The canonical model property</td>
</tr>
<tr>
<td>$\mathcal{O}_\mathcal{P}$</td>
<td>The ontology for the canonical model property</td>
</tr>
<tr>
<td>$\mathcal{P}_{\text{ini}}$</td>
<td>The initialization property</td>
</tr>
<tr>
<td>$\mathcal{P} \rightarrow$</td>
<td>The successor property</td>
</tr>
<tr>
<td>$\mathcal{P}^\circ$</td>
<td>The concatenation property</td>
</tr>
<tr>
<td>$\mathcal{P} \Leftrightarrow$</td>
<td>The transfer property</td>
</tr>
<tr>
<td>$\mathcal{P} \neq$</td>
<td>The solution property</td>
</tr>
<tr>
<td>$\sigma_{a,b}$</td>
<td>The scaling function $\sigma_{a,b}: [0,1] \rightarrow [a,b]$</td>
</tr>
<tr>
<td>$\Rightarrow$</td>
<td>Constructor simulating the residuum in $\mathcal{L}^{(0,1)}_{\text{gEL}}$</td>
</tr>
<tr>
<td>$C^n$</td>
<td>The $n$-ary conjunction of a concept $C$ with itself</td>
</tr>
<tr>
<td>$g$</td>
<td>Superscript indicating the use of general model semantics</td>
</tr>
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**Chapter 6. Infinite Lattices**

<table>
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<tr>
<th>Symbol</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$L_Z$</td>
<td>The infinite total order over $\mathbb{Z} \cup {-\infty, \infty}$</td>
</tr>
<tr>
<td>$L_\infty$</td>
<td>The infinite total order over $[0,1] \cup {-\infty, -2, 2, \infty}$</td>
</tr>
<tr>
<td>$4$</td>
<td>The reduction function $4: L_\infty \rightarrow 4$</td>
</tr>
</tbody>
</table>
1 Introduction

Research in the field of Knowledge Representation is concerned with developing formalisms to describe domain-specific knowledge pertinent to a given application. The information is stored as a collection of logical expressions, a so-called knowledge base. Automated reasoning techniques aid users in application-specific tasks by deriving useful inferences from the knowledge base. The first crucial step when modeling real-world applications is the choice of a language that is able to express all relevant pieces of information and for which the desired inferences can be computed efficiently. Subsequent concerns include the maintenance of large knowledge bases and the clear and concise presentation of the stored and inferred information to the user.

This thesis is concerned with the combination of two such formalisms, Description Logics and Fuzzy Logics, which are briefly introduced in the following. We then describe the history of research on Fuzzy Description Logics and motivate the work presented in the subsequent chapters. A more comprehensive discussion of this topic is deferred until Section 2.4, when all relevant notions have been defined. We conclude this chapter with an overview over the structure of this thesis and the obtained results.

1.1 Description Logics

The term Description Logics (DLs) encompasses a large family of logical formalisms that aim to combine expressivity, efficient reasoning, and readability (Baader, Calvanese, McGuinness, Nardi, and Patel-Schneider 2007). To this end, many DLs have been proposed, ranging from the inexpressive $\mathcal{EL}$ and $\mathcal{DL}-\text{Lite}$ that support efficient implementations of certain reasoning tasks (Baader, Brandt, and Lutz 2005; Calvanese, De Giacomo, Lembo, Lenzerini, and Rosati 2005; Kazakov, Krötzsch, and Simančík 2012; Rosati and Almatelli 2010), over the prototypical expressive $\mathcal{ALC}$ (Schild 1991; Schmidt-Schauß and Smolka 1991), to the very expressive $\mathcal{SROIQ(D)}$, for which nevertheless highly optimized reasoning systems have been implemented and are used successfully (Motik, Shearer, and Horrocks 2009; Steigmiller, Liebig, and Glimm 2012). They share an easily memorized syntax that is loosely based on natural language constructs.

The basic building blocks of this syntax are *concepts* that represent sets of objects (e.g. Human), *roles* relating objects to objects via binary relations (e.g. hasParent), and *individuals* representing concrete objects (e.g. bob). A DL knowledge base or ontology typically consists of a *TBox* and an *ABox*. The former is a collection of terminological axioms like

\[ \text{Human} \sqsubseteq \forall \text{hasParent}.\text{Human}, \]

saying that a human being can have only human parents, or

\[ \text{Grandfather} \equiv \text{Human} \sqcap \text{Male} \sqcap \exists \text{hasParent} \neg \exists \text{hasParent} \neg \text{.Human} \]
1 Introduction

defining the notion of a grandfather as a man that has a child which in turn also has a child. Such axioms are called general concept inclusions (GCIs) and concept definitions, respectively. In contrast, the ABox contains assertions that represent data about specific objects. For example, \texttt{bob:Male} states that the individual named \texttt{bob} is male.

Standard reasoning tasks over DL ontologies include

- \textit{ontology consistency}, i.e. checking whether the ontology is non-contradictory;
- \textit{concept satisfiability}, i.e. checking whether a given concept is non-contradictory in the ontology;
- \textit{subsumption}, i.e. checking whether a specific concept is included in a more general one; and
- \textit{instance checking}, i.e. checking whether a new assertion follows from the ontology.

Additionally, many non-standard reasoning tasks like finding least common subsumers or most specific concepts, matching, unification, axiom pinpointing, and conjunctive query answering have been considered in the literature (Baader and Narendran 2001; Calvanese et al. 2005; Küsters 2001; Schlobach and Cornet 2003).

A variety of algorithms has been developed to solve these problems, such as automata-based procedures (Sattler and Vardi 2001), tableau algorithms (Baader and Sattler 2001), resolution-based procedures (Motik 2006), hypertableau algorithms (Motik, Shearer, and Horrocks 2009), and rule-based completion of the ontology (Baader, Brandt, and Lutz 2005; Krötzsch 2011). Most of these approaches try to construct a (counter-)model for the given inference question. This search for a model is aided by the fact that many DLs enjoy the \textit{tree model property}, or even the \textit{finite model property}, which reduce the size of the search space. These algorithms have different advantages and drawbacks—some are better suited for an efficient implementation, others allow to derive tight bounds on the computational complexity of the decision problems.

Applications of DLs range from standardization efforts like the Web Ontology Language OWL 2\footnote{\url{http://www.w3.org/TR/owl2-overview/}} to the formalization of many kinds of domain knowledge, most prominently in the biomedical domain.\footnote{\url{http://bioportal.bioontology.org/}, \url{http://obofoundry.org/}}

1.2 Fuzzy Logics

Classical logical formalisms are inherently unsuited to deal with quantitative information, e.g. involving \textit{vague} concepts like “tall” or “healthy”, for which no exact definition exists. The problem with these concepts is that the question “Is person \texttt{x} tall?” has no yes-or-no answer, but can only be answered, e.g. by “She is very tall.” or “She is taller than person \texttt{y}.” This is in contrast to \textit{uncertain} information like “Person \texttt{x} has disease \texttt{z} with a probability of 0.95.”, which might be the result of a medical test. While \texttt{z} has a clear definition—person \texttt{x} definitely either has the disease or not—we cannot be certain who actually has the disease due to uncertainty inherent in the procedures used to obtain the diagnosis. Uncertainty as a source of quantitative information has been considered in the context of DLs by Lukasiewicz (2008) and Lutz and Schröder (2010) and led to the
development of different kinds of *probabilistic* description logics. In this thesis, we are only concerned with vague information, which can be formalized using so-called *fuzzy* logics.

The first time the word “fuzzy” was used in a formal mathematical context was in the seminal paper of Zadeh (1965) on *fuzzy set theory*. Envisioned to be an extension of classical set theory, it is based on the central notion of a *fuzzy set*. The intuition underlying fuzzy set theory is that an element of a domain does not need to be either outside or inside a set, as in the classical case, but it can also be, e.g. “a little”, “somewhat”, or “mostly” inside this set.

Such imprecise *degrees of membership* are modeled by Zadeh using values from the real unit interval $[0, 1]$. A fuzzy set is then a mapping $A: \Delta \to [0, 1]$ from some domain $\Delta$ into this interval, where $A(x)$ for $x \in \Delta$ specifies the degree to which $x$ belongs to $A$. Zadeh also proposes the following fuzzy versions of set intersection, union, and complement:

$$
(A \cap B)(x) := \min\{A(x), B(x)\} \\
(A \cup B)(x) := \max\{A(x), B(x)\} \\
(\overline{A})(x) := 1 - A(x)
$$

The origin of Mathematical Fuzzy Logic as described by Hájek (2001), however, can be found much earlier in the development of *many-valued logics*. As in fuzzy set theory, the basic idea is the extension of the classical truth values *true* and *false* by other *degrees of truth* that allow a graded transition between these two values. Formulae of a many-valued logic often have the same syntax as those of the corresponding classical logic, but are evaluated by many-valued interpretations to one of the postulated truth degrees. Given the same set of truth values, one can obtain different logics by varying how the logical connectives are evaluated. The first many-valued logic, a three-valued extension of classical propositional logic, was published by Łukasiewicz (1920). It was later generalized to $n$-valued logics for any finite $n \in \mathbb{N}$ and infinite-valued logics by Hay (1963). Another three-valued logic, differing from Łukasiewicz’s approach only in the definition of the implication function, was proposed by Kleene (1952).

Zadeh’s fuzzy set theory is often adapted for many-valued logics over the truth values in $[0, 1]$, where conjunction is interpreted as the minimum of the truth degrees of the two subformulae, disjunction is evaluated by taking the maximum, and negation by the function $x \mapsto 1 - x$. Following an approach introduced by Kleene (1952), implication is defined in analogy to the classical equivalence of $\varphi \rightarrow \psi$ and $\neg\varphi \vee \psi$ as the maximum between the consequent and the negation of the antecedent. This implication function is often called the *Kleene-Dienes-implication* (Dienes 1949; Kleene 1952).

However, this is by far not the only way to interpret the logical connectives over the truth values in $[0, 1]$. In fact, Mathematical Fuzzy Logic allows any *triangular norm* (or *t-norm*), which must only satisfy some basic properties, to interpret the conjunction (Hájek 2001; Klement, Mesiar, and Pap 2000). Triangular norms were originally introduced in the context of statistical metric spaces to generalize the classical triangle inequality (Menger 1942; Schweizer and Sklar 1960), and have first been used to combine fuzzy degrees by Klement (1982). One such t-norm is the above-mentioned minimum function used by Zadeh, but there are infinitely many others.
Hájek (2001) motivates the properties of t-norms \( \otimes: [0, 1] \times [0, 1] \rightarrow [0, 1] \) as desirable properties of any multi-valued operator interpreting the conjunction:

- A t-norm should generalize classical conjunction, and thus it must satisfy the boundary conditions \( 0 \otimes 0 = 0 \otimes 1 = 1 \otimes 0 = 0 \) and \( 1 \otimes 1 = 1 \).
- A large truth value for the conjunction should indicate that both conjuncts are true to a large degree, and vice versa. This means that \( \otimes \) is monotone in both arguments.
- The order of the conjuncts should be irrelevant, i.e. \( \otimes \) is commutative and associative.
- The conjunction with a tautology should not influence the truth value, i.e. 1 is neutral w.r.t. \( \otimes \).

Based on triangular norms, other functions can be defined to interpret the other logical constructors like disjunction, implication, and negation.

More generally, one can also consider arbitrary lattices as truth domains instead of the interval \([0, 1]\). Since these need not be totally ordered, it is possible that the truth degree of the assertion “Mary likes John” is incomparable to the truth degree of “Mary likes Chris”, i.e. Mary likes both in different ways and is unable to state a preference between them. We introduce the related notions in more detail in Section 2.4.1.

1.3 Fuzzy Description Logics

The two formalisms were first combined in (Yen 1991) to a basic fuzzy description logic by taking the syntax of Description Logics and enriching the semantics with notions from Fuzzy Logics. In particular, concepts are then unary fuzzy predicates, which are interpreted as fuzzy sets, and roles are binary fuzzy predicates. This research direction did not receive much further attention until in 1998 two papers described tableau algorithms for fuzzy extensions of ALC (Straccia 1998; Tresp and Molitor 1998). These first fuzzy DLs employed only the operators introduced by Zadeh for the fuzzy semantics.

Later, Hájek (2005b) considered fuzzy description logics from the point of view of Mathematical Fuzzy Logic and described the first reasoning algorithm for t-norm-based fuzzy DLs. Subsequently, many tableau algorithms were developed to deal with these new semantics (Bobillo and Straccia 2009; Haarslev, Pai, and Shiri 2009). Another popular approach to reasoning in fuzzy description logics is to rewrite fuzzy DL ontologies into classical ones and reuse existing highly optimized reasoners. This works either if the semantics is sufficiently simple, e.g. for Zadeh semantics (Straccia 2004a), or if the set of truth degrees is restricted a priori to only finitely many values (Straccia 2006).

It was recently discovered that many tableau algorithms proposed before are incorrect in the presence of GCI axioms (Baader and Peñaloza 2011a; Bobillo, Bou, and Straccia 2011). After this revelation, it was shown that in some t-norm-based fuzzy DLs consistency is even undecidable (Baader and Peñaloza 2011a,b; Cerami and Straccia 2013). For a more detailed discussion of these results, see Section 2.4.

This thesis provides a principled investigation of reasoning in fuzzy description logics, in particular in the presence of GCIs. The goal is to determine the boundary between decidability and undecidability of consistency and other reasoning problems. In case of
decidability, we derive tight bounds on the computational complexity, something which has been neglected in the literature on t-norm-based fuzzy DLs so far.

To achieve this, we study existing reasoning algorithms and proofs of undecidability to determine how far they can be extended. Additionally, new approaches are presented with the aim to complete the picture of the reasoning landscape of fuzzy description logics with general concept inclusions.

1.4 Structure of the Thesis

We give a brief overview of the methods used in this thesis and the results obtained. After introducing the main notions of fuzzy logics and fuzzy description logics, we first consider fuzzy description logics over finite lattices and show that they mostly behave like classical description logics in terms of the complexity of their reasoning problems. We then turn our attention to fuzzy description logics over the interval $[0, 1]$ and show decidability of consistency for some special classes of t-norms. Subsequently, we extend previous undecidability results to cover most of the remaining fuzzy DLs. We then briefly report on some results obtained for fuzzy description logics over infinite lattices.

In the following, we list the contents of the separate chapters in more detail and provide references to related own publications. Most of the reported results arose from joint work with Dr. rer. nat. Rafael Peñaloza Nyssen and Dr. rer. nat. Felix Distel.

- Chapter 2 introduces the main notions of the thesis, in particular fuzzy description logics. We systematically introduce all relevant facets, starting with the concept constructors and the notation for fuzzy DLs. We then cover different classes of interpretations relevant for fuzzy reasoning and relations between them. Finally, we introduce axioms, ontologies, and the main reasoning problems. Then follow some examples illustrating the use of fuzzy DLs in the literature, and finally a detailed account of related work in the area of fuzzy description logics.

The notation introduced in this chapter and used throughout the thesis was developed over the course of several previous publications and is based on notations used by other fuzzy DL researchers, in particular Cerami, Garcia-Cerdaña, and Esteva (2010).

- Chapter 3 analyzes the complexity of reasoning in fuzzy description logics where the set of truth values forms a finite lattice. First the so-called local consistency, then consistency and other reasoning problems are shown to have the same complexity as the corresponding problems in the underlying classical DLs. For GCIs, the complexity is \text{ExpTime} even for quite expressive logics, while in some sublogics we are able to show \text{PSPACE} upper bounds when the TBox is restricted to be acyclic. The results are obtained using a combination of automata-based and tableau algorithms, which are generalizations of classical techniques (Baader, Hladik, and Peñaloza 2008; Hollunder 1996; Horrocks and Sattler 1999).

The automata-based approach for deciding local consistency from Section 3.1 has previously been published as (Borgwardt and Peñaloza 2013c), and preliminary versions have appeared in (Borgwardt and Peñaloza 2011a,b,c).
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The tableau algorithm of Section 3.2 and the reductions in Section 3.3 have already been described in (Borgwardt and Peñaloza 2014), an extension of (Borgwardt and Peñaloza 2012a).

- Chapter 4 presents fuzzy description logics over the interval $[0,1]$ for which the consistency problem is decidable, due in most part to the properties of the t-norms involved. In particular, when the t-norm has no so-called zero divisors, then consistency of fuzzy ontologies can often be trivially reduced to classical reasoning by simply ignoring the fuzzy values in the ontology. For the particular case of the G"{o}del t-norm, the decidability is not so easily obtained in all cases, but requires techniques similar to those of Chapter 3. Finally, we provide examples showing that the approach used for t-norms without zero divisors does not work for entailment problems such as subsumption and instance checking.

The reduction of Section 4.1 and the examples in Section 4.3 have previously been described in (Borgwardt, Distel, and Peñaloza 2012b). The contents of Section 4.2 and the first part of Section 4.3 have been published as (Borgwardt, Distel, and Peñaloza 2014a).

- Chapter 5 contains the most difficult proofs, showing undecidability of (local) consistency in many fuzzy description logics over the interval $[0,1]$. It starts by introducing a framework for showing undecidability by reducing the Post correspondence problem. The framework consists of several properties that a fuzzy DL must satisfy in order for this reduction to work. Many undecidability results are already obtained using this basic framework, and subsequently some adaptations of the framework are developed to extend these results to even larger classes of logics. Finally, we very briefly comment on possible extensions of these results to other reasoning problems.

The framework from Sections 5.1.1 and 5.1.2 and the results of Sections 5.1.3 and 5.1.4 have already been described in (Borgwardt and Peñaloza 2012c). Together with the contents of Sections 4.1 and 5.1.5, they are currently under submission to a journal. The proof in Section 5.2.2 has appeared in (Borgwardt and Peñaloza 2012b); the one in Section 5.2.3 has not been published before.

- Chapter 6 contains another undecidability proof and two examples of families of fuzzy DLs over infinite lattices over which consistency is decidable and undecidable, respectively. These have previously appeared as parts of (Borgwardt and Peñaloza 2012a, 2014).

2 Preliminaries

This chapter introduces the main notions used in this thesis. After an introduction to t-norm-based fuzzy logics, the theory is generalized to allow residuated De Morgan lattices as truth domains. Subsequently, fuzzy description logics over such lattices are defined in general and the notation used in the following chapters is fixed. At the end, this chapter contains a survey of related work on fuzzy description logics.

2.1 Fuzzy Logics

We now introduce the basic notions of Mathematical Fuzzy Logic. Starting from t-norms, which are used to interpret conjunction, various other operators are defined to interpret the other logical connectives. For a broader introduction to this topic, refer to Cintula, Hájek, and Noguera (2011), Hájek (2001), and Klement, Mesiar, and Pap (2000).

2.1.1 Conjunction and Triangular Norms

Rather than using the minimum to interpret the conjunction—as Zadeh does—a more general approach is taken in Mathematical Fuzzy Logic, using triangular norms.

Definition 2.1 (triangular norm) A triangular norm (t-norm) is a binary operator $\otimes: [0, 1] \times [0, 1] \rightarrow [0, 1]$ that satisfies the following properties:

- It is commutative, i.e. $x \otimes y = y \otimes x$ holds for all $x, y \in [0, 1]$.
- It is associative, i.e. we have $x \otimes (y \otimes z) = (x \otimes y) \otimes z$ for all $x, y, z \in [0, 1]$.
- It is monotone, i.e. for all $x, y, z \in [0, 1]$ with $x \leq y$ we have $x \otimes z \leq y \otimes z$.
- It has 1 as identity element, i.e. $x \otimes 1 = x$ holds for all $x \in [0, 1]$.

Since every t-norm is commutative, it is monotone in both arguments and 1 is a left identity as well as a right identity. These properties are in accordance with the behavior of classical conjunction on the set $\{0, 1\}$. The definition of triangular norms includes Zadeh’s minimum, but it also allows other operators.

In this thesis, we only consider continuous t-norms, which means that they are continuous as a function. This assumption makes the class of t-norms a lot more manageable since they can be reduced to three particular continuous t-norms. The Gödel, Product, and Łukasiewicz t-norms, which will be denoted throughout this thesis by $G$, $\Pi$, and $\mathcal{L}$, respectively, are listed in Table 2.1, together with some associated operators that are introduced in the following sections. The graphs of these t-norms are depicted in Figure 2.1. Note that the Gödel t-norm is simply the minimum function proposed by Zadeh. These three fundamental continuous t-norms can be used to build all continuous t-norms by the following construction.
Table 2.1: Three fundamental t-norms, their residua, residual negations, and t-conorms

<table>
<thead>
<tr>
<th>name</th>
<th>$x \otimes y$</th>
<th>$x \Rightarrow y \ (x &gt; y)$</th>
<th>$\odot x \ (x &gt; 0)$</th>
<th>$x \oplus y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gödel (G)</td>
<td>$\min{x, y}$</td>
<td>$y$</td>
<td>$0$</td>
<td>$\max{x, y}$</td>
</tr>
<tr>
<td>Product (Π)</td>
<td>$x \cdot y$</td>
<td>$y/x$</td>
<td>$0$</td>
<td>$x + y - x \cdot y$</td>
</tr>
<tr>
<td>Łukasiewicz (Ł)</td>
<td>$\max{0, x + y - 1}$</td>
<td>$1 - x + y$</td>
<td>$1 - x$</td>
<td>$\min{1, x + y}$</td>
</tr>
</tbody>
</table>

Definition 2.2 (ordinal sum) Let $(a_i, b_i, \otimes_i)_{i \in I}$ be a family of tuples over the index set $I$, where $(a_i, b_i)_{i \in I}$ are disjoint, non-empty, open subintervals of $[0, 1]$ and $(\otimes_i)_{i \in I}$ are t-norms. The ordinal sum of this family is the t-norm $\otimes$ defined by

$$x \otimes y := \begin{cases} 
  a + (b - a) \left( \frac{x - a}{b - a} \otimes_i \frac{y - a}{b - a} \right) & \text{if } x, y \in [a_i, b_i] \text{ for some } i \in I, \\
  \min\{x, y\} & \text{otherwise}.
\end{cases}$$

Intuitively, an ordinal sum behaves like its constituent t-norms in the designated intervals, and like the Gödel t-norm everywhere else. Note that the index set $I$ is not required to be finite.

It is relatively easy to show that the ordinal sum of continuous t-norms is again a continuous t-norm. Moreover, every continuous t-norm is basically an ordinal sum of copies of the Product and Lukasiewicz t-norms. However, these copies need not be exactly $\Pi$ or $Ł$, but only up to an order isomorphism $h$ on $[0, 1]$, i.e., a strictly increasing mapping $h \colon [0, 1] \to [0, 1]$ with $h(0) = 0$ and $h(1) = 1$. This result, which is stated below, follows already from results about topological semigroups by Faucett (1955) and Mostert and Shields (1957).

Proposition 2.3 A t-norm is continuous iff it is an ordinal sum of t-norms isomorphic to $\Pi$ or $Ł$. \qed

For ease of presentation and in a slight abuse of notation, we will from now on use the name continuous t-norm to mean only those t-norms $\otimes$ that are represented by an ordinal sum of $(a_i, b_i, \otimes_i)_{i \in I}$, where all $\otimes_i$ are equal to either $\Pi$ or $Ł$. It is easy to see that all results can be transferred to other continuous t-norms by appropriate application of the (unique) isomorphisms given by Proposition 2.3.

The tuples $(a_i, b_i, \otimes_i)$ in this representation are called the components of $\otimes$. The Gödel t-norm is not allowed in the components since otherwise a t-norm could have several such representations. For example, it is easy to verify that $G$ itself is the ordinal sum of, e.g., $(0, 0.5, G)$ and $(0.5, 1, G)$. However, the Gödel t-norm is implicitly present in all intervals between the component intervals $[a_i, b_i]$. We say that $\otimes$ contains $\otimes'$ if it has a component of the form $(a, b, \otimes')$. Similarly, $\otimes$ starts with (resp. ends with) $\otimes'$ if it contains $\otimes'$ in an interval of the form $[0, b]$ (resp. $[a, 0]$).

A distinguishing property of the elements of $[0, 1]$ that lie outside of the components of a continuous t-norm $\otimes$ is their idempotence. An element $x \in [0, 1]$ is called idempotent (w.r.t. $\otimes$) if $x \otimes x = x$. If $(a_i, b_i, \otimes_i)_{i \in I}$ are the components of $\otimes$ according to Proposition 2.3, then the set of its idempotent elements is

$$[0, 1] \setminus \left( \bigcup_{i \in I} (a_i, b_i) \right)$$

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2.1 Fuzzy Logics

(Klement, Mesiar, and Pap 2000). In other words, all borders of the component intervals are idempotent, as well as all elements between the component intervals. These are exactly those elements on which $\otimes$ behaves like the Gödel t-norm. The idea of characterizing "Gödel elements" using their idempotence will often be useful.

**Example 2.4** The t-norm defined by

$$x \otimes_1 y := \begin{cases} 2xy & \text{if } x, y \in (0, 0.5), \\
\max\{x + y - 1, 0.5\} & \text{if } x, y \in (0.5, 1), \\
\min\{x, y\} & \text{otherwise}
\end{cases}$$

contains $\Pi$ in the interval $[0, 0.5]$, and $Ł$ in $[0.5, 1]$. Its idempotent elements are 0, 0.5, and 1.

The t-norm

$$x \otimes_2 y := \begin{cases} \max\left\{\frac{1}{2^n}, x + y - \frac{1}{2^n}\right\} & \text{if } x, y \in \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right) \text{ for some } n \geq 1, \\
\min\{x, y\} & \text{otherwise}
\end{cases}$$

is the infinite ordinal sum of the family $\left(\frac{1}{2^n}, \frac{1}{2^{n-1}}, Ł\right)_{n \geq 1}$. The set of its idempotent elements is $\left\{\frac{1}{2^n} \mid n \geq 0\right\} \cup \{0\}$. Plots of $\otimes_1$ and $\otimes_2$ can be found in Figure 2.2.

2.1.2 Implication and Residua

Now that we have a fuzzy generalization of the classical conjunction, we can derive fuzzy versions of the other propositional connectives.
Definition 2.5 (residuum) Given a t-norm $\otimes$, a residuum of $\otimes$ is a binary operator $\Rightarrow: [0, 1] \times [0, 1] \to [0, 1]$ such that

$$x \Rightarrow y \leq z \text{ iff } x \otimes z \leq y$$

holds for all $x, y, z \in [0, 1]$. ♦

For a continuous t-norm $\otimes$, the residuum is unique and satisfies

$$x \Rightarrow y = \sup\{z \in [0, 1] \mid x \otimes z \leq y\}$$

for all $x, y \in [0, 1]$ (Klement, Mesiar, and Pap 2000). Since we only deal with continuous t-norms, we will often use this alternative representation to compute the residuum. The existence of a residuum of $\otimes$ is in fact equivalent to the left-continuity of the t-norm (Klement, Mesiar, and Pap 2000).

On the truth values 0 and 1, the residuum behaves exactly like classical implication, e.g., $0 \Rightarrow x$ is always 1. Note that by definition we have $x \Rightarrow y = 1$ whenever $x \leq y$; therefore, the third column of Table 2.1 lists the residua of the fundamental continuous t-norms only for values $x > y$.

Consider now any ordinal sum $\otimes$ of $(a_i, b_i, \otimes_i)_{i \in I}$ for continuous t-norms $\otimes_i$, $i \in I$. If $\Rightarrow_i$ is the residuum of $\otimes_i$, $i \in I$, then it is readily checked that the residuum $\Rightarrow$ of $\otimes$ can be expressed as follows:

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ a + (b - a) \left( \frac{x-a}{b-a} \Rightarrow_i \frac{y-a}{b-a} \right) & \text{if } a_i \leq y < x \leq b_i \text{ for some } i \in I, \\ y & \text{otherwise.} \end{cases}$$

This means that, just as for the t-norm, the residuum of an ordinal sum behaves like the residua of the component t-norms in the associated intervals, and like the residuum of the Gödel t-norm everywhere else (except in the case where $x \leq y$).

2.1.3 Negation Functions

Given a residuum, one can define the residual negation $\ominus: [0, 1] \to [0, 1]$ as

$$\ominus x := x \Rightarrow 0.$$
Again, in the case that \( x = 0 \), we have \(-x = 1\), and only for \( x > 0 \) do the residual negations differ among the t-norms (see the fourth column of Table 2.1). More precisely, the shape of the residual negation depends on the presence of so-called zero divisors. A zero divisor (w.r.t. \( \otimes \)) is an element \( x \in (0, 1) \) for which there is another \( y \in (0, 1) \) such that \( x \otimes y = 0 \). It turns out that a t-norm has zero divisors iff it starts with \( \mathfrak{t} \).

**Proposition 2.6 (Klement, Mesiar, and Pap 2000)** A continuous t-norm has zero divisors iff it starts with the Lukasiewicz t-norm. \( \blacksquare \)

Consequently, every t-norm \( \otimes \) that has zero divisors contains \( \mathfrak{t} \) in an interval of the form \([0, b]\). In this case, the residual negation of \( \otimes \) is given by

\[
\otimes x = \begin{cases} 
1 & \text{if } x = 0, \\
b - x & \text{if } x \in (0, b], \\
0 & \text{otherwise}
\end{cases}
\]

for \( x \in [0, 1] \). For \( \mathfrak{t} \) in particular, the residual negation is equal to the involutive negation \( \sim : [0, 1] \to [0, 1] \), defined as \( \sim x := 1 - x \), that was originally used by Zadeh (1965). The name of this operation reflects that it is involutive, which means that \( \sim \sim x = x \) holds for all \( x \in [0, 1] \). On the other hand, if \( \otimes \) has no zero divisors, then its residual negation is given by

\[
\otimes x = \begin{cases} 
1 & \text{if } x = 0, \\
0 & \text{otherwise.}
\end{cases}
\]

### 2.1.4 Disjunction and Triangular Conorms

Finally, the operator corresponding to the classical disjunction is the **triangular conorm** (t-conorm) \( \oplus : [0, 1] \times [0, 1] \to [0, 1] \), which is uniquely determined by the De Morgan law

\[
x \oplus y = \sim(\sim x \otimes \sim y)
\]

w.r.t. the involutive negation \( \sim \). The t-conorms of the fundamental continuous t-norms are listed in the fifth column of Table 2.1.

Inspired by the classical equivalence between \( \varphi \to \psi \) and \( \neg \varphi \lor \psi \), in the literature sometimes so-called S-implications \( \Rightarrow_S \) are used instead of the residuum to interpret implication. For a given t-norm \( \otimes \), the S-implication is defined, for all \( x, y \in [0, 1] \), by

\[
x \Rightarrow_S y := \sim x \oplus y = \sim(x \otimes \sim y).
\]

The letter \( S \) indicates the use of the t-conorm to define this operation, because t-conorms can also be defined via their properties (similar to Definition 2.1), and are then sometimes called s-norms (Klement, Mesiar, and Pap 2000). The Kleene-Dienes-implication is the S-implication associated with the Gödel t-norm (Dienes 1949; Kleene 1952).

In this context, implication functions that are the residuum of some t-norm are also called R-implications, where \( R \) stands for the residuum (Klement, Mesiar, and Pap 2000).
2.1.5 Triangular Norms over Lattices

The introduced fuzzy connectives can also be defined more generally over other structured sets of truth values. The most basic structure we need is that of a partially ordered set, but for meaningful definitions of continuous t-norms and residua we make additional assumptions on this order, and require it to form a complete lattice.

Definition 2.7 (complete lattice) A lattice is an algebraic structure \((L, \wedge, \vee)\) with two binary operations infimum (or meet) \(\wedge\) and supremum (or join) \(\vee\) that satisfy the following identities for all \(x, y, z \in L\):

\[
(x \wedge y) \wedge z = x \wedge (y \wedge z) \quad (x \vee y) \vee z = y \vee (y \vee z) \quad \text{(associativity)}
\]
\[
x \wedge y = y \wedge x \quad x \vee y = y \vee x \quad \text{(commutativity)}
\]
\[
x \wedge x = x \quad x \vee x = x \quad \text{(idempotence)}
\]
\[
x \wedge (x \vee y) = x \quad x \vee (x \wedge y) = x \quad \text{(absorption)}
\]

The natural partial order on such a lattice is defined by \(x \leq y\) iff \(x \wedge y = x\) for all \(x, y \in L\). We call \((L, \wedge, \vee)\) a total order if \(\leq\) is a total order, i.e. we have \(x \leq y\) or \(y \leq x\) for all \(x, y \in L\). An antichain in \((L, \wedge, \vee)\) is a set \(S \subseteq L\) of incomparable elements, i.e. where either \(x = y\) or \(x \not\leq y\) holds for all \(x, y \in S\). The width of \((L, \wedge, \vee)\) is the maximum cardinality of all its antichains.

The lattice \((L, \wedge, \vee)\) is finite if its carrier set \(L\) is finite. It is distributive if, for all \(x, y, z \in L\), it holds that \(x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)\) and \(x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)\). It is bounded if there are two elements \(0, 1 \in L\) such that \(0 \leq x \leq 1\) holds for all \(x \in L\). It is complete if suprema and infima of all subsets of \(L\) exist, i.e. for all \(S \subseteq L\) there is a smallest element \(\bigvee S \in L\) greater than or equal to all elements of \(S\), and a largest element \(\bigwedge S \in L\) smaller than or equal to all elements of \(S\).

Since we usually deal with a fixed lattice, we refer to it simply by its carrier set \(L\), omitting the full signature. The lattice operations \(\wedge\) and \(\vee\) and elements \(0\) and \(1\) are always implicitly associated with \(L\). Observe that every finite lattice is also complete, and every complete lattice is bounded by \(0 := \bigvee \emptyset\) and \(1 := \bigwedge \emptyset\). We will never consider singleton lattices, but always assume that a bounded lattice contains at least the two different elements \(0\) and \(1\).

The interval \([0, 1]\), which we will often call the standard interval, is a complete distributive lattice with the usual order. We now define additional operators on any lattice, which correspond to t-norms and residua over the standard interval. To generalize the involutive negation, we further require the lattice to be distributive.

Definition 2.8 (residuated De Morgan lattice) For a bounded lattice \(L\), a (generalized) t-norm is a binary operator \(\otimes: L \times L \to L\) that is associative, commutative, monotone w.r.t. the lattice order, and has \(1\) as its unit. A residuated lattice is a lattice \(L\) endowed with a t-norm \(\otimes\) and a (generalized) residuum \(\Rightarrow: L \times L \to L\) such that, for all \(x, y, z \in L\), we have \(x \otimes y \leq z\) iff \(y \leq x \Rightarrow z\). A De Morgan lattice is a distributive lattice with an involutive unary operator \(\sim\) such that \(\sim(x \wedge y) = \sim x \vee \sim y\) and \(\sim(x \vee y) = \sim x \wedge \sim y\) hold for all \(x, y \in L\).

As expected, the definitions of generalized t-norms and residua are exactly the same as for the standard interval, and the De Morgan negation can be seen as a generalization of
the involutive negation \( x \mapsto 1 - x \). Hence, any \( t \)-norm over the standard interval that has a residuum induces a complete residuated De Morgan lattice over \([0, 1]\). We only consider complete residuated De Morgan lattices \( L \), as they generalize the canonical example of \([0, 1]\).

As for the standard interval, for any complete residuated De Morgan lattice \( L \) we define the residual negation by \( \ominus x := x \Rightarrow 0 \) and the \( t \)-conorm by \( x \oplus y := \sim (\sim x \otimes \sim y) \) for all \( x, y \in L \). Similarly, the notions of idempotent elements and zero divisors are defined as before, i.e. \( x \in L \) is idempotent (w.r.t. \( \otimes \)) if \( x \otimes x = x \), and it is a zero divisor (w.r.t. \( \otimes \)) if \( x > 0 \) and there is a \( y \in L \setminus \{0\} \) such that \( x \otimes y = 0 \). Unfortunately, no characterizations of generalized \( t \)-norms in the spirit of Propositions 2.3 and 2.6, i.e. in terms of their idempotent elements, zero divisors, or any fundamental \( t \)-norms, are known. However, as for the standard interval, the following characterization of the residual negation is valid in all lattices without zero divisors.

**Proposition 2.9** (Galatos, Jipsen, Kowalski, and Ono 2007) If \( L \) is a complete residuated De Morgan lattice without zero divisors, then, for all \( x, y \in L \), we have 
\[
\ominus x = \begin{cases} 0 & \text{if } x > 0, \\ 1 & \text{otherwise.} \end{cases}
\]

For a more thorough introduction to lattices, and in particular residuated lattices, we refer the reader to Galatos, Jipsen, Kowalski, and Ono (2007) and Grätzer (2003). Note that the notion of residuated lattices of Galatos et al. (2007) is weaker than the one introduced here; in particular, their residuated lattices need not be bounded, commutative, or distributive, and the unit of \( \otimes \) is not necessarily 1. The definition given here is closer to the one usually employed in Mathematical Fuzzy Logic (Cintula, Hájek, and Noguera 2011; Hájek 2001); in particular, the requirements that the fuzzy conjunction \( \otimes \) should be commutative and have as its unit the maximal truth value 1 are central to fuzzy logics.

In this more general setting, the continuity of a \( t \)-norm on a complete lattice \( L \) corresponds to \( \otimes \) being join-preserving and meet-preserving, which means that it satisfies 
\[
x \otimes \bigvee_{y \in S} y = \bigvee_{y \in S} (x \otimes y) \quad \text{and} \quad x \otimes \bigwedge_{y \in S} y = \bigwedge_{y \in S} (x \otimes y)
\]

for all \( x \in L \) and \( S \subseteq L \). As before, we usually consider only \( t \)-norms satisfying these properties. For a join-preserving and meet-preserving \( t \)-norm \( \otimes \), the residuum is unique and can be computed by
\[
x \Rightarrow y = \bigvee \{z \in L \mid x \otimes z \leq y\}
\]

for all \( x, y \in [0, 1] \). In fact, a \( t \)-norm \( \otimes \) on a complete lattice \( L \) has a residuum iff it is join-preserving (Galatos et al. 2007).

We only mention here that fuzzy set theory also has a generalization where the membership degrees come from a lattice \( L \). The resulting structures \( A : \Delta \to L \) are called \( L \)-fuzzy sets and have first been studied by Goguen (1967).
2 Preliminaries

2.2 Fuzzy Description Logics

Classical description logics can be extended with fuzzy set theory to deal with vague notions occurring in many application domains. They are denoted by expressions of the form \( L\mathcal{-L} \), where \( L \) is a complete residuated De Morgan lattice and \( \mathcal{L} \) roughly corresponds to the classical DL underlying the new logic. The second component determines the syntax by specifying which constructors can be used to build expressions, while the first component describes the semantics of the logic. We often use the special notation \( \otimes\mathcal{-L} \) in the case that \( L \) is the standard interval \([0, 1]\) with t-norm \( \otimes \). In particular, if \( \otimes \) is the Gödel, Product, or Łukasiewicz t-norm, then we write \( G\mathcal{-L} \), \( \Pi\mathcal{-L} \), or \( \Ł\mathcal{-L} \), respectively.

Instead of introducing classical DLs first, we just mention that the usual definitions for classical DLs can be obtained by instantiating \( L \) with the two-element lattice \( 2 := \{0, 1\} \) with the order determined by \( 0 < 1 \). In this setting, the t-norm and residuum correspond to classical conjunction and implication over the two truth values \textit{false} (0) and \textit{true} (1).

We refer to (Baader, Calvanese, et al. 2007) for a more comprehensive introduction to DLs from a purely classical perspective. For the remainder of this chapter, \( (L, \wedge, \vee) \) denotes a complete residuated De Morgan lattice with t-norm \( \otimes \), residuum \( \Rightarrow \), t-conorm \( \oplus \), residual negation \( \ominus \), and De Morgan negation \( \sim \), and we assume that \( \otimes \) is join- and meet-preserving. In the case of \( L = [0, 1] \), this means that \( \otimes \) is continuous.

2.2.1 Constructors

As mentioned before, in fuzzy description logics, concepts usually look exactly like those in classical DLs, but they are interpreted as fuzzy sets over some interpretation domain \( \Delta \). Likewise, roles are interpreted as \textit{fuzzy binary relations} over this domain, i.e. fuzzy sets over \( \Delta \times \Delta \). In the following, let \( \mathbb{N}_C, \mathbb{N}_R, \) and \( \mathbb{N}_I \) be mutually disjoint sets of \textit{concept names}, \textit{role names}, and \textit{individual names}, respectively. We will usually denote generic concept names by \( A \) or \( B \), role names by \( r \) or \( s \), and individual names by \( c \) or \( d \), possibly with sub- or superscripts.

\textbf{Definition 2.10 (fuzzy DL syntax and semantics)} A \textit{(complex) role} is either a role name or an \textit{inverse role} of the form \( r^{-} \), where \( r \) is a role name. \textit{(Complex) concepts} are built from concept names by applying the constructors listed in Table 2.2, where \( C, D \) denote arbitrary concepts, \( r \) is a role, and \( c \) an individual name.

An \textit{interpretation} \( I = (\Delta^I, \cdot^I) \) consists of a \textit{domain} \( \Delta^I \), which is a non-empty (classical) set, and an \textit{interpretation function} \( \cdot^I \) that assigns to each concept name \( A \) a fuzzy set \( A^I : \Delta^I \rightarrow L \), to each role name \( r \) a fuzzy binary relation \( r^I : \Delta^I \times \Delta^I \rightarrow L \), and to each individual name \( c \) a domain element \( c^I \in \Delta^I \). This function is extended to complex roles by defining \( (r^{-})^I(x, y) := r^I(y, x) \) for all \( r \in \mathbb{N}_R \) and \( x, y \in \Delta^I \). The interpretation function is also extended to complex concepts \( C \), mapping them to fuzzy sets \( C^I : \Delta^I \rightarrow L \). In Table 2.2, the semantics column specifies the value of the concept in the syntax column at an arbitrary element \( x \) of \( \Delta^I \).

For a complex role \( r \), we denote by \( r^{-} \) its \textit{inverse}, i.e. \( r := r^{-} \) if \( r \in \mathbb{N}_R \), and \( r := s^{-} \) if \( r = s^{-} \) for some \( s \in \mathbb{N}_R \).

We now define various description logics \( \mathcal{L} \), each determined by a set of allowed constructors and denoted by a combination of letters that roughly correspond to these
constructors (see the symbol column of Table 2.2). The basic DL $\mathcal{EL}$ allows the top constructor, conjunction, and existential restrictions, but no inverse roles. The logic $\mathcal{AL}$ extends $\mathcal{EL}$ by universal restrictions. The presence of the involutive negation, disjunction, nominals, and inverse roles is denoted by appending $C$, $U$, $O$, and $I$, respectively. A prefixed $\mathfrak{R}$ indicates the presence of the residual negation, while $\mathfrak{J}$ stands for implication and bottom. In the first part of Table 2.3, we summarize this nomenclature for some logics relevant for this thesis. The letters $S$ and $H$ will be explained in Section 2.2.3.

The origin of this nomenclature lies in the established names for classical DLs (Baader, Calvanese, et al. 2007), the main difference being the need to distinguish two different negations ($C$, $\mathfrak{R}$) and the implication constructor ($\mathfrak{I}$). The additional prefix $\mathfrak{J}$ for the implication constructor was introduced in (Cerami 2012; Cerami, Garcia-Cerdaña, and Esteva 2010). However, there they use as basic logic a variant of $\mathcal{AL}$ that already includes the involutive negation on concept names and unqualified existential restrictions of the form $\exists r$. Above that, they distinguish between adding full existential restrictions (using the letter $\mathcal{E}$), disjunction, involutive negation, and implication.

In logics allowing conjunction and implication the so-called weak conjunction and weak disjunction are definable, interpreted as the point-wise infimum and supremum, respectively, of two interpreted concepts (Cerami 2012; Cerami, Garcia-Cerdaña, and Esteva 2010). We will not consider these constructors here.

Note that in the presence of the involutive negation, one can define the disjunction constructor by $C \sqcup D := \neg(\neg C \sqcap \neg D)$ (see Section 2.1.4). Similarly, the implication and bottom constructors together can simulate the residual negation via $\sqcap C := C \sqcap \bot$. As a consequence, the logic $L-\mathcal{ELC}$ has the same expressivity as $L-\mathcal{ELU}$, and $L-\mathfrak{R}$ is always at least as expressive as $L-\mathfrak{R}$ of $\mathcal{ELC}$. Thus, we will usually not consider fuzzy DLs with the constructor combinations $\mathcal{CU}$ or $\mathfrak{R}$.

Throughout this thesis, we often refer to classical DLs, where $L = 2$, by their conventional names. For example, the equivalent logics $2-\mathcal{ELC}$, $2-\mathcal{ALC}$, $2-\mathfrak{R}$, $2-\mathfrak{N}$, $2-\mathfrak{A}$.
Table 2.3: Some relevant DLs and their expressivity

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✓ ... constructor is present  (✓) ... constructor can be simulated

2-IEL, and 2-IACL are denoted simply by ALC. The reason for this is that in 2 the involutive and residual negation are the same and we have \((C \rightarrow D)^I = (\neg C \sqcup D)^I\) in all interpretations. Furthermore, existential and value restrictions are dual to each other, i.e. \((\neg \exists r.C)^I = (\forall r.\neg C)^I\).

Unfortunately, these equivalences do not hold in arbitrary fuzzy DLs. For example, the R-implication \(x \Rightarrow y\) is in general not equivalent to the S-implication \(\neg x \oplus y\) (see Section 2.1.4). An important exception is the Łukasiewicz t-norm over \([0,1]\), where all mentioned equivalences hold, i.e. the residual negation is the same as the involutive negation, the R-implication is equal to the S-implication, and the duality between existential and value restrictions holds.

2.2.2 Witnessed Models

The semantics of the existential and value restrictions deserves some more explanation. The origin of these definitions can be found in the first-order translation of the classical DL constructors. For example, the existential restriction \(\exists r.C\) can be viewed as the first-order formula \(\exists y. r(x,y) \land C(y)\) with free variable \(x\) (Baader, Calvanese, et al. 2007), thus representing the set of all elements having an \(r\)-connection to an element satisfying \(C\). For fuzzy first-order logics, the classical quantifier \(\exists\) is usually interpreted by the supremum, owing to its connotation as an “infinite disjunction” over the domain (Hájek 2001, 2005a).

Similarly, the first-order equivalent \(\forall y. r(x,y) \rightarrow C(y)\) of \(\forall r.C\) is interpreted using the infimum. This directly yields the semantics listed in Table 2.2. These definitions have the counter-intuitive effect that an existential restriction can be satisfied to some degree without there being a single role successor justifying that degree, as shown in the following example.
Example 2.11 Consider the lattice $[0, 1]$ with any t-norm $\otimes$ and an interpretation $\mathcal{I} = (\mathbb{N}, \mathcal{I})$ such that $A^\mathcal{I}(n) := \frac{n-1}{n}$ and $r^\mathcal{I}(0, n) := 1$ for all $n \geq 1$, and $\mathcal{I}$ maps everything else to 0. Then we can compute the value of $\exists r.A$ at 0 as

$$(\exists r.A)^\mathcal{I}(0) = \bigvee_{n \in \mathbb{N}} (r^\mathcal{I}(0, n) \otimes A^\mathcal{I}(n)) = \bigvee_{n \geq 1} \frac{n-1}{n} = 1.$$ 

Note, however, that there is no single domain element $n \in \mathbb{N}$ where $r^\mathcal{I}(0, n) \otimes A^\mathcal{I}(n)$ actually takes the value 1.

It is often argued in the fuzzy DL literature that this behavior of the existential restrictions runs counter to the intuition that an existential restriction should actually force the existence of a single domain element that satisfies the restriction. This led to the introduction of a class of restricted interpretations that explicitly satisfy this intuition, called witnessed interpretations (Hájek 2005b).

We introduce here a more general definition that also covers the case where finitely many domain elements may together satisfy an existential restriction. This is relevant for lattices $L$ that are not totally ordered.

Definition 2.12 ($n$-witnessed interpretation) Given a natural number $n \geq 1$, we say that an interpretation $\mathcal{I}$ is $n$-witnessed if for every concept $C$, role name $r$, and domain element $x$, there are $2n$ witnesses $y_1, \ldots, y_n, y'_1, \ldots, y'_n \in \Delta^\mathcal{I}$ such that

$$(\exists r.C)^\mathcal{I}(x) = \bigvee_{i=1}^{n} (r^\mathcal{I}(x, y_i) \otimes C^\mathcal{I}(y_i)) \text{ and}$$

$$(\forall r.C)^\mathcal{I}(x) = \bigwedge_{i=1}^{n} (r^\mathcal{I}(x, y'_i) \Rightarrow C^\mathcal{I}(y'_i)).$$

For $n = 1$, the values of existential and universal restrictions are maxima and minima instead of potentially infinite suprema and infima, and we call a 1-witnessed interpretation $\mathcal{I}$ simply witnessed.

If $L$ is finite, then all interpretations are $|L|$-witnessed since every element of $L$ can be represented as the supremum of all elements below it. We can even give a smaller bound on the number of required witnesses in this case.

Lemma 2.13 If $L$ is finite, then every interpretation is $n$-witnessed, where $n$ is the width of $L$.

Proof. Consider an interpretation $\mathcal{I}$, an element $x \in \Delta^\mathcal{I}$, a concept $C$, and a role name $r$. Since $L$ is finite, there must be finitely many domain elements that satisfy the existential restriction $\exists r.C$ at $x$, i.e. there exist $y_1, \ldots, y_m \in \Delta^\mathcal{I}$ with

$$(\exists r.C)^\mathcal{I}(x) = \bigvee_{y \in \Delta^\mathcal{I}} (r^\mathcal{I}(x, y) \otimes C^\mathcal{I}(y)) = \bigvee_{i=1}^{m} (r^\mathcal{I}(x, y_i) \otimes C^\mathcal{I}(y_i)).$$

If we consider the smallest such $m$, we know that this supremum cannot be reached using only $m - 1$ different values of the form $r^\mathcal{I}(x, y_i) \otimes C^\mathcal{I}(y_i)$. Thus, removing any
element from \( \{ y_1, \ldots, y_m \} \) decreases this supremum. This can only be the case if the values \( r^I(x, y_i) \otimes C^I(y_i) \) are all incomparable, i.e. for every \( i \neq j \) it holds that
\[
r^I(x, y_i) \otimes C^I(y_i) \nleq r^I(x, y_j) \otimes C^I(y_j);
\]
otherwise, removing \( y_i \) would yield the same supremum. This means that the set
\[
\{ r^I(x, y_i) \otimes C^I(y_i) \mid 1 \leq i \leq m \}
\]
forms an antichain of cardinality \( m \). By assumption, we know that \( m \leq n \). To find the \( n \) witnesses required by Definition 2.12, we can add, e.g. the element \( y_1 \) \( n - m \) times to the sequence \( y_1, \ldots, y_m \). Similar arguments show the same for universal restrictions.

From this proof we can also see that, even if \( L \) is infinite, one does not need to distinguish \( n \)-witnessed interpretations for which \( n \) is larger than the width of \( L \). In particular, for total orders, which have width 1 (e.g. \([0, 1]\)), every \( n \)-witnessed interpretation is also \((1-)\)witnessed.

There are several other classes of interpretation that are interesting for fuzzy DLs.

**Definition 2.14 (finite, finitely valued, crisp, finitely branching)** Consider an interpretation \( I = (\Delta^I, \cdot^I) \).
- \( I \) is finite if \( \Delta^I \) is finite.
- \( I \) is finitely valued if there is a finite set \( S \subseteq L \) such that the image of each \( A^I \), \( A \in \mathbb{N}_C \), and \( r^I, r \in \mathbb{N}_R \), is contained in \( S \).
- \( I \) is crisp if it is finitely valued with \( S = \{0, 1\} \).
- \( I \) is finitely branching if for every \( x \in \Delta^I \) and role name \( r \) there are only finitely many \( y \in \Delta^I \) such that \( r^I(x, y) > 0 \).

Note that crisp interpretations exactly correspond to the classical semantics of DLs, as each concept must have value 0 (false) or 1 (true).

It is easy to see that every finite interpretation \( I \) is finitely valued, \( n \)-witnessed for some \( n \geq 1 \), and finitely branching. Furthermore, any of the latter three conditions implies the following property of \( I \), which we call finitely witnessed: for every \( x \in \Delta^I \), role name \( r \), and concept \( C \), the supremum over \( \{ r^I(x, y) \otimes C^I(y) \mid y \in \Delta^I \} \) can be computed as the supremum over a finite subset of this set, and likewise the infimum over \( \{ r^I(x, y) \Rightarrow C^I(y) \mid y \in \Delta^I \} \) can be computed as the infimum over a finite subset.

The relationships between the introduced properties are summarized in Figure 2.3. In a total order, any of these conditions is enough to force an interpretation to be witnessed.

**Lemma 2.15** For every interpretation \( I \), the following three properties hold:
- If \( I \) is \( n \)-witnessed, finitely branching, or finitely valued, then it is finitely witnessed.
- If \( I \) is crisp, then it is witnessed.
- If \( I \) is finitely witnessed and \( L \) is a total order, then \( I \) is witnessed.

**Proof.** Let \( I \) be an interpretation and consider an element \( x \in \Delta^I \), a concept \( C \), and a role name \( r \). We only consider existential restrictions in the following proofs; similar arguments apply for value restrictions.
2.2 Fuzzy Description Logics

Figure 2.3: The implications between several properties of interpretations

If \( I \) is \( n \)-witnessed, then the supremum over \( \{ r^I(x, y) \otimes C^I(y) \mid y \in \Delta^I \} \) can be computed as the supremum of a subset of cardinality \( n \), and dually for value restrictions.

If \( I \) is finitely branching, then for each \( x \in \Delta^I \) there are only finitely many non-zero values of the form \( r^I(x, y) \otimes C^I(y) \), and therefore their supremum can obviously be computed as the supremum of a finite subset of these values.

If \( I \) is finitely valued with some finite \( S \subseteq L \), then we show by induction on the structure of concepts that \( C^I \) can only take values from a finite subset of \( L \) (that depends on \( C \)). This is true for all concept names, and if it is the case for \( D \) and \( E \), then it certainly holds for \( D \sqcap E \), \( D \sqcup E \), \( D \rightarrow E \), \( \neg D \), and \( \sqsubseteq D \). Similarly, there are only finitely many values that can be computed as \( r^I(x, y) \otimes C^I(y) \), and therefore only finitely many suprema over such values. This shows that again the set \( \{ r^I(x, y) \otimes C^I(y) \mid y \in \Delta^I \} \) is finite for each \( x \in \Delta^I \).

If \( I \) is crisp, then \( (\exists r.C)^I(x) \) can only take the values \( 0 \) or \( 1 \). In either case, there must be at least one \( y \in \Delta^I \) such that \( r^I(x, y) \otimes C^I(y) \), which must also be either \( 0 \) or \( 1 \), has the same value.

If \( I \) is finitely witnessed and \( L \) is a total order, then \( L \) has width 1. Thus, the set \( \{ r^I(x, y) \otimes C^I(y) \mid y \in \Delta^I \} \) can have at most one maximal element. But it also must have at least one maximal element since otherwise its supremum could not be computed as the supremum over a finite subset. Any \( y \in \Delta^I \) corresponding to this maximum can be chosen as the witness for \( \exists r.C \) at \( x \).

In this thesis, we usually fix a class of interpretations at the beginning of each chapter or section, e.g. crisp or witnessed interpretations. If we want to talk about all interpretations without any restriction, then we say general interpretations. All subsequent considerations should then be understood relative to this fixed class. In special cases or for emphasis we may also mention the class of interpretations explicitly.

2.2.3 Axioms

As usual in fuzzy logics, fuzzy DL axioms are simply classical DL axioms associated with a degree to which they must be satisfied. This is usually stated as a lower bound on the actual degree of the statement in an interpretation. For example, we might assert that \((\text{mary}, \text{john}) : \text{likes} \) always has a degree \( \geq 0.5 \) in the lattice \([0, 1] \). Sometimes we also allow
Table 2.4: A list of axioms

<table>
<thead>
<tr>
<th>Name</th>
<th>Syntax</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>General concept inclusion (GCI)</td>
<td>( \langle C \sqsubseteq D \geq p \rangle )</td>
<td>( (C^I(x) \Rightarrow D^I(x)) \geq p ) for all ( x \in \Delta^I )</td>
</tr>
<tr>
<td>Concept definition</td>
<td>( \langle A \equiv C \geq p \rangle )</td>
<td>( (A^I(x) \Rightarrow C^I(x)) \geq p ) and ( (C^I(x) \Rightarrow A^I(x)) \geq p ) for all ( x \in \Delta^I )</td>
</tr>
<tr>
<td>Concept assertion</td>
<td>( \langle c : C \triangleright p \rangle )</td>
<td>( C^I(c) \triangleright p )</td>
</tr>
<tr>
<td>Role assertion</td>
<td>( \langle (c, d) : r \triangleright p \rangle )</td>
<td>( r^I(c^I, d^I) \triangleright p )</td>
</tr>
<tr>
<td>Role inclusion</td>
<td>( r \sqsubseteq s )</td>
<td>( r^I(x, y) \leq s^I(x, y) ) for all ( x, y \in \Delta^I )</td>
</tr>
<tr>
<td>Transitivity axiom</td>
<td>trans ( (r) )</td>
<td>( r^I(x, y) \otimes r^I(y, z) \leq r^I(x, z) ) for all ( x, y, z \in \Delta^I )</td>
</tr>
</tbody>
</table>

To enforce an exact degree by an axiom. Additionally, we consider axioms that state (crisp) properties about roles, namely subrole-superrole relationships and transitivity.

As in classical DLs, a collection of axioms is called an ontology and usually consists of three parts: terminological statements about concepts (the TBox), knowledge about individual names (the ABox), and role axioms (the RBox).

**Definition 2.16 (fuzzy DL axioms and ontologies)** An axiom is one of the expressions listed in Table 2.4, where \( C, D \) are arbitrary concepts, \( p \in L \), \( A \) is a concept name, \( c, d \) are individual names, \( r, s \) are roles, and \( \triangleright \in \{ =, \geq \} \). An assertion is either a concept assertion or a role assertion. An assertion is called equality assertion if it uses the relation \( = \), and inequality assertion if it uses \( \geq \).

A general TBox is a finite set of GCIIs. A finite set \( T \) of concept definitions is called an acyclic TBox if

(i) for every concept name \( A \) there is at most one concept definition of the form \( \langle A \equiv C \geq p \rangle \) in \( T \), and

(ii) the transitive closure of the following relation is irreflexive:

\[
\{(A, B) \in N_C \times N_C \mid \langle A \equiv C \geq p \rangle \in T, \ B \text{ occurs in } C\}.
\]

A concept name \( A \) is defined w.r.t. an acyclic TBox \( T \) if \( T \) contains a concept definition of the form \( \langle A \equiv C \geq p \rangle \). In this case, \( C \) is called the definition of \( A \) in \( T \). A concept name that is not defined in \( T \) is called primitive w.r.t. \( T \).

An ABox is a finite set of assertions. The ABox \( A \) is called local if there is an individual name \( c \in N_I \) such that all assertions in \( A \) are of the form \( \langle c : C = p \rangle \) for some concept \( C \) and \( p \in L \).

An RBox is a finite set of role inclusions and transitivity axioms. An ontology is a triple \( O = (A, T, R) \) consisting of an ABox \( A \), a (general or acyclic) TBox \( T \), and an RBox \( R \). An axiom is satisfied by an interpretation \( I \) if the condition listed in the semantics column of Table 2.4 holds in \( I \). An ontology or a set of axioms is satisfied by \( I \) if all its axioms are satisfied by \( I \). In this case, we call \( I \) a model of the axiom/set of axioms/ontology.

An ontology or a set of axioms is called crisp if all the axioms contained in it are crisp, i.e. they contain only the value \( p = 1 \). In this case, we may also omit the suffix \( \geq 1 \)
2.2 Fuzzy Description Logics

or = 1 from the axioms, and simply write, e.g. \( C \sqsubseteq D \). We often use the expression \( C \equiv D \) to abbreviate the two crisp axioms \( C \sqsubseteq D \) and \( D \sqsubseteq C \).

A crisp acyclic TBox is sometimes called an unfoldable TBox in the literature on fuzzy DLs, e.g. in (Bobillo, Bou, and Straccia 2011). The unfolding \( C^T \) of a concept \( C \) w.r.t. an unfoldable TBox \( T \) is obtained from \( C \) by recursively replacing each defined concept name by its definition. Note that the size of \( C^T \) may be exponential in the size of \( C \) and \( T \) (Nebel 1990). To avoid this exponential blow-up, often a technique called lazy unfolding is employed, which replaces defined concept names by their definitions only when it becomes necessary in the course of a computation. We will present examples of this technique in the automata-based algorithms in Section 3.1 and Appendix A.

Observe that general TBoxes are indeed more general than acyclic ones since a concept definition \( \langle A \equiv C \geq p \rangle \) is satisfied by an interpretation \( I \) iff \( I \) is a model of the two GCIs \( \langle A \sqsubseteq C \geq p \rangle \) and \( \langle C \sqsubseteq A \geq p \rangle \). Furthermore, inequality assertions can always be simulated by equality assertions and crisp GCIs: the assertion \( \langle c : C \geq p \rangle \) can be expressed by the axioms \( \langle c : A = p \rangle \) and \( \langle A \sqsubseteq C \geq 1 \rangle \), where \( A \) is a new concept name that does not yet occur in the ontology. It is easy to see that the latter two axioms impose the same restrictions on the interpretations of \( c \) and \( C \) as the original axiom. This means that the variant of a fuzzy DL allowing equality assertions and crisp GCIs is more expressive than the same logic with inequality assertions instead.

Example 2.17 Consider the lattice \([0, 1]\) with any continuous t-norm. The ABox

\[
A := \{\langle sphinx, chimera \rangle: hasMother, \langle sphinx, orthrus \rangle: hasFather, \langle sphinx: Female \rangle, \langle sphinx: Lion = \frac{1}{3} \rangle, \langle sphinx: Human = \frac{1}{3} \rangle, \langle sphinx: Bird = \frac{1}{3} \rangle, \langle chimera: Lion = \frac{1}{3} \rangle, \langle chimera: Goat = \frac{1}{3} \rangle, \langle chimera: Snake = \frac{1}{3} \rangle, \langle orthrus: Dog \rangle\}
\]

states some facts about beasts from Greek mythology. The TBox

\[
T := \{\langle T \sqsubseteq \forall hasMother. Female \rangle, \langle T \sqsubseteq \forall hasFather. Male \rangle, \langle Dog \sqsubseteq \forall hasChild. Dog \geq 0.5 \rangle, \langle Lion \sqsubseteq Dangerous \geq 0.7 \rangle, \langle Dog \sqsubseteq Dangerous \geq 0.5 \rangle\}
\]

describes general knowledge about animals and inheritance. Finally, the RBox

\[
R := \{\text{hasMother} \sqsubseteq \text{hasParent}, \text{hasFather} \sqsubseteq \text{hasParent}, \text{hasParent} \sqsubseteq \text{hasAncestor}, \text{trans(hasAncestor)}, \text{hasParent} \sqsubseteq \text{hasChild}^\neg, \text{hasChild}^\neg \sqsubseteq \text{hasParent}\}
\]

relates some of the used role names to each other.

When describing a fuzzy description logic, it will be mentioned explicitly which class of TBoxes is considered, i.e. acyclic, unfoldable, crisp general, or fuzzy general TBoxes. Likewise, we explicitly distinguish the presence of crisp assertions, inequality assertions, and equality assertions in the ABox.

The role axioms traditionally receive a special treatment in DLs in that they are denoted in the name of the DL under consideration. In general, the presence of transitivity axioms is denoted by a subscript \( R^+ \) at the logic name, e.g. \( L^\cdot \mathcal{E}_R^{R^+} \). However, for
classical $\mathcal{ALC}$ extended with transitive roles, the shortcut $S$ has become customary, which was introduced due to its close connection to the multi-modal logic $\mathcal{S4}_m$ (Horrocks, Sattler, and Tobies 2000). It is important to note that we will here use $L-S$ to denote only $L-\mathcal{ALC}R^+$ since we need to distinguish two kinds of negation and the implication constructor. For example, $L-\mathcal{ALC}R^+$ becomes $L-\mathcal{SC}$, and $L-\mathcal{I}ALR^+$ is abbreviated to $L-IS$.

Finally, the presence of role inclusions is denoted by the letter $H$ in the name of the logic (see Table 2.3).

Hence, we may talk about

“$L-ISCHOI$ with an acyclic TBox and inequality assertions”

and hence we may talk about

“subsumption w.r.t. $n$-witnessed models”

Now that we have represented the knowledge of a particular domain of interest using fuzzy DL axioms, we want to draw inferences that give us new insights into the domain.

### 2.2.4 Reasoning

We now fix a class $\mathcal{C}$ of interpretations. When we say $\mathcal{C}$-interpretation, we mean an arbitrary interpretation of this class, and similarly for $\mathcal{C}$-models. A basic inference problem in any logical formalism is the entailment of single axioms by a set of axioms. Formally, an axiom $\alpha$ is entailed by a set of axioms or an ontology $\mathcal{O}$ if all $\mathcal{C}$-models of $\mathcal{O}$ satisfy $\alpha$. Many reasoning problems in (fuzzy) DLs are of this form, e.g. subsumption and instance checking.

**Definition 2.18 (fuzzy DL reasoning problems)** Let $\mathcal{O} = (A, T, R)$ be an ontology, $C, D$ be concepts, $p \in L$, and $c$ be an individual name.

- $\mathcal{O}$ is **consistent** if it has a $\mathcal{C}$-model.
- If $A$ is a local ABox, then $\mathcal{O}$ is **locally consistent** if it is consistent.
- $C$ is $p$-satisfiable w.r.t. $\mathcal{O}$ if there is a $\mathcal{C}$-model $I$ of $\mathcal{O}$ and an element $x \in \Delta^I$ such that $C^I(x) \geq p$.
- $C$ is $p$-subsumed by $D$ w.r.t. $\mathcal{O}$ if the GCI $\langle C \sqsubseteq D \geq p \rangle$ is entailed by $\mathcal{O}$.
- $c$ is a $p$-instance of $C$ w.r.t. $\mathcal{O}$ if the assertion $\langle c; C \geq p \rangle$ is entailed by $\mathcal{O}$.
- The **best satisfiability degree** of $C$ w.r.t. $\mathcal{O}$ is the supremum of all $p' \in L$ such that $C$ is $p'$-satisfiable w.r.t. $\mathcal{O}$.
- The **best subsumption degree** of $C$ and $D$ w.r.t. $\mathcal{O}$ is the supremum of all $p' \in L$ such that $C$ is $p'$-subsumed by $D$ w.r.t. $\mathcal{O}$.
- The **best instance degree** of $c$ in $C$ w.r.t. $\mathcal{O}$ is the supremum of all $p' \in L$ such that $c$ is a $p'$-instance of $C$.

When we want to talk about reasoning w.r.t. a specific class of models, we may say, e.g. “subsumption w.r.t. $n$-witnessed models”.

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Observe that the above list contains decision problems as well as the corresponding computation problems that ask for the best fuzzy degree of a given inference. We usually consider the decision problems first. In case of decidability, we also discuss how to compute the associated optimal degree.

**Example 2.19** Consider again the ontology $O = (A, T, R)$ from Example 2.17 and the Product t-norm for the semantics. One consequence of these axioms is that orthrus is an instance of Male to degree 1 due to the assertion that he is the father of sphinx and all fathers are male.

Similarly, one can derive that sphinx is an instance of Dangerous to degree $\frac{7}{10}$ since she is one third lion and lions are dangerous to degree at least 0.7. However, sphinx is also an instance of Dog to degree 0.5 since her father is a dog to degree 1, and thus the best instance degree of sphinx in the concept Dangerous is $\frac{1}{2}$, which exceeds $\frac{7}{10}$. Under the Łukasiewicz t-norm, this degree is $\frac{1}{2}$ since $\frac{1}{2} + \frac{7}{10} - 1 = \frac{1}{20} > 0 = \frac{1}{2} + \frac{1}{2} - 1$.

This illustrates that in general one has to consider all possible derivations of a consequence in order to determine the best degree to which it holds. All this does not explain how a dog and a hybrid between a lion, a goat, and a snake can have (partly) human offspring, but that lies beyond the scope of this thesis. \hfill \spadesuit 

It is interesting to note that the best subsumption degree is actually a maximum instead of a supremum, as is the best instance degree.

**Lemma 2.20** Let $C, D$ be two concepts, $c \in N$, and $p \in L$. If $p$ is the best subsumption degree of $C$ and $D$ w.r.t. $O$, then $C$ is $p$-subsumed by $D$ w.r.t. $O$. Likewise, if $p$ is the best instance degree of $c$ in $C$ w.r.t. $O$, then $c$ is a $p$-instance of $C$ w.r.t. $O$.

**Proof.** Consider the set $S$ of all elements $p' \in L$ for which $C$ is $p'$-subsumed by $D$ w.r.t. $O$. Then $C^I(x) \Rightarrow D^I(x) \geq p'$ holds for all $p' \in S$, all $C$-models $I$ of $O$, and all $x \in \Delta^I$. This implies that for all $C$-models $I$ of $O$ and all $x \in \Delta^I$, we have $C^I(x) \Rightarrow D^I(x) \geq \bigvee S$, i.e. the best subsumption degree $\bigvee S$ of $C$ and $D$ w.r.t. $O$ is itself a subsumption degree. A similar argument shows the claim for the best instance degree. \hfill \qed

However, for the best satisfiability degree this need not be true: there may be several $C$-models satisfying a concept to degrees whose supremum is $p$, but no $C$-model that actually satisfies the concept to degree $p$.

**Example 2.21** Consider the lattice $L_4$ from Figure 2.4, as t-norm the infimum of this lattice, and the TBox

$$T = \{ \langle \top \sqsubseteq (A \sqcap \neg A) \sqcup (B \sqcap \neg B) \geq t \rangle \}.$$

The interpretation $I_0 = \{ (x_1, x_2), I_0 \}$ with

$$A^{I_0}(x_1) = B^{I_0}(x_2) = u \text{ and } B^{I_0}(x_1) = A^{I_0}(x_2) = d$$

is a model of $T$ that proves the $u$- and $d$-satisfiability of $A$ w.r.t. $T$. Thus, the best satisfiability degree of $A$ w.r.t. $T$ is $t$.

However, since we know that $p \land \neg p \neq 1$ for every $p \in L_4$, the TBox can only be satisfied by an interpretation $I$ if for every $x \in \Delta^I$ it holds that $\{ A^I(x), B^I(x) \} = \{ u, d \}$. Thus, we always have $A^I(x) < t$. \hfill \spadesuit
Among the introduced decision problems, the most basic one is ontology consistency: if the underlying ontology is inconsistent, then the other reasoning problems become trivial. Moreover, in the presence of the bottom constructor, these reasoning problems are always at least as hard as (in)consistency since for every ontology $\mathcal{O}$, the following are equivalent:

- $\mathcal{O}$ is inconsistent.
- $\bot$ is 1-satisfiable w.r.t. $\mathcal{O}$.
- $\top$ is 1-subsumed by $\bot$ w.r.t. $\mathcal{O}$.
- $c$ is a 1-instance of $\bot$ w.r.t. $\mathcal{O}$, where $c$ is a fresh individual name.

Similarly, a concept $C$ is $p$-satisfiable w.r.t. $\mathcal{O} = (A, T, R)$ iff $(A \cup \{c : C \geq p\}, T, R)$ is consistent, where $c$ is again a fresh individual name. Subsumption and instance checking can, however, not so easily be reduced to ontology consistency since our language does not allow for axioms of the form $\langle c : C < p \rangle$ (except in Section 4.2).

**Reasoning without ABoxes**

In the absence of nominals, subsumption and satisfiability are often analyzed without considering ABoxes. The reason for this is that an ABox can only change the answers to such questions by making a previously consistent ontology inconsistent.

**Lemma 2.22** Let $L, \mathcal{L}$ be a fuzzy DL without nominals, $\mathcal{O} = (A, T, R)$ be an ontology, $C,D$ be concepts and $p \in L$. Then $C$ is $p$-subsumed by $D$ w.r.t. $\mathcal{O}$ iff $\mathcal{O}$ is inconsistent or $C$ is $p$-subsumed by $D$ w.r.t. $(\emptyset, T, R)$. Similarly, $C$ is $p$-satisfiable w.r.t. $\mathcal{O}$ iff $\mathcal{O}$ is consistent and $C$ is $p$-satisfiable w.r.t. $(\emptyset, T, R)$.

**Proof.** Note first that our logics are monotonic, and thus a GCI that is already entailed by $(\emptyset, T, R)$ is also entailed by $\mathcal{O}$. Conversely, assume that a consistent $\mathcal{O}$ entails $\langle C \sqsubseteq D \geq p \rangle$, while $(\emptyset, T, R)$ does not. Then there must be a model $I$ of $T$ and $R$ with $C^I(x) \Rightarrow D^I(x) < p$ for some $x \in \Delta^I$. By assumption, $I$ cannot be a model of $A$. However, since $\mathcal{O}$ is consistent, it has a model $I'$. By adding $I$ to $I'$ (see the construction and Lemma 2.23 below) we obtain a model of $\mathcal{O}$ that contradicts $\langle C \sqsubseteq D \geq p \rangle$, which contradicts the assumptions.

Similarly, if $C$ is $p$-satisfiable w.r.t. $\mathcal{O}$, then it is $p$-satisfiable w.r.t. $(\emptyset, T, R)$ and $\mathcal{O}$ must be consistent. Conversely, assume that $\mathcal{O}$ has a model $I'$ and there is a model $I$ of $T$ and $R$ with $C^I(x) \geq p$ for some $x \in \Delta^I$. Again, we can add $I$ to $I'$ to construct a model of $\mathcal{O}$ that satisfies $C$ to degree $p$. \hfill \Box

---

Figure 2.4: The finite De Morgan lattice $L_4$
This in particular implies that in the absence of nominals the best satisfiability and subsumption degrees w.r.t. a consistent ontology are independent of the ABox. On the other hand, if the ontology is inconsistent, then these degrees are always 0 and 1, respectively.

To complete the proof of Lemma 2.22, it remains to present the construction by which we can merge separate interpretations into one. Let $I_1 = (\Delta_1, I_1)$ and $I_2 = (\Delta_2, I_2)$ be two interpretations for which we assume without loss of generality that their domains are disjoint. We now define a new interpretation $I_+$ over the domain $\Delta_+ := \Delta_1 \cup \Delta_2$ as follows:

- For all $c \in N_I$, we define $c^{I_+} := c^{I_1}$.
- For all $A \in N_C$ and $x \in \Delta_+$, we define $A^{I_+}(x) := \begin{cases} A^{I_1}(x) & \text{if } x \in \Delta_1, \\
A^{I_2}(x) & \text{otherwise.} \end{cases}$
- For all $r \in N_R$ and $x, y \in \Delta_+$, we define $r^{I_+}(x, y) := \begin{cases} r^{I_1}(x, y) & \text{if } x, y \in \Delta_1, \\
r^{I_2}(x, y) & \text{if } x, y \in \Delta_2, \\
0 & \text{otherwise.} \end{cases}$

Intuitively, we “add” the domain elements of $I_2$ to $I_1$ while keeping the interpretations of the individual names and interpreting the concept and role names as in the original interpretations. It is easy to verify that if $I_1$ and $I_2$ are both $n$-witnessed, finite, finitely valued, crisp, finitely branching, or finitely witnessed, then so is $I_+$.

We now show that $I_+$ satisfies exactly those GCIs, concept definitions, role inclusions, and transitivity axioms that are satisfied by both $I_1$ and $I_2$. Furthermore, $I_+$ satisfies exactly those assertions that $I_1$ satisfies. All of this follows directly from the following lemma.

**Lemma 2.23** Let $L$ be a fuzzy DL without nominals, $r$ be a role, $C$ a concept, and $x, y \in \Delta_+$. Then

\begin{align*}
&\text{a)} \ C^{I_+}(x) = \begin{cases} C^{I_1}(x) & \text{if } x \in \Delta_1, \\
C^{I_2}(x) & \text{otherwise; and} \end{cases} \\
&\text{b)} \ r^{I_+}(x, y) = \begin{cases} r^{I_1}(x, y) & \text{if } x, y \in \Delta_1, \\
r^{I_2}(x, y) & \text{if } x, y \in \Delta_2, \\
0 & \text{otherwise.} \end{cases}
\end{align*}

**Proof.** Claim b) follows directly from the definition of $I_+$ and the semantics of inverse roles. We prove a) by induction on the structure of $C$. For concept names, the top concept, and the bottom concept, it easily follows from the definition $I_+$.

Consider now a conjunction $C = D \cap E$ and assume that a) holds for $D$ and $E$. If $x \in \Delta_1$, then $C^{I_+}(x) = D^{I_+}(x) \cap E^{I_+}(x) = D^{I_1}(x) \cap E^{I_1}(x) = C^{I_1}(x)$. Otherwise, we get $C^{I_+}(x) = C^{I_2}(x)$ by similar arguments. The proofs for the other concept constructors not involving roles are analogous.
If \( C \) is of the form \( \exists r.D \) and \( x \in \Delta_1 \), then by b) and the induction hypothesis we have

\[
C^I_+(x) = \bigvee_{y \in \Delta_+} r^I_+(x, y) \otimes D^I_+(y)
\]

\[
= \bigvee_{y \in \Delta_1} r^I_+(x, y) \otimes D^I_1(y)
\]

\[
= C^I_1(x),
\]

and similarly for \( x \in \Delta_2 \) and for value restrictions.

This lemma obviously does not hold for nominals, as the interpretation of the individual names under \( I_2 \) is discarded by the construction.

On the other hand, if the logic allows nominals, then inequality assertions can be eliminated from all reasoning problems by replacing them by GCIs as follows. It is easy to check that any concept inequality assertion \( \langle c: C \geq p \rangle \) has the same semantics as the GCI \( \langle \{c\} \sqsubseteq C \geq p \rangle \). Similarly, every role inequality assertion \( \langle (c, d): r \geq p \rangle \) can be replaced by \( \langle \{c\} \sqsubseteq \exists r.\{d\} \geq p \rangle \). Note that crisp assertions can thus be translated into crisp GCIs. We will present a different way to remove crisp assertions from an ontology in Appendix A.

If we are allowed to use the involutive negation, then also equality assertions can be simulated by GCIs. The assertion \( \langle c: C = p \rangle \) can be replaced by a GCI as above, together with \( \langle \{c\} \sqsubseteq \neg C \geq \neg p \rangle \); similarly, for a role assertion \( \langle (c, d): r = p \rangle \) we need the additional GCI \( \langle \{c\} \sqsubseteq \neg \exists r.\{d\} \geq \neg p \rangle \).

### Entailment of Role Axioms

We have already considered two entailment problems for fuzzy DLs, namely subsumption and instance checking, which ask about entailment of GCIs and concept assertions, respectively. It is thus natural to also study the entailment of role axioms. Since these are basically crisp, we can use methods known from classical DLs to deal with them. Similar to what was shown in Lemma 2.22 for subsumption and satisfiability, the entailment of role axioms is in essence a task that involves only the RBox. In the following, we introduce several notions that are helpful to determine the properties of roles that are implied by an RBox.

It is easy to compute all role inclusions following from an RBox \( R \) as the closure of the role inclusions in \( R \) under inverse roles, reflexivity, and transitivity. More formally, we define the role hierarchy \( \sqsubseteq_R \) as the reflexive transitive closure of the binary relation

\[
\{(r, s) \mid r \sqsubseteq s \in \mathcal{R} \text{ or } \bar{r} \sqsubseteq \bar{s} \in \mathcal{R}\},
\]

on complex roles, where inverse roles are only considered if the fuzzy DL under consideration supports them. This relation is defined in the same way as for classical DLs and can be computed in polynomial time in the size of \( \mathcal{R} \) (Horrocks, Sattler, and Tobies 2000).

The transitivity of a role \( r \) can only be enforced by an RBox by making it equivalent to (the inverse of) a role that is explicitly marked as transitive. As in (Horrocks, Sattler, and Tobies 2000), we will call a role \( r \) transitive (w.r.t. an RBox \( \mathcal{R} \)) if \( \mathcal{R} \) contains either trans\((r)\) or trans\((\bar{r})\).
Complexity

The goal of this thesis is to analyze the computational complexity of the introduced reasoning problems for fuzzy DLs. We assume in the following that the reader is familiar with the basics of complexity theory as described, among others, by Papadimitriou (1994).

In order to measure the complexity of a decision procedure, we will often talk about the size of a concept $C$ or an ontology $O$, by which we mean the number of symbols it takes to write down $C$ or $O$. Regarding the size of an element $p \in L$, we define it as the number of bits it takes to express $p$, which of course depends on the precise encoding of $L$ and $p$.

For fuzzy DLs over the standard interval $[0, 1]$, we assume values in the ontologies to be rational numbers in binary representation. However, this is not a great restriction as most fuzzy DLs are undecidable (sometimes even for crisp ontologies; see Chapter 5) or easily decidable (and then we only need to be able to check order relations between the values of the ontology; see Chapter 4).

For our results about finite lattices (Chapter 3), we assume that the lattice $L$ of truth degrees is given as a list of its elements and that all lattice operations are computable in polynomial time in the size of the input elements. In particular, it is important for the result in Lemma 3.21 that the cardinality $|L|$ is polynomial in the size of the input. All other results of Chapter 3 remain valid even if the cardinality of $L$ is exponential in the size of the input encoding of $L$.

2.3 Examples

We now provide some examples of complete residuated De Morgan lattices and describe possible use cases.

Consider first the lattice $L_4$ with the elements $t$ (true), $f$ (false), $d$ (disagreement), and $u$ (unknown), depicted in Figure 2.4. This lattice can be used to resolve inconsistencies between conflicting pieces of information (Belnap 1977). Suppose for example that we want to consolidate information obtained from several sources, e.g. medical textbooks or doctors. The books only provide information on a certain area like anatomy, and do not contain much about other topics. Furthermore, a doctor might disagree with a textbook on a certain issue because of personal experience or access to recent results from clinical trials. Given a statement, we can now classify it according to whether there is no information about it available ($u$), there are sources asserting its truth and no other sources disagree ($t$), at least one source refutes it and no source provides positive evidence ($f$), or the sources disagree ($d$). In classical logic, statements can only be true, false, or unknown, but using $L_4$, we can additionally distinguish contradicting beliefs. Even more, we can say that the truth degree of a statement is $\geq u$, which means that nothing refutes it, but we do not know whether it actually holds. Consider now the t-norm defined by $x \otimes y := x \land y$. Given two predicates $A$ and $B$, and an interpretation $\mathcal{I}$ such that $A^\mathcal{I}(x) = u$ and $B^\mathcal{I}(x) = d$ for some $x \in \Delta^\mathcal{I}$, we obtain $(A \cap B)^\mathcal{I}(x) = f$ since at least one source believes the conjunction to be false, and no other source disagrees.

A common issue with fuzzy DLs is the concern of how to obtain the fuzzy values for the axioms. Straccia (2005) proposes to use fuzzy concrete domains to automatically
infer the value of a concept like Tall from information about the height of an individual. For this, classical DL semantics is augmented by an additional concrete domain like \( \mathbb{N} \) and concrete features like \texttt{hasHeight} assigning each abstract domain element a concrete domain value, together with concrete domain predicates, e.g. \( \geq 180 \), that express properties of concrete values. One can then talk about all individuals of height \( \geq 180 \) cm using the concept \( \exists \texttt{hasHeight} \geq 180 \). For a survey of classical DLs with concrete domains, see (Lutz 2003). Straccia considers only unary fuzzy concrete domain predicates like Tall that are interpreted as fuzzy sets over the concrete domain. Thus, \( \exists \texttt{hasHeight}.\text{Tall} \) is interpreted as a fuzzy set over the abstract domain that describes the degree of tallness of all individuals. To obtain computationally manageable logics, the interpretation of the fuzzy concrete predicates should have a simple representation, e.g. as piece-wise linear or polynomial functions from the concrete domain to \([0, 1]\).

It is common to consider restrictions of the Gödel and Łukasiewicz t-norms to a finite set of truth values (Bobillo and Straccia 2011, 2013b). This is motivated by the observation that the set \( [0, 1]_n := \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\} \) is closed under these two t-norms, for any \( n \geq 2 \). Together with the negation \( \frac{i}{n-1} \mapsto \frac{n-1-i}{n-1} \), this set forms a complete De Morgan lattice, and the Gödel and Łukasiewicz residuum are exactly those in Table 2.1, restricted to \([0, 1]_n\). The resulting logics are denoted by \( G_n-\mathcal{L} \) or \( \mathcal{L}_n-\mathcal{L} \), respectively. There is no equivalent of the Product t-norm over a finite total order since by multiplying repeatedly one can create infinitely many distinct values. Although \([0, 1]_n\) is closed under the fuzzy operators induced by the Gödel and Łukasiewicz t-norms, the semantics of \( \mathcal{L}_n-\mathcal{L} \) and \( \mathcal{L}-\mathcal{L} \) are different even when only values from \([0, 1]_n\) occur in the ontology (cf. Section 4.2.1). For example, the equation \( x \otimes x = 0 \) has different sets of solutions under these two semantics, namely \( \{\frac{i}{n-1} \mid 0 \leq i \leq \frac{n-2}{2}\} \) and \([0, \frac{1}{2}]\), respectively. Therefore, an axiom like \( \langle c : A \sqcap A = 0 \rangle \) can introduce values outside of \([0, 1]_n\) to a model in \( \mathcal{L}-\mathcal{L} \).

2.4 Related Work

The main interest of this thesis lies in the decidability and complexity of reasoning problems in fuzzy DLs w.r.t. general concept inclusions. In particular, the focus will be on deciding ontology consistency under \((n-)\)witnessed model semantics. Before we describe the results obtained in this investigation, we will first give an overview of the relevant related work in the fields of mathematical fuzzy logic and fuzzy description logics.

2.4.1 Mathematical Fuzzy Logic for Fuzzy Description Logics

From a more mathematical point of view, formulae of fuzzy logic are usually formed using only the constructors conjunction and implication, together with constants for 0 and 1. Common problems in this setting are to decide whether a given formula is a tautology w.r.t. a class of residuated lattices (roughly, whether a concept \( C \) is \( 1 \)-subsumed by \( \top \) w.r.t. an empty ontology in all lattices of this class), or whether it is \((1-)\)satisfiable.

A distinction is made between standard semantics, where formula are interpreted over t-norms on \([0, 1]\), or general semantics, which considers the class of all BL-algebras. A
BL-algebra is a residuated lattice satisfying the two additional properties of divisibility and prelinearity (Hájek 2001; Klement, Mesiar, and Pap 2000). The intuition is that these are desirable properties since they are satisfied in particular by each continuous t-norm over the standard interval.

Apart from determining the computational complexity of deciding tautologies and satisfiability, a large interest lies in finding axiomatizations of such logics, i.e. finite sets of tautologies that suffice to derive all other tautologies using inference rules such as modus ponens. For example, Hájek (2001) provides axiomatizations of all fuzzy propositional tautologies over BL-algebras and some subclasses of BL-algebras capturing the properties of the t-norms Ł, Π, and G over [0, 1]. For instance, the class of MV-algebras consists of those BL-algebras having an involutive residual negation, and can be shown to have the same tautologies as propositional fuzzy logic over Ł. Similarly, the distinguishing property of G is the idempotence of conjunction. The framework is also extended to fuzzy predicate logics, but there a sound and complete axiomatization is not always possible. For a more comprehensive treatment of these issues, see (Cintula, Hájek, and Noguera 2011; Hájek 2001).

This point of view was first applied to fuzzy DLs in (Hájek 2005b), where the author proposes to use mathematical fuzzy logic to enrich the formalism of fuzzy DLs, which so far had considered only Zadeh semantics (Gödel t-norm and t-conorm, Kleene-Dienes-implication, and involutive negation). In the course of his investigation, Hájek introduced witnessed models, which he later considered also for predicate logics in general (Hájek 2007a,b, 2010).

In (Hájek 2005b), it was shown that 1-satisfiability and 1-subsumption for \( \otimes \mathcal{A} \mathcal{L} \) under any t-norm \( \otimes \) over [0, 1] are decidable w.r.t. witnessed models; however, Hájek did not consider a background ontology. These results also hold for Ł without the restriction to witnessed models since he proved that in this case the two semantics coincide.

In (Cerami, Esteva, and Bou 2010), it is proved that 1-subsumption w.r.t. general models is also decidable under the Product t-norm, but still without a background ontology. Decidability of 1-satisfiability under the restriction to so-called quasi-witnessed models is also shown there. Recently, a first implementation of the algorithms of Cerami, Esteva, and Bou (2010) using an SMT (SAT modulo theories) approach was reported (Alsinet, Barroso, Béjar, Bou, Cerami, and Esteva 2013).

García-Cerdaña, Armengol, and Esteva (2010) extend the approach of Hájek (2005b) to deal with ontologies, and axiomatizations of t-norm-based fuzzy DLs are investigated. Axiomatizations of fuzzy DLs over finite total orders are developed in (Cerami, García-Cerdaña, and Esteva 2010; Cerami, García-Cerdaña, and Esteva 2014).

2.4.2 Fuzzy Description Logics over the Standard Interval

The first treatment of fuzzy DLs can be found in (Yen 1991), where a fuzzy extension of the basic DL \( \mathcal{FL}^- \) is investigated. This logic, which allows for conjunctions, value restrictions, and unqualified existential restrictions (\( \exists r \)), was first considered by Brachman and Levesque (1984). As mentioned before, the fuzzy semantics is basically that proposed by Zadeh over the interval [0, 1]; we will in the following refer to this using the prefix Z. Yen presents an algorithm for deciding concept subsumption in Z-\( \mathcal{FL}^- \), an extension of
the algorithm of Brachman and Levesque (1984). This algorithm allows for unfoldable TBoxes to define concept names.

Research on fuzzy DLs rested until in 1998 two tableau algorithms were presented for deciding consistency in variants of $\mathcal{Z}$-$\mathcal{ALC}$ (Straccia 1998; Tresp and Molitor 1998). No TBoxes were considered in the beginning, but the algorithm was soon extended to deal with unfoldable TBoxes by replacing all concepts by their unfolding before applying the tableau algorithm (Straccia 2001). This approach yields $\text{EXPSPACE}$ as the best known upper bound on the complexity of consistency, which leaves a large gap to the best known lower bound of $\text{PSPACE}$—the complexity of deciding consistency in classical $\mathcal{ALC}$ with acyclic TBoxes (Lutz 1999; Schmidt-Schauß and Smolka 1991). This tableau algorithm was later generalized to deal with crisp GCIs (Stoilos, Straccia, Stamou, and Pan 2006), the DL $\mathcal{Z}$-$\mathcal{SCHIN}$, which allows so-called (unqualified) number restrictions on roles (Stoilos, Stamou, Pan, Tzouvaras, and Horrocks 2007), and even DLs with qualified number restrictions such as $\mathcal{Z}$-$\mathcal{ALCIQ}$ (Stoilos, Stamou, and Kollias 2008) and $\mathcal{Z}$-$\mathcal{SCHOIQ}$ (Stoilos and Stamou 2013).

Only in 2005 was it noticed that an essential assumption necessary for the correctness of these algorithms had been overlooked in the beginning (Hájek 2005b). In the same paper in which he introduced witnessed interpretations to address this issue, Hájek started a whole new line of investigation by restating the semantics of fuzzy DLs in terms of arbitrary t-norms.

Subsequently, tableau algorithms have been developed to decide consistency in $\otimes$-$\mathcal{ALC}$ with unfoldable TBoxes, where $\otimes$ is any finite ordinal sum of $\Pi$ and $\mathfrak{L}$ (Bobillo and Straccia 2009). The main idea is to construct an abstract representation of a model (called tableau) of a given ontology in which the concrete values of concepts at domain elements are encoded by variables ranging over $[0, 1]$. According to the semantics of the fuzzy DL, the values of these variables are then restricted by polynomial inequations to obtain the correct behavior. In this way, a system of inequations is constructed that is of exponential size and that has a solution iff the input ontology has a model. For the Łukasiewicz t-norm, one obtains a system of linear equations over continuous variables (evaluated in $[0, 1]$) and integer variables (evaluated as either 0 or 1), which is solvable in nondeterministic polynomial time (Salkin and Mathur 1989). For the Product t-norm, the result is a system of quadratic inequations over such variables, which is solvable in $\text{PSPACE}$ (Canny 1988). Even with lazy unfolding of the TBox, one obtains a worst-case behavior of $\text{NEXP}$ or even $\text{EXPSPACE}$. A tableau algorithm for Zadeh, Gödel, and Łukasiewicz semantics has been implemented in the fuzzy reasoner $\text{fuzzyDL}$, which uses an external optimization library to solve the generated systems of inequations (Bobillo and Straccia 2008a).

In classical tableau algorithms, the presence of GCIs (or other language elements like transitive roles) can lead to infinite tableaux. This problem is resolved by appropriate blocking techniques that formulate conditions to detect when a finite part of a tableau is already sufficient to indicate the existence of a (possibly infinite) model (Baader and Sattler 2001; Horrocks and Sattler 1999). For fuzzy tableau algorithms, naive adaptations

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1For the Łukasiewicz t-norm, the restriction to unfoldable TBoxes can be relaxed to acyclic TBoxes: It was shown by Bobillo, Bou, and Straccia (2011) that under $\mathfrak{L}$ every acyclic TBox can be simulated by an unfoldable TBox.
of classical blocking conditions were claimed to be sufficient to show decidability of consistency in the presence of fuzzy GCIs in Ł-\(\mathcal{ALC}\) (Straccia and Bobillo 2007), \(\Pi-\mathcal{ALC}\) (Bobillo and Straccia 2007), \(\otimes-\mathcal{SC}\)T (Stoilos and Stamou 2009), and \(\otimes-\mathcal{ALC}\) with S-implication (Haarslev, Pu, and Shiri 2009), where \(\otimes\) is an arbitrary continuous t-norm.

However, it was demonstrated independently by Baader and Peñaloza (2011a) and Bobillo, Bou, and Straccia (2011) that the developed algorithms were not sound. To remedy this, in (Baader, Borgwardt, and Peñaloza 2014; Peñaloza 2011) a tableau algorithm was developed that reduces consistency w.r.t. witnessed models in \(\otimes-\mathcal{ALC}\) to the satisfiability problem for a certain kind of finitely represented, but infinite, systems of polynomial inequations. Unfortunately, the decidability of the latter problem is unknown. In fact, it was then shown by Baader and Peñaloza (2011a) that consistency w.r.t. witnessed models in \(\Pi-\mathcal{ALC}\) with strict GCIs of the form \((C \sqsubseteq D > p)\) and inequality assertions is undecidable. Following the same idea, undecidability was later shown for consistency w.r.t. witnessed models in \(\otimes-\mathcal{ALC}\) with fuzzy general TBoxes and equality assertions if \(\otimes\) starts with \(\Pi\) (Baader and Peñaloza 2011b) and in Ł-\(\mathcal{ELC}\) with fuzzy GCIs and inequality assertions (Cerami and Straccia 2013).

All results mentioned so far focused on deciding consistency of ontologies. In \(\mathcal{EL}\), where consistency is trivial since this logic is too weak to express contradictions, subsumption is the central reasoning problem (Baader 2003; Baader, Brandt, and Lutz 2005). Vojtáš (2006) presents a first fuzzy extension of \(\mathcal{EL}\) based on the Zadeh semantics, but restricts roles to be \textit{crisp}, i.e. take only the values 0 and 1. Stoilos, Stamou, and Pan (2008) and Mailis, Stoilos, Simou, Stamou, and Kollias (2012) develop subsumption algorithms for extensions of \(\mathcal{EL}\) using the Gödel t-norm and show that under reasonable assumptions the complexity is the same as for classical \(\mathcal{EL}\), namely P.

### 2.4.3 Fuzzy Description Logics over Finite Lattices

The investigation of fuzzy description logics over finite lattices \(L\) was started by Straccia (2004b), presenting a tableau algorithm for consistency w.r.t. witnessed models in \(L-\mathcal{ALC}\) with unfoldable TBoxes and equality assertions under a generalized Zadeh semantics (the t-norm is the lattice infimum and S-implication is used instead of the residual implication). As for the standard interval, the obtained upper bound on the complexity is \(\text{ExpSpace}\) due to the unfolding of the concepts in advance. In (Jiang, Tang, Wang, Deng, and Tang 2010), an extension of this tableau algorithm using a \textit{pair-wise blocking} condition is presented for consistency w.r.t. witnessed models in \(L-\mathcal{SHIN}\) with unfoldable TBoxes and equality assertions under generalized Zadeh semantics.

A popular approach to deal with finite \textit{total orders}, also pioneered by Straccia (2004a), is to reduce the fuzzy ontology to a classical one and then employ optimized decision procedures for the classical reasoning problems. This approach can even deal with fuzzy GCIs. The main idea is to translate every concept name \(A\) into finitely many \textit{crisp} concept names \(A_{\geq p}\), one for each element \(p\) of \(L\), with the intention that the interpretation of \(A_{\geq p}\) collects all those individuals that belong to \(A\) with a membership degree \(\geq p\). Then, for every concept name \(A\) and every adjacent pair \((p_1, p_2)\) of the total order \(L\), one has to introduce a classical GCI \(A_{\geq p_2} \sqsubseteq A_{\geq p_1}\) to express the order structure. A similar translation is done for role names using role inclusions and all fuzzy axioms are then recursively translated into classical axioms that employ the introduced crisp
2 Preliminaries

concept and role names. The resulting classical ontology is consistent in the classical sense iff the original fuzzy ontology is consistent.

Under generalized Zadeh semantics, the reduction is polynomial (Straccia 2004a, 2006), thereby showing ExpTime-completeness of consistency and other reasoning problems in \textit{L-ALCH} with fuzzy general TBoxes and equality assertions. However, for other total orders, e.g. based on the finite-valued Łukasiewicz or Gödel t-norms, such translations cause an exponential blowup in the size of the ontology, yielding suboptimal complexity upper bounds. A series of papers detailing these reductions was published: for finite-valued Łukasiewicz semantics in \textit{Ł lance} (Bobillo and Straccia 2011), for \textit{Gödel} (Bobillo, Delgado, Gómez-Romero, and Straccia 2009), and combinations of Gödel and Łukasiewicz in \textit{L-ALCH} (Bobillo and Straccia 2010, 2013b) and Gödel and Zadeh in \textit{L-SCROIQ} (Bobillo, Delgado, Gómez-Romero, and Straccia 2012).

A different setting was considered by Bobillo, Delgado, and Gómez-Romero (2009), which combines the Kleene-Dienes-implication for the concept constructors \( \forall \) and \( \rightarrow \) with Gödel implication for interpreting GCIs. However, this paper considers infinite-valued semantics over \([0,1]\), which leads to incorrectness of the reduction. Basically, the authors assume that it suffices to consider models using a fixed finite set of truth degrees, which is not true (see Section 4.2 for details).

In (Bou, Cerami, and Esteva 2011), the authors consider a many-valued modal logic over the finitely many truth degrees of \( Ł_n \) and show that \( 1 \)-satisfiability in this logic, in which formulae may contain truth constants and the \( \Delta \) operator, can be decided in \textit{PSpace}. This implies that \( 1 \)-satisfiability in \( Ł_n-\text{ALC} \) without background ontology (and with only one role name) is \textit{PSpace}-complete.

2.4.4 Different Constructors and Semantics

We have already seen several fuzzy DLs that include fuzzy number restrictions on role successors (\( N/\mathbb{Q} \); see also (Bobillo and Straccia 2008b; Stoilos, Stamou, and Kollias 2008)). However, their semantics is somewhat controversial as it is not clear whether two role successors to degree 0.5 should count as one “full” successor.

We want to briefly mention other constructors and semantics that have been considered in the literature on fuzzy description logics. In (Bobillo and Straccia 2013a; Vojtáš 2006), so called aggregation operators are proposed that generalize t-norms and allow to compute, e.g. weighted sums of a set of fuzzy values.

Kulacka, Pattinson, and Schröder (2013) consider a different kind of ABoxes, which do not allow to compare membership degrees to constants, but rather (sums of) different membership degrees to one another, e.g. \( (c:A + (c,d):r > e:A) \). We will consider a restricted variant of such ABoxes in Section 4.2.

In (Bobillo, Delgado, and Gómez-Romero 2008), a fuzzy variant of the nominal constructor is considered, allowing to express the concept of German speaking countries as \{1/\text{germany}, 1/\text{austria}, 0.67/\text{switzerland}\}.

So-called concept modifiers or linguistic hedges are often found in fuzzy description logics (Hölldobler, Nga, and Khang 2005; Straccia 2005; Tresp and Molitor 1998) and were originally proposed by Zadeh (1972). For example, one could express the concept \textit{very}(Tall) that transforms the value of Tall according to some function on the membership degrees, e.g. piecewise linear or polynomial functions, such as \( \text{very}^T : [0,1] \rightarrow [0,1] : x \mapsto x^2 \).
3 Fuzzy Description Logics over Finite Lattices

We start the investigation of the precise complexity of the reasoning problems in the presence of fuzzy general TBoxes by considering fuzzy description logics over finite residuated De Morgan lattices. We show that under such semantics, in many fuzzy DLs the reasoning problems introduced in Section 2.2.4 have the same complexity as in the corresponding classical DLs. For the following considerations, let \((L, \land, \lor)\) be a finite residuated De Morgan lattice with De Morgan negation \(\sim\), a join- and meet-preserving t-norm \(\otimes\), and the associated operators \(\Rightarrow\) and \(\oplus\) as introduced in Section 2.1.5. Recall that we assume \(L\) to be given as a list of its elements and all lattice operations to be computable in polynomial time in the size of the input elements.

We will see in this chapter that even for the expressive fuzzy DL \(L\text{-}\mathcal{SHI}\) consistency, satisfiability, subsumption, and instance checking w.r.t. fuzzy general TBoxes are Exp-Time-complete, as they are in classical \(\mathcal{SHI}\) (\(2\text{-}\mathcal{SHI}\)). This fuzzy DL can express all constructors introduced in Section 2.2.1 except nominals. On the question of the class of interpretation to consider, recall that every interpretation over a finite lattice of truth values is \(n\)-witnessed, where \(n\) is the width of \(L\) (see Lemma 2.13). In the following, we mainly investigate 1-witnessed semantics, and afterwards comment on straightforward extensions of the algorithms to deal with \(n\)-witnessed interpretations.

In Section 3.1, we present an automata-based procedure that allows us to prove ExpTime-completeness of local consistency w.r.t. fuzzy general TBoxes. Moreover, when restricted to acyclic TBoxes the complexity drops from ExpTime to PSpace in some fragments of classical \(\mathcal{SHI}\), and we can derive matching upper bounds for most of the corresponding fuzzy DLs. The automata-based algorithm is an adaptation of a known construction for classical DLs (Baader, Hladik, and Peñaloza 2008), and the original formulation can be obtained by setting \(L = 2\).

In Section 3.2, we develop a tableau algorithm that is used to lift the results from local consistency to consistency. Following a common approach, we first define the notion of tableaux, which are abstract representations of models that simplify the complex interplay between transitive roles, role hierarchies, and existential and value restrictions. We then describe a general algorithm for local consistency that tries to construct a tableau by exhaustively applying certain completion rules. This algorithm does not yield an optimal worst-case complexity. The completion rules, however, can be used to pre-complete the input ABox in order to reduce the general consistency problem to several local consistency problems. The tableau algorithm is very closely related to similar procedures developed for classical DLs (see (Baader and Sattler 2001) for an overview), and in fact for \(L = 2\) it is essentially the algorithm described by Horrocks and Sattler (1999). The pre-completion technique is based on ideas of Hollunder (1996).

At the end of this chapter, we briefly describe how to reduce satisfiability, subsumption, and instance checking to ontology consistency in order to obtain corresponding complexity results for these problems.
3 Fuzzy Description Logics over Finite Lattices

3.1 Local Consistency

We show that deciding local consistency in L-\textsc{Schi} is \textsc{ExpTime}-complete, which matches the complexity of satisfiability in classical \textsc{Shi} (Tobies 2001).\footnote{For 2-valued DLs, the local consistency of an ontology with the assertions \((c:C_1), \ldots, (c:C_n), (c:D_1 = 0), \ldots, (c:D_m = 0)\) is equivalent to the 1-satisfiability of \(C_1 \sqcap \cdots \sqcap C_n \sqcap \neg D_1 \sqcap \cdots \sqcap \neg D_m\) w.r.t. this ontology.} In the following, we consider an ontology \(\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})\) with a local ABox \(\mathcal{A}\). Note that, in contrast to classical \textsc{Shi}, general TBoxes cannot be internalized, i.e. integrated into the local ABox (Horrocks and Sattler 1999). This is due to the fact that our language does not allow for truth constants, and hence there is no concept that expresses the satisfaction of a GCI \(\langle C \sqsubseteq D \geq p \rangle\) at all domain elements.\footnote{To simulate a crisp GCI \(\langle C \sqsubseteq D \rangle\), one can add \(\langle c:((C \rightarrow D) \sqcap \forall u.(C \rightarrow D))\rangle\) to the local ABox, where \(u\) is a universal role, i.e. a transitive superrole of all other roles and their inverses (Horrocks and Sattler 1999).}

Our algorithm for deciding local consistency exploits the fact that \(\mathcal{O}\) has a (witnessed) model if it has a well-structured tree-shaped model, called a Hintikka tree. Intuitively, Hintikka trees are abstract representations of models that only store the membership values of “relevant” concepts. We construct automata that have Hintikka trees as their runs, thereby reducing consistency to the emptiness problem of these automata.

3.1.1 Hintikka trees

We will show that it mainly suffices to consider the values of all concepts occurring in the ontology \(\mathcal{O}\). However, due to the interaction between role inclusions and transitivity axioms, the values of additional existential and value restrictions (not appearing in \(\mathcal{O}\)) may become relevant. For simplicity, we will in the following consider these to also be subconcepts of \(\mathcal{O}\). For the following definition, recall the notions of the role hierarchy \(\sqsubseteq_R\) and transitive roles w.r.t. \(\mathcal{R}\) from Section 2.2.4.

Definition 3.1 (subconcept) The set \(\text{sub}_R(C)\) of subconcepts of a concept \(C\) w.r.t. an RBox \(\mathcal{R}\) is defined recursively as follows:

- \(\text{sub}_R(A) := \{A\}\) if \(A\) is a concept name, \(\top\), or \(\bot\),
- \(\text{sub}_R(C) := \{C\} \cup \text{sub}_R(D) \cup \text{sub}_R(E)\) if \(C\) is of the form \(D \sqcap E\) or \(D \rightarrow E\),
- \(\text{sub}_R(\neg C) := \{-C\} \cup \text{sub}_R(C)\),
- \(\text{sub}_R(\exists r.C) := \{\exists r.C\} \cup \text{sub}_R(C) \cup \{\exists s.C | s \sqsubseteq_R r, s\ \text{transitive}\}\),
- \(\text{sub}_R(\forall r.C) := \{\forall r.C\} \cup \text{sub}_R(C) \cup \{\forall s.C | s \sqsubseteq_R r, s\ \text{transitive}\}\).

For an ontology \(\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})\), the set \(\text{sub}(\mathcal{O})\) of subconcepts of \(\mathcal{O}\) is the union of the sets \(\text{sub}_R(C)\) for all concepts \(C\) occurring in axioms of \(\mathcal{O}\). \(\Diamond\)

The nodes of Hintikka trees are labeled with so-called Hintikka functions, which are fuzzy sets over the domain \(\text{sub}(\mathcal{O}) \cup \{\varrho\}\) that specify the values of all relevant concepts of a certain individual. The special element \(\varrho\) will be used to express the degree with which the role relation to the parent node in the Hintikka tree holds. Note that a Hintikka function need not specify values for all concepts, a possibility which will become important once we need to be careful how much space is used (see Section 3.1.3).
Definition 3.2 (Hintikka function) A Hintikka function for $\mathcal{O}$ is a partial function $H : \text{sub}(\mathcal{O}) \cup \{g\} \to L$ such that:

(i) $H(g)$ is defined;
(ii) if $H(\top)$ is defined, then $H(\top) = 1$;
(iii) if $H(\bot)$ is defined, then $H(\bot) = 0$;
(iv) if $H(C \cap D)$ is defined, then $H(C)$ and $H(D)$ are also defined and it holds that $H(C \cap D) = H(C) \otimes H(D)$;
(v) if $H(C \to D)$ is defined, then $H(C)$ and $H(D)$ are also defined and it holds that $H(C \to D) = H(C) \Rightarrow H(D)$; and
(vi) if $H(\neg C)$ is defined, then $H(C)$ is defined and $H(\neg C) = \neg H(C)$.

A Hintikka function $H$ is compatible with a concept definition $\langle A \equiv C \geq p \rangle$ if, whenever $H(A)$ is defined, then $H(C)$ is also defined, and it holds that $H(A) \Rightarrow H(C) \geq p$ and $H(C) \Rightarrow H(A) \geq p$. It is compatible with a GCI $\langle C \subseteq D \geq p \rangle$ if $H(C)$ and $H(D)$ are defined and $H(C) \Rightarrow H(D) \geq p$, and it is compatible with a TBox if it is compatible with all its axioms. It is compatible with a local ABox $\mathcal{A}$ if, for all $c : C = p \in \mathcal{A}$, the value $H(C)$ is defined and $H(C) = p$.

The support of $H$ (denoted by $\text{supp}(H)$) is the set of all concepts $C \in \text{sub}(\mathcal{O})$ for which $H(C)$ is defined.

The compatibility condition for concept definitions implements the technique of lazy unfolding mentioned earlier. The idea is that the definition $C$ of a concept name $A$ only needs to have a defined value if $A$ itself is defined by the Hintikka function. If $H(A)$ is undefined, this means that the value of $A$ is irrelevant for the individual described by $H$, and therefore the concept definition imposes no restriction on the value of $C$.

Hintikka trees have a fixed arity $k$ that is defined as the total number of existential and value restrictions contained in $\text{sub}(\mathcal{O})$. Intuitively, each successor will act as the witness for one of these restrictions. We define $K$ to be the index set $\{1, \ldots, k\}$ of all successors. Since we need to know which successor in the tree is the witness of which restriction, we fix an arbitrary bijection

$$\varphi : \{C \mid C \in \text{sub}(\mathcal{O})\} \to K.$$

For a given role $r$, we further define $\varphi_r(\mathcal{O})$ as the set of all indices $i \in K$ such that $i = \varphi(C)$ for a $C \in \text{sub}(\mathcal{O})$ of the form $\exists r.D$ or $\forall r.D$.

Definition 3.3 (Hintikka condition) A tuple $(H_0, H_1, \ldots, H_k)$ of Hintikka functions for $\mathcal{O}$ satisfies the Hintikka condition if the following hold:

1. For every existential restriction $\exists r.C \in \text{sub}(\mathcal{O})$:
   a) If $\exists r.C \in \text{supp}(H_0)$ and $i = \varphi(\exists r.C)$, then we also have $C \in \text{supp}(H_i)$ and it holds that $H_0(\exists r.C) = H_i(g) \otimes H_i(C)$.
   b) If $\exists r.C \in \text{supp}(H_0)$, then for all $r' \sqsubseteq_R r$ and $i \in \varphi_{r'}(\mathcal{O})$, we have $C \in \text{supp}(H_i)$ and $H_0(\exists r.C) \geq H_i(g) \otimes H_i(C)$.

   Further, for all transitive roles $s$ with $r' \sqsubseteq_R r \sqsubseteq_R s$, we have $\exists s.C \in \text{supp}(H_i)$ and $H_0(\exists r.C) \geq H_i(g) \otimes H_i(\exists s.C)$.
c) For all $r' \sqsubseteq_R \tau$ and $i \in \varphi_{r'}(\mathcal{O})$ with $\exists r.C \in \text{supp}(H_i)$, we have $C \in \text{supp}(H_0)$ and $H_i(\exists r.C) \geq H_i(\emptyset) \otimes H_0(C)$.

Further, for all transitive subroles $s$ with $r' \sqsubseteq_R s \sqsubseteq_R \tau$, we have $\exists s.C \in \text{supp}(H_0)$ and $H_i(\exists s.C) \geq H_i(\emptyset) \otimes H_0(\exists s.C)$.

2. For every value restriction $\forall r.C \in \text{sub}(\mathcal{O})$:

a) If $\forall r.C \in \text{supp}(H_0)$ and $i = \varphi(\forall r.C)$, then we also have $C \in \text{supp}(H_i)$ and it holds that $H_0(\forall r.C) = H_i(\emptyset) \Rightarrow H_i(C)$.

b) If $\forall r.C \in \text{supp}(H_0)$, then for all $r' \sqsubseteq_R r$ and $i \in \varphi_{r'}(\mathcal{O})$, we have $C \in \text{supp}(H_i)$ and $H_0(\forall r.C) \leq H_i(\emptyset) \Rightarrow H_i(C)$.

Further, for all transitive roles $s$ with $r' \sqsubseteq_R s \sqsubseteq_R r$, we have $\forall s.C \in \text{supp}(H_i)$ and $H_0(\forall r.C) \leq H_i(\emptyset) \Rightarrow H_i(\forall s.C)$.

c) For all $r' \sqsubseteq_R \tau$ and $i \in \varphi_{r'}(\mathcal{O})$ with $\forall r.C \in \text{supp}(H_i)$, we have $C \in \text{supp}(H_0)$ and $H_i(\forall r.C) \leq H_i(\emptyset) \Rightarrow H_0(\forall s.C)$.

Further, for all transitive roles $s$ with $r' \sqsubseteq_R s \sqsubseteq_R \tau$, we have $\forall s.C \in \text{supp}(H_0)$ and $H_i(\forall r.C) \leq H_i(\emptyset) \Rightarrow H_0(\forall s.C)$.

Intuitively, Condition 1.a) checks that an existential restriction $\exists r.C$ is witnessed by its designated successor $\varphi(\exists r.C)$. Condition 1.b) ensures that the degree of the existential restriction is indeed the maximum of the degrees of all $r$-successors. Furthermore, for all transitive subroles $s$ of $r$ the restriction $\exists s.C$ has to be propagated since by transitivity every $s$-successor of this $s$-successor must also be an $s$-successor. Finally, Condition 1.c) deals with the analogous consequences of the restriction $\exists r.C$ along inverse role connections. Conditions 2.a)–2.c) express the dual conditions for value restrictions.

In the following, we view the set $K^*$ as an infinite $k$-ary tree: the empty word $\varepsilon$ represents the root node, and $u_i$ represents the $i$-th successor of the node $u$. A path is a sequence $u_1, \ldots, u_m$ of nodes such that $u_1 = \varepsilon$ and, for every $i$, $1 \leq i \leq m - 1$, $u_{i+1}$ is a successor of $u_i$.

**Definition 3.4 (Hintikka tree)** A **Hintikka tree** for $\mathcal{O}$ is a mapping $T$ that assigns a Hintikka function for $\mathcal{O}$ to each node of $K^*$ such that

- $T(\varepsilon)$ is compatible with $A$,
- for every $u \in K^*$, $T(u)$ is compatible with $T$, and
- for every $u \in K^*$, $(T(u), T(u_1), \ldots, T(u_k))$ satisfies the Hintikka condition.

Compatibility ensures that $T$ is satisfied at any node of the Hintikka tree, while the Hintikka condition makes sure that the tree is in fact a witnessed model.

**Example 3.5** Consider the lattice $L_4$ from Figure 2.4 with the infimum as the t-norm, and the ontology $\mathcal{O} = (A, T, R)$, where $A = \{c:B \sqcap \exists r.A = d\}$, $T = \{A \equiv \forall s.\neg B\}$, and $R = \{r \subseteq s\}$. The set $\text{sub}(\mathcal{O})$ consists of the elements $A$, $B$, $\exists r.A$, $\forall s.\neg B$, and $B \sqcap \exists r.A$. The arity $k$ of the Hintikka trees is 2 since we only have the restrictions $\exists r.A$ and $\forall s.\neg B$. We fix the mapping $\varphi$ by setting $\varphi(\exists r.A) := 1$ and $\varphi(\forall s.\neg B) := 2$.

Figure 3.1 depicts the beginning of the Hintikka tree $T$ for $\mathcal{O}$. Each node that is not shown only assigns $t$ to $\emptyset$. The Hintikka function $T(\varepsilon)$ at the root must be compatible.
with $A$, i.e. we must have $T(\epsilon)(B \sqcap \exists r.A) = d$. Definition 3.2 forces us to assign to $B$ and $\exists r.A$ two values whose infimum is $d$. Here, we guess $T(\epsilon)(B) = T(\epsilon)(\exists r.A) = d$. Thus, the Hintikka function $T(1)$ labeling the first successor must satisfy Condition 1.a) of Definition 3.3, which implies that $T(1)(\varphi) \otimes T(1)(A)$ must be $d$. If we guess $T(1)(A) = d$, then to achieve compatibility with $T$, we need to set also $T(1)(\forall r. \neg A) = d$. Finally, since $r \sqsubseteq R$, Condition 2.c) requires that $T(1)(\varphi) \sqsubseteq T(1)(A) = d$. This can be satisfied by setting $T(\epsilon)(\neg B) = d$. This assignment satisfies Definition 3.2 since $T(\epsilon)(\neg B) = d = \sim d = \sim T(\epsilon)(B)$. The remaining requirements of Definitions 3.2 and 3.3 are trivially satisfied.

On the other hand, when using the ABox $\{(c:B \sqcap \exists r.A)\}$ instead of $A$, it is not possible to construct a Hintikka tree anymore since the values of $B$ and $\neg B$ at the root node cannot both be $t$.

\[\Diamond\]

The proof of the following lemma uses arguments that generalize those used in (Baader, Hladik, and Peñaloza 2008) for classical $\mathcal{S}I$. The Hintikka condition in that paper is simpler since Hintikka functions are only sets of subconcepts, no successors witnessing the value restrictions are needed, and there is no role hierarchy.

**Lemma 3.6** $O$ has a witnessed model iff there is a Hintikka tree for $O$.

**Proof.** Given a Hintikka tree $T$ for $O$, we define a model $I$ of $O$ over the domain $\Delta^T := K^*$ as follows. For a role name $r$, we first define the fuzzy binary relation $r^T$ on $\Delta^T$ by

1. $r^T(x, xi) := T(xi)(\varphi)$ for every $x \in \Delta^T$ and every $i \in \varphi_r(O)$ with $r' \sqsubseteq_R r$;
2. $r^T(x, xi) := T(xi)(\varphi)$ for every $x \in \Delta^T$ and every $i \in \varphi_r(O)$ with $r' \sqsubseteq_R r$;
3. $r^T(x, y) := 0$ for all other $x, y \in \Delta^T$.

We further set $(r^{-1})^T(x, y) := r^T(y, x)$ for all $x, y \in \Delta^T$. For an arbitrary complex role $r$ and any sequence $x_1, \ldots, x_n \in \Delta^T$ with $n \geq 2$, we define

$$r^T(x_1, \ldots, x_n) := r^T(x_1, x_2) \otimes \cdots \otimes r^T(x_{n-1}, x_n).$$

We can now define the interpretation of a role name $r$ under $I$ as follows:

$$r^I(x, y) := r^T(x, y) \lor \bigvee_{s \sqsubseteq_R \epsilon} \bigvee_{s \text{ transitive}} \bigvee_{n \geq 1} \bigvee_{z_1, \ldots, z_n \in \Delta^T} s^T(x, z_1, \ldots, z_n, y).$$
This complex expression is necessary to correctly account for the transitive subroles of $r$. It was inspired by a similar construction in (Glimm, Horrocks, Lutz, and Sattler 2008). It is easy to show that the same equation holds for inverse roles. Furthermore, if $r$ is transitive, then $r^T$ is the transitive closure of $r^T$, and therefore a transitive fuzzy relation. Consider now a role inclusion $r \subseteq s \in \mathcal{R}$ and $x, y \in \Delta^r$. By definition of $r^T$ and $s^T$, we have $r^T(x, y) \leq s^T(x, y)$. Since every transitive subrole of $r$ is also a transitive subrole of $s$, we obtain $r^T(x, y) \leq s^T(x, y)$, and thus $\mathcal{I}$ is already a model of $\mathcal{R}$.

It remains to define the interpretations of concepts under $\mathcal{I}$. We set

$$A^T(x) := \begin{cases} T(x)(A) & \text{if } T(x)(A) \text{ is defined}, \\ 0 & \text{otherwise}, \end{cases}$$

for all primitive concept names $A$ and all $x \in \Delta^T$. To show that $\mathcal{I}$ can be extended to defined concept names such that it agrees with $T$ on complex concepts, we define a weight function $o(C)$ that maps concepts to natural numbers:

- $o(\top) := o(\bot) := o(A) := 0$ for a primitive concept name $A$;
- $o(A) := o(C) + 1$ if $\langle A \equiv C \geq p \rangle \in \mathcal{T}$;
- $o(\neg C) := o(C) + 1$;
- $o(C \sqcap D) := o(C \rightarrow D) := \max\{o(C), o(D)\} + 1$;
- $o(\exists r.C) := o(\forall r.C) := o(C) + 1$.

If there are concept definitions in $\mathcal{T}$, then they have an acyclic dependency relation. Thus, the order on concepts induced by their weights is well-founded. We now show the following claim by induction on $o(C)$: if $T(x)(C)$ is defined for some $x \in \Delta^T$, then $C^T(x) = T(x)(C)$. Primitive concept names, $\top$, and $\bot$ are interpreted correctly by the definition of $\mathcal{I}$ and Hintikka sets.

Consider now a defined concept name $A$. If $\langle A \equiv C \geq p \rangle \in \mathcal{T}$ and $T(x)(A)$ is defined, then, since $T(x)$ is compatible with $\langle A \equiv C \geq p \rangle$, we know that $T(x)(C)$ must also be defined. Furthermore, we have $T(x)(A) = T(x)(C) \geq p$ and $T(x)(C) = T(x)(A) \geq p$. Since $o(C) < o(A)$, we get $C^T(x) = T(x)(C)$ by induction. Thus, by defining the value of $A^T(x)$ as $T(x)(A)$ we ensure that $\mathcal{I}$ satisfies the concept definition $\langle A \equiv C \geq p \rangle$ at $x$. Whenever $T(x)(A)$ is not defined, we can set $A^T(x) := C^T(x)$ to satisfy this concept definition without violating the claim.

If $T(x)(\neg C)$ is defined, then $T(x)(C)$ is also defined. Moreover, by induction we obtain $\neg C^T(x) = \neg C^T(x) = \neg T(x)(C) = T(x)(\neg C)$. The claims for $C \sqcap D$ and $D \rightarrow E$ follow similarly.

We now come the more interesting case of existential restrictions. Due to the dual nature of the Hintikka conditions for value restrictions, these can be handled by similar arguments. If $T(x)(\exists r.C) = p$, let $y := x\varphi(\exists r.C)$. By Condition 1.a) of Definition 3.3, $T(y)(C)$ is defined, and by induction we have $C^T(y) = T(y)(C)$. Moreover,

$$p = T(y)(g) \otimes T(y)(C) = r^T(x, y) \otimes C^T(y) \leq r^T(x, y) \otimes C^T(y).$$

We now show that $r^T(x, z) \otimes C^T(z) \leq p$ holds for every $z \in \Delta^T$. This in particular implies that $y$ is a witness for $(\exists r.C)^T(x)$, and thus $\mathcal{I}$ is a witnessed interpretation.
By definition of $r^T$ and the fact that $\otimes$ is join-preserving, it suffices to show that

(a) $r^T(x, z) \otimes C^T(z) \leq p$ and
(b) $s^T(x, y_1, \ldots, y_n, z) \otimes C^T(z) \leq p$ for all transitive roles $s \sqsubseteq_R r$ and all $z_1, \ldots, z_n \in \Delta^T$, $n \geq 1$.

If $r^T(x, z) = 0$, the claim is trivial; otherwise, by definition of $r^T$ there must be an index $i \in \varphi_r(O)$ such that either $z = xi$ and $r' \sqsubseteq_R r$ or $x = zi$ and $r' \sqsubseteq_R r$. In the first case, we can apply Condition 1.b) of Definition 3.3 and the induction hypothesis to show that the value $T(z)(C)$ is defined and $p = T(x)(\exists r.C) \geq T(z)(\varrho) \otimes T(z)(C) = r^T(x, z) \otimes C^T(z)$. Similarly, in the second case Condition 1.c) yields that $T(z)(C)$ is defined and we have

$$p = T(x)(\exists r.C) \geq T(y)(\varrho) \otimes T(z)(C) = r^T(x, z) \otimes C^T(z).$$

By monotonicity of $T$, we conclude $p = T(x)(\exists r.C) \geq s^T(x, y_1, \ldots, y_n, z) \otimes C^T(z)$.

Thus, $I$ is an interpretation that satisfies all concept definitions in $T$. For the case that $T$ is a general TBox, consider any GCI $(C \subseteq D \geq p) \in T$. Since every Hintikka set in $T$ must be compatible with this GCI, $T(x)(C)$ and $T(x)(D)$ are always defined and we have $T(x)(C) \Rightarrow T(x)(D) \geq p$ for every $x \in K^*$. By the above claim, it follows that $D^T(x) \Rightarrow E^T(x) \geq p$ for every $x \in \Delta^T$, and thus $I$ satisfies this GCI.

Finally, for every assertion $(c: C = p) \in A$, we have $T(z)(C) = p$ by Definitions 3.2 and 3.4, and therefore $C^T(z) = p$. This shows that $I$ is a model of $O$ if we set $c^E := c$ for every individual name $c$.

Conversely, we show that every witnessed model $I$ of $O$ can be “unraveled” into a Hintikka tree $T$ for $O$. For this, we inductively define a mapping $g: K^* \rightarrow \Delta^T$ that will specify which elements of $\Delta^T$ are represented by the nodes of $T$. We begin by setting $g(\varepsilon) := \varepsilon$, where $c$ is the unique individual name occurring in the local ABox $A$.

Let now $u \in K^*$ be such that $g(u)$ has already been defined. For each $C \in \text{sub}(O)$, we set $T(u)(C) := C^T(g(u))$. Since $I$ satisfies $T$, this obviously defines a Hintikka function that is compatible with $T$. Consider now any existential restriction $\exists r.C \in \text{sub}(O)$. There must be a witness $y \in \Delta^T$ with $(\exists r.C)^T(g(u)) = r^T(g(u), y) \otimes C^T(y)$. We set $g(u \varphi(\exists r.C)) := y$ and $T(u \varphi(\exists r.C))(g(y)) := r^T(g(u), y)$. Similarly, for every $\forall r.C \in \text{sub}(O)$, there is a $y \in \Delta^T$ such that $(\forall r.C)^T(g(y)) = r^T(g(y), y) \Rightarrow C^T(y)$, and we define $g(u \varphi(\forall r.C)) := y$ and $T(u \varphi(\forall r.C))(g(y)) := r^T(g(u), y)$.

We show that every tuple $(T(u), T(u_1), \ldots, T(u_k))$ with $u \in K^*$ satisfies the Hintikka condition. Consider first any $\exists r.C \in \text{sub}(O)$ and set $v := u \varphi(\exists r.C)$. For Condition 1.a) of Definition 3.3, we know that $T(v)(C)$ is defined by construction and

$$T(u)(\exists r.C) = (\exists r.C)^T(g(u)) = r^T(g(u), v) \otimes C^T(g(u)) = T(v)(g(u) \otimes T(v)(C)).$$
Let now \( i \in \varphi^r(\mathcal{O}) \) with \( r' \sqsubseteq_R r \). Then \( T(u_i)(C) \) is defined by construction and we have

\[
T(u_i)(\exists r.C) = (\exists r.C)^T(g(u))
\]

\[
\geq r'^T(g(u), g(\cup_i)) \otimes C^T(g(\cup_i))
\]

\[
\geq r'^T(g(u), g(\cup_i)) \otimes C^T(g(\cup_i))
\]

\[
= T(u_i)(\varnothing) \otimes T(u_i)(C).
\]

Furthermore, if \( s \) is a transitive role with \( r' \sqsubseteq_R s \sqsubseteq_R r \), then

\[
T(u_i)(\exists r.C) = (\exists r.C)^T(g(u))
\]

\[
= \bigvee_{y \in \Delta^x} r'^T(g(u), y) \otimes C^T(y)
\]

\[
\geq \bigvee_{y \in \Delta^x} s'^T(g(u), y) \otimes C^T(y)
\]

\[
\geq \bigvee_{y \in \Delta^x} s'^T(g(u), g(\cup_i)) \otimes s^T(g(\cup_i), y) \otimes C^T(y)
\]

\[
= s'^T(g(u), g(\cup_i)) \otimes (\exists s.C)^T(g(\cup_i))
\]

\[
\geq r'^T(g(u), g(\cup_i)) \otimes (\exists s.C)^T(g(\cup_i))
\]

\[
= T(u_i)(\varnothing) \otimes T(u_i)(\exists s.C)
\]

since \( \otimes \) is join-preserving, which shows that Condition 1.b) of Definition 3.3 is satisfied by \( T \). The remaining conditions can be shown using analogous arguments.

Finally, for every \( \langle c : C = p \rangle \in A \) we have \( T(\varepsilon)(C) = C^T(g(\varepsilon)) = C^T(c^T) = p \). \( \square \)

It remains to show how to decide the existence of Hintikka trees for \( \mathcal{O} \).

### 3.1.2 Hintikka automata

By building an automaton whose runs correspond to Hintikka trees for \( \mathcal{O} \), we reduce local consistency in \( L-\text{FSCHL} \) to the emptiness problem of such automata. The states of this automaton contain the Hintikka functions compatible with \( T \), and the transition relation ensures that the Hintikka condition is satisfied, while the initial states may only contain Hintikka functions that satisfy the assertions in \( A \). In Section 3.1.5, we will additionally need to know the index of each node relative to its siblings in the tree. Thus, we actually use as states pairs of the form \((H, i)\), where \( H \) is a compatible Hintikka function and \( i \in K \). For this, we assume without loss of generality that \( k \geq 1 \). If this is not the case, then by Lemma 3.6 we can easily check local consistency of \( \mathcal{O} \) in \( \text{NP} \) by guessing a Hintikka function for \( \mathcal{O} \) that is compatible with \( A \).

**Definition 3.7 (Hintikka automaton)** A looping (tree) automaton is defined as a tuple \( A = (Q, I, \Delta) \), where \( Q \) is a finite set of states, \( I \subseteq Q \) is a set of initial states, and \( \Delta \subseteq Q^{k+1} \) is the transition relation. A run of \( A \) is a mapping \( r : K^* \rightarrow Q \) such that \( r(\varepsilon) \in I \) and for every \( u \in K^* \), we have \( (r(u), r(u1), \ldots, r(uk)) \in \Delta \). The emptiness problem for looping automata is to decide whether a given looping automaton has a run.

The **Hintikka automaton** for \( \mathcal{O} \) is the looping automaton \( A_{\mathcal{O}} = (Q_{\mathcal{O}}, I_{\mathcal{O}}, \Delta_{\mathcal{O}}) \), where
• \( Q_O \) is the set of all pairs \((H,i)\) of Hintikka functions \(H\) for \(O\) that are compatible with \(T\) and indices \(i \in K\),

• \( I_O \) is the set of all pairs \((H,i) \in Q_O\) where \(H\) is compatible with \(A\), and

• \( \Delta_O \) is the set of all tuples \(((H_0,i_0),(H_1,1),\ldots,(H_k,k))\) such that \((H_0,H_1,\ldots,H_k)\) satisfies the Hintikka condition.

The first components of the runs of \(A_O\) form exactly the Hintikka trees for \(O\). The second component simply stores the index of the existential or universal restriction for which a node acts as a witness for its parent, but does not influence the transition relation. Thus, \(O\) is locally consistent iff \(A_O\) is not empty.

Recall that \(k\) is the number of existential and value restrictions in \(\text{sub}(O)\), and is thus linear in the size of \(O\). Since there are at most \((|L| + 1)^{|\text{sub}(O)|+1}\) Hintikka functions, the size of the automaton \(A_O\) is therefore exponential in the input. The emptiness of looping automata can be decided in (deterministic) polynomial time using a bottom-up approach that finds all the states that can appear in a run (Vardi and Wolper 1986). Hence, local consistency of \(O\) can be decided in exponential time. This upper bound is tight since deciding concept satisfiability w.r.t. general TBoxes is already \(\text{ExpTime}\)-hard for classical \(\text{ALC}\) (Schild 1991).

Recall that we have restricted ourselves so far to witnessed interpretations. However, one can build analogous automata to decide local consistency w.r.t. \(n\)-witnessed interpretations with \(n > 1\). To do this, one needs to consider \((nk)\)-ary Hintikka trees, where \(n\) successors are used to witness each of the quantified concepts in \(\text{sub}(O)\). Thus, the arity of the Hintikka automata grows polynomially in \(n\). Since \(n\) can be bounded by \(|L|\) (see Lemma 2.13), we obtain the same complexity for local consistency w.r.t. general interpretations.

**Theorem 3.8** Let \(L\) be a finite residuated De Morgan lattice. Then local consistency w.r.t. general models in \(L-\text{ISC}HI\) with fuzzy general TBoxes is decidable in \(\text{ExpTime}\). It is \(\text{ExpTime}\)-hard already in \(2-\text{REL}\) and \(2-\text{ELC}\). \(\square\)

Concept satisfiability is in fact \(\text{ExpTime}\)-complete for classical \(\text{SH}\) even if the TBox is empty (Horrocks, Sattler, and Tobies 2000). It thus follows that local consistency is \(\text{ExpTime}\)-complete for \(L-\text{ISC}H\) and \(L-\text{ISC}HI\) with acyclic or empty TBoxes. On the other hand, when restricted to acyclic TBoxes, concept satisfiability in classical \(\text{SI}\) becomes \(\text{PSPACE}\)-complete (Baader, Hladik, and Peñaloza 2008; Horrocks, Sattler, and Tobies 2000).

In the following, we show a corresponding upper bound for local consistency in \(L-\text{ISC}C\) with acyclic TBoxes, where the subscript \(c\) denotes that all roles are restricted to be crisp, i.e. we consider only those interpretations \(I\) that satisfy \(r^I(x,y) \in \{0,1\}\) for all role names \(r\) and \(x,y \in \Delta^I\). We also show that the same complexity result holds for \(L-\text{ALCHI}\) with acyclic TBoxes, even without the restriction to crisp roles.

### 3.1.3 \(\text{PSPACE}\) on-the-fly constructions

The proofs of these results use the notion of a \(\text{PSPACE}\) on-the-fly construction that was developed by Baader, Hladik, and Peñaloza (2008) for classical \(\text{SI}\). We briefly recall the relevant definitions and results here. The idea is that one can check the emptiness of a
looping automaton using a nondeterministic top-down approach, which relies on the fact that if there is a run, then there is also a periodic run. This method guesses a period and verifies that it does correspond to a run. To speed up this search, the period should be as short as possible. This motivates the notion of blocking automata.

**Definition 3.9 (m-blocking)** Let $A = (Q, I, \Delta)$ be a looping automaton and $\rightarrow$ a binary relation over $Q$ called the blocking relation. The automaton $A$ is called $\rightarrow$-invariant if for all $p, q \in Q$ with $q \rightarrow p$ and $(q_0, q_1, \ldots, q_{i-1}, q, q_{i+1}, \ldots, q_k) \in \Delta$ we have $(q_0, q_1, \ldots, q_{i-1}, p, q_{i+1}, \ldots, q_k) \in \Delta$. $A$ is called $m$-blocking (w.r.t. $\rightarrow$) for $m \geq 1$ if every path $u_1, \ldots, u_m$ of length $m$ in a run $r$ of $A$ contains two nodes $u_i$ and $u_j$ ($i < j$) such that $r(u_j) \rightarrow r(u_i)$.

Every looping automaton is $\rightarrow$-invariant and $m$-blocking for every $m > |Q|$. However, the main interest in blocking automata arises when one can find a smaller bound on $m$. Although this is not always possible, one can try to reduce this limit with the help of a so-called faithful family of functions.

**Definition 3.10 (faithful)** Let $A = (Q, I, \Delta)$ be a looping automaton. The family of functions $f_q : Q \rightarrow Q$ for $q \in Q$ is faithful w.r.t. $A$ if for all $q, q_0, q_1, \ldots, q_k \in Q$,

- if $(q, q_1, \ldots, q_k) \in \Delta$, then $(q, f_q(q_1), \ldots, f_q(q_k)) \in \Delta$, and
- if $(q_0, q_1, \ldots, q_k) \in \Delta$, then $(f_q(q_0), f_q(q_1), \ldots, f_q(q_k)) \in \Delta$.

The subautomaton $A^S = (Q, I, \Delta^S)$ of $A$ induced by this family has the transition relation

$$\Delta^S := \{(q, f_q(q_1), \ldots, f_q(q_k)) \mid (q, q_1, \ldots, q_k) \in \Delta\}.$$

The name faithful reflects the fact that the resulting subautomaton simulates all runs of $A$. In particular, the following equivalence between the emptiness of the two automata holds.

**Proposition 3.11 (Baader, Hladik, and Peñaloza 2008)** Let $A$ be a looping automaton and $A^S$ its subautomaton induced by a faithful family of functions. Then $A$ has a run iff $A^S$ has a run. $\square$

We have already shown how to construct looping automata of exponential size to decide local consistency. If we can modify this construction such that the resulting automata are $m$-blocking for some $m$ bounded polynomially in the size of the input (that is, logarithmically in the size of the automaton), then the emptiness test requires only polynomial space.

**Definition 3.12 (PSpace on-the-fly construction)** Assume that we have a set $I$ of inputs and a construction that yields, for every $i \in I$, an $m_i$-blocking automaton $A_i = (Q_i, I_i, \Delta_i)$ working on $k_i$-ary trees. This construction is a PSpace on-the-fly construction if there is a polynomial $P$ such that, for every input $i$ of size $n$,

(i) $m_i \leq P(n)$ and $k_i \leq P(n)$,

(ii) every element of $Q_i$ has size bounded by $P(n)$, and

(iii) one can nondeterministically guess in time bounded by $P(n)$ an element of $I_i$, and, for a state $q \in Q_i$, a transition from $\Delta_i$ with first component $q$. $\diamond$
3.1 Local Consistency

As hinted at by the name, these elaborate conditions guarantee the following complexity result for checking emptiness of the constructed automata.

**Proposition 3.13 (Baader, Hladik, and Peñaloza 2008)** If the looping automata \( A_i \) are obtained from the inputs \( i \in I \) by a \( \text{PSPACE} \) on-the-fly construction, then emptiness of \( A_i \) can be decided in \( \text{PSPACE} \) in the size of \( i \).

We now illustrate these definitions on a simple decision problem.

**Example 3.14** Suppose we want to determine the complexity of the following number theoretical problem: given a finite set \( N \) of positive integers, can it be partitioned into two subsets \( A_1, A_2 \) such that \( 2 \sum A_1 = \sum A_2 \), and \( A_1 \) and \( A_2 \) can also be recursively partitioned in this way, unless the cardinality of the set is smaller than 3?

This problem can be solved by deciding the emptiness of the looping automaton \( A_N = (Q_N, I_N, \Delta_N) \) over binary trees, where

- \( Q_N := 2^N \),
- \( I_N := \{N\} \), and
- \( \Delta_N := \{(A, B, C) \mid |A| \leq 2 \text{ or } B \cap C = \emptyset, B \cup C = A, 2 \sum B = \sum C\} \).

Figure 3.2 depicts a run of \( A_N \) for the input \( N = \{1, 3, 4, 5, 6, 8\} \), where each node that is not shown is assigned an arbitrary state of cardinality 2. It is clear that such an automaton has a run if the problem stated above has a solution. The number of states is exponential in the size of \( N \), and thus our problem is decidable in \( \text{EXPTIME} \).

However, we can find the following faithful family of functions \( f_A : Q_N \rightarrow Q_N \) for \( A \in Q_N \):

\[
f_A(A') := \begin{cases} A' & \text{if } |A| > 2, \\
\emptyset & \text{if } |A| \leq 2. 
\end{cases}
\]

The example run from Figure 3.2 can easily be transformed into a run of the induced subautomaton \( A^S_N \) by labeling the nodes that are not depicted by \( \emptyset \).

Furthermore, we can show that the construction of \( A^S_N \) from the input \( N \) is a \( \text{PSPACE} \) on-the-fly construction:

(i) With equality as the blocking relation, \( A^S_N \) is \(|N|+1\)-blocking since every transition must reduce the cardinality of the set by at least 1, and thus after at most \(|N| \) transitions the empty set must be reached. The arity of \( A^S_N \) is always 2.
(ii) Every element of $Q_N$ has size polynomial in the size of $N$.

(iii) We do not have to guess elements of $I_N$ since it contains only one element. For a given state $A \in Q_N$ of cardinality greater than 2, one can guess a partition of $A$ into two subsets $B, C$ and check whether they satisfy the conditions of $\Delta_N$ in polynomial time.

By Proposition 3.13, this shows that the problem is also in PSPACE.

This example illustrates that a naive modeling of a problem using looping automata can be easy to describe, but might not yield a good complexity bound. By a subsequent faithful reduction to a PSPACE on-the-fly-construction, the complexity bound can be improved.

In the following, we consider as input a finite residuated De Morgan lattice $L$ and an ontology $O = (A, \mathcal{T}, \mathcal{R})$, where $A$ is a local ABox and $\mathcal{T}$ is an acyclic TBox. Our goal is to obtain PSPACE-decision procedures by modifying the construction of the Hintikka automata $A_O$ from Definition 3.7 into a PSPACE on-the-fly construction. Notice that this construction already satisfies all but one of the conditions of Definition 3.12:

(i) the arity $k$ of the automata is given by the number of existential and value restrictions in $\text{sub}(O)$,

(ii) every Hintikka function (and hence every state of the automaton) consists of $|\text{sub}(O)| + 1$ lattice values, and

(iii) building a state or a transition requires only to guess $|\text{sub}(O)| + 1$ or $k(|\text{sub}(O)| + 1)$ lattice values, respectively, and then verifying that this is indeed a valid state or transition of the automaton, which can be done in time polynomial in $|\text{sub}(O)|$ and in the size of the lattice values.

However, one can easily find runs of $A_O$ where blocking occurs only after exponentially many transitions, violating the first condition of PSPACE on-the-fly constructions. We will use a faithful family of functions to obtain a reduced automaton that guarantees blocking after at most polynomially many transitions, thus obtaining the claimed PSPACE upper bound.

### 3.1.4 Acyclic TBoxes in $L$-\textit{\textsc{ALCHI}}

In the case of $L$-\textit{\textsc{ALCHI}}, the faithful family of functions only needs to guarantee that the maximal role depth decreases with each transition. For the acyclic TBox $\mathcal{T}$, the role depth of concepts w.r.t. $\mathcal{T}$ ($\text{rd}_\mathcal{T}$) is recursively defined as follows:

- $\text{rd}_\mathcal{T}(A) := \text{rd}_\mathcal{T}(\top) := \text{rd}_\mathcal{T}(\bot) := 0$ for each primitive concept name $A$,
- $\text{rd}_\mathcal{T}(A) := \text{rd}_\mathcal{T}(C)$ for every concept definition $A \equiv C \geq p \in \mathcal{T}$,
- $\text{rd}_\mathcal{T}(C \cap D) := \text{rd}_\mathcal{T}(C \rightarrow D) := \max\{\text{rd}_\mathcal{T}(C), \text{rd}_\mathcal{T}(D)\}$,
- $\text{rd}_\mathcal{T}(\neg C) := \text{rd}_\mathcal{T}(C)$, and
- $\text{rd}_\mathcal{T}(\exists r.C) := \text{rd}_\mathcal{T}(\forall r.C) := \text{rd}_\mathcal{T}(C) + 1$.

The acyclicity of $\mathcal{T}$ ensures that this is well-defined. We use $\text{rd}_\mathcal{T}(H)$ to denote the maximal role depth $\text{rd}_\mathcal{T}(C)$ of a concept $C$ in $\text{supp}(H)$. For $n \geq 0$, we denote by $\text{sub}^{\leq n}(O)$ the set of all concepts in $\text{sub}(O)$ with role depth less than or equal to $n$. 
3.1 Local Consistency

Definition 3.15 (functions \(f_{(H,i)}\)) Let \((H,i)\) and \((H',i')\) be two states of \(A_O\) and set \(n := rd_T(H)\). We define the function \(f_{(H,i)}(H',i') := (H'',i'')\) for all \(C \in \text{sub}(O)\) as follows:

\[
H''(C) := \begin{cases} 
  H'(C) & \text{if } C \in \text{sub}^{\leq n-1}(O), \\
  \text{undefined} & \text{otherwise},
\end{cases}
\]

\[
H''(\varnothing) := \begin{cases} 
  0 & \text{if } \text{supp}(H) = \varnothing, \\
  H'(\varnothing) & \text{otherwise}.
\end{cases}
\]

Since \(T\) is acyclic, \(H''\) is still a Hintikka function for \(O\) and compatible with \(T\).

Lemma 3.16 In \(L\text{-\(\Delta\)ALCHI}\text{L}\), the family \(f_{(H,i)}\) is faithful w.r.t. \(A_O\).

Proof. Let \(H, H_0, \ldots, H_k\) be Hintikka functions and \(i, i_0 \in K\) and consider the pairs \((H_0, i_0) := f_{(H,i)}(H_0, i_0)\) and \((H'_j, j) := f_{(H,i)}(H_j, j)\) for all \(j, 1 \leq j \leq k\). We show that if \((H, H_1, \ldots, H_k)\) satisfies the Hintikka condition, then \((H, H'_1, \ldots, H'_k)\) also satisfies it. We only check the conditions for the existential restrictions (Conditions 1.a)–1.c) of Definition 3.3). The conditions for the value restrictions can be shown by dual arguments.

For Condition 1.a), let \(\exists r.C \in \text{sub}(O)\) and \(j = \varphi(\exists r.C)\) and assume that \(H(\exists r.C)\) is defined. Since we have \(rd_T(C) \leq rd_T(H)\) and \(H_j(C)\) is defined, the value \(H'_j(C)\) is defined and equal to \(H_j(C)\). Moreover, \(\text{supp}(H) \neq \varnothing\), and thus \(H'_j(\varnothing)\) is equal to \(H_j(\varnothing)\). This shows that the equality in Condition 1.a) remains satisfied.

To show Condition 1.b), let \(\exists r.C \in \text{supp}(H)\) and \(j \in \varphi'(\exists r.C)\) with \(r' \subseteq_R r\). We can show as above that \(H'_j(C)\) and \(H'_j(\varnothing)\) are defined and equal to \(H_j(C)\) and \(H_j(\varnothing)\), respectively. Thus, the required inequality is still satisfied after applying \(f_{(H,i)}\). Since in \(L\text{-\(\Delta\)ALCHIL}\) there are no transitive roles, the rest of this condition is trivially satisfied.

For Condition 1.c), consider \(j \in \text{sub}(O)\) with \(r' \subseteq_R \top\) and \(\exists r.C \in \text{supp}(H'_j)\). Thus, \(H_j(\exists r.C)\) is defined and equal to \(H'_j(\exists r.C)\), which implies that \(C \in \text{supp}(H_j)\). This in turn implies \(\text{supp}(H) \neq \varnothing\), which yields \(H'_j(\varnothing) = H_j(\varnothing)\). This shows that all relevant values are the same as before applying \(f_{(H,i)}\), i.e. the inequality is still satisfied.

To show the second condition of Definition 3.10, assume that \((H_0, H_1, \ldots, H_k)\) satisfies the Hintikka condition. We show that \((H'_0, H'_1, \ldots, H'_k)\) also satisfies it.

Let \(\exists r.C \in \text{supp}(H'_0)\) and \(j = \varphi(\exists r.C)\). By the definition of \(f_{(H,i)}\), we know that \(H_0(\exists r.C) = H'_0(\exists r.C)\) and \(rd_T(C) < rd_T(\exists r.C) < rd_T(H)\). Thus, \(H_j(C)\) is also defined and equal to \(H'_j(C)\). Moreover, \(\text{supp}(H) \neq \varnothing\), and thus \(H'_j(\varnothing) = H_j(\varnothing)\). This shows that Condition 1.a) is still satisfied.

Condition 1.b) can again be shown by similar arguments, replacing \(\varphi(\exists r.C)\) by an element of \(\varphi'(O)\) and the equality condition by an inequality.

For Condition 1.c), consider \(j \in \varphi'(O)\) with \(r' \subseteq_R \top\) and \(\exists r.C \in \text{supp}(H'_j)\). In this case, \(H_j(\exists r.C)\) must also be defined and equal to \(H'_j(\exists r.C)\). This implies that \(H_0(C)\) is defined and \(rd_T(C) < rd_T(\exists r.C) < rd_T(H)\), and thus \(H'_0(C)\) is also defined and equal to \(H_0(C)\). Since \(\text{supp}(H) \neq \varnothing\), we again have \(H'_j(\varnothing) = H_j(\varnothing)\). \(\square\)

By Proposition 3.11, it now follows that \(A_O\) is empty iff the subautomaton \(A^S_0\) induced by the family \(f_{(H,i)}\) is empty. It remains to show that this latter problem can be decided in PSPACE. For this, we show that the construction of \(A^S_0\) is a PSPACE on-the-fly construction, where we consider as blocking relation the equality relation on \(Q_O\).
Lemma 3.17 The construction of $A^3_S$ from $L$ and $O$ is a PSPACE on-the-fly construction.

Proof. We show that the automata $A^3_S$ are $m$-blocking for

$$m := \max\{\text{rd}_T(C) \mid C \in \text{sub}(O)\} + k + 3.$$ 

The other conditions of Definition 3.12 have already been shown above.

By definition of $A^3_S$, every transition decreases the maximal role depth of the support of the state. Hence, after at most $\max\{\text{rd}_T(C) \mid C \in \text{sub}(O)\} + 1$ transitions, we must reach a state $(H, i)$ for which $H(C)$ is undefined for all concepts $C \in \text{sub}(O)$, and hence $\text{supp}(H) = \emptyset$. From the next transition on, all Hintikka functions additionally assign $0$ to $\varrho$. Hence, after at most $m$ transitions, we find two states that are equal. Since $m$ is bounded by a polynomial in the size of $O$, the automata $A^3_S$ satisfy Definition 3.12. $\square$

Proposition 3.13 yields the desired PSPACE upper bound for local consistency w.r.t. acyclic TBoxes in the lattice-based description logic $L$-$\mathcal{IALCI}$. 

3.1.5 Acyclic TBoxes in $L$-$\mathcal{ISCIC}_{c}$

In the logic $L$-$\mathcal{ISCIC}$, we cannot directly reduce the role depth as in the previous section, due to the conditions on transitive roles. However, if we restrict to crisp roles only, we can still provide a PSPACE upper bound using a faithful family of functions.

Since the interpretations of roles are restricted to have values from $\{0, 1\}$, all Hintikka functions $H$ now need to satisfy the additional condition that $H(\varrho) \in \{0, 1\}$. It is easy to see that Lemma 3.6 also holds in the presence of this modification. Given a Hintikka function $H$ and a role $r$, we define the sets

$$H|_r := \{C \in \text{supp}(H) \mid C = \exists r.F \text{ or } C = \forall r.F\},$$  

$$H^{-r} := \{C \in \text{supp}(H) \mid \exists r.C \text{ or } \forall r.C \in \text{sub}(O)\}.$$ 

Definition 3.18 (functions $g_{(H,i)}$) Let $(H, i)$ and $(H', i')$ be two states of $A_O$ and consider $n := \text{rd}_T(H)$. We define the function $g_{(H,i)}(H', i') := (H'', i'')$, where $i'' \in \varphi_r(O)$, for all $C \in \text{sub}(O)$ as follows:

$$P := \begin{cases} \text{sub}^{\leq n}(O) \cap H'|_r & \text{if } r \text{ is transitive,} \\ \emptyset & \text{otherwise,} \end{cases}$$  

$$H''(C) := \begin{cases} H'(C) & \text{if } C \in \text{sub}^{\leq n-1}(O) \cup P, \\ \text{undefined} & \text{otherwise}, \end{cases}$$  

$$H''(\varrho) := \begin{cases} 0 & \text{if } \text{supp}(H) = \emptyset, \\ H'(\varrho) & \text{otherwise.} \end{cases}$$ 

These functions are a natural generalization of the functions used in (Baader, Hladik, and Peñaloza 2008) to provide a PSPACE upper bound for classical $\mathcal{S}L$.

Lemma 3.19 In $L$-$\mathcal{ISCIC}_{c}$, the family $g_{(H,i)}$ is faithful w.r.t. $A_O$. 

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To see that the automata

Thus, we have $H_0(∃r.C) = H_j(∃r.C)$.

Finally, we have $supp(H) ≠ ∅$, which implies that $H_j(∃r.C) = H_j(φ)$.

Let now $∃r.C ∈ supp(H_j)$ for some $j ∈ ψ(0)$ and $r$ be transitive. Thus, we have $∃r.C ∈ sub^{≤n−1}(O)$ and the value $H_j(∃r.C)$ is defined and equal to $H_j′(∃r.C)$. By the Hintikka condition, we have $∃r.C ∈ supp(H)$ and $H_j′(φ) = H_j(φ)$, and thus the value of $H_j′(φ)$ is still satisfied. Finally, the values $H_j(φ)$ and $H_j′(φ)$ must also be equal. □

To show that the automata $A^S_o$ can be built by a PSPACE on-the-fly construction, we employ the following blocking relation $→_{ι_{3ΣC}e}$.

**Definition 3.20 ($→_{ι_{3ΣC}e}$)** Let $(H, i)$ and $(H′, i′)$ be two states of $A^S_o$. We define the blocking relation $→_{ι_{3ΣC}e}$ by $(H, i) →_{ι_{3ΣC}e} (H′, i′)$ iff $i = i′ = ϕ(E)$ for $E ∈ sub(0)$ of the form $∃r.F$ or $∀r.F$ and either

(i) $H = H′$,

(ii) $H(φ) = H′(φ) = 0$ and the sets $H|_r ∪ H′|_r$ and $H|_r ∪ H′|_r$ are equal, or

(iii) 1. $r$ is transitive, $H(φ) = H′(φ) = 1$, $H(F) = H′(F)$,

2. $H(C) = H′(C)$ for every concept $C$ in

$Q(H, H′, r) := H|_r ∪ H′|_r ∪ H|_r ∪ H′|_r$, and

3. we have $H′(C) ≤ H′(∃r.C)$ for every $∃r.C ∈ H|_r$ and $H′(C) ≥ H′(∀r.C)$ for every $∀r.C ∈ H|_r$.

$\diamond$

To see that the automata $A^S_o$ are $→_{ι_{3ΣC}e}$-invariant, we analyze the three conditions above:
(i) The equality relation trivially satisfies the definition of \(\sim_{\mathcal{FLL}}\)-invariance.

(ii) Observe that if \(H(\varrho) = 0\), then all the inequalities in the Conditions 1.b), 1.c), 2.b), and 2.c) of Definition 3.3 are satisfied. Furthermore, Conditions 1.a) and 2.a) remain satisfied when replacing one successor \(H\) of \(H_0\) with \(H(\varrho) = 0\) by another \(H'\) which also satisfies \(H'(\varrho) = 0\). Thus, one only needs to ensure that \(H'\) is defined for the relevant concepts, which is expressed by the second part of this condition.

(iii) The first condition ensures that Conditions 1.a) and 2.a) of Definition 3.3 remain satisfied. The second condition restricts all the quantified concepts that are transferred by the transitive role \(r\) to be evaluated by identical values. Thus, Conditions 1.c) and 2.c) and the last inequalities of Conditions 1.b) and 2.b) of Definition 3.3 are still satisfied. Finally, the third condition ensures that the first inequalities of Conditions 1.b) and 2.b) are satisfied: We already know that \(H_0(\forall r.C) \leq H'(\varrho) \Rightarrow H'(\forall r.C)\) holds, and thus the additional condition \(H'(\forall r.C) \leq H'(C)\) ensures that also \(H_0(\forall r.C) \leq H'(\varrho) \Rightarrow H'(C)\) is satisfied, and dually for the existential restrictions.

We can now proceed to the last proof of this section.

**Lemma 3.21** The construction of \(\mathcal{A}_\mathcal{O}^\mathcal{S}\) from \(L\) and \(\mathcal{O}\) is a \(\text{PSpace}\) on-the-fly construction.

**Proof.** We have to show that the automata \(\mathcal{A}_\mathcal{O}^\mathcal{S}\) induced by the functions in Definition 3.18 are polynomially blocking w.r.t. the blocking relation \(\sim_{\mathcal{FLL}}\). For this, consider the states \((H_0, i_0), (H_1, i_1), (H_2, i_2)\) of three consecutive nodes in a path of a run of \(\mathcal{A}_\mathcal{O}^\mathcal{S}\) and let \(r_0, r_1, r_2\) be the roles of the restrictions designated by the indices \(i_0, i_1, i_2\), respectively. Recall first that the faithful family of functions ensures that \(rd_T(H_0) \geq rd_T(H_1) \geq rd_T(H_2)\).

If \(r_1\) is not transitive, then \(rd_T(H_0) > rd_T(H_1)\). Moreover, if \(r_1 \neq r_2\), then we have \(rd_T(H_0) > rd_T(H_2)\), regardless of whether \(r_1\) and \(r_2\) are transitive or not. Thus, a path in a run can have at most \(\max\{rd_T(C) \mid C \in \text{sub}(\mathcal{O})\} + 1\) states using a non-transitive role, or using different roles for consecutive transitions before reaching a state \((H, i)\) with \(\text{supp}(H) = \emptyset\).

If \(r_1 = r_2\) is a transitive role, then the role depth of the Hintikka functions may remain constant through both transitions. From the Hintikka condition, \(H_1 | r_1 \subseteq H_2 | r_1\), \(H_1^{-r_1} \subseteq H_2^{-r_1}\), and \(H_2^{\overline{r_1}} \subseteq H_1^{\overline{r_1}}\) must hold. This means that there can be at most \(|\text{sub}(\mathcal{O})|\) many states \((H, i)\) involving the same transitive role with \(H(\varrho) = 0\) before Condition (ii) of the blocking relation triggers.

Finally, if \(H_1(\varrho) > 0\), then \(H_1(\varrho) = 1\) since we assumed that all roles are crisp. In this case, the Hintikka condition implies that

\[H_0(\forall r_1.C) \leq H_1(\varrho) \Rightarrow H_1(\forall r_1.C) = H_1(\forall r_1.C)\]

for any \(\forall r_1.C \in \text{supp}(H_0)\), and dually

\[H_0(\exists r_1.C) \geq H_1(\varrho) \land H_1(\exists r_1.C) = H_1(\exists r_1.C)\]

whenever \(\exists r_1.C \in \text{supp}(H_0)\). Thus, after a chain of at most \(|L|\cdot|\text{sub}(\mathcal{O})|\) transitions with role \(r_1\) to degree 1, we find two states \((H, i), (H', i')\) with \(H(C) = H'(C)\) for every \(C \in Q(H, H', r_1)\). By the Hintikka condition, \(H_0(\forall r_1.C) \leq H_1(\varrho) \Rightarrow H_1(C) = H_1(C)\)

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3.2 Consistency

and $H_0(\exists r_1. C) \geq H_1(C)$, which shows that the last condition is also satisfied after at most $|L|\text{sub}(O)$ such transitions.

An additional factor of $|L|\text{sub}(O)$ enables us to ensure the existence of two nodes $(H, i)$ and $(H', i')$ that satisfy the remaining condition of $\neg \exists_{\text{SCI}}$, namely that we have $i = i' = \varphi(E)$ for some $E \in \text{sub}(O)$ of the form $\exists s.F$ or $\forall s.F$ and that $H(F) = H'(F)$.

Hence, in total, every path of length at least $(|L|\text{sub}(O))^5$ will contain two nodes that are in the blocking relation. This number is polynomial in the size of the input. $\square$

As before, this yields a PSPACE upper bound for local consistency in $L-\text{ISCHI}_c$. Thus, local consistency in $L-\text{ALCHI}$ and $L-\text{SCI}_c$ with acyclic TBoxes is decidable in PSPACE. A corresponding lower bound follows from PSPACE-hardness of concept satisfiability in classical $\mathcal{ALC}$ w.r.t. the empty TBox (Schmidt-Schauß and Smolka 1991).

Observe that these results also hold if we consider $n$-witnessed interpretations with $n > 1$. This is because the blocking relations and the bounds on the lengths paths may have before blocking is triggered depend only on the structure of the concepts in the Hintikka sets, and not on the arity of the Hintikka automaton.

**Theorem 3.22** Let $L$ be a finite residuated De Morgan lattice. Then local consistency w.r.t. general models in $L-\text{ISCHI}_c$ and $L-\text{ALCHI}$ with acyclic TBoxes is decidable in PSPACE. It is PSPACE-hard already in $2\text{-\mathcal{NEL}}$ and $2\text{-\mathcal{ALC}}$. $\square$

3.2 Consistency

As a preliminary step to deciding consistency in $L-\text{ISCHI}$, we will first describe another algorithm for local consistency, this time a tableau algorithm. While it does not have optimal worst-case behavior, the approach can be generalized to a decision procedure for full consistency.

Another reason for developing a tableau algorithm is the hope that it is amenable to optimizations used for classical tableau algorithms, whereas the automata-based construction is always of exponential complexity since one has to construct an automaton of exponential size. While the following tableau algorithm involves a great deal of nondeterminism due to the fact that most rules guess at least one value from the lattice, it is nevertheless better able to exploit the structure of the TBox and of the concepts appearing in it.

### 3.2.1 Tableaux for Local Consistency

We again consider an ontology $O = (A, T, R)$, where $A$ is a local ABox that contains only the individual name $c$, and $T$ is a fuzzy general TBox. For the following, it is convenient to view acyclic TBoxes as general TBoxes by replacing every definition $\langle A \equiv C \geq p \rangle$ by $\langle A \sqsubseteq C \geq p \rangle$ and $\langle C \sqsubseteq A \geq p \rangle$. For the following constructions, we do not need to distinguish acyclic TBoxes, as we have already established the complexity of local consistency in this case.

To decide whether $O$ is consistent, the basic idea is to construct an abstract description of a model of $O$. To this end, we first show that $O$ has a model iff we can find a **tableau**, which intuitively corresponds to a (possibly infinite) “completed version” of $A$. 

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We denote by \( C \subseteq T \) the set of individual names occurring in a tableau \( T \).

### Table 3.1: The tableaux conditions for \( L\mathcal{A}SCI \)

<table>
<thead>
<tr>
<th>((\text{trigger}))</th>
<th>((\text{values}))</th>
<th>((\text{assertions}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \top \langle x:T = p \rangle )</td>
<td>( \langle x:T = 1 \rangle )</td>
<td></td>
</tr>
<tr>
<td>( \bot \langle x: \bot = p \rangle )</td>
<td>( \langle x: \bot = 0 \rangle )</td>
<td></td>
</tr>
<tr>
<td>( \bigwedge (x:C_1 \cap C_2 = p) )</td>
<td>( p_1, p_2 \in L \text{ with } p_1 \otimes p_2 = p )</td>
<td>( \langle x:C_1 = p_1 \rangle, \langle x:C_2 = p_2 \rangle )</td>
</tr>
<tr>
<td>( \rightarrow (x:C_1 \rightarrow C_2 = p) )</td>
<td>( p_1, p_2 \in L \text{ with } p_1 \Rightarrow p_2 = p )</td>
<td>( \langle x:C_1 = p_1 \rangle, \langle x:C_2 = p_2 \rangle )</td>
</tr>
<tr>
<td>( \neg (x:\neg C = p) )</td>
<td>( p_1, p_2 \in L \text{ with } p_1 \otimes p_2 = p )</td>
<td>( \langle x = \sim p \rangle )</td>
</tr>
<tr>
<td>( \exists (x: \exists r.C = p) )</td>
<td>( p_1, p_2 \in L \text{ with } p_1 \otimes p_2 = p )</td>
<td>( \langle x, y : r = p_1 \rangle, \langle y:C = p_2 \rangle )</td>
</tr>
<tr>
<td>( \exists \subseteq (x: \exists r.C = p), \langle (x,y): r = p_1 \rangle )</td>
<td>( p_2 \in L \text{ with } p_1 \otimes p_2 \leq p )</td>
<td>( \langle y:C = p_2 \rangle )</td>
</tr>
<tr>
<td>( \exists + (x: \exists s.C = p), \langle (x,y): r = p_1 \rangle )</td>
<td>( p_2 \in L \text{ with } p_1 \otimes p_2 \leq p )</td>
<td>( \langle y: \exists r.C = p_2 \rangle )</td>
</tr>
<tr>
<td>( \forall (x: \forall r.C = p) )</td>
<td>( p_1, p_2 \in L \text{ with } p_1 \Rightarrow p_2 = p, \langle (x,y): r = p_1 \rangle, \langle y:C = p_2 \rangle )</td>
<td></td>
</tr>
<tr>
<td>( \forall \subseteq (x: \forall r.C = p), \langle (x,y): r = p_1 \rangle )</td>
<td>( p_2 \in L \text{ with } p_1 \Rightarrow p_2 \geq p )</td>
<td>( \langle y:C = p_2 \rangle )</td>
</tr>
<tr>
<td>( \forall + (x: \forall s.C = p), \langle (x,y): r = p_1 \rangle )</td>
<td>( p_2 \in L \text{ with } p_1 \Rightarrow p_2 \geq p )</td>
<td>( \langle y: \forall r.C = p_2 \rangle )</td>
</tr>
<tr>
<td>( \text{inv} \langle (x,y): r = p_1 \rangle )</td>
<td>( p_2 \in L \text{ with } p_1 \leq p_2 )</td>
<td>( \langle (y,x): r = p_1 \rangle )</td>
</tr>
<tr>
<td>( \subseteq_R \langle (x,y): r = p_1 \rangle, r \subseteq_R s )</td>
<td></td>
<td>( \langle (x,y): s = p_2 \rangle )</td>
</tr>
<tr>
<td>( \subseteq_T (C_1 \subseteq C_2 \geq p) \text{ in } T )</td>
<td>( p_1, p_2 \in L \text{ with } p_1 \Rightarrow p_2 \geq p )</td>
<td>( \langle x:C_1 = p_1 \rangle, \langle x:C_2 = p_2 \rangle )</td>
</tr>
</tbody>
</table>

Afterwards, we describe an algorithm that tries to construct a finite representation of such a tableau.

**Definition 3.23 (tableau)** A tableau for \( \mathcal{O} \) is a set \( T \) of equality assertions of the form \( \langle x:C = p \rangle \) or \( \langle (x,y): r = p \rangle \), where \( x, y \in \mathbb{N}_1 \), \( C \in \text{sub}(\mathcal{O}) \), \( r \in \mathbb{N}_R \), and \( p \in L \), such that \( A \subseteq T \) and the following conditions are satisfied for all \( C, C_1, C_2 \in \text{sub}(\mathcal{O}) \), \( x, y \in \mathbb{N}_1 \), \( r, s \in \mathbb{N}_R \), and \( p \in L \):

- **T is clash-free:** If \( \langle x:C = p \rangle \in T \) or \( \langle (x,y): r = p \rangle \in T \), then there is no \( p' \in L \) such that \( p' \neq p \) and \( \langle x:C = p' \rangle \in T \) or \( \langle (x,y): r = p' \rangle \in T \), respectively.

- **T is complete:** For every row of Table 3.1, the following condition holds: “If \( \langle \text{trigger} \rangle \) is in \( T \), then there are \( \langle \text{values} \rangle \) such that \( \langle \text{assertions} \rangle \) are in \( T \).”

We denote by \( \text{Ind}(T) \) the set of individual names occurring in a tableau \( T \).

In classical DLs, a clash is defined as the simultaneous presence of two assertions of the form \( a:C \) and \( a:\neg C \). Our definition generalizes this to fuzzy assertions: if \( \langle a:C = 1 \rangle \) and \( \langle a:\neg C = 1 \rangle \) are contained in \( T \), then by completeness \( T \) also contains \( \langle a:C = 0 \rangle \), and clearly \( \bot \neq 1 \).

The conditions in Table 3.1 concerning the basic constructors, inverse roles, role inclusions, and GCIIs are quite straightforward. For example, the condition \( \top \) requires that individuals never belong to \( \top \) to a degree smaller than \( 1 \), while the condition \( \subseteq_T \) ensures that a GCI \( \langle C_1 \subseteq C_2 \geq p \rangle \) is satisfied at every individual \( x \) by asserting appropriate values for \( C_1 \) and \( C_2 \) at \( x \). The conditions for the existential and value...
restrictions deserve some more explanation. First, note that the semantics of ∀ is dual to that of ∃, and thus every rule for ∃ must have a dual counterpart for ∀ where the order is reversed and ⊗ is replaced by ⇒.

In contrast to classical SHI, where only the conditions ∃ and ∃ are needed to deal with existential restrictions (Horrocks and Sattler 1999), we need three rules in the fuzzy setting. The reason lies in the witnessed semantics of an assertion ⟨x:∃r.C = p⟩. The condition ∃ ensures that a witness y with the correct value rT(x,y) ⊗ CT(y) exists (if we view T as an abstract description of an interpretation), while ∃ is needed to restrict all other individuals y′ to not exceed this value. Finally, the conditions ∃+ and ∀+ specify how existential and value restrictions should be propagated along chains of successors through a transitive role, as shown in the following example.

**Example 3.24** Consider the lattice L4 from Figure 2.4, the transitive role contains, and the individual names my_apartment and living_room. Assume that the following assertions are in our tableau T:

⟨my_apartment:∀contains.(Wall → White) = d⟩,

⟨(my_apartment, living_room):contains = t⟩.

The condition ∀+ transports the value restriction to living_room in order to ensure that all transitive sub-parts of my_apartment also satisfy the restriction, in particular all walls of the living room. Thus, an assertion

⟨living_room:∀contains.(Wall → White) = p⟩,

where p′ is either t or d, must also be in T.

The common notation T is not the only similarity between the notions of tableaux and Hintikka trees from the previous section. There is also a close connection between the tableau rules of Table 3.1 and Definitions 3.2 and 3.3 for Hintikka functions and the Hintikka condition. For instance, Conditions ∃, ∃, and ∃ express exactly the same restrictions as Conditions 1.a) and 1.b) of Definition 3.3. This connection between tableau algorithms and automata-based approaches has been formalized for classical DLs by Baader, Hladik, Lutz, and Wolter (2003). The main advantage of the formalization using sets of equality assertions instead of Hintikka trees will become apparent later when we apply the conditions of Table 3.1 to the general consistency problem, where we have to work with structures that are not tree-shaped.

The following lemma shows that the conditions of Definition 3.23 are sufficient to detect whether O has a model. Due to the strong correspondence between tableaux and Hintikka trees, its proof is closely related to the one of Lemma 3.6. However, it is easier to deal with inverse roles and role inclusions due to the tableau conditions inv and ⊑R, as opposed to the very intricate Hintikka condition. Furthermore, we do not explicitly consider acyclic TBoxes here.

**Lemma 3.25** O is locally consistent w.r.t. witnessed models iff it has a tableau.

**Proof.** Let T be a tableau for O. For each role name r, we define a fuzzy binary relation rT over Ind(T) as follows:

\[ r^T(x,y) := \begin{cases} p & \text{if } \langle (x,y):r = p \rangle \in T, \\ 0 & \text{otherwise.} \end{cases} \]
Note that these values are either unique or undefined since $T$ is clash-free. If they are undefined in $T$, then we set them to $0$ for now. In this way, $T$ immediately defines a rudimentary interpretation of the role names. As in the proof of Lemma 3.6, we interpret inverse roles by $(r^\rightarrow)^T(x, y) := r^T(y, x)$ for all $x, y \in \text{Ind}(T)$, and we denote by $r^T(z_1, \ldots, z_n)$ for a complex role $r$ the value $r^T(z_1, z_2) \otimes \cdots \otimes r^T(z_{n-1}, z_n)$ for any sequence $z_1, \ldots, z_n \in \text{Ind}(T)$ with $n \geq 2$.

We now construct a proper model $I$ of $O$ as in the proof of Lemma 3.6:

- $A^I(x) := \text{Ind}(T)$;
- $d^I := c$ for every $d \in N$;
- for all concept names $A$ and $x \in \text{Ind}(T)$,
  $$A^I(x) := \begin{cases} p & \text{if } \langle x : A = p \rangle \in T, \\ 0 & \text{otherwise}; \end{cases}$$
- for all role names $r$ and $x, y \in \text{Ind}(T)$,
  $$r^I(x, y) := r^T(x, y) \lor \bigvee_{s \sqsubseteq r \text{ transitive}} \bigvee_{n \geq 1, z_1, \ldots, z_n \in \text{Ind}(T)} s^T(x, z_1, \ldots, z_n, y).$$

By the condition inv, it is easy to show that the same equation holds for all inverse roles.

Furthermore, if $r$ is transitive, then $r^I$ is the transitive closure of $r^T$, and therefore a transitive fuzzy relation. Consider now a role inclusion $r \subseteq s \in R$. By condition $\subseteq_R$, we know that $r^T(x, y) \leq s^T(x, y)$ holds for all $x, y \in \text{Ind}(T)$. Moreover, every transitive subrole of $r$ is also a transitive subrole of $s$. Thus, we obtain $r^I(x, y) \leq s^I(x, y)$ for all $x, y \in \text{Ind}(T)$, and hence $I$ is a model of $R$.

We now show that for every $C \in \text{sub}(O)$, $x \in \text{Ind}(T)$, and $p \in L$, we have $C^I(x) = p$ whenever $\langle x : C = p \rangle \in T$. Together with the condition $\subseteq_T$ and the fact that $A \subseteq T$, this proves that $I$ satisfies all axioms of $O$. We show the claim by induction on the structure of $C$.

- The claim for $\top$, $\bot$, and concept names follows from the conditions $\top$, $\bot$, clash-freeness of $T$, and definition of $I$.
- If $\langle x : \neg C = p \rangle \in T$, then by condition $\neg$ and induction we have
  $$(\neg C)^I(x) = \neg C^I(x) = \neg \neg p = p.$$ The claims for $C \cap D$ and $C \rightarrow D$ follow by similar arguments.

- If $\langle x : \exists r : C = p \rangle \in T$, then by condition $\exists$ there must be $y \in \text{Ind}$ and $p_1, p_2 \in L$ such that $p_1 \otimes p_2 = p$ and $\langle (x, y) : r = p_1 \rangle, \langle y : C = p_2 \rangle \in T$. By induction, we have $p = r^T(x, y) \otimes C^I(y) \leq r^I(x, y) \otimes C^I(y)$. We now show that for every $z \in \text{Ind}$ we have $r^I(x, z) \otimes C^I(z) \leq p$, which in particular implies that $y$ is a witness for $(\exists r : C)^I(x)$, and that $I$ is witnessed.

By definition of $r^I$ and the fact that $\otimes$ is join-preserving, it suffices to show that we have (a) $r^I(x, z) \otimes C^I(z) \leq p$ and (b) $s^I(x, y_1, \ldots, y_n, z) \otimes C^I(z) \leq p$ for all transitive roles $s \subseteq_R r$ and all $y_1, \ldots, y_n \in \text{Ind}$, $n \geq 1$.

\[3\text{ Recall that } c \text{ is the unique individual name occurring in } A.\]
We now present a tableau algorithm for deciding local consistency. The algorithm starts with the local ABox \( A \) and nondeterministically expands it to a tree-like ABox \( A' \) that represents a model of \( O \). It uses the tableau conditions from Table 3.1 and reformulates them into expansion rules of the form:

"If there is \( \langle \text{trigger} \rangle \) in \( A' \) and there are no \( \langle \text{values} \rangle \) such that \( \langle \text{assertions} \rangle \) are in \( A' \), then introduce \( \langle \text{values} \rangle \) and add \( \langle \text{assertions} \rangle \) to \( A' \)."

The rules \( \exists \) and \( \forall \) always introduce new individuals \( y \) that do not appear in \( A' \). Initially, the ABox \( A \) contains the single individual \( c \). This ABox is expanded by the rules in a

### 3.2.2 Tableau Algorithm for Local Consistency

We present a tableau algorithm for deciding local consistency. The algorithm starts with the local ABox \( A \) and nondeterministically expands it to a tree-like ABox \( A' \) that represents a model of \( O \). It uses the tableau conditions from Table 3.1 and reformulates them into expansion rules of the form:

"If there is \( \langle \text{trigger} \rangle \) in \( A' \) and there are no \( \langle \text{values} \rangle \) such that \( \langle \text{assertions} \rangle \) are in \( A' \), then introduce \( \langle \text{values} \rangle \) and add \( \langle \text{assertions} \rangle \) to \( A' \)."

The rules \( \exists \) and \( \forall \) always introduce new individuals \( y \) that do not appear in \( A' \). Initially, the ABox \( A \) contains the single individual \( c \). This ABox is expanded by the rules in a
tree-like way: role connections are only created by adding new successors to existing individuals. If an individual \( y \) was created by a rule \( \exists \) or \( \forall \) that was applied to an assertion involving an individual \( x \), then we say that \( y \) is a successor of \( x \), and \( x \) is the predecessor of \( y \); ancestor is the transitive closure of predecessor. Note that the presence of an assertion \( (\langle x, y \rangle; r = p) \) in \( A^* \) does not imply that \( y \) is a successor of \( x \)—it could also be the case that this assertion was introduced by the inv-rule, which would mean that \( x \) is actually a successor of \( y \).

We further denote by \( A^*_x \) the set of all concept assertions from \( A^* \) that involve the individual \( x \), i.e. are of the form \( \langle x; C = p \rangle \) for some concept \( C \in \text{sub}(O) \) and \( p \in L \). As is standard in DL, to ensure that the application of the rules terminates, we need to add a blocking condition. Here, we use anywhere blocking (Motik, Shearer, and Horrocks 2007), which is based on the idea that it suffices to examine each set \( A^*_x \) only once in the whole ABox \( A^* \).

Let \( \succ \) be a total order on the individuals of \( A^* \) such that whenever \( y \) is a successor of \( x \), then \( y \succ x \). An individual \( y \) is directly blocked if for some other individual \( x \) in \( A^* \) with \( y \succ x \), \( A^*_x \) is equal to \( A^*_y \) modulo the individual names used; in this case, we write \( A^*_x \equiv A^*_y \) and also say that \( x \) blocks \( y \). It is indirectly blocked if its predecessor is either directly or indirectly blocked. An individual is blocked if it is either directly or indirectly blocked. The rules \( \exists \) and \( \forall \) are applied to \( A^* \) only if the individual \( x \) that triggers their execution is not blocked. All other rules are applied only if \( x \) is not indirectly blocked.

The total order \( \succ \) is used to avoid cycles in the blocking relation in which two individuals are mutually blocking each other. One way to build this order is to simply use the order in which the individuals were created by the expansion rules. Note that the only individual \( c \) that occurs in \( A \), which is the root of the tree-like structure represented by \( A^* \), cannot be blocked since it is an ancestor of all other individuals in \( A^* \). With this blocking condition, we can show that the size of \( A^* \) is bounded exponentially in the size of \( A \), as in the crisp case (Motik, Shearer, and Horrocks 2007).

**Lemma 3.26** Every sequence of applications of expansion rules to \( A \) terminates after at most exponentially many rule applications.

**Proof.** Every rule application expands \( A^* \) in a tree-like manner, where every individual is a node in this tree. Note that there are at most \( |L||\text{sub}(O)| \) possible concept assertions for one individual \( x \). Thus, every node in this tree has at most \( |L||\text{sub}(O)| \) successors: one for each possible assertion involving an existential or value restriction. Moreover, there can be at most \( 2^{L||\text{sub}(O)|} \) non-blocked nodes in \( A^* \) at any time, and thus, when a node becomes blocked, at most exponentially many nodes become indirectly blocked.

This bounds the total number of possible non-blocked, directly blocked, and indirectly blocked nodes by an exponential in the size of the input. Thus, we obtain a tree of at most exponential size before every rule application is disallowed by the blocking condition. The claim now follows from the fact that every rule application adds at least one assertion to \( A^* \) and cannot remove assertions from \( A^* \).

We say that \( A^* \) contains a clash if it contains two assertions that are equal except for their lattice value (cf. Definition 3.23). \( A^* \) is complete if it contains a clash or none of the expansion rules are applicable. We now show that the algorithm is correct in the sense that it produces a clash iff \( O \) is not locally consistent. As expected, the proof uses
Lemma 3.27 \( O \) is locally consistent w.r.t. witnessed models iff some sequence of applications of the expansion rules to \( A \) yields a complete and clash-free ABox.

**Proof.** By Lemma 3.25, \( O \) is locally consistent w.r.t. witnessed models iff it has a tableau. Assume first that \( T \) is a tableau for \( O \). We show how to guide the application of the expansion rules in such a way that no clash is produced. Observe that the initial ABox \( A \) is included in \( T \) by definition. We will ensure that the expansion rules add only assertions to \( A^* \) that are also in \( T \). Assume that, for some row of Table 3.1, an expansion rule is applicable, i.e. \( \langle \text{trigger} \rangle \) is in \( A^* \) and there are no \( \langle \text{values} \rangle \) such that \( \langle \text{assertions} \rangle \) are in \( A^* \) and the blocking condition does not apply. Since \( \langle \text{trigger} \rangle \) is also in the tableau \( T \), there must be \( \langle \text{values} \rangle \) such that \( \langle \text{assertions} \rangle \) are in \( T \), and thus we can add \( \langle \text{assertions} \rangle \) to \( A^* \). Since \( T \) is clash-free, this process cannot create any clashes in \( A^* \). Lemma 3.26 shows that at some point \( A^* \) must also be complete.

For the other direction, assume now that the expansion rules have produced a complete and clash-free ABox \( A^* \). It is easy to see from Table 3.1 that \( A^* \) can only contain concepts from \( \text{sub}(O) \). We can thus define a tableau \( T \) for \( O \) over the set

\[
\text{Ind} := \{ x \in \mathbb{N}_1 \mid x \text{ occurs in } A^* \text{ and is not blocked} \}
\]

of individuals as follows:

\[
T := \{ \langle x:C = p \rangle \in A^* \mid x \in \text{Ind} \} \cup
\{ \langle (x,y):r = p \rangle \in A^* \mid x,y \in \text{Ind} \} \cup
\{ \langle (x,y):r = p \rangle \mid x,y \in \text{Ind}, \langle (x,z):r = p \rangle \in A^*, \text{ and } y \text{ blocks } z \} \cup
\{ \langle (x,y):r = p \rangle \mid x,y \in \text{Ind}, \langle (z,y):r = p \rangle \in A^*, \text{ and } x \text{ blocks } z \}.
\]

Thus, whenever \( y \) blocks \( z \) and \( z \) is not indirectly blocked, then all incoming role connections of \( z \) are “re-routed” back to \( y \). Since the root \( c \) of the tree-like structure \( A^* \) has no predecessors, it cannot be blocked, and thus the initial ABox \( A \) is still contained in \( T \). Furthermore, since \( A^* \) is clash-free, \( T \) is also clash-free.

It remains to show completeness of \( T \). For any row of Table 3.1, we distinguish three cases based on the form of \( \langle \text{trigger} \rangle \).

a) If \( \langle \text{trigger} \rangle \) involves only assertions from \( A^* \), then the corresponding expansion rule was applied at some point and introduced \( \langle \text{values} \rangle \) and \( \langle \text{assertions} \rangle \). If no new individual was introduced, all \( \langle \text{assertions} \rangle \) must also be in \( T \). We consider now the case of the rule \( \exists \); the rule \( \forall \) can be handled similarly.

Assume that \( \langle x: \exists r:C = p \rangle \in A^* \) and \( x \) is not blocked. Then a new individual \( y \) was introduced, together with the assertions \( \langle (x,y):r = p_1 \rangle \) and \( \langle y:C = p_2 \rangle \), where \( p_1 \otimes p_2 = p \). If \( y \) is not blocked, these assertions are also in \( T \). If \( y \) is blocked by an individual \( z \), then the assertion \( \langle (x,z):r = p_2 \rangle \) is in \( T \). Additionally, we have \( A^*_y \equiv A^*_z \), and thus also \( \langle z:C = p_2 \rangle \) is in \( T \).

b) If \( \langle \text{trigger} \rangle \) involves a role assertion \( \langle (x,y):r = p_1 \rangle \) where \( \langle (x,z):r = p_1 \rangle \in A^* \) and \( y \) blocks \( z \), then \( x \) is not blocked and the corresponding expansion rule was applied to \( A^* \) with \( z \) instead of \( y \).
Consider the rule $\exists \xi$. Then the assertions $\langle x: \exists r.C = p \rangle$ and $\langle z: C = p_2 \rangle$ must be in $A^*$ with $p_1 \otimes p_2 \leq p$. Since $A^*_x \equiv A^*_y$, we have $\langle y: C = p_2 \rangle$ in $A^*$ and also in $T$. The rules $\exists_+, \forall_\geq$, and $\forall_+$ behave similarly.

If the rule inv was applied, then $\langle (x, z): p = p_1 \rangle \in A^*$, and thus $\langle (y, x): r = p_1 \rangle \in T$.

If the rule $\sqsubseteq_R$ was applied with $r \sqsubseteq_R s$, then $\langle (x, z): s = p_2 \rangle \in A^*$ with some $p_2 \in L$ such that $p_1 \leq p_2$. Thus, we have $\langle (x, y): s = p_2 \rangle \in T$.

c) If $\langle \text{trigger} \rangle$ involves a role assertion $\langle (x, y): r = p_1 \rangle$ where $\langle (z, y): r = p_1 \rangle \in A^*$ and $x$ blocks $z$, then we consider the concrete condition concerned.

If it is $\exists \xi$, then we have $\langle x: \exists r.C = p \rangle$ in $T$ and also in $A^*$. Since $A^*_x \equiv A^*_y$, this implies that $\langle z: \exists r.C = p \rangle$ is in $A^*$. Since $z$ must be a successor of $y$, $z$ is not indirectly blocked, and thus by the rule $\exists \xi$ there is $\langle y: C = p_2 \rangle$ in $A^*$ with $p_1 \otimes p_2 \leq p$. The same assertion must also be present in $T$ since $y$ is not blocked. Again, the conditions $\exists_+, \forall_\geq$, and $\forall_+$ can be handled similarly.

If it is inv, since $z$ is not indirectly blocked, we have $\langle (y, z): r = p_1 \rangle \in A^*$, and thus $\langle (y, x): r = p_1 \rangle \in T$.

If it is $\sqsubseteq_R$ with $r \sqsubseteq_R s$, then, since $z$ is not indirectly blocked, there must be a $p_2 \geq p_1$ such that $\langle (z, y): s = p_2 \rangle$ is in $A^*$, and thus $\langle (x, y): s = p_2 \rangle$ is in $T$. 

\[\Box\]

Since the expansion rules are nondeterministic, Lemmata 3.26 and 3.27 together imply that the tableau algorithm decides local consistency w.r.t. witnessed models in nondeterministic exponential time.

Again, this algorithm is easily adapted for $n$-witnessed interpretations where $n > 1$, and thus to general models. For $n > 0$, it does not suffice to generate only one successor for every existential and universal restriction, but one must produce $n$ different successors to ensure that the degrees guessed for these complex concepts are indeed witnessed by the model. The only required change to the algorithm is in the rules $\exists$ and $\forall$ (see Table 3.1), where we have to introduce $n$ individuals $y_1, \ldots, y_n$, and $2n$ values $p_1^1, p_1^2, \ldots, p_n^1, p_n^2 \in L$ that satisfy $\bigvee_{i=1}^n p_i^1 \otimes p_i^2 = p$ or $\bigwedge_{i=1}^n p_i^1 \Rightarrow p_i^2 = p$, respectively. The complexity of the algorithm as analyzed in Lemma 3.26 remains the same under this modification, as the number of successors of a node is still bounded polynomially, namely by $n|L||\text{sub}(\mathcal{O})|$.

### 3.2.3 Pre-Completion for Consistency

We now present the promised decision procedure for general consistency in $L:\text{\sc\text{\texttt{H}}}L$. Let $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$ be an ontology, where $\mathcal{A}$ is an arbitrary ABox, and let $\text{Ind}(\mathcal{A})$ denote the set of individual names occurring in $\mathcal{A}$.

We first make sure that the information contained in $\mathcal{A}$ is consistent in itself, i.e. that the knowledge that can be inferred about the individuals appearing in $\mathcal{A}$ without considering additional domain elements is not contradictory. It then suffices to check a local consistency condition for each of these individuals. This procedure is based on a similar idea developed for classical description logics, called pre-completion (Hollunder 1996).

**Definition 3.28 (pre-completion)** An ABox $\mathcal{A}^*$ is a pre-completion of $\mathcal{A}$ w.r.t. $(\mathcal{T}, \mathcal{R})$ if

...
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Figure 3.3: Consistency checking by pre-completion and local consistency tests

- it contains only equality assertions of the forms $\langle c:C = p \rangle$ and $\langle (c,d):r = p \rangle$, where $c,d \in \text{Ind}(\mathcal{A})$, $C \in \text{sub}(\mathcal{O})$, $p \in L$, and $r$ is a role name occurring in $\mathcal{O}$;
- for every $\langle \alpha > p \rangle \in \mathcal{A}$, there is a $p' \in L$ such that $p' > p$ and $\langle \alpha = p' \rangle \in \mathcal{A}^*$;
- it is clash-free; and
- it satisfies the tableaux conditions of Table 3.1, except $\exists$ and $\forall$.

Observe that we can guess a pre-completion of $\mathcal{A}$ w.r.t. $(\mathcal{T}, \mathcal{R})$ in nondeterministic polynomial time and polynomial space in the size of $\mathcal{O}$ and $L$.

**Lemma 3.29** $\mathcal{O}$ has a witnessed model iff there is a pre-completion $\mathcal{A}^*$ of $\mathcal{A}$ w.r.t. $(\mathcal{T}, \mathcal{R})$ such that, for every $c \in \text{Ind}(\mathcal{A})$, the ontology $\mathcal{O}_c := (\mathcal{A}^*, \mathcal{T}, \mathcal{R})$ has a witnessed model.

**Proof.** Let $\mathcal{I}$ be a witnessed model of $\mathcal{O}$ and $\mathcal{A}^*$ be the set of all assertions of the form $\langle c:C = p \rangle$ or $\langle (c,d):r = p \rangle$ for $c,d \in \text{Ind}(\mathcal{A})$, $r \in N_R$, and $C \in \text{sub}(\mathcal{O})$. Using the same arguments as in the proof of Lemma 3.25, we can show that $\mathcal{A}^*$ is a pre-completion of $\mathcal{A}$ w.r.t. $(\mathcal{T}, \mathcal{R})$. Furthermore, by construction $\mathcal{I}$ satisfies $\mathcal{O}_c$ for any $c \in \text{Ind}(\mathcal{A})$.

Conversely, let $\mathcal{A}^*$ be a pre-completion of $\mathcal{A}$ w.r.t. $(\mathcal{T}, \mathcal{R})$ and $\mathcal{O}_c$ be locally consistent w.r.t. witnessed models for every $c \in \text{Ind}(\mathcal{A})$. By Lemma 3.25, for each $c \in \text{Ind}(\mathcal{A})$ there is a tableau $\mathcal{T}_c$ for $\mathcal{O}_c$ over a set $\text{Ind}_c$ of individuals with $c \in \text{Ind}_c$. We can assume that the sets $\text{Ind}_c$ are mutually disjoint.

We now define $C^\mathcal{T}(x) := p$ whenever $\langle x:C = p \rangle \in \mathcal{T}_c$ for some $c \in \text{Ind}(\mathcal{A})$. Similarly, we set $r^\mathcal{T}(x,y) := p$ if $\langle (x,y):r = p \rangle \in \mathcal{T}_c$ for some $c \in \text{Ind}(\mathcal{A})$. Note that, since the tableaux $\mathcal{T}_c$ are clash-free and the sets $\text{Ind}_c$ are disjoint, these values are uniquely defined. To reconnect the individuals of $\text{Ind}(\mathcal{A})$, we additionally define $r^\mathcal{T}(c,d) := p$ whenever $\langle (c,d):r = p \rangle \in \mathcal{A}^*$.

As in the proof of Lemma 3.25, we can now define an interpretation $\mathcal{I}$ from these values by constructing the transitive closure of $r^\mathcal{T}$ if $r$ is transitive. Then, we have $C^\mathcal{T}(x) = p$ for every assertion $\langle x:C = p \rangle$ occurring in $\mathcal{A}$ and the tableaux $\mathcal{T}_c$. Note that $\mathcal{I}$ is witnessed since each of the tableaux satisfies the witnessing conditions $\exists$ and $\forall$. The second condition of Definition 3.28 ensures that $\mathcal{I}$ satisfies $\mathcal{A}$ and by the conditions $\sqsubseteq_T$ and $\sqsubseteq_R$, $\mathcal{I}$ satisfies $\mathcal{T}$ and $\mathcal{R}$. 

Figure 3.3 illustrates this approach of constructing a model for $\mathcal{O}$ by combining a pre-completion $\mathcal{A}^*$ with models for each $\mathcal{O}_c$, $c \in \text{Ind}(\mathcal{A})$, which can be assumed to be in the shape of tree-like tableaux $\mathcal{T}_c$ rooted in $c$.  

This shows that we can decide consistency by first guessing a pre-completion and then deciding linearly many local consistency problems (of polynomial size). Again, the generalization to n-witnessed models is straightforward. Together with Theorems 3.8 and 3.22, this implies the following complexity results.

**Theorem 3.30** Let L be a finite residuated De Morgan lattice. Then consistency w.r.t. general models in L-3SCHI with fuzzy general TBoxes and equality assertions is decidable in ExpTime. When restricted to either L-3ALCHI or L-3SCLc and acyclic TBoxes, the problem is in PSPACE. Corresponding hardness results hold already in 2-\(\mathcal{RE}\) and \(2-\mathcal{ELC}\).

### 3.3 Satisfiability and Entailment

To decide whether a concept \(C\) is \(p\)-satisfiable w.r.t. an ontology \(O = (A, T, R)\), we can simply check whether \((A \cup \{\{c : C \geq p\}\}, T, R)\) is consistent, where \(c\) is a fresh individual name not occurring in \(A\). Thus, satisfiability has the same complexity as consistency. Moreover, we can compute the best satisfiability degree of \(C\) as the supremum of all values \(p \in L\) such that the ontology \((A \cup \{\{c : C \geq p\}\}, T, R)\) is consistent. For this, we have to call the decision procedure for consistency a linear number of times, i.e. once for each \(p \in L\). By exploiting the structure of \(L\), this number could be decreased; for example, over a total order we could use a binary search strategy to find the best satisfiability degree of \(C\) using logarithmically many consistency tests.

To check \(p\)-instances, we can exploit the fact that \(c\) is not a \(p\)-instance of \(C\) w.r.t. \(O\) iff there is a model \(I\) of \(O\) and a domain element \(x \in \Delta^I\) such that \(C^I(c^I) \not\leq p\). This is the case iff there is a value \(p' \not\geq p\) such that the ontology \((A \cup \{\{c : C = p'\}\}, T, R)\) is consistent. Thus, \(p\)-instances can be decided by calling the decision procedure for consistency a linear number of times, namely at most once for each \(p' \in L\) with \(p' \not\geq p\). We can also compute the best instance degree for \(c\) and \(C\) is the infimum of all \(p' \in \mathcal{L}\) since

\[
\bigvee \{p \in L \mid c\text{ is a }p\text{-instance of }C\} = \bigvee \{p \in L \mid \forall p' \not\geq p : p' \not\in \mathcal{L}\} = \bigvee \{p \in L \mid \forall p' \in \mathcal{L} : p \leq p'\} = \bigwedge \mathcal{L}.
\]

Finally, note that \(C\) is \(p\)-subsumed by \(D\) iff \(c\) is a \(p\)-instance of \(C \rightarrow D\), where \(c\) is a fresh individual name. Thus, deciding subsumption and computing the best subsumption degree can be done using the same approach as above. By Theorem 3.30, we now get the following complexity results.

**Theorem 3.31** Let L be a finite residuated De Morgan lattice. Then satisfiability, subsumption, and instance checking w.r.t. general models in L-3SCHI with fuzzy general TBoxes and equality assertions are decidable in ExpTime. When restricted to L-3ALCHI or L-3SCLc and acyclic TBoxes, these problems are in PSPACE. Corresponding hardness results hold already in 2-\(\mathcal{RE}\) and 2-\(\mathcal{ELC}\). □

In (Bobillo et al. 2009; Bobillo and Straccia 2011), reasoning algorithms were presented for the finite-valued fuzzy DLs \(G_n\text{-SCROIQ}\) and \(\text{\#}_n\text{-SCROIQ}\) with fuzzy GCIs. They
rely on a reduction to crisp $SROIQ$-ontologies (cf. Section 2.4). However, even when restricted to the constructors of $SCHI$, the reduction results in an exponential blow-up in the size of the ontology, thereby giving only a 2-EXPTime upper bound for the complexity of reasoning in these fuzzy DLs. The above results improve these bounds for the sublogics $G_n$-$SCHI$ and $Ł_n$-$SCHI$ to match the complexity of crisp $SHI$. Correspondingly, we obtain PSPACE-completeness for the sublogics $G_n$-$ALCHI$, $G_n$-$SCI_c$, $Ł_n$-$ALCHI$, and $Ł_n$-$SCI_c$ with acyclic TBoxes.
4 Decidable Fuzzy Description Logics over the Standard Interval

We now consider reasoning in fuzzy DLs over the standard interval $[0, 1]$, mostly w.r.t. witnessed interpretations. We start our investigation by providing tight complexity results for some logics in which consistency is decidable. In Chapter 5, we will take a look at undecidable fuzzy DLs over $[0, 1]$.

In Section 4.1, we analyze fuzzy DLs using t-norms without zero divisors. We will prove that under this assumption, consistency w.r.t. witnessed models is decidable even for $\boxtimes\mathcal{-suhoi}$ with fuzzy general TBoxes and inequality assertions. We will show in Chapter 5 that adding equality assertions or the involutive negation makes this problem undecidable, except if $\boxtimes$ is the Gödel t-norm. Likewise, consistency is undecidable in $\boxtimes\mathcal{-suhoi}$ if $\boxtimes$ has zero divisors, i.e. it starts with $\bot$ (see Theorem 5.11). Thus, we cannot extend the considered logic in any direction (within the limits of Chapter 2) without losing decidability. Even more, we can show decidability of consistency for arbitrary complete residuated De Morgan lattices $L$ without zero divisors, and the complexity we obtain is the same as for classical $SHOIQ$. Indeed, we will show that our consistency problem can be reduced to consistency in classical $SHOIQ (2\boxtimes\mathcal{-suhoi})$ in linear time, and is thus ExpTime-complete.

In Section 4.2, we consider the particular case of the Gödel t-norm. We show that in this case decidability is preserved even in the presence of equality assertions and the involutive negation constructor. Surprisingly, consistency in $G\mathcal{-alcc}$ cannot be decided over the class of finitely valued models, in contrast to the previous decidability results. This faculty to enforce infinitely valued models is essential for the undecidability proofs in Chapter 5. However, for the inexpressive Gödel t-norm, we can show decidability of consistency using techniques similar to those of Chapter 3. The main insight is that to construct a model it suffices to consider the order between membership degrees for all relevant concepts, instead of their precise values.

Finally, we discuss in Section 4.3 how the presented constructions apply to reasoning problems other than consistency.

4.1 Consistency without Zero Divisors

Let $L$ be a complete residuated De Morgan lattice without zero divisors and consider the logic $L\boxtimes\mathcal{-suhoi}$ with fuzzy general TBoxes and inequality assertions under witnessed model semantics. Our reduction to crisp reasoning is based on the monotone function $2: L \to 2$ that maps fuzzy truth degrees to crisp truth degrees and is compatible with all lattice operations relevant for $L\boxtimes\mathcal{-suhoi}$. We define, for all $x \in L$,

$$2(x) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$
Thus, by Proposition 2.9 we have that \(2(x) = \theta \otimes x\) for all \(x \in L\).

**Lemma 4.1** For all \(x, y \in L\) and all non-empty sets \(X \subseteq L\), it holds that

- \(\forall x, y \in L:\ \theta (x \otimes y) = \theta (x) \otimes \theta (y)\),
- \(\forall x, y \in L:\ \theta (x \oplus y) = \theta (x) \oplus \theta (y)\),
- \(\forall x \in L, y \in L:\ 2(x \Rightarrow y) = 2(x) \Rightarrow 2(y)\),
- \(\forall x \in L:\ \theta (\bigvee_{x \in X} x) = \bigvee_{x \in X} 2(x)\), and
- \(\exists x \in X: x > 0\Rightarrow x \otimes x = 0\).

**Proof.** Since \(\otimes\) does not have zero divisors, it holds that \(x \otimes y = 0\) if \(x = 0\) or \(y = 0\). This yields \(2(x \otimes y) = 0\) if \(2(x) = 0\) or \(2(y) = 0\). Because there are no zero divisors, this shows that \(2(x \otimes y) = 0\) if \(2(x) \otimes 2(y) = 0\). Since both \(2(x \otimes y)\) and \(2(x) \otimes 2(y)\) are either 0 or 1, the claim follows.

Since 0 is a unit for \(\oplus\), we have \(x \oplus y = 0\) if \(x = y = 0\), and thus \(2(x \oplus y) = 0\) holds if \(2(x) \oplus 2(y) = 0\), which similarly proves the claim for \(\oplus\).

Furthermore, by Proposition 2.9 we get \(2(x \Rightarrow y) = 0\) if \(x > 0\) and \(y = 0\). This is equivalent to \(2(x) = 1\) and \(2(y) = 0\), i.e. \(2(x) \Rightarrow 2(y) = 0\).

Observe now that \(\bigvee_{x \in X} x = 0\) if \(X = \{0\}\), which yields that \(\bigvee_{x \in X} 2(x) = 0\) if \(2(x) = 0\) for all \(x \in X\), or equivalently \(\bigvee_{x \in X} 2(x) = 0\).

Assume now that \(X\) has a least element \(x_0\). Then we have \(\bigvee_{x \in X} 2(x) = 0\) if \(x_0 = 0\) if \(\bigvee_{x \in X} 2(x) = 0\).

Notice that in general \(2\) is not compatible with the lattice infimum. Consider for example the set \(X = \{\frac{1}{n} | n \in \mathbb{N}\} \subseteq [0, 1]\). Then \(\bigvee X = 0\) and hence \(2(\bigvee X) = 0\), but \(\bigvee \{\frac{1}{n} | n \in \mathbb{N}\} = 1\).

### 4.1.1 Reduction to Crisp Ontologies

The undecidability results of Chapter 5 all rely on the fact that one can design ontologies that allow only witnessed models with infinitely many truth values. We will now use the function \(2\) to show that one cannot construct such an ontology in \(L_{\text{FSUHOL}}\) with fuzzy general TBoxes and inequality assertions if \(L\) has no zero divisors. Even more, all consistent ontologies in this logic have a crisp (and finite) model.

Consider an ontology \(\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})\) over \(L_{\text{FSUHOL}}\), where \(\mathcal{T}\) is a general TBox and \(\mathcal{A}\) contains only inequality assertions. Given a witnessed model \(\mathcal{I}\) of \(\mathcal{O}\), we construct the crisp interpretation \(\mathcal{J}\) over the domain \(\Delta^\mathcal{J} := \Delta^\mathcal{I}\) by defining, for all concept names \(A \in \mathcal{N}_C\), role names \(r \in \mathcal{N}_R\), individual names \(c \in \mathcal{N}_I\), and \(x, y \in \Delta^\mathcal{I}\),

\[
\mathcal{A}^\mathcal{J}(x) := 2(\mathcal{A}^\mathcal{I}(x)), \quad \mathcal{R}^\mathcal{J}(x, y) := 2(\mathcal{R}^\mathcal{I}(x, y)), \quad \text{and} \quad \mathcal{C}^\mathcal{J} := \mathcal{C}^\mathcal{I}.
\]

**Lemma 4.2** If \(\mathcal{I}\) is a witnessed model of \(\mathcal{O}\), then \(\mathcal{J}\) is also a witnessed model of \(\mathcal{O}\).

**Proof.** By Lemma 2.15, since \(\mathcal{J}\) is crisp, it is also witnessed. We now consider the role axioms in \(\mathcal{O}\). Observe first that \(\mathcal{R}^\mathcal{J}(x, y) = 2(\mathcal{R}^\mathcal{I}(x, y))\) also holds for all complex roles \(r\).
Let now \( \text{trans}(r) \in \mathcal{R} \) and consider any \( x, y, z \in \Delta^\mathcal{I} \). By Lemma 4.1, the fact that \( \mathcal{I} \) is a model of this axiom, and monotonicity of \( \Sigma \), we obtain

\[
\begin{align*}
\text{trans}(x, y) \otimes \text{trans}(y, z) &= 2(\text{trans}(x, y)) \otimes 2(\text{trans}(y, z)) \\
&= 2(\text{trans}(x, y) \otimes \text{trans}(y, z)) \\
&\leq 2(\text{trans}(x, z)) \\
&= \text{trans}(x, z).
\end{align*}
\]

Likewise, for any role inclusion \( r \sqsubseteq s \) in \( \mathcal{R} \) and all \( x, y \in \Delta^\mathcal{I} \), we get

\[
\text{trans}(x, y) = 2(\text{trans}(x, y)) \leq 2(\text{trans}(x, y)) = \text{trans}(x, y).
\]

To prove that \( \mathcal{J} \) also satisfies \( \mathcal{A} \) and \( \mathcal{T} \), we first need to show that \( C^{\mathcal{J}}(x) = 2(C^{\mathcal{I}}(x)) \) holds for all concepts \( C \) and \( x \in \Delta^\mathcal{I} \). We do this by induction on the structure of \( C \).

The claim obviously holds for \( \bot \) and \( \top \). For all \( A \in \mathcal{N} \), it follows immediately from the definition of \( \mathcal{J} \). It also holds for nominals \( \{c\} \) with \( c \in \mathcal{N} \), because \( \{c\}^{\mathcal{I}}(x) \) can only take the values \( 0 \) or \( 1 \) for any \( x \in \Delta^\mathcal{I} \).

Assume now that \( C \) and \( D \) satisfy the claim and consider \( C \sqcap D \). By Lemma 4.1, we have, for all \( x \in \Delta^\mathcal{I} \),

\[
\begin{align*}
(C \sqcap D)^{\mathcal{J}}(x) &= C^{\mathcal{J}}(x) \otimes D^{\mathcal{J}}(x) \\
&= 2(C^{\mathcal{I}}(x)) \otimes 2(D^{\mathcal{I}}(x)) \\
&= 2(C^{\mathcal{I}}(x) \otimes D^{\mathcal{I}}(x)) \\
&= 2((C \sqcap D)^{\mathcal{I}}(x)).
\end{align*}
\]

Likewise, the compatibility of \( \Sigma \) with the t-conorm and residuum entails the result for concepts of the forms \( C \sqcup D \) and \( C \rightarrow D \).

For concepts of the form \( \exists r.C \), we similarly obtain from Lemma 4.1 that, for all \( x \in \Delta^\mathcal{I} \),

\[
\begin{align*}
2(\exists r.C)^{\mathcal{I}}(x) &= 2(\bigvee_{y \in \Delta^\mathcal{I}} \text{trans}(x, y) \otimes C^{\mathcal{I}}(y)) \\
&= \bigvee_{y \in \Delta^\mathcal{I}} 2(\text{trans}(x, y)) \otimes 2(C^{\mathcal{I}}(y)) \\
&= \bigvee_{y \in \Delta^\mathcal{I}} \text{trans}(x, y) \otimes C^{\mathcal{I}}(y) \\
&= (\exists r.C)^{\mathcal{J}}(x).
\end{align*}
\]

For \( \forall r.C \), we have

\[
\begin{align*}
2(\forall r.C)^{\mathcal{I}}(x) &= 2(\bigwedge_{y \in \Delta^\mathcal{I}} \text{trans}(x, y) \Rightarrow C^{\mathcal{I}}(y)) \\
&= 2(\bigwedge_{y \in \Delta^\mathcal{I}} \text{trans}(x, y) \Rightarrow C^{\mathcal{I}}(y)).
\end{align*}
\]

Since \( \mathcal{I} \) is witnessed, there must be some \( y_0 \in \Delta^\mathcal{I} \) such that

\[
\text{trans}(x, y_0) \Rightarrow C^{\mathcal{I}}(y_0) = \bigwedge_{y \in \Delta^\mathcal{I}} \text{trans}(x, y) \Rightarrow C^{\mathcal{I}}(y);\]

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that is, the set \( \{ r^I(x, y) \Rightarrow C^I(y) \mid y \in \Delta^I \} \) has a least element. Thus, as in the case for \( \exists r.C \), we can apply Lemma 4.1 to derive that \( 2((\forall r.C)^I(x)) = (\forall r.C)^I(x) \).

This concludes the proof of the claim. We can now show that \( J \) satisfies all axioms in \( A \) and \( T \). Observe first that axioms with value \( p = 0 \) are trivially satisfied. Let now \( \langle c : C \geq p \rangle \) be a concept assertion in \( A \) with \( p > 0 \). Since it is satisfied by \( I \), we have \( C^I(c) \geq p > 0 \). The above claim yields that \( C^J(c) = 1 \geq p \). The same argument can be used for role assertions. Let now \( \langle C \sqsubseteq D \geq p \rangle \) be a GCI in \( T \) with \( p > 0 \) and consider any \( x \in \Delta^I \). As the GCI is satisfied by \( I \), we have \( C^I(x) \Rightarrow D^I(x) \geq p > 0 \).

By Lemma 4.1 and the claim above, we obtain
\[
C^J(x) \Rightarrow D^J(x) = 2(C^I(x)) \Rightarrow 2(D^I(x)) = 2(C^I(x) \Rightarrow D^I(x)) = 1 \geq p,
\]
and thus \( J \) satisfies the GCI.

We now come to the anticipated reduction from \( L\text{-}\text{SUHOL} \)-ontologies to \( 2\text{-}\text{SUHOL} \). We construct from \( O \) a crisp ontology \( \text{crisp}(O) := (A', T', R) \) by replacing all truth degrees appearing in the axioms according to 2:
\[
A' := \{ \langle \alpha \geq 2(p) \rangle \mid \langle \alpha \geq p \rangle \in A \} \quad \text{and}
T' := \{ \langle C \sqsubseteq D \geq 2(p) \rangle \mid \langle C \sqsubseteq D \geq p \rangle \in T \}.
\]

**Lemma 4.3** \( O \) is consistent in \( L\text{-}\text{SUHOL} \) iff \( \text{crisp}(O) \) is consistent in \( 2\text{-}\text{SUHOL} \).

**Proof.** Assume first that \( J \) is a (crisp) model of \( \text{crisp}(O) \) and consider \( \langle C \sqsubseteq D \geq p \rangle \in T \). By assumption, \( C^J(x) \Rightarrow D^J(x) \geq 1 \geq p \) holds for all \( x \in \Delta^J \), and thus \( J \) satisfies \( \langle C \sqsubseteq D \geq p \rangle \). The proof for the assertions in \( A \) is analogous. Thus, \( J \) is also a model of \( O \).

Conversely, assume that \( I \) is a witnessed model of \( O \). By Lemma 4.2, the corresponding crisp interpretation \( J \) also satisfies \( O \). Consider any GCI \( \langle C \sqsubseteq D \geq p \rangle \) from \( T \). If \( p = 0 \), then \( 2(p) \) is also 0 and the GCI is trivially satisfied by \( J \). If \( p > 0 \), we infer \( C^J(x) \Rightarrow D^J(x) = 1 = 2(p) \) for all \( x \in \Delta^J \) since \( J \) is crisp. Thus, \( J \) satisfies the crisp GCI \( \langle C \sqsubseteq D \geq 2(p) \rangle \). A similar argument can be used for the assertions.

We can now use reasoning algorithms for classical \( \text{SHOL} \) to decide consistency of \( L\text{-}\text{SUHOL} \)-ontologies. Consistency in classical \( \text{ALC} \) and \( \text{SHOL} \) is known to be \text{ExpTime}-complete (Hladik 2007; Schild 1991).

**Theorem 4.4** Let \( L \) be a complete residuated De Morgan lattice without zero divisors. Then consistency w.r.t. witnessed models in \( L\text{-}\text{SUHOL} \) with fuzzy general TBoxes and inequality assertions is decidable in \text{ExpTime}. It is \text{ExpTime}-hard already in \( 2\text{-}\text{ALC} \).

Of course, the above reductions also work for the sublogics \( L\text{-}\text{SUHO} \) and \( L\text{-}\text{SUZ} \). Since the classical DLs \( \text{SHO} \) and \( \text{SZ} \) have the finite model property, i.e. every consistent ontology has a finite model (Horrocks, Sattler, and Tobies 1998; Lutz, Areces, Horrocks, and Sattler 2005), and every crisp model of \( \text{crisp}(O) \) is also a model of \( O \), this also holds for \( L\text{-}\text{SUHO} \) and \( L\text{-}\text{SUZ} \). This contradicts the result from Theorem 3.8 in (Bobillo, Bou, and Straccia 2011) that \( \Pi\text{-}\text{ALC} \) with fuzzy general TBoxes and inequality assertions does not have the finite model property. Indeed, the proof from Bobillo, Bou, and Straccia (2011) is based on the erroneous claim that every model \( I \) of \( \langle c: A \geq 0.5 \rangle \) must
be such that $A^T(c^T) = 0.5$. The case of an interpretation with $A^T(c^T) = 1$, which also satisfies this assertion, is overlooked in the proof.

Finally, note that a concept definition $\langle A \equiv C \geq p \rangle$ (with $p > 0$) is equivalent to the GCIs $\langle A \sqsubseteq C \geq p \rangle$ and $\langle C \sqsubseteq A \geq p \rangle$, which would be translated by the above construction to $\langle A \sqsubseteq C \geq 1 \rangle$ and $\langle C \sqsubseteq A \geq 1 \rangle$, which in turn are equivalent to $\langle A \equiv C \geq 1 \rangle$. Thus, it is easy to see that the translation from $O$ to $\text{crisp}(O)$ also works for acyclic TBoxes since the acyclicity condition is maintained. This means that we can lift the PSPACE-completeness results for sublogics of $\text{SHOI}$ with acyclic TBoxes to the fuzzy setting. These logics include $\text{ALCHI}$ (see Section 3.1.4), $\text{SI}$ (Baader, Hladik, and Peñaloza 2008; Horrocks, Sattler, and Tobies 2000), $\text{ALCHO}$, and $\text{SO}$ (see Appendix A). On the other hand, in $\text{ALCOI}$ and $\text{SH}$ consistency w.r.t. acyclic TBoxes is already EXPTIME-hard (Horrocks 1997; Tobies 2000). This situation is summarized in Figure 4.1.

**Theorem 4.5** Let $L$ be a complete residuated De Morgan lattice without zero divisors. Then consistency w.r.t. witnessed models in $L\text{-\text{ALUHI}}$, $L\text{-\text{SU}}$, $L\text{-\text{ALUHO}}$, and $L\text{-\text{SUO}}$ with acyclic TBoxes and inequality assertions is decidable in PSPACE. It is PSPACE-hard already in $2\text{-\text{REL}}$.

If we want to lift the restriction to witnessed models, it is straightforward to apply the above construction to the logic $L\text{-\text{SUHOI}}^{-\forall}$, which disallows value restrictions. Indeed,
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the properties of witnessed interpretations were only needed in the proof of Lemma 4.2 to ensure that a certain least element always exists. Thus, consistency w.r.t. general models is \text{ExpTime}-complete in \(L\mathcal{UHOT}^\forall\) with fuzzy general TBoxes and inequality assertions, and \text{PSPACE}-complete in the relevant sublogics (see (Borgwardt, Distel, and Peñaloza 2012a) for details).

4.2 Consistency under the Gödel t-norm

The simplest of the three fundamental continuous t-norms is the Gödel t-norm, and consistency in fuzzy DLs of the form \(\text{G-}\mathcal{L}\) is widely believed to be decidable, although in the literature strangely no proof of this can be found. The previous section yields decidability of consistency only in case no equality assertions or involutive negation are allowed. The only known results (Bobillo et al. 2009, 2012; Bobillo and Straccia 2013b) for similar fuzzy DLs with equality assertions and involutive negation restrict reasoning a priori to a finite total order, in which case the results of Chapter 3 yield tight complexity bounds. This restriction is sometimes justified by the “limited precision of computers” (Bobillo et al. 2009). In (Bobillo, Delgado, and Gómez-Romero 2009), the authors consider the Gödel residuum for the semantics of GCIs in \(\text{Z-SCROT}\); however, the proof of correctness of the presented reduction to classical \(\text{SROT}\) uses the premature assumption that reasoning can be restricted to finitely valued models.

We will show that, somewhat surprisingly, reasoning in \(\text{G-}\mathcal{ALC}\) with fuzzy general TBoxes and equality assertions cannot be restricted to finitely valued models without loss of generality, in contrast to results for fuzzy DLs under the related Zadeh semantics (Bobillo, Delgado, and Gómez-Romero 2008; Straccia 2004a). This holds even if we allow only witnessed models. However, we can show decidability using techniques similar to those of Chapter 3. This provides the only known fuzzy DL for which consistency is decidable despite not being able to restrict to finitely many values.

4.2.1 Models Need Infinitely Many Values

We now show that, even for the simple fuzzy DLs \(\text{G-}\mathcal{ALC}\) and \(\text{G-}\mathcal{EL}\) with crisp general TBoxes and equality assertions, the problem of deciding consistency cannot be restricted to finitely valued models without loss of generality.

Example 4.6 Consider the fuzzy DL \(\text{G-}\mathcal{ALC}\) and the ontology \(\mathcal{O}_1 := (A_1, \mathcal{T}_1, \emptyset)\), where

\[
A_1 := \{(c: A = 0.5)\} \quad \text{and} \\
\mathcal{T}_1 := \{\forall r. A \sqsubseteq A, \exists r. \top \sqsubseteq A\}.
\]

We show that \(\mathcal{O}_1\) is consistent, and indeed has a witnessed model, but has no finitely valued model.

For the former, we construct a witnessed model \(I_1\) of \(\mathcal{O}_1\) as follows (see Figure 4.2). We define \(\Delta I_1 := \mathbb{N}_+\) to be the set of all positive natural numbers. Furthermore, we set \(A^I_1(n) := r^I_1(n, n + 1) := \frac{1}{n+1}\) for all \(n \in \mathbb{N}_+\) and \(r^I_1(n, m) := 0\) if \(m \neq n + 1\). It is straightforward to check that this is indeed a model of \(\mathcal{O}_1\). It is witnessed since it is finitely branching (see Lemma 2.15).
As in the previous example, we show that this ontology has a witnessed model, but no contradiction to the minimality of Example 4.7.

Consider the ontology $\mathcal{A}_3 := (A_3, \mathcal{E}_3, \emptyset)$. Since $\mathcal{I}$ uses only finitely many truth values, there is element $x_0 \in \Delta^\mathcal{I}$ for which $A^\mathcal{I}(x_0)$ is minimal, i.e. we have $A^\mathcal{I}(x_0) \leq A^\mathcal{I}(x)$ all $x \in \Delta^\mathcal{I}$. Since $\mathcal{I}$ satisfies $\mathcal{A}_1$, we in particular have $A^\mathcal{I}(x_0) < 1$. The first axiom of $\mathcal{I}_1$ entails

$$\inf_{x \in \Delta^\mathcal{I}} r^\mathcal{I}(x, x) \Rightarrow A^\mathcal{T}(x) \leq A^\mathcal{I}(x_0) < 1,$$

and thus there must be an element $y \in \Delta^\mathcal{I}$ such that $r^\mathcal{I}(x_0, y) \Rightarrow A^\mathcal{I}(y) < 1$, which means that $A^\mathcal{I}(y) < r^\mathcal{I}(x_0, y)$. The second axiom from $\mathcal{I}_1$ now yields

$$A^\mathcal{I}(y) < r^\mathcal{I}(x_0, y) = \min(r^\mathcal{I}(x_0, y), 1) \leq (\exists y) r^\mathcal{I}(x_0) \leq A^\mathcal{I}(x_0),$$

in contradiction to the minimality of $A^\mathcal{I}(x_0)$. ♦

A similar example shows that the restriction to finitely valued models is also not without loss of generality for $\mathcal{G}$-$\mathcal{FL}$.}

**Example 4.7** Consider the ontology $\mathcal{O}_2 := (\mathcal{A}_2, \mathcal{E}_2, \emptyset)$, where

$$\mathcal{A}_2 := \{ (c:A = 0.5) \} \quad \text{and} \quad \mathcal{E}_2 := \{ (B \subseteq A), \ (A \rightarrow B \subseteq B), \ (\top \subseteq \exists y) (\exists y) (\exists r.A \subseteq B) \}.$$ 

As in the previous example, we show that this ontology has a witnessed model, but no finitely valued one.

A witnessed model $\mathcal{I}_2$ of $\mathcal{E}_2$ can be built as follows (see Figure 4.3). Let $\Delta^\mathcal{I}_2$ be the set $\mathbb{N}_+$ of all positive natural numbers, and define $A^\mathcal{I}_2(n) := \frac{1}{n + 1}$, $B^\mathcal{I}_2(n) := \frac{1}{n + 2}$, and $r^\mathcal{I}_2(n, n + 1) := 1$ for all $n \in \mathbb{N}_+$ and $r^\mathcal{I}_2(n, m) := 0$ if $m \neq n + 1$. It is straightforward to check that this is indeed a witnessed model of $\mathcal{O}_2$.

Assume now that there is a finitely valued model $\mathcal{I}$ of $\mathcal{O}_2$. Let $x_0 \in \Delta^\mathcal{I}$ be such that $A^\mathcal{I}(x_0)$ is minimal. As in the previous example, we know that $A^\mathcal{I}(x_0) < 1$ since $\mathcal{I}$ satisfies $\mathcal{A}_2$. From the first axiom of $\mathcal{E}_2$, we obtain $B^\mathcal{I}(x_0) \leq A^\mathcal{I}(x_0) < 1$. The second axiom yields $A^\mathcal{I}(x_0) \Rightarrow B^\mathcal{I}(x_0) \leq B^\mathcal{I}(x_0) < 1$, and therefore $B^\mathcal{I}(x_0) < A^\mathcal{I}(x_0)$. By the third axiom of $\mathcal{E}_2$, we have

$$\sup_{x \in \Delta^\mathcal{I}} r^\mathcal{I}(x_0, x) = 1,$$

and thus there must be a $y \in \Delta^\mathcal{I}$ such that $r^\mathcal{I}(x_0, y) > A^\mathcal{I}(x_0)$. Finally, we obtain from $(\exists y) r.A \subseteq B$ that

$$\min(r^\mathcal{I}(x_0, y), A^\mathcal{I}(y)) \leq \sup_{x \in \Delta^\mathcal{I}} \min\{r^\mathcal{I}(x_0, x), A^\mathcal{I}(x)\} \leq B^\mathcal{I}(x_0) < A^\mathcal{I}(x_0).$$

Since $r^\mathcal{I}(x_0, y) > A^\mathcal{I}(x_0)$, this implies that $A^\mathcal{I}(y) < A^\mathcal{I}(x_0)$, which contradicts the minimality of $A^\mathcal{I}(x_0)$. ♦
Since every finite model is also finitely valued, these examples also show that equality assertions destroy the finite model property (recall the discussion after Theorem 4.4). This indicates that some of the standard techniques used for reasoning in fuzzy DLs cannot be directly applied to any logic that contains $\mathcal{G}$-ALC or $\mathcal{G}$-IEL. Indeed, for most known algorithms to work, one must either

- restrict the semantics to a finite set of truth degrees (see (Bobillo et al. 2009, 2012; Bobillo and Straccia 2011, 2013b; Straccia 2006) and Chapter 3),
- prove that reasoning can be restricted to a finite set of degrees (see (Bobillo, Delgado, and Gómez-Romero 2008; Straccia 2001) and Section 4.1), or
- prove that models can be built from a finite pattern, e.g. an ABox completed by tableaux rules that can be unraveled into a model (see (Stoilos, Stamou, Pan, et al. 2007; Straccia and Bobillo 2007)).

On the other hand, the undecidability proofs in (Baader and Peñaloza 2011a,b; Cerami and Straccia 2013) and Chapter 5 all rely on the fact that one can force models to have infinitely many values. One could thus be inclined to believe that consistency in $\mathcal{G}$-IELC with fuzzy general TBoxes and equality assertions is also undecidable. In the following sections, we show that this is not the case, and indeed consistency is ExpTime-complete. We focus here on witnessed model semantics. In Section 4.3, we will extend this result to the problems of deciding satisfiability and subsumption under witnessed models.

**4.2.2 Order Structures and Ordered ABoxes**

The following constructions are similar to the ones from Chapter 3. In particular, we will start by deciding local consistency using an automata-based approach, and then extend this to consistency by a modified version of pre-completion. However, the basic data structure of these algorithms (the Hintikka trees) are quite different from those of Chapter 3. Although in principle we could adapt the techniques of Chapter 3 for $\mathcal{G}$-IELC, and even show PSPACE-completeness results for the case of acyclic TBoxes, we aim to illustrate the main idea behind the constructions on the smaller logic $\mathcal{G}$-IALC without role axioms and inverse roles, and consider only fuzzy general TBoxes.

The key observation is that the axioms and the semantics of the constructors of $\mathcal{G}$-IALC only introduce restrictions on the order of the values that models can assign to concepts, not on the values themselves. For example, an interpretation $\mathcal{I}$ satisfies the assertion $\langle c : (A \to B) = p \rangle$ with $p < 1$ if $A^\mathcal{I}(c^\mathcal{I}) > B^\mathcal{I}(c^\mathcal{I})$ and $B^\mathcal{I}(c^\mathcal{I}) = p$. Thus, rather than building a model directly, we create an abstract representation of a model that encodes for each domain element only the order between the membership degrees for all relevant concepts.

Figure 4.3: The model $I_2$ from Example 4.7
We now formalize this approach using several auxiliary notions.

Consider again the TBox \( \mathcal{T}_1 = \{ (\forall r. A \sqsubseteq A), (\exists r. \top \sqsubseteq A) \} \) from Example 4.6. When trying to construct a model satisfying \( (c: A = 0.5) \), we start with a domain element satisfying the restriction that the value of \( A \) is equal to 0.5 (see Figure 4.4).

The second axiom of \( \mathcal{T}_2 \) implies that the degree of any outgoing \( r \)-connection is bounded by the value of \( A \). Moreover, the first axiom states that the witness of \( \forall r. A \) must satisfy \( A \) to a degree strictly smaller than the value of the \( r \)-connection, and thus strictly smaller than the value of \( A \) at the original element.

This yields an abstract description of two domain elements in terms of order relations between values of concepts at the current node and the parent node (denoted by a subscript \( \uparrow \)). Applying the same argument to the new element yields another element with the same restrictions. However, in order for this construction to yield a model, it is easy to see that the value of \( A \) at all considered elements has to be strictly greater than 0—once the value of \( A \) is 0, there can be no successors with smaller values for \( A \).

Note that it suffices to consider order relations between concepts of neighboring elements, which are directly connected by some role to a degree greater than 0. \( \Diamond \)

We now formalize this approach using several auxiliary notions.

A total preorder over a set \( S \) is a transitive and total binary relation \( \preceq_s \subseteq S \times S \).

For \( x, y \in S \), we write \( x \equiv_s y \) if \( x \preceq_s y \) and \( y \preceq_s x \). Note that \( \equiv_s \) is an equivalence relation on \( S \). Similarly, we write \( x <_s y \) if \( x \preceq_s y \) but not \( y \preceq_s x \). By the symbol \( \bowtie \) we denote an arbitrary element of \( \{ =, \geq, >, \leq, < \} \), and by \( \bowtie_s \) the corresponding relation induced by the total preorder \( \preceq_s \), i.e. \( \equiv_s, \geq_s, >_s, \leq_s, \bowtie_s \). We will use subscripts to distinguish different total preorders over the same carrier set \( S \).

**Definition 4.9 (order structure)** An order structure is a finite set \( S \) containing at least the real numbers \( 0, 0.5, 1 \), together with an involutive unary operation \( \text{inv}_S \) \( S \rightarrow S \) such that \( \text{inv}_S(x) = 1 - x \) for all \( x \in S \cap [0, 1] \). Given an order structure \( S \), \( \text{order}(S) \) is the set of all total preorders \( \preceq_s \) over \( S \) such that

- for all \( x \in S \), we have \( 0 \preceq_s x \preceq_s 1 \);  
- for all \( x, y \in S \cap [0, 1] \), we have \( x \preceq_s y \) if \( x \leq y \); and  
- for all \( x, y \in S \), we have \( x \preceq_s y \) if \( \text{inv}_S(y) \preceq_s \text{inv}_S(x) \).

This means that the elements of \( \text{order}(S) \) are those total preorders over \( S \) that are compatible with the order of the real numbers in \( [0, 1] \) and with the involution \( \text{inv}_S \).

Given \( \preceq_s \in \text{order}(S) \), the following functions on \( S \) that mimic the Gödel t-norm and residuum are well-defined since \( \preceq_s \) is total:

\[
\text{min}_s(x, y) := \begin{cases} 
  x & \text{if } x \preceq_s y, \\
  y & \text{otherwise},
\end{cases} \quad \text{res}_s(x, y) := \begin{cases} 
  1 & \text{if } x \preceq_s y, \\
  y & \text{otherwise}.
\end{cases}
\]

Figure 4.4: An abstract description of \( \mathcal{T}_1 \) from Example 4.6
It is easy to see that these operators agree with min and \( \Rightarrow \) on the set \( S \cap [0,1] \).

We will consider in the following a more expressive form of ABoxes that generalize those introduced in Section 2.2.3 by allowing to express arbitrary order relations between concepts.

**Definition 4.10 (ordered ABox)** An order assertion is an expression of the form \( \langle \alpha \triangleright\triangleleft \beta \rangle \), where \( \alpha \) is of the form \( c:C \) or \( (c,d):r \) for a concept \( C \), \( r \in \mathbb{N}_R \), and \( c,d \in \mathbb{N}_1 \), and \( \beta \) is either also of this form or a value from \([0,1]\). An interpretation \( I \) satisfies an order assertion \( \langle \alpha \triangleright\triangleleft \beta \rangle \) if \( \alpha^I \triangleright\triangleleft \beta^I \), where \( (c:C)^I := C^I(c^I) \), \( ((c,d):r)^I := r^I(c^I,d^I) \), and \( p^I := p \) for all \( p \in [0,1] \).

An ordered ABox is a finite set of order assertions. An interpretation is a model of an ordered ABox \( A \) if it satisfies all order assertions in \( A \).

An ordered ABox is called local if it contains no role assertions and only one individual name appears in it. In the following, we consider ontologies \( O = (A,T) \), where \( A \) is a (local) ordered ABox, and \( T \) is a fuzzy general TBox. It is clear that all decidability results also apply to ordinary ABoxes. The set \( \text{sub}(O) \) is defined as in Definition 3.1 as the set of all concepts occurring in \( O \), but we now need to consider the closure of this set under the involutive negation constructor.

For ease of presentation, in the following the expressions \( \neg\neg C \) and \( C \) are regarded as the same concept. The negation closure of \( \text{sub}(O) \) is the set

\[
\text{cl}(O) := \{C, \neg C \mid C \in \text{sub}(O)\}.
\]

We further denote by \( \mathcal{V}_O \) the set

\[
\{0, 0.5, 1\} \cup \{1-p \mid p \in [0,1]\} \text{ occurs in } O\}.
\]

Since \( \text{sub}(O) \) is finite, its closure is also finite. Similarly, since only finitely many values can occur in \( O \) and the involutive negation is involutive, \( \mathcal{V}_O \) contains finitely many values.

In the following, we often denote the elements of \( \mathcal{V}_O \) as \( 0 = p_0 < p_1 < \cdots < p_n = 1 \).

### 4.2.3 Local Consistency

In this section, we consider only the special case where the ontology \( O = (A,T) \) is such that \( A \) is a local ordered ABox which uses only the individual name \( c \). We construct Hintikka trees similar to those of Section 3.1. In Section 4.2.4, we extend the approach to handle arbitrary ontologies.

To formally represent the order relationships between all relevant concepts, we consider the order structure

\[
U := \mathcal{V}_O \cup \text{cl}(O) \cup \text{cl}_I(O) \cup \{\varrho, \neg\varrho\},
\]

where we define \( \text{cl}_I(O) := \{C_\uparrow \mid C \in \text{cl}(O)\} \), \( \text{inv}_U(\varrho) := \neg\varrho \), \( \text{inv}_U(C) := \neg C \), and \( \text{inv}_U(C_\uparrow) := \neg C_\uparrow \) for all \( C \in \text{cl}(O) \).

The idea is that total preorders from \( \text{order}(U) \) describe the relationships between all the subconcepts from \( O \) and the truth degrees from \( \mathcal{V}_O \) at given domain elements. One can think of such a preorder as the type of a domain element, from which a tree-shaped interpretation can be built. As illustrated in Example 4.8, in order to handle the semantics of the existential and value restrictions, we also need to know the type of the
parent node in the tree, as well as the degree of the role relation connecting them. For that reason, we introduce \( cl_\tau(O) \) and \( \varrho \), respectively.

As in Section 3.1, we consider the number \( k \) of quantified concepts in \( \text{sub}(O) \) and an arbitrary but fixed bijection \( \varphi \) between the set of all quantified concepts in \( \text{sub}(O) \) and \( K := \{1, \ldots, k\} \) that specifies which quantified concept is witnessed by which successor in the Hintikka tree. As before, \( \varphi_r(O) \) contains those indices corresponding to existential or value restrictions using the role name \( r \). Our algorithm will try to decide the existence of a \( k \)-ary infinite tree whose nodes are labeled with a preorder from \( \text{order}(U) \), such that the semantics of the constructors and all the axioms in \( O \) are preserved.

**Definition 4.11 (Hintikka ordering)** An element \( \lessdot_H \in \text{order}(U) \) is called a Hintikka ordering for \( O \) if it satisfies the following conditions for every \( C \in \text{cl}(O) \):

- \( C = \top \) implies \( C \equiv_H 1 \),
- \( C = \bot \) implies \( C \equiv_H 0 \),
- \( C = D_1 \cap D_2 \) implies \( C \equiv_H \min_H(D_1, D_2) \), and
- \( C = D_1 \rightarrow D_2 \) implies \( C \equiv_H \res_H(D_1, D_2) \).

This preorder is compatible with the TBox \( T \) if for every GCI \( \langle C \sqsubseteq D \geq p \rangle \in T \) we have \( \res_H(C, D) \geq_H p \). It is compatible with the ABox \( A \) if for every assertion \( \langle c : C \bowtie p \rangle \) or \( \langle c : C \bowtie c : D \rangle \) in \( A \), we have \( C \bowtie_H p \) or \( C \bowtie_H D \), respectively.

The conditions imposed on Hintikka orderings ensure that they preserve the semantics of all the propositional constructors. For every quantified concept, we still need to ensure the existence of a successor that serves as its witness. As in Section 3.1, this is achieved by the bijection \( \varphi \) and the Hintikka condition.

**Definition 4.12 (ordered Hintikka condition)** A tuple \( (\lessdot_0, \lessdot_1, \ldots, \lessdot_k) \) of \( k + 1 \) Hintikka orderings for \( O \) satisfies the ordered Hintikka condition if:

- for every \( 1 \leq i \leq k \) and all \( \alpha, \beta \in \text{V}_O \cup \text{cl}(O) \), we have \( \alpha \lessdot_0 \beta \) iff \( \alpha \lessdot_i \beta \), where we set \( p_i := p \) for all \( p \in \text{V}_O \);

- for every existential restriction \( \exists r.D \in \text{sub}(O) \), we have
  - \( (\exists r.D) \equiv_i \min_i(\varrho, D) \) for \( i = \varphi(\exists r.D) \), and
  - \( (\exists r.D) \equiv_i \res_i(\varrho, D) \) for all \( i \in \varphi_r(O) \); and

- for every value restriction \( \forall r.D \in \text{sub}(O) \), we have
  - \( (\forall r.D) \equiv_i \res_i(\varrho, D) \) for \( i = \varphi(\forall r.D) \), and
  - \( (\forall r.D) \equiv_i \res_i(\varrho, D) \) for all \( i \in \varphi_r(O) \).

We now combine Hintikka orderings using the ordered Hintikka condition into an ordered Hintikka tree.

**Definition 4.13 (ordered Hintikka tree)** An ordered Hintikka tree for \( O \) is a mapping \( \lessdot_* \), assigning to every node \( u \in K^* \) a Hintikka ordering \( \lessdot_u \) for \( O \) such that

- \( \lessdot_* \) is compatible with \( A \),
- for every \( u \in K^* \), \( \lessdot_u \) is compatible with \( T \), and
Figure 4.5: An ordered Hintikka tree for Example 4.6

- for every \( u \in K^* \), \((\preceq_u, \preceq_{u1}, \ldots, \preceq_{un})\) satisfies the ordered Hintikka condition. \( \diamond \)

Figure 4.5 shows an ordered Hintikka tree for the ontology \( \mathcal{O}_1 \) from Example 4.6. This tree is invariant w.r.t. the choice of \( \varphi \) and every node below depth 1 is assigned the same Hintikka ordering \( \preceq_{11} \). We now show that the existence of a Hintikka tree for an ontology \( \mathcal{O} \) characterizes the local consistency of \( \mathcal{O} \) (cf. Lemma 3.6).

**Lemma 4.14** \( \mathcal{O} \) has a witnessed model if and only if there is an ordered Hintikka tree for \( \mathcal{O} \).

**Proof.** Given an ordered Hintikka tree \( \preceq \), we construct a witnessed model of \( \mathcal{O} \) in two steps. In the first step, we recursively define a function \( v : \mathcal{U} \times K^* \to [0, 1] \) satisfying the following conditions for all nodes \( u \in K^* \): 

(P1) for all values \( p \in \mathcal{V}_\mathcal{O} \) we have \( v(p, u) = p \),

(P2) for all \( \alpha, \beta \in \mathcal{U} \) we have \( v(\alpha, u) \leq v(\beta, u) \) iff \( \alpha \preceq_{u} \beta \),

(P3) for all \( \alpha \in \mathcal{U} \) we have \( v(\mathit{inv}_\mathcal{U}(\alpha), u) = 1 - v(\alpha, u) \), and

(P4) for all \( C \in \mathcal{C}(\mathcal{O}) \) and all \( i \in K \) we have \( v(C, u) = v(C_i, u_i) \).

In the second step, we construct an interpretation \( \mathcal{I}_v \) over the domain \( K^* \) that satisfies \( C^{\mathcal{I}_v}(u) = v(C, u) \) for all \( C \in \mathcal{C}(\mathcal{O}) \) and all nodes \( u \), and show that \( \mathcal{I}_v \) is indeed a witnessed model of \( \mathcal{O} \).

The function \( v \) is defined recursively, starting from the root node \( \varepsilon \). Let \( U/\equiv_\varepsilon \) be the set of all equivalence classes of \( \equiv_\varepsilon \). Then \( \preceq_\varepsilon \) yields a total order \( \leq_\varepsilon \) on \( U/\equiv_\varepsilon \). In particular, since \( \preceq_\varepsilon \) preserves the order of real numbers on \( \mathcal{V}_\mathcal{O} \), we have

\[
[0]_\varepsilon \prec_\varepsilon [p_1]_\varepsilon \prec_\varepsilon [p_2]_\varepsilon \prec_\varepsilon \cdots \prec_\varepsilon [p_n-1]_\varepsilon \prec_\varepsilon [1]_\varepsilon .
\]

For an equivalence class \( [\alpha]_\varepsilon \), we set \( \mathit{inv}_\mathcal{U}([\alpha]_\varepsilon) := [\mathit{inv}_\mathcal{U}(\alpha)]_\varepsilon \), which is well-defined since \( \preceq_\varepsilon \) is an element of \( \text{order}(\mathcal{U}) \).

We first define an auxiliary function \( \tilde{v}_\varepsilon : \mathcal{U}/\equiv_\varepsilon \to [0, 1] \). For all \( p \in \mathcal{V}_\mathcal{O} \), we define \( \tilde{v}_\varepsilon([p]_\varepsilon) := p \). It remains to define a value for all equivalence classes that do not contain
a value from \( \mathcal{V}_O \). Notice that, because of the minimality of \([0]_\epsilon\) and maximality of \([1]_\epsilon\), every such class must be strictly between \([q_i]_\epsilon\) and \([q_{i+1}]_\epsilon\) for two adjacent truth degrees \(q_i\) and \(q_{i+1}\). For every \(i \in \{0, \ldots, n - 1\}\), let \(\nu_i\) be the number of equivalence classes that are strictly between \([p_i]_\epsilon\) and \([p_{i+1}]_\epsilon\). We assume that these classes are denoted by \(E_j^i\) such that
\[
[p_i]_\epsilon <_\epsilon E_1^i <_\epsilon E_2^i <_\epsilon \cdots <_\epsilon E_{\nu_i}^i <_\epsilon [p_{i+1}]_\epsilon.
\]
We then define values \(s_j^i\), \(1 \leq j \leq \nu_i\), such that \(p_i < s_1^i < s_2^i < \cdots < s_{\nu_i}^i < p_{i+1}\) by setting \(s_j^i := p_i + \frac{j - 1}{\nu_i - 1}(p_{i+1} - p_i)\) and define \(\bar{v}_\epsilon(E_j^i) := s_j^i\) for every \(j\), \(1 \leq j \leq \nu_i\). Finally, we define \(v(\alpha, \epsilon) := \bar{v}_\epsilon([\alpha]_\epsilon)\) for all \(\alpha \in \mathcal{U}\). This construction ensures that (P1) and (P2) hold at the node \(\epsilon\). To see that (P3) is also satisfied, note that \(1 - p_{i+1}\) and \(1 - p_i\) are also adjacent in \(\mathcal{V}_O\) and have exactly the inverses \(\text{inv}_\mathcal{U}(E_j^i)\) between them in reversed order.

For the recursion step, assume that we have already defined \(v\) for a node \(u\), such that (P1)–(P3) are satisfied at \(u\) and consider any \(i \in K\). We initialize the auxiliary function \(\bar{v}_{ui} : \mathcal{U}/\equiv_{ui} \to [0, 1]\) by setting \(\bar{v}_{ui}([p]_{ui}) := p\) for all \(p \in \mathcal{V}_O\) and \(\bar{v}_{ui}([C]_{ui}) := v(C, u)\) for all \(C \in \mathcal{C}(\mathcal{O})\). To see that this is well-defined, consider \([C]_{ui} = [D]_{ui}\), i.e. \(C \equiv_{ui} D\). From the ordered Hintikka condition it follows that \(C \equiv_{ui} D\), and from (P2) at \(u\) we obtain \(v(C, u) = v(D, u)\). A similar argument can be used to show that \([p]_{ui} = [C]_{ui}\) implies \(v(p, u) = v(C, u)\). For the remaining equivalence classes, we can use a construction analogous to the case for \(\epsilon\) by considering the two unique neighboring equivalence classes that contain an element of \(\mathcal{V}_O \cup \mathcal{C}(\mathcal{O})\). We now define \(v(\alpha, ui) := \bar{v}_{ui}([\alpha]_{ui})\). This construction ensures that (P1)–(P3) hold at \(ui\), and that (P4) holds for \(u\).

Given an ordered Hintikka tree and a function \(v\) that satisfies (P1)–(P4), we define the interpretation \(I_v = (K^*, \mathcal{T}_v)\) as follows. For all \(A \in \mathcal{N}_C\) and \(u \in K^*\), we set
\[
A^I_v(u) := \begin{cases} v(A, u) & \text{if } A \in \mathcal{C}(\mathcal{O}), \\ 0 & \text{otherwise}. \end{cases}
\]
For every role name \(r \in \mathcal{N}_R\) and all domain elements \(u, w\), we likewise define
\[
r^I_v(u, w) := \begin{cases} v(\varphi_r, u) & \text{if } w = ui \text{ with } i \in \mathcal{V}(\mathcal{O}), \\ 0 & \text{otherwise}. \end{cases}
\]
Finally, we define \(e^I_v := \epsilon\) for the individual name \(e\).

We now show by induction on the structure of \(C\) that for all \(C \in \mathcal{C}(\mathcal{O})\) and \(u \in K^*\) it holds that \(C^I_v(u) = v(C, u)\). The claim for \(C \in \mathcal{N}_C\) follows from the definition of \(I_v\). If \(C = \top\), we get \(\top \equiv_{ui} 1\) since \(\equiv_u\) is a Hintikka ordering. From (P1) and (P2), we have \(v(\top, u) = v(1, u) = 1\), and thus \(\top^I_v(u) = 1 = v(\top, u)\). Similarly, we can show that \(\perp^I_v(u) = v(\perp, u)\). For \(C = \neg D\), we have \(\text{inv}_\mathcal{U}(D) = C\), and thus \(C^I_v(u) = 1 - D^I_v(u) = 1 - v(D, u) = v(C, u)\) by induction hypothesis and (P3).

Consider now the case \(C = D \cap E\). Because \(\equiv_u\) is a Hintikka ordering, by (P2) we have
\[
C \equiv_{u} \min_u(D, E) = \begin{cases} D & \text{if } v(D, u) \leq v(E, u) \\ E & \text{otherwise}. \end{cases}
\]
By (P2) and the induction hypothesis, we get
\[ v(C, u) = \min\{ v(D, u), v(E, u) \} = \min\{ D^{\mathcal{I}_v}(u), E^{\mathcal{I}_v}(u) \} = C^{\mathcal{I}_v}(u). \]

The case of \( D \rightarrow E \) can be treated similarly.

Let \( C = \exists r. D \). For \( i = \varphi(\exists r. D) \), we get \((\exists r. D)_I \equiv_{\mathcal{U}} \min_{\mathcal{U}}(p, D)\) from the ordered Hintikka condition. As in the case for \( C = D \cap E \), the condition (P2) yields \( v((\exists r. D)_I, ui) = \min\{ v(p, ui), v(D, ui) \} \). Using (P4) and the induction hypothesis, we obtain \( v((\exists r. D, u) = \min\{ v^{\mathcal{I}_v}(u, ui), D^{\mathcal{I}_v}(ui) \} \). Similarly, for every \( i \in \varphi_r(\mathcal{O}) \) we can show that \( v(\exists r. D, u) \geq \min\{ v^{\mathcal{I}_v}(u, ui), D^{\mathcal{I}_v}(ui) \} \). Thus, we conclude
\[ ((\exists r. D)_I^{\mathcal{I}_v}(u) = \sup_{w \in K^*} \min\{ v^{\mathcal{I}_v}(u, w), D^{\mathcal{I}_v}(w) \} = \max_{i \in \varphi_r(\mathcal{O})} \min\{ v^{\mathcal{I}_v}(u, ui), D^{\mathcal{I}_v}(ui) \} = v(\exists r. D, u). \]

The case of \( C = \forall r. D \) can be treated analogously, which finishes the proof of the claim.

It remains to show that \( \mathcal{I}_v \) is indeed a witnessed model of \( \mathcal{O} \). It is witnessed since it is finitely branching (see Lemma 2.15). For every \( \langle c : C \triangleright p \rangle \in \mathcal{A} \), the ordered Hintikka tree satisfies \( C \triangleright p \), and thus we obtain \( C^{\mathcal{I}_v}(c^{\mathcal{I}_v}) = v(C, \varepsilon) \triangleright v(p, \varepsilon) = p \) from the above claim, (P1), and (P2), and similarly for assertions of the form \( \langle a : C \triangleright a : D \rangle \).

Now, let \( \langle C \subseteq D \geq p \rangle \in \mathcal{T} \) be a GCI and \( u \in K^* \) a domain element of \( \mathcal{I}_v \). Since \( p \in \mathcal{V}_\mathcal{O} \) and \( \lesssim_u \) is compatible with \( \mathcal{U} \), by (P2) it holds that
\[ p \lesssim_u \text{res}_u(C, D) = \begin{cases} 1 & \text{if } v(C, u) \leq v(D, u) \\ D & \text{if } v(D, u) < v(C, u). \end{cases} \]

Thus, (P1), (P2), and the claim from above yield
\[ p = v(p, u) \leq v(C, u) \Rightarrow v(D, u) = C^{\mathcal{I}_v}(u) \Rightarrow D^{\mathcal{I}_v}(u), \]

which shows that \( \mathcal{I}_v \) satisfies the GCI.

Conversely, let \( \mathcal{I} \) be a witnessed model of \( \mathcal{O} \). We use this model to guide the construction of an ordered Hintikka tree \( \preceq_* \) for \( \mathcal{O} \). During this construction, we will recursively generate a mapping \( g : K^* \rightarrow \Delta^\mathcal{I} \) specifying which domain elements correspond to the nodes in the tree. This mapping will satisfy the following condition:

(P5) For all \( \alpha, \beta \in \mathcal{V}_\mathcal{O} \cup \mathcal{C}(\mathcal{O}) \) and all \( u \in K^* \), we have
\[ \alpha \lesssim_u \beta \iff \alpha^{\mathcal{I}}(g(u)) \leq \beta^{\mathcal{I}}(g(u)), \]

where \( p^\mathcal{I}(x) := p \) for all \( p \in \mathcal{V}_\mathcal{O} \) and \( x \in \Delta^\mathcal{I} \).

We first consider the root node \( \varepsilon \) of the tree. Recall that the local ABox \( \mathcal{A} \) uses only the individual name \( c \). We define \( g(\varepsilon) := c^{\mathcal{I}} \) and the Hintikka ordering \( \preceq_\varepsilon \) as follows for all \( \alpha, \beta \in \mathcal{V}_\mathcal{O} \cup \mathcal{C}(\mathcal{O}) \):
\[ \alpha \preceq_\varepsilon \beta \iff \alpha^{\mathcal{I}}(c^{\mathcal{I}}) \leq \beta^{\mathcal{I}}(c^{\mathcal{I}}). \]

We extend this order to the elements in \( \mathcal{C}(\mathcal{O}) \cup \{ p, \neg p \} \) arbitrarily, in such a way that for all \( \alpha, \beta \in \mathcal{U} \) we have \( \alpha \preceq_\varepsilon \beta \iff \text{inv}_\mathcal{U}(\beta) \preceq_\varepsilon \text{inv}_\mathcal{U}(\alpha) \). Such an extension is possible
since \( \lnot \) is interpreted as the involutive negation. It is clear that this defines a total preorder satisfying (P5). In particular, it preserves the natural order on \( \mathcal{V}_O \) and has 0 and 1 as least and greatest element, respectively. Thus, it is an element of \( \text{order}(\mathcal{U}) \).

We show that \( \preceq_e \) is a Hintikka ordering. Let \( C \in \text{cl}(\mathcal{O}) \). If \( C = T \), we have \( T^T(c^T) = 1 \), and thus \( T \equiv_e 1 \), and similarly we get \( \perp \equiv_e 0 \). If \( C = D \cap E \), then

\[
C^T(c^T) = \min\{D^T(c^T), E^T(c^T)\}
\]

Thus, by definition of \( \preceq_e \), we get \( C \equiv_e \min_e(D, E) \). Analogous arguments can be used for \( C = D \rightarrow E \). Furthermore, \( \preceq_e \) is compatible with \( T \) since for every \( \langle C \subseteq D \geq p \rangle \in T \) we have \( p \leq C^T(c^T) \Rightarrow D^T(c^T) \), and thus \( p \preceq_e \text{res}_e(C, D) \) by similar arguments as above.

Assume now that \( g(u) \) and \( \preceq_u \) are already defined for a node \( u \in K^* \) such that (P5) is satisfied. For all \( i \in K \), we now define \( \preceq_{ui} \) in such a way that the tuple \( (\preceq_u, \preceq_{u1}, \ldots, \preceq_{uk}) \) satisfies the ordered Hintikka condition. We consider only the case that \( i = \varphi(\exists r.D) \); value restrictions can be handled using similar arguments. Since \( I \) is witnessed, there must be a domain element \( y_i \in \Delta^T \) such that \( (\exists r.D)^T(g(u)) = \min\{r^T(g(u), y_i), D^T(y_i)\} \).

We define \( g(ui) := y_i \) and \( \preceq_{ui} \) for all \( \alpha, \beta \in \mathcal{U} \) by

\[
\alpha \preceq_{ui} \beta \Leftrightarrow \alpha^T(g(ui)) \leq \beta^T(g(ui)),
\]

where \( g^T(g(ui)) := r^T(g(u), g(ui)) \) and \( (C_1)^T(g(ui)) := C^T(g(u)) \) for all concepts \( C \in \text{cl}(\mathcal{O}) \). It is clear that \( \preceq_{ui} \) behaves on \( \mathcal{V}_O \cup \text{cl}(\mathcal{O}) \) exactly as \( \preceq_u \) does on \( \mathcal{V}_O \cup \text{cl}(\mathcal{O}) \).

Following the same arguments used for the root node, it is easy to show that \( \preceq_{ui} \) is actually a Hintikka ordering compatible with \( T \).

We show the ordered Hintikka condition for \( (\preceq_u, \preceq_{u1}, \ldots, \preceq_{uk}) \). If \( i = \varphi(\exists r.D) \), then by construction of \( g \) we have \( (\exists r.D)^T(g(u)) = \min\{r^T(g(u), g(ui)), D^T(g(ui))\} \), and thus

\[
((\exists r.D)^T)^T(g(u)) = \min\{g^T(g(ui)), D^T(g(ui))\}.
\]

By the definition of \( \preceq_{ui} \), we obtain \( (\exists r.D)^T \equiv_{ui} \min_{ui}(\varphi, D) \), as required. Furthermore, for all \( i \in \varphi_r(\mathcal{O}) \), it holds that

\[
(\exists r.D)^T(g(u)) = \sup_{y \in \Delta^T} \min\{r^T(g(u), y), D^T(y)\}
\geq \min\{r^T(g(u), g(ui)), D^T(g(ui))\},
\]

which similarly shows that \( (\exists r.D)^T \preceq_{ui} \min_{ui}(\varphi, D) \) holds. Analogous arguments apply to the value restrictions in \( \text{sub}(\mathcal{O}) \).

Finally, for every \( \langle c:C \bowtie p \rangle \in \mathcal{A} \), we have \( C^T(c^T) \bowtie p \), and thus \( C \bowtie p \) by definition of \( \preceq_e \), and similarly for assertions of the form \( \langle c:C \bowtie c:D \rangle \). Hence, the tree defined by \( \preceq_u \), for \( u \in K^* \), is an ordered Hintikka tree for \( \mathcal{O} \).

This lemma shows that deciding the existence of an ordered Hintikka tree for \( \mathcal{O} \) suffices for deciding local consistency of \( \mathcal{O} \). As in Section 3.1, we can solve the former problem in exponential time in the size of \( \mathcal{O} \) using a looping tree automaton.
Definition 4.15 (ordered Hintikka automaton) The ordered Hintikka automaton for $\mathcal{O}$ is the looping automaton $A_\mathcal{O} := (Q_\mathcal{O}, I_\mathcal{O}, \Delta_\mathcal{O})$, where

- $Q_\mathcal{O}$ is the set of all Hintikka orderings for $\mathcal{O}$ compatible with $\mathcal{T}$,
- $I_\mathcal{O}$ is the set of all Hintikka orderings for $\mathcal{O}$ compatible with $\mathcal{A}$ and $\mathcal{T}$, and
- $\Delta_\mathcal{O}$ is the set of all tuples from $Q_{\mathcal{O}}^{k+1}$ that satisfy the Hintikka condition.

It is easy to see that the runs of $A_\mathcal{O}$ are exactly the ordered Hintikka trees for $\mathcal{O}$. Thus, $\mathcal{O}$ is locally consistent iff $A_\mathcal{O}$ is not empty.

Observe that the number of Hintikka orderings for $\mathcal{O}$ is bounded by $2^{|\mathcal{U}|}$ and the cardinality of $\mathcal{U} = V_\mathcal{O} \cup \text{cl}(\mathcal{O}) \cup \text{cl}_T(\mathcal{O}) \cup \{\varrho, \neg \varrho\}$ is linear in the size of $\mathcal{O}$. Likewise, the arity $k$ of the automaton $A_\mathcal{O}$ is exponential in the size of $\mathcal{O}$. Since (non-)emptiness of looping tree automata can be decided in polynomial time (Vardi and Wolper 1986), we obtain an ExpTime-decision procedure for local consistency in G-$\mathcal{IALC}$ with fuzzy general TBoxes and order assertions. The lower bound follows from the equivalence to classical satisfiability shown in Section 4.1 since the Gödel t-norm has no zero divisors.

Theorem 4.16 In G-$\mathcal{IALC}$ with fuzzy general TBoxes and order assertions, local consistency w.r.t. witnessed models is decidable in ExpTime. It is ExpTime-hard already in G-$\mathcal{INEL}$ with inequality assertions.

Recently, a slightly different Hintikka condition has been used to show a similar result about reasoning in G-$\mathcal{IALC}$ w.r.t. general models (Borgwardt, Distel, and Peñaloza 2014b). The idea is to circumvent the absence of a single proper witness for an existential restriction $\exists r.C$ at a node $u$ by generating a “prototypical” witness $u_i$, and enforcing that $(\exists r.C)_u > u_i \min_{u_i}(\varrho, C)$ (cf. Definition 4.12). When constructing a model, this prototype is replaced by infinitely many domain elements, each closer to the overall supremum.

4.2.4 Consistency

To decide consistency of ontologies containing more that one individual name, we again do a pre-completion of the input ordered ABox (Hollunder 1996). However, instead of guessing an explicit value for each relevant concept at all named individuals as in Section 3.2, we guess a total preorder $\preceq_A$ between all these values. This preorder represents the nucleus of a model of the ontology. To extend this to a full model, we check a local consistency condition for each of the individual names, and use $\preceq_A$ to combine the resulting interpretations.

More formally, let $\mathcal{O} = (\mathcal{A}, \mathcal{T})$ be an ontology with an ordered ABox $\mathcal{A}$ and a fuzzy general TBox $\mathcal{T}$. Let further $rcl(\mathcal{O})$ denote the set $\{r, \neg r \mid r \in \mathbb{N}_R \text{ occurs in } \mathcal{O}\}$. As for concepts, we consider the expressions $\neg \neg r$ and $r$ to be identical. We use the order structure

$$W := V_\mathcal{O} \cup \{c:C \mid c \in \text{Ind}(\mathcal{A}), C \in \text{cl}(\mathcal{O})\}$$
$$\cup \{(c, d):r \mid c, d \in \text{Ind}(\mathcal{A}), r \in rcl(\mathcal{O})\}$$

with $\text{inv}_W(c:C) := c: \neg C$ and $\text{inv}_W((c, d):r) := (c, d): \neg r$ for all $c, d \in \text{Ind}(\mathcal{A}), C \in \text{cl}(\mathcal{O})$, and $r \in rcl(\mathcal{O})$. 
4.2 Consistency under the Gödel t-norm

**Definition 4.17** (ordered pre-completion) An ordered pre-completion of \( \mathcal{A} \) w.r.t. \( \mathcal{T} \) is a total preorder \( \preceq_A \in \text{order}(\mathcal{W}) \) such that:

a) for every \( C \in \text{sub}(\mathcal{O}) \) and all \( c \in \text{Ind}(\mathcal{A}) \),
   - if \( C = \top \), then \( c:C \equiv_A 1 \),
   - if \( C = \bot \), then \( c:C \equiv_A 0 \),
   - if \( C = D \cap E \), then \( c:C \equiv_A \min_A(c:D, c:E) \),
   - if \( C = D \rightarrow E \), then \( c:C \equiv_A \max_A(c:D, c:E) \);

b) for every \( \forall r.C \in \text{sub}(\mathcal{O}) \) and all \( c, d \in \text{Ind}(\mathcal{A}) \), we have \( c:R.C \succeq_A \min_A((c,d):r,d:C) \);

c) for every \( \exists r.C \in \text{sub}(\mathcal{O}) \) and all \( c, d \in \text{Ind}(\mathcal{A}) \), we have \( c:R.C \succeq_A \max_A((c,d):r,d:C) \);

d) for every \( \forall \langle C \subseteq D \geq p \rangle \in \mathcal{T} \) and all \( c \in \text{Ind}(\mathcal{A}) \), we have \( \max_A(c:C,c:D) \geq_A p \); and

e) for every \( \langle \alpha \triangleright \beta \rangle \in \mathcal{A} \), we have \( \alpha \triangleright_A \beta \).

\( \diamond \)

This definition generalizes the local conditions of Definitions 3.3 and 4.11 to handle several named individuals simultaneously. As in Definition 3.28, the main difference is that we do not create witnesses for the quantified concepts here. This will be taken care of by testing the following local ordered ABoxes for consistency.

Given a pre-completion \( \preceq_A \) of \( \mathcal{A} \) w.r.t. \( \mathcal{T} \), we define the local ordered ABoxes \( \mathcal{A}_c \) for each \( c \in \text{Ind}(\mathcal{A}) \) as the set of all order assertions \( \langle \alpha \triangleright \beta \rangle \) over \( c \) and \( \text{cl}(\mathcal{O}) \) for which \( \alpha \triangleright_A \beta \) holds. Formally,

\[
\mathcal{A}_c := \{ (c:C \triangleright p) \mid C \in \text{cl}(\mathcal{O}), p \in \text{V}_{\mathcal{O}}, c:C \triangleright_A p \} \cup \\
\{ (c:C \triangleright c:D) \mid C, D \in \text{cl}(\mathcal{O}), c:C \triangleright_A c:D \}.
\]

Note that it actually suffices to consider only the relations \( \triangleright, =, < \) to fully characterize the local preorders induced by \( \preceq_A \).

**Lemma 4.18** \( \mathcal{O} \) has a witnessed model iff there is a pre-completion \( \preceq_A \) of \( \mathcal{A} \) w.r.t. \( \mathcal{T} \) such that, for every \( c \in \text{Ind}(\mathcal{A}) \), the ontology \( \mathcal{O}_c := (\mathcal{A}_c, \mathcal{T}) \) has a witnessed model.

**Proof.** Let \( \mathcal{I} \) be a model of \( \mathcal{O} \). We define the total preorder \( \preceq_A \) for all \( \alpha, \beta \in \mathcal{W} \) by

\[
\alpha \preceq_A \beta \text{ iff } \alpha^\mathcal{I} \leq \beta^\mathcal{I},
\]

where we set \( ((c,d):r)^\mathcal{I} := 1 - r^\mathcal{I}(c^\mathcal{I}, d^\mathcal{I}) \). In particular, \( \preceq_A \) preserves the natural order on \( \mathcal{V}_\mathcal{O} \) and has 0 and 1 as least and greatest element, respectively. Furthermore, it satisfies \( \alpha \preceq_A \beta \text{ iff } \text{inv}_\mathcal{W}(\beta) \preceq_A \text{inv}_\mathcal{W}(\alpha) \) for all \( \alpha, \beta \in \mathcal{W} \), i.e. it is an element of \( \text{order}(\mathcal{W}) \).

Since \( \mathcal{I} \) satisfies \( \mathcal{A} \), for every \( \langle \alpha \triangleright \beta \rangle \in \mathcal{A} \), we have \( \alpha^\mathcal{I} \triangleright \beta^\mathcal{I} \), and thus the preorder \( \preceq_A \) satisfies Condition e) of Definition 3.28. For Condition b), consider any \( c, d \in \text{Ind}(\mathcal{A}) \) and \( \exists r.C \in \text{sub}(\mathcal{O}) \). By the semantics of \( \exists \), we have \( (\exists r.C)^\mathcal{I}(c^\mathcal{I}) \geq \min\{r^\mathcal{I}(c^\mathcal{I}, d^\mathcal{I}), C^\mathcal{I}(d^\mathcal{I})\} \), which already shows the claim. The remaining conditions of Definition 3.28 can be shown using similar arguments. Finally, it is easy to see that \( \mathcal{I} \) is also a model of \( (\mathcal{A}_c, \mathcal{T}) \) for each \( c \in \text{Ind}(\mathcal{A}) \).

Conversely, let \( \preceq_A \) be a pre-completion of \( \mathcal{A} \) w.r.t. \( \mathcal{T} \) and each \( (\mathcal{A}_c, \mathcal{T}) \) be consistent. By Lemma 4.14, there is a Hintikka tree \( \preceq^*_A \) for every \( (\mathcal{A}_c, \mathcal{T}) \), consisting of Hintikka

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orderings $\leq_{\varphi}$ for all $u \in K^*$. As in the proof of Lemma 4.14, we first construct a function $v: \mathcal{W} \cup (\text{Ind}(\mathcal{A}) \times \mathcal{U} \times K^*) \to [0, 1]$ such that

- for all values $p \in \mathcal{V}_O$, we have $v(p) = p$,
- for all $\alpha, \beta \in \mathcal{W}$, we have $v(\alpha) \leq v(\beta)$ iff $\alpha \leq_{\mathcal{A}} \beta$,
- for all $\alpha \in \mathcal{W}$, we have $v(\text{inv}_\mathcal{W}(\alpha)) = 1 - v(\alpha)$,
- for every $C \in \text{cl}(\mathcal{O})$ and all $c \in \text{Ind}(\mathcal{A})$, we have $v(c; C) = v(c, C, \varepsilon)$,
- for all $u \in K^*$ and all $c \in \text{Ind}(\mathcal{A})$,
  - for all values $p \in \mathcal{V}_O$, we have $v(c, p, u) = p$,
  - for all $\alpha, \beta \in \mathcal{U}$, we have $v(c, \alpha, u) \leq v(c, \beta, u)$ iff $\alpha \leq_{\mathcal{U}} \beta$,
  - for all $\alpha \in \mathcal{U}$, we have $v(c, \text{inv}_\mathcal{U}(\alpha), u) = 1 - v(c, \alpha, u)$, and
  - for all $C \in \text{cl}(\mathcal{O})$ and all $i \in \{1, \ldots, n\}$, we have $v(c, C, u) = v(c, C_i, u_i)$.

We will then use this function to define a model of $\mathcal{O}$.

Using the technique from the proof of Lemma 4.14, we first define $v$ on $\mathcal{W}$. On the set $\mathcal{W}/\equiv_{\mathcal{A}}$ of all equivalence classes of $\equiv_{\mathcal{A}}$, $\leq_{\mathcal{A}}$ induces a total order $<_{\mathcal{A}}$ such that $[0]_{\mathcal{A}} <_{\mathcal{A}} [p_1]_{\mathcal{A}} <_{\mathcal{A}} \cdots <_{\mathcal{A}} [p_{n-1}]_{\mathcal{A}} <_{\mathcal{A}} [1]_{\mathcal{A}}$. We first define the auxiliary function $\hat{v}_{\mathcal{A}}: \mathcal{W}/\equiv_{\mathcal{A}} \to [0, 1]$, starting with $\hat{v}_{\mathcal{A}}([p]_{\mathcal{A}}) := p$ for each $p \in \mathcal{V}_O$. For every $i \in \{0, \ldots, k - 1\}$, let now $E^i_1, \ldots, E^i_{\nu_i}$ be all equivalence classes strictly between $[p_i]_{\mathcal{A}}$ and $[p_{i+1}]_{\mathcal{A}}$ such that

$$[p_i]_{\mathcal{A}} <_{\mathcal{A}} E^i_1 <_{\mathcal{A}} \cdots <_{\mathcal{A}} E^i_{\nu_i} <_{\mathcal{A}} [p_{i+1}]_{\mathcal{A}}.$$  

We define $\hat{v}_{\mathcal{A}}(E^i_j) := p_i + \frac{j}{\nu_i + 1}(p_{i+1} - p_i)$ for all $j, 1 \leq j \leq \nu_i$, and then $v(\alpha) := \hat{v}_{\mathcal{A}}([\alpha]_{\mathcal{A}})$ for all $\alpha \in \mathcal{W}$.

For each $c \in \text{Ind}(\mathcal{A})$ and $C \in \text{cl}(\mathcal{O})$, we now set $v(c, C, \varepsilon) := v(c; C)$. The values of $v(c, \alpha, \varepsilon)$ for elements $\alpha \in \text{cl}(\mathcal{O}) \cup \{\varnothing, \neg \varnothing\}$ are irrelevant for the properties we seek and can be fixed arbitrarily, as long as we have $v(c, \text{inv}_\mathcal{U}(\alpha), \varepsilon) = 1 - v(c, \alpha, \varepsilon)$ and $v(c, \alpha, \varepsilon) \leq v(c, \beta, \varepsilon)$ iff $\alpha \leq_{\mathcal{U}} \beta$ for all $\alpha, \beta \in \mathcal{U}$. This can be ensured using the technique from above since by the definition of $\mathcal{A}_{\varepsilon}$ we have, for all $\alpha, \beta \in \mathcal{V}_O \cup \text{cl}(\mathcal{O})$, $\alpha \leq_{\mathcal{U}} \beta$ iff $c; \alpha \leq_{\mathcal{A}} c; \beta$, where $c; p := p$ for every $p \in \mathcal{V}_O$. The definition of $v(c, \alpha, \varepsilon)$ can now proceed as in the proof of Lemma 4.14 based on the ordered Hintikka trees $\leq_{\varphi}$ for $(\mathcal{A}_{\varepsilon}, \mathcal{T})$. This construction ensures that $v$ has the desired properties.

We now define the interpretation $\mathcal{I}$ as follows:

- $\Delta^\mathcal{I} := \text{Ind}(\mathcal{A}) \times K^*$,
- $c^\mathcal{I} := (c, \varepsilon)$ for each $c \in \text{Ind}(\mathcal{A})$,
- $A^\mathcal{I}(c, u) := v(c, A, u)$ for all $A \in \text{N}_C \cap \text{sub}(\mathcal{O})$, $c \in \text{Ind}(\mathcal{A})$, and $u \in K^*$, and
- for all $r \in \text{N}_R$, $c, d \in \text{Ind}(\mathcal{A})$, and $u, u' \in K^*$,

$$v^\mathcal{I}((c, u), (d, u')) := \begin{cases} v(c, \varnothing, u) & \text{if } c = d \text{ and } u' = u \text{ with } i \in \varphi_r(\mathcal{O}), \\ v(c, d; r) & \text{if } u = u' = \varepsilon \text{ and } r \text{ occurs in } \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$$
4.3 Satisfiability and Entailment

The interpretation of the remaining individual and concept names is irrelevant and can be fixed arbitrarily. As in Lemma 4.14, we can show by induction on the structure of $C$ that $C^I(a, u) = v(a, C, u)$ holds for all $C \in \text{cl}(\mathcal{O})$, $a \in \text{ind}(\mathcal{A})$, and $u \in K^*$. The claim for $\top$, $\bot$, $\neg C$, $C \cap D$, and $C \rightarrow D$ follows as before from Condition a) of Definition 3.28 and the fact that each $\leq^a_r$ is a Hintikka ordering.

Consider now an existential restriction $\exists r.C \in \text{sub}(\mathcal{O})$ and the domain element $(c, \varepsilon)$ for some $c \in \text{ind}(\mathcal{A})$. By the Hintikka condition and the induction hypothesis, we obtain that $v(c, \exists r.C, u) = \min\{r^I((c, \varepsilon), (c, i)), C^I(c, i)\}$, where $i = \varphi(\exists r.C)$, as in the proof of Lemma 4.14. Likewise, $v(c, \exists r.C, u) \geq \min\{r^I((c, \varepsilon), (c, i)), C^I(c, i)\}$ for all $i \in \varphi_r(\mathcal{O})$. Finally, for each $d \in \text{ind}(\mathcal{A})$, we have $v(c, \exists r.C, u) \geq \min\{r^I((c, \varepsilon), (d, \varepsilon)), C^I(d, \varepsilon)\}$ by Condition b) of Definition 3.28. Since $(c, \varepsilon)$ does not have any other relevant $r$-successors, this shows the claim for $\exists r.C$ at $(c, \varepsilon)$. At the other domain elements, it can be shown as for Lemma 4.14. Similar arguments apply for any $\forall r.C \in \text{sub}(\mathcal{O})$.

Finally, $\mathcal{I}$ is witnessed since it is finitely branching (see Lemma 2.15) and it is a model of $\mathcal{O}$ because of the compatibility of all Hintikka orderings with $T$ and Conditions d) and e) of Definition 3.28.

The cardinality of $\text{order}(\mathcal{W})$ is exponential in the size of $\mathcal{O}$, and all elements of $\text{order}(\mathcal{W})$ are of polynomial size. We can thus enumerate $\text{order}(\mathcal{W})$, check for each element whether it satisfies Definition 3.28 in polynomial time, and then execute the polynomially many local consistency tests as described by Lemma 4.18. This yields the following complexity result.

**Theorem 4.19** In $\mathcal{G\-A\-L\-C}$ with fuzzy general TBoxes and order assertions, consistency w.r.t. witnessed models is decidable in ExpTime. It is ExpTime-hard already in $\mathcal{G\-R\-L\-C}$ with inequality assertions.

### 4.3 Satisfiability and Entailment

We now direct our attention to the problems of deciding concept satisfiability, subsumption, and instance checking, and computing the best degrees to which these inferences hold. We show that in $\mathcal{G\-A\-L\-C}$ with fuzzy general TBoxes and order assertions, all these problems can be solved in exponential time in the size of the input ontology. Note that Lemma 2.22 also holds for ordered ABoxes since the result of Lemma 2.23 is independent of the precise shape of the assertions. Thus, we can assume without loss of generality that the input ABox is empty.

Recall from Section 2.2.4 that concept satisfiability w.r.t. an ontology $\mathcal{O} = (\emptyset, T)$ can be reduced in polynomial time to local consistency. Since we are now allowed to use order assertions, similar reductions work for subsumption and instance checking. More precisely, for any two concepts $C, D$, $c \in N_1$, and $p \in [0, 1]$,

- $C$ is $p$-satisfiable w.r.t. $\mathcal{O}$ iff $\{\{c:C \geq p\}\}, T$ is consistent,
- $C$ is $p$-subsumed by $D$ w.r.t. $\mathcal{O}$ iff $\{\{c:C \rightarrow D < p\}\}, T$ is inconsistent, and
- $c$ is a $p$-instance of $C$ w.r.t. $\mathcal{O}$ iff $\{\{c:C < p\}\}, T$ is inconsistent.

We thus obtain the following results from Theorem 4.16.
Theorem 4.20 In G-$\mathcal{ALC}$ with fuzzy general TBoxes and order assertions, satisfiability, subsumption, and instance checking w.r.t. witnessed models are decidable in ExpTime. They are ExpTime-hard already in G-$\mathcal{RCL}$ and G-$\mathcal{ELC}$. □

Regarding the best degrees to which these inferences hold, we observe that the local consistency checks required for deciding $p$-satisfiability, $p$-subsumption, and $p$-instances only depend on the position of $p$ relative to the values occurring in $T$, but not on the precise value of $p$. To prove this, we again use the preorders of the previous sections, and in particular ordered Hintikka trees.

Lemma 4.21 Let $p,p' \in (p_i,p_{i+1})$ for two adjacent values $p_i,p_{i+1} \in \mathcal{V}_O$, $C$ be a concept, and $c$ an individual name. Then $\langle \{ \langle c:C \bowtie p \rangle \}, \mathcal{T} \rangle$ has a witnessed model iff $\langle \{ \langle c:C \bowtie p' \rangle \}, \mathcal{T} \rangle$ has a witnessed model.

Proof. By Lemma 4.14, both consistency conditions are equivalent to the existence of ordered Hintikka trees, albeit over different order structures. We denote by $\mathcal{U}_p$ the order structure defined in Section 4.2.3 over the set $\mathcal{V}_p := \mathcal{V}_O \cup \{p,1-p\}$, and by $\mathcal{U}_{p'}$ the one over $\mathcal{V}_{p'} := \mathcal{V}_O \cup \{p',1-p'\}$. Observe that the bijection $\iota: \mathcal{V}_p \rightarrow \mathcal{V}_{p'}$ that simply maps $p$ to $p'$ and $1-p$ to $1-p'$ and leaves the other values as they are, can be extended to a bijection between $\mathcal{U}_p$ and $\mathcal{U}_{p'}$ by defining it as the identity on all elements outside of $\mathcal{V}_p$. Furthermore, it is compatible with the involutive operators of the two order structures, i.e. we have $\iota(\text{inv}_{\mathcal{U}_p}(\alpha)) = \text{inv}_{\mathcal{U}_{p'}}(\iota(\alpha))$ for all $\alpha \in \mathcal{U}_p$.

We now lift this bijection to $\text{order}(\mathcal{U}_p)/\text{order}(\mathcal{U}_{p'})$ by setting, for any $\preceq_p \in \text{order}(\mathcal{U}_p)$, $\alpha \preceq_{p'} \beta$ iff $\iota(\alpha) \preceq_p \iota(\beta)$ for all $\alpha,\beta \in \mathcal{U}_{p'}$. It is easy to see that this defines an element of $\text{order}(\mathcal{U}_{p'})$ and that every element of $\text{order}(\mathcal{U}_{p'})$ can be obtained in this way (simply apply the inverse of $\iota$). In particular, $\preceq_{p'}$ preserves the order of the real numbers on $\mathcal{V}_{p'}$ since $p$ and $p'$ are in the same relative position w.r.t. the elements of $\mathcal{V}_O$. Furthermore, we have $\iota(\text{min}_p(\alpha,\beta)) = \min_{p'}(\iota(\alpha),\iota(\beta))$ and $\iota(\text{res}_p(\alpha,\beta)) = \text{res}_{p'}(\iota(\alpha),\iota(\beta))$ for all $\alpha,\beta \in \mathcal{U}_p$.

Moreover, if $\preceq_p$ is a Hintikka ordering, then $\preceq_{p'}$ is also a Hintikka ordering, and vice versa, since this notion only depends on the order between the concepts in $\mathcal{U}_p/\mathcal{U}_{p'}$. Compatibility with $\mathcal{T}$ is also equivalent for the two preorders. Similarly, by definition of $\preceq_{p'}$, $\preceq_p$ is compatible with $\{\langle c:C \bowtie p \rangle\}$ iff $C \bowtie_p p$ iff $C \bowtie_{p'} p'$ iff $\preceq_{p'}$ is compatible with $\{\langle c:C \bowtie p' \rangle\}$.

From the above arguments and similar ones for the ordered Hintikka condition, it follows that there is an ordered Hintikka tree for $\{\langle c:C \bowtie p \rangle\}$ iff there is an ordered Hintikka tree for $\{\langle c:C \bowtie p' \rangle\}$. Lemma 4.14 now yields the claim. □

This shows that satisfiability, subsumption, and instance checking either hold for all values in an interval $(p_i,p_{i+1})$, or for none of them. In particular, the best satisfiability degree of $C$ w.r.t. $\mathcal{O}$ is always in $\mathcal{V}_O$, and likewise for the best subsumption and instance degrees.

By Lemma 2.20, the best subsumption degree $p$ of $C$ and $D$ is always a subsumption degree, and thus it suffices to check subsumption w.r.t. the values from $\mathcal{V}_O$ in order to determine the best subsumption degree. Thus, we only have to execute linearly many (in-)consistency checks to compute the best subsumption degree. The same approach can be used for computing the best instance degree of $c$ in $C$.

However, Example 2.21 shows that $C$ may be $p$-satisfiable for every $p \in (p_i,p_{i+1})$, but not $p_{i+1}$-satisfiable. Therefore, to compute the best satisfiability degree, we have to check
satisfiability for all values $p_{i+1}$. The best satisfiability degree is then the largest $p_{i+1}$ for which this check succeeds (or 0 if it never succeeds). Again, this means that we have to execute linearly many consistency checks to compute the best satisfiability degree.

By combining these reductions with Theorem 4.16, we obtain the following result.

**Theorem 4.22** In $G\mathcal{ALC}$ with fuzzy general TBoxes and order assertions, the best satisfiability, subsumption, and instance degrees w.r.t. witnessed models can be computed in exponential time.

We now consider again the logics of Section 4.1, i.e. $L\mathcal{SUHOL}$ with fuzzy general TBoxes and inequality assertions, where $L$ has no zero divisors. It is easy to see that the reduction to crisp reasoning also works for satisfiability of concepts w.r.t. witnessed models, and thus in particular not to finite or crisp ones. To see this, consider the witnessing interpretation $I_3$ of Example 4.6 (see Figure 4.2). This is indeed a model of $O_3$ since $\langle \top \sqsubseteq A \rangle = p$. Moreover, $I_3$ satisfies $\langle \top \sqsubseteq \neg\neg A \rangle$. In fact, the best subsumption degree of $\top$ and $A$ w.r.t. $O_3$ is 0, which is smaller than 1.

For the next example, we increase the expressivity of the DL by adding the residual negation constructor, but allow only crisp general TBoxes and consider only t-norms over $[0,1]$ without zero divisors.

**Example 4.23** We show that the ontology $O_3$ containing only the GCI $\langle \top \sqsubseteq A \rangle = p$ for some $p$, $0 < p < 1$, entails $\langle \top \sqsubseteq \neg\neg A \rangle$ when reasoning is restricted to crisp models, but $\top$ is not 1-subsumed by $A$ w.r.t. $O_3$ in general. The former is easy to see from the fact that for every crisp model $I$ of $O_3$ and $x \in A^I$, we have $A^I(x) = 1$.

To see the latter, observe that the (witnessed) interpretation $I_3 = (\{x\}, \mathcal{I}_3)$, where $A^{I_3}(x) := p$, is also a model of $O_3$, but violates the axiom $\langle \top \sqsubseteq \neg\neg A \rangle$. In fact, the best subsumption degree of $\top$ and $A$ w.r.t. $O_3$ is 0, which is smaller than 1.

For the next example, we increase the expressivity of the DL by adding the residual negation constructor, but allow only crisp general TBoxes and consider only t-norms over $[0,1]$ without zero divisors.

**Example 4.24** Consider the ontology $O_4$ containing only the axiom $\langle \top \sqsubseteq \equiv\equiv A \rangle$. As in Example 4.23, it is easy to see that every crisp model of $O_4$ also satisfies $\langle \top \sqsubseteq \equiv\equiv A \rangle$. On the other hand, the best subsumption degree of $\top$ and $A$ w.r.t. $O_4$ is 0.

To show this, consider the witnessed interpretation $I_4$ constructed in Example 4.6 (see Figure 4.2). This is indeed a model of $O_4$ since $A^{I_4}(n) > 0$, and hence $(\neg\neg A)^{I_4}(n) = 1$ holds for every $n \in \mathbb{N}$. However, for every $p > 0$ there is an $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < p$. Thus, the best subsumption degree of $\top$ and $A$ w.r.t. $O_4$ is 0.
Finally, for the Product t-norm, we can provide a similar example as for the Gödel t-norm to show that subsumption in \( \Pi\mathcal{RL} \) cannot be decided using only finitely valued models.

**Example 4.25** Consider the ontology
\[
\mathcal{O}_5 := \{\langle \top \sqsubseteq \top \top A \rangle, \langle \top \sqsubseteq \top r \top \rangle, \langle \top r A \sqsubseteq A \cap A \rangle\}.
\]
We show that every finitely valued model of \( \mathcal{O}_5 \) also satisfies the GCI \( \langle \top \sqsubseteq A \rangle \), but the best subsumption degree of \( \top \) and \( A \) w.r.t. \( \mathcal{O}_5 \) is 0.

Let first \( I \) be a model of \( \mathcal{O}_5 \) that violates \( \langle \top \sqsubseteq A \rangle \). We show that \( I \) cannot be finitely valued. To do this, we show by induction that for every \( n \geq 1 \) there exist \( x_1, \ldots, x_n \in \Delta_I \) such that \( 1 > A^I(x_1) > \ldots > A^I(x_n) > 0 \).

For the induction base, since \( I \) violates \( \langle \top \sqsubseteq A \rangle \), there must be an \( x \in \Delta_I \) such that \( A^I(x) < 1 \). As \( I \) satisfies the first axiom of \( \mathcal{O} \), it follows that \( A^I(x) > 0 \). Thus, if we set \( x_1 := x \), then the claim holds for \( n = 1 \). Supposing that it holds for \( n \geq 1 \), we show that it also holds for \( n + 1 \). Since \( A^I(x_n) < 1 \), the second axiom implies that there must exist a \( y \in \Delta_I \) such that \( r^I(x_n, y) > A^I(x_n) \). The third axiom then implies that
\[
(A^I(x_n))^2 > (\exists r A)^I(x_n) \geq r^I(x_n, y) \cdot A^I(y) > A^I(x_n) \cdot A^I(y),
\]
and thus \( A^I(x_n) > A^I(y) \). Since \( I \) satisfies the first axiom, we have \( A^I(y) > 0 \). Thus, setting \( x_{n+1} := y \) yields the result.

It only remains to show that the best subsumption degree of \( \top \) and \( A \) w.r.t. \( \mathcal{O}_5 \) is 0. We build a (witnessed) model \( I_5 \) of \( \mathcal{O}_5 \) that violates \( \langle \top \sqsubseteq A \rangle \) for every \( p > 0 \). Let \( I_5 = \{\{2^n \mid n \in \mathbb{N}\}, \Delta_I\} \) be given for all \( x, y \in \Delta_I \) by \( A^{I_5}(x) := 2^{-x} \) and \( r^{I_5}(x, y) := 1 \) if \( y = 2x \), and 0 otherwise (see Figure 4.6).

We verify that \( I_5 \) is a witnessed model of \( \mathcal{O}_5 \). First, since \( 2^{-2^n} > 0 \) for every \( n \in \mathbb{N} \), it follows that \( A^{I_5}(x) > 0 \) for all \( x \in \Delta_I \). Thus, \( I_5 \) satisfies the first axiom of \( \mathcal{O} \). For every \( x \in \Delta_I \), it also holds that \( (\exists r \top)^{I_5}(x) = r^{I_5}(x, 2x) = 1 \) and
\[
(\exists r A)^{I_5}(x) = r^{I_5}(x, 2x) \otimes A^{I_5}(2x) = 2^{-2x} = 2^{-x} \cdot 2^{-x} = A^{I_5}(x) \otimes A^{I_5}(x),
\]
satisfying the remaining two axioms of the ontology. Finally, this model is witnessed since it is finitely branching (see Lemma 2.15).

This means that \( \top \) is not \( p \)-subsumed by \( A \) w.r.t. \( \mathcal{O}_5 \) for any \( p > 0 \), but \( \top \) is 1-subsumed by \( A \) in every finitely valued model of \( \mathcal{O}_5 \).

This means that, regardless of whether we consider witnessed or general models, subsumption cannot be decided over crisp models in the following logics:

- \( L\mathcal{L} \) with fuzzy general TBoxes if \( \mathcal{L} \) allows the top constructor and \( L \) has at least three elements;
4.3 Satisfiability and Entailment

- \( \otimes \mathcal{K} \) with crisp general TBoxes if \( \mathcal{L} \) allows the top constructor and \( \otimes \) has no zero divisors; and
- \( G \mathcal{A} \mathcal{L} \) with crisp general TBoxes.

In \( \Pi \mathcal{K} \), \( G \mathcal{A} \mathcal{L} \), and \( G \mathcal{K} \mathcal{L} \) with crisp general TBoxes, subsumption cannot even be decided over finitely valued or finite models. All examples equally apply to instance checking since \( \top \) is \( p \)-subsumed by \( A \) w.r.t. \( \mathcal{O} = (\emptyset, \mathcal{T}) \) iff an arbitrary individual name \( c \) is a \( p \)-instance of \( A \) w.r.t. \( \mathcal{O} \). This demonstrates that subsumption and instance checking in the listed logics cannot be decided over crisp models, as was done in Lemma 4.3 for consistency.

Regarding subsumption in fuzzy DLs with fuzzy general TBoxes, not much else is known. Some preliminary results on the complexity of subsumption in \( \otimes \mathcal{L} \) have been reported in (Borgwardt and Peñaloza 2013b). In particular, \( p \)-subsumption for any t-norm containing \( \mathcal{L} \) is \( \text{co-NP} \)-hard in this logic, but no upper bounds are known. An extension of the completion-based algorithm for crisp \( \mathcal{L} \) from (Baader, Brandt, and Lutz 2005) to \( G \mathcal{L} \) has been presented in (Mailis et al. 2012), yielding \( P \) as upper bound for the complexity of subsumption. A similar polynomial-time algorithm was developed in (Borgwardt and Peñaloza 2013a) for \( \otimes \mathcal{L} \), but this only applies to a very limited form of ontologies and only to 1-subsumption when all roles are restricted to be crisp. In \( G \mathcal{L}_0 \) with so-called cyclic TBoxes, subsumption between concept names can be decided in \( \text{PSPACE} \), matching the complexity of the corresponding classical problem (Baader 1996; Borgwardt, Leyva Galano, and Peñaloza 2014).
5 Undecidable Fuzzy Description Logics over the Standard Interval

We now show that, apart from the results of the previous chapter, the combination of GCIs and the infinitely many truth degrees in \[0,1\] easily leads to undecidability of the consistency problem. All logics for which we will show this in the following are restricted to crisp general TBoxes and may use only rational numbers in the ABoxes. Sometimes undecidability arises even with completely crisp ontologies in inexpressive fuzzy DLs like NEL or ELC (see Theorems 5.11 and 5.16).

As shown in Chapter 4, such strong undecidability results cannot apply to the Gödel t-norm, and neither to t-norms without zero divisors when considering sublogics of \(\otimes\)-SUHOI with inequality assertions. On the other hand, in Section 5.1 most of the remaining logics will be shown to have an undecidable consistency problem under witnessed model semantics. These proofs are based on a reduction from the Post correspondence problem (Post 1946) and generalize previous proofs that show undecidability for particular fuzzy DLs (Baader and Peñaloza 2011a; Cerami and Straccia 2013) (see Section 2.4 for details).

Section 5.2 applies similar ideas to show undecidability of consistency in several fuzzy DLs under general model semantics. In Section 5.3, we conclude this chapter with a few comments on how the undecidability proofs could be transferred to other reasoning problems like satisfiability and subsumption.

5.1 Consistency under Witnessed Model Semantics

We describe a general approach for proving that the (local) consistency problem w.r.t. witnessed models is undecidable in a given fuzzy DL \(\otimes\)-L. In the following, we always implicitly consider crisp general TBoxes, and we use the term crisp ontologies for ontologies with crisp general TBoxes and crisp assertions. This approach is based on a reduction from a variant of the Post correspondence problem which is known to be undecidable (Post 1946).

**Definition 5.1 (PCP)** Let \(P = \{(v_1, w_1), \ldots, (v_n, w_n)\}\) be a finite set of pairs of words over the alphabet \(\Sigma = \{1, \ldots, s\}\) with \(s > 1\). The Post correspondence problem (PCP) asks whether there is a finite non-empty sequence \(i_1 \ldots i_k \in \{1, \ldots, n\}^*\) such that \(v_1 v_{i_1} \ldots v_{i_k} = w_1 w_{i_1} \ldots w_{i_k}\). Such a sequence is called a solution for \(P\).

We abbreviate \(\{1, \ldots, n\}\) by \(N\), and with a slight abuse of notation we define the abbreviations \(v_\nu := v_1 v_{i_1} \ldots v_{i_k}\) and \(w_\nu := w_1 w_{i_1} \ldots w_{i_k}\) for all \(\nu = i_1 \ldots i_k \in N^*\). Note that this allows the expressions \(v_1\) and \(w_1\) to be interpreted in two different ways, but it is usually clear from the context which of them is meant.

In order to solve an instance \(P = \{(v_1, w_1), \ldots, (v_n, w_n)\}\) of the PCP, we consider its search tree, which has one node for every \(\nu \in N^*\), where \(\varepsilon\) is the root, and \(vi\) is the
5 Undecidable Fuzzy Description Logics over the Standard Interval

Figure 5.1: The search tree for an instance $P$ of the PCP

$i$-th successor of $\nu$ for each $i \in \mathbb{N}$. Every node $\nu$ in this tree is labeled with the words $v_\nu, w_\nu \in \Sigma^*$, as shown in Figure 5.1. The instance $P$ has a solution iff its search tree contains a node labeled by two equal words.

Following this idea, our reduction of the PCP to the consistency problem of a fuzzy DL consists of two parts. Given an instance $P$ of the PCP, we first construct an ontology $O_P$ that describes the search tree of $P$. The second step is to check whether this tree contains a solution for $P$. More precisely, we will enforce that for every model $\mathcal{I}$ of $O_P$ and every $\nu \in \mathbb{N}^*$, there is a domain element $x_\nu \in \Delta^\mathcal{I}$ such that $V^\mathcal{I}(x_\nu) = \text{enc}(v_\nu)$ and $W^\mathcal{I}(x_\nu) = \text{enc}(w_\nu)$, where $\text{enc}: \Sigma^* \rightarrow [0,1]$ is an injective function that encodes words over $\Sigma$ into the interval $[0,1]$ (see Theorem 5.3). Once we have encoded the words $v_\nu$ and $w_\nu$ using $V$ and $W$, we add axioms that restrict the models to those that satisfy $V^\mathcal{I}(x_\nu) \neq W^\mathcal{I}(x_\nu)$ for all $\nu \in \mathbb{N}^*$. This ensures that $P$ has a solution if and only if the ontology is inconsistent (see Theorem 5.4).

5.1.1 An Example

We first describe the construction on the relatively easy example of the fuzzy DL $\Pi$-$\mathcal{AL}$ with equality assertions. In Section 5.1.2, we present a general framework that allows us to prove undecidability of many fuzzy DLs at the same time. This framework consists of several properties that a fuzzy DL can have, which together lead to undecidability. We label each part of the following construction by the name of the property of the general framework it corresponds to.

Let in the following $P = \{(v_1, w_1), \ldots, (v_n, w_n)\}$ be an instance of the PCP over the alphabet $\Sigma$. Recall that $\Sigma$ consists of the first $s$ positive integers. We can thus view every word in $\Sigma^*$ as a natural number represented in base $s + 1$. On the other hand, every natural number $n$ has a unique representation in base $s + 1$, which can be seen as a word over the alphabet $\Sigma_0 := \Sigma \cup \{0\} = \{0, \ldots, s\}$. This is not a bijection since, e.g. the words 001202 and 1202 represent the same number. However, it is a bijection between the set $\Sigma \Sigma_0^*$ and the positive natural numbers. In the following, we interpret the empty word $\varepsilon$ as 0, thereby extending this bijection to $\{\varepsilon\} \cup \Sigma \Sigma_0^*$ and all non-negative integers.
5.1 Consistency under Witnessed Model Semantics

In our constructions and proofs, we will view elements of $\Sigma^*_\alpha$ both as words and as natural numbers in base $s + 1$. It is usually clear from the context which interpretation is used. However, to avoid confusion, we sometimes use the notation $\nu$ to express that $u$ is seen as a word. Thus, for instance, if $s = 3$, then $3 \cdot 2^2 = 30$ (in base 4), but $3 \cdot 2^2 = 322$. Furthermore, $000$ is a word of length 3, whereas $000$ is simply the number 0. We extend this notation to rational numbers, and may use, e.g. the expression $0.0^3 \cdot 1$ to denote the number 0.0001 (again in base 4). For a word $u = \alpha_1 \cdots \alpha_m$ with $\alpha_i \in \Sigma_0$, $1 \leq i \leq m$, we denote by $\nu u$ the word $\alpha_m \cdots \alpha_1 \in \Sigma^*_\alpha$ resulting from $u$ by reading it backwards.

For the purposes of the current example of $\Pi-\mathcal{AL}$ with equality assertions, we use the encoding function $\text{enc}: \Sigma^* \rightarrow [0, 1]$ given by $\text{enc}(u) := 2^{-u}$ to encode words as values from the interval $[0, 1]$. For example, we have $\text{enc}(\varepsilon) = 2^{-0} = 1$ and $\text{enc}(2) = 2^{-2} = 1/4$.

Recall that we need to encode the search tree of $\mathcal{P}$ depicted in Figure 5.1 in a fuzzy DL ontology $\mathcal{O}_P$ such that all models contain an encoding of this tree.

The Initialization Property

The first step in constructing $\mathcal{O}_P$ is to initialize the root of the search tree. The root is represented by the individual name $c_r$, and there we initialize the values for $V$ and $W$, as well as several other auxiliary concept names. This can be expressed using equality assertions:

$$\langle c_r : V = \text{enc}(v_1), \langle c_r : W = \text{enc}(w_1), \langle c_r : M = 1/2, \langle c_r : V_1 = \text{enc}(v_1), \ldots, \langle c_r : V_n = \text{enc}(v_n), \langle c_r : W_1 = \text{enc}(w_1), \ldots, \langle c_r : W_n = \text{enc}(w_n) \rangle. \quad (5.1)$$

The concept names $V_1, \ldots, V_n, W_1, \ldots, W_n$ are intended to be constants that hold the above values at every node of the search tree, and are used in each step to concatenate the words $v_1, \ldots, v_n, w_1, \ldots, w_n$ to the words currently encoded by $V$ and $W$. Similarly, the value of $M$ is constant throughout the search tree, and is used to compare the values of $V$ and $W$ at each node (see axiom (5.5) below).

The Concatenation Property

The next step is to compute the values $\text{enc}(v_1v_i)$ and $\text{enc}(w_1w_i)$ for the successors $i \in N$ of the root node. We introduce additional auxiliary concept names $D_{Vv_i}$ and $D_{Ww_i}$ to hold these values. We can achieve the correct concatenation using the equivalence

$$\langle D_{Vv_i} \equiv V^{(s + 1)|v_i| \cap V_i} \rangle \quad (5.2)$$

for every $i \in N$, and similarly for $D_{Ww_i}$. Indeed, since $V$ has the value $\text{enc}(v_1) = 2^{-v_1}$ and $V_i$ has the value $\text{enc}(v_i) = 2^{-v_i}$ at $c_r$, $D_{Vv_i}$ will be evaluated to

$$2^{-(v_1(s + 1)|v_i| + v_i)} = 2^{-v_1v_i} = \text{enc}(v_1v_i).$$

In general, whenever $V$ has the value $\text{enc}(v_\nu)$ for some $\nu \in N^*$, then $D_{Vv_\nu}$ will have the value $\text{enc}(v_\nu)$. 

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The Successor Property

We now construct the successors of the root node, which will be identified by the role names \(r_1,\ldots,r_n\), using the axioms

\[
\langle \top \sqsubseteq \exists r_1.\top \rangle, \ldots, \langle \top \sqsubseteq \exists r_n.\top \rangle.
\]

(5.3)

In fact, this forces every domain element to have an \(r_i\)-successor with degree 1 for each \(i \in \mathbb{N}\).

The Transfer Property

To finish the construction of the search tree of \(P\), it remains to transfer the values of \(D_{V_{ow}}\circ v\) to the value of \(V\) at the \(r_i\)-successors. We also have to transfer the values of \(D_{W_{ow}}\) and the auxiliary constants \(M,V_1,\ldots,V_n,W_1,\ldots,W_n\). This is accomplished using the axioms

\[
\langle \exists r_1.V \sqsubseteq D_{V_{ow}} \rangle, \langle D_{V_{ow}} \sqsubseteq \forall r_1.V \rangle \\
\langle \exists r_1.W \sqsubseteq D_{W_{ow}} \rangle, \langle D_{W_{ow}} \sqsubseteq \forall r_1.W \rangle \\
\langle \exists r_1.M \sqsubseteq M \rangle, \langle M \sqsubseteq \forall r_1.M \rangle \\
\ldots
\]

(5.4)

for each \(i \in \mathbb{N}\) (cf. Lemma 5.9). It can be shown that the axioms in (5.1)–(5.4) restrict all their models to “embed” an encoding of the search tree of \(P\). This is summarized in the canonical model property in the next section (for details, see the proof of Theorem 5.3).

The Solution Property

Finally, to ensure that \(V\) and \(W\) always encode different words, we employ the axiom

\[
\langle \top \sqsubseteq ((V \to W) \cap (W \to V)) \to M \rangle.
\]

(5.5)

This ensures that at each node \(\nu \in \mathbb{N}^*\) of the search tree one of the concepts \(V \to W\) or \(W \to V\) has a value smaller than or equal to that of \(M\), i.e. 1/2. This means that \(\text{enc}(v_\nu)\) and \(\text{enc}(w_\nu)\) differ by at least a factor of 2, which is equivalent to the fact that \(v_\nu \neq w_\nu\) (for details, see Lemmata 5.5 and 5.10).

Thus, if we collect all the axioms in (5.1)–(5.5), the resulting ontology is consistent iff \(P\) has no solution. Therefore, consistency w.r.t. witnessed models in \(\Pi\)-\(\Delta\) with crisp GCIs and equality assertions is undecidable. For other fuzzy DLs, different steps of this construction are more or less difficult, depending on the \(t\)-norm and the allowed constructors. In the next section, we present a generalized description of how to show undecidability by a reduction from the PCP, which is instantiated in the subsequent sections to yield undecidability results for a variety of fuzzy description logics.

5.1.2 The Framework

In the following, let \(P\) be an instance of the PCP and \(\otimes\)-\(\mathcal{L}\) be any fuzzy DL as introduced in Section 2.2. We first formalize the requirements for the encoding function \(\text{enc}\). Recall
from the previous section that we have to be able to concatenate constant words \((v_i)\) to already computed encodings of words \((v_\nu)\). Furthermore, we need to be able to test equality of words by comparing the residua of their encodings. When \(\text{enc}\) satisfies the latter property, we call it a **valid encoding function**. The former requirement will be formalized later in the concatenation property.

Recall that by the properties of continuous t-norms, for every \(p, q \in [0, 1]\), we have \(p = q \iff q \Rightarrow p \quad \text{and} \quad q \Rightarrow p \) are both 1 (see Section 2.1.2). Thus, to decide whether \(\mathcal{P}\) has a solution, it suffices to check whether \(\text{enc}(v_\nu) \Rightarrow \text{enc}(w_\nu) < 1 \) or \(\text{enc}(w_\nu) \Rightarrow \text{enc}(v_\nu) < 1\) holds for every \(\nu \in N^*\). In the special case in Section 5.1.1, it is clear that these residua are either 1 or smaller or equal to 1/2. Thus, the test simplifies to checking whether \(\text{enc}(v_\nu) \Rightarrow \text{enc}(w_\nu) \leq 1/2 \) or \(\text{enc}(w_\nu) \Rightarrow \text{enc}(v_\nu) \leq 1/2\) holds. However, in general we cannot put a constant bound on these residua in case they are smaller than 1. Instead, we can often construct a word whose encoding bounds these residua. Clearly, the precise word and encoding depend on the t-norm used.

Another difference to the previous section is that we allow a word \(u\) to be encoded by a set of values \(\text{Enc}(u) \subseteq [0, 1]\). This simplifies some of the proofs. However, we have to ensure that these encodings remain unique, i.e. that no two words can be encoded by the same value.

**Definition 5.2 (valid encoding function)** A function \(\text{Enc}: \Sigma_0^* \rightarrow 2^{[0,1]}\) is called a **valid encoding function** for \(\otimes\) if

a) for every \(u \in \{\varepsilon\} \cup \Sigma \Sigma_0^*\) and every \(u' \in \{0\}^*\), we have \(\text{Enc}(u' u) = \text{Enc}(u)\),

b) the sets \(\text{Enc}(u_1)\) and \(\text{Enc}(u_2)\) are nonempty and disjoint for any two different words \(u_1, u_2 \in \{\varepsilon\} \cup \Sigma \Sigma_0\), and

c) there exist two words \(u_\varepsilon, u_+ \in \Sigma_0^*\) such that for every \(\nu \in N^*\), \(p \in \text{Enc}(v_\nu)\), \(q \in \text{Enc}(w_\nu)\), and \(m \in \text{Enc}(u_\varepsilon \cdot u_+\nu|\nu|)\) it holds that \(u_\varepsilon \cdot u_+\nu|\nu| \in \{\varepsilon\} \cup \Sigma \Sigma_0^*\) and \(v_\nu \neq w_\nu \iff \min\{p \Rightarrow q, q \Rightarrow p\} \leq m\).

Condition a) reflects the fact that we often view the words of \(\Sigma_0^*\) as natural numbers in base \(s + 1\) (cf. Section 5.1.1), and thus words that differ only in the number of leading zeros should have the same encoding. Condition b) ensures that one can uniquely identify a word from its encoding—at least modulo any leading zeros. Finally, Condition c) requires that every value in \(\text{Enc}(u_\varepsilon \cdot u_+\nu|\nu|)\) can be used to check whether encodings of \(v_\nu\) and \(w_\nu\) are equal by comparing the above residua to this value.

In the following, let \(\text{Enc}\) be a valid encoding function for \(\otimes\), and \(u_\varepsilon, u_+\) be the words required by Condition c). We additionally assume that we are given a function \(\text{enc}: \Sigma_0^* \rightarrow [0, 1]\) that chooses a representative \(\text{enc}(u) \in \text{Enc}(u)\) for each \(u \in \Sigma_0^*\). Such a function must always exist because of Conditions a) and b) of Definition 5.2.

As in the previous section, we use the concept names \(V, W\) to represent the values of the words \(v_\nu\) and \(w_\nu\) at the nodes of the search tree for \(\mathcal{P}\). We designate the concept name \(M\) to encode the bounding word \(u_\varepsilon \cdot u_+\nu|\nu|\) from Definition 5.2, and \(M_+\) to store \(u_+\). We also use the concept names \(V_i, W_i\) to encode the words \(v_i, w_i\) from \(\mathcal{P}\), and the role names \(r_i\) to distinguish the different successors in the search tree, for each \(i \in N\). The individual name \(c_\nu\) is used to specify the root node.

Formally, the search tree for \(\mathcal{P}\) is represented by the **canonical model** \(\mathcal{I}_\mathcal{P} = (N^*, I_\mathcal{P})\) of the ontology \(\mathcal{O}_\mathcal{P}\) we will construct. It is defined as follows for every \(\nu \in N^*\) and \(i \in N\):
\[ V: \text{enc}(v_1) \]
\[ W: \text{enc}(w_1) \]
\[ M: \text{enc}(u_\varepsilon) \]

\[ V: \text{enc}(v_1 v_1) \]
\[ W: \text{enc}(w_1 w_1) \]
\[ M: \text{enc}(u_\varepsilon u_+) \]

\[ V: \text{enc}(v_1 v_2) \]
\[ W: \text{enc}(w_1 w_2) \]
\[ M: \text{enc}(u_\varepsilon u_+) \]

\[ V: \text{enc}(v_1 v_n) \]
\[ W: \text{enc}(w_1 w_n) \]
\[ M: \text{enc}(u_\varepsilon u_+ | \nu|) \]

\[ V: \text{enc}(v_\nu v_1) \]
\[ W: \text{enc}(w_\nu w_1) \]
\[ M: \text{enc}(u_\varepsilon u_+ | \nu| + 1) \]

\[ V: \text{enc}(v_\nu v_n) \]
\[ W: \text{enc}(w_\nu w_n) \]
\[ M: \text{enc}(u_\varepsilon u_+ | \nu| + 1) \]

\[ r_1 \]
\[ r_2 \]
\[ \ldots \]
\[ r_n \]

Figure 5.2: The canonical model \( \mathcal{I}_P \) for an instance \( P \) of the PCP

- \( c_{I_P} := \varepsilon \),
- \( V_{I_P}(\nu) := \text{enc}(v_\nu) \), \( W_{I_P}(\nu) := \text{enc}(w_\nu) \),
- \( V_{I_P}^i(\nu) := \text{enc}(v_i) \), \( W_{I_P}^i(\nu) := \text{enc}(w_i) \),
- \( M_{I_P}(\nu) := \text{enc}(u_\varepsilon \cdot u_+ | \nu|) \), \( M_{I_P}^+(\nu) := \text{enc}(u_+) \),
- \( r_{I_P}^i(\nu, \nu) := 1 \), and \( r_{I_P}^i(\nu, \nu') := 0 \) for all \( \nu' \in N^* \setminus \{ \nu_i \} \).

Since every element of \( N^* \) has exactly one \( r_i \)-successor with degree greater than 0, \( \mathcal{I}_P \) is finitely branching, and thus witnessed (see Lemma 2.15). This model is depicted in Figure 5.2 and clearly represents the search tree for \( P \) (cf. Figure 5.1).

Recall that our goal is to construct an ontology \( \mathcal{O}_P \) that can only be satisfied by interpretations that “include” the search tree of \( P \). Given that the interpretation \( \mathcal{I}_P \) represents this tree, we want the logic to satisfy the following property. Here, we use the expression \( p \sim q \) for \( p, q \in [0, 1] \) to denote the fact that \( p, q \in \text{Enc}(u) \) for some word \( u \in \Sigma_0^* \). By Conditions a) and b) of Definition 5.2, this word is unique except for the number of leading zeros. But Condition a) ensures that leading zeros are irrelevant for the encoding, and thus from \( p \sim q \) and \( p \in \text{Enc}(u) \) for some \( u \in \Sigma_0^* \), we can always infer that \( q \in \text{Enc}(u) \).
The Canonical Model Property \((P_\Delta)\)

The logic \(\otimes \mathcal{L}\) has the canonical model property if there is an ontology \(\mathcal{O}_P\) such that for every model \(\mathcal{I}\) of \(\mathcal{O}_P\) there is a mapping \(g : \Delta^\mathcal{I}_P \rightarrow \Delta^\mathcal{I}\) with

\[
A^\mathcal{I}_P(\nu) \sim A^\mathcal{I}(g(\nu))
\]

for every \(A \in \{V,W,M,M_+\} \cup \bigcup_{i \in \mathbb{N}} \{V_i, W_i\}\) and \(\nu \in \nu^*\).

As in the previous section, rather than trying to prove this property directly for some fuzzy DL, we provide several simpler properties that together imply the canonical model property. We will often motivate the following constructions using only the concept \(\nu\) and the words \(\nu_i\); however, all the arguments apply analogously to \(W, w,\) and \(M, u_+, u_+^{[\nu]}\).

As illustrated in Section 5.1.1, we construct the search tree in an inductive way. First, we restrict every interpretation \(\mathcal{I}\) to satisfy that \(A^\mathcal{I}_P(\nu) \sim A^\mathcal{I}(c^\mathcal{I}_P)\) for every relevant concept name. This makes sure that the root \(\nu\) of the search tree is properly represented at the individual \(g(\nu) := c^\mathcal{I}_P\). Let now \(g(\nu)\) be a node satisfying this property, and \(i \in \mathbb{N}\). We ensure that there is a node \(g(\nu_i)\) that also satisfies the property in three steps: first, we force the existence of an individual \(y\) with \(r^\mathcal{I}_I(g(\nu), y) = 1\) and set \(g(\nu_i) := y\). Then we compute a value in \(\text{Enc}(v_i, v_i)\) from \(V^\mathcal{I}(g(\nu)) \in \text{Enc}(v_i)\) and \(V^\mathcal{I}_j(g(\nu)) \in \text{Enc}(v_i)\). Finally, we transfer this value to the previously created successor to ensure that \(V^\mathcal{I}_j(g(\nu_i)) \sim \text{enc}(v_i, v_i)\). The value of \(V^\mathcal{I}_j(g(\nu))\) for every \(j \in \mathbb{N}\) is similarly transferred to \(V^\mathcal{I}_j(g(\nu_i))\).

Each step of the previous construction is guaranteed by a property of the logic \(\otimes \mathcal{L}\). These properties, which will ultimately be used to construct the ontology \(\mathcal{O}_P\), are described next.

The Initialization Property \((P_{\text{ini}})\)

The logic \(\otimes \mathcal{L}\) has the initialization property if for every concept \(C\), individual name \(c_i\), and \(u \in \Sigma_v^0\) there is an ontology \(\mathcal{O}_{C(c) = u}\) such that for every model \(\mathcal{I}\) of \(\mathcal{O}_{C(c) = u}\) it holds that \(C^\mathcal{I}(c^\mathcal{I}) \in \text{Enc}(u)\).

Assume now that \(\otimes \mathcal{L}\) satisfies \(P_{\text{ini}}\). Then, to initialize the search tree, we can set the values of \(V\) and \(W\) at \(c_i\) to encodings of \(v_1\) and \(w_1\), respectively, and the value of \(M\) to an encoding of \(u_+\). Moreover, the concept name \(M_+\) should encode \(u_+\) and every \(V_i\) and \(W_i\) should encode the words \(v_i\) and \(w_i\), respectively, for every \(i \in \mathbb{N}\). To this end, we define the ontology

\[
\mathcal{O}_{P,\text{ini}} := \mathcal{O}_{M(c) = u_+} \cup \mathcal{O}_{M_+(c) = u_+} \cup \mathcal{O}_{V(c) = v_1} \cup \mathcal{O}_{W(c) = w_1} \cup \bigcup_{i \in \mathbb{N}} \mathcal{O}_{V_i(c) = v_i} \cup \mathcal{O}_{W_i(c) = w_i}.
\]

We write \(\cup\) here in an abuse of notation to express that the ABoxes, TBoxes, and RBoxes of the respective ontologies are merged. This is an abstract version of the axioms (5.1) presented in Section 5.1.1 for \(\Pi\mathcal{L}\). Note that there we had \(u_+ = \varepsilon\), and thus the concept name \(M_+\) was unnecessary.
The Transfer Property (P\textsubscript{o})

The logic \(\otimes\mathcal{L}\) has the *concatenation property* if for all words \(u \in \Sigma_0^+\) and concepts \(C\) and \(C_u\) there is an ontology \(\mathcal{O}_{C_0u}\) and a concept name \(D_{C_0u}\) such that for every model \(I\) of \(\mathcal{O}_{C_0u}\) and every \(x \in \Delta^I\), if \(C^I_u(x) \in \text{Enc}(u)\) and \(C^I(x) \in \text{Enc}(u')\) for some \(u' \in \{\varepsilon\} \cup \Sigma_0\), then \(D^I_{C_0u}(x) \in \text{Enc}(u'u)\).

The goal of this property is to ensure that at every domain element with \(V^I(x) \in \text{Enc}(v)\) for some \(v \in N^+\) and \(C^I_u(x) \in \text{Enc}(v)\), we also have \(D^I_{V^Iu}(x) \in \text{Enc}(v)\), and similarly for \(W, w_i\) and \(M, u_i\). Thus, we define the ontology

\[
\mathcal{O}_{P_o} := \mathcal{O}_{M_{ou}} \cup \bigcup_{i \in N} \mathcal{O}_{V^Iu_i} \cup \mathcal{O}_{W^Iw_i}.
\]

To simplify the notation, we use the concept names \(V_i, W_i, M_i\) instead of \(C_{v_i}, C_{w_i}, C_{u_i}\) in this ontology. Thus, \(D_{V^Iu_i}\) now encodes the concatenation of the words encoded by \(V\) and \(V_i\) for every \(i \in N\). This corresponds to the axioms given for \(\Pi\mathcal{AL}\) in (5.2).

Note that by construction, the values of \(V^I(x), W^I(x)\), and \(M^I(x)\) should always be encodings of words from \(\{\varepsilon\} \cup \Sigma_0^+\).

The Successor Property (P\textsubscript{\rightarrow})

The logic \(\otimes\mathcal{L}\) has the *successor property* if for all role names \(r\) there is an ontology \(\mathcal{O}_{\exists r}\) such that for every model \(I\) of \(\mathcal{O}_{\exists r}\) and every \(x \in \Delta^I\) there is an element \(y \in \Delta^I\) with \(r^I(x, y) = 1\).

If a logic satisfies this property, then the ontology

\[
\mathcal{O}_{P_{\rightarrow}} := \bigcup_{i \in N} \mathcal{O}_{\exists r_i}
\]

ensures the existence of an \(r_i\)-successor with degree 1 for every node of the search tree and every \(i \in N\), corresponding to the \(r_i\)-connections in the canonical model. For our initial example of \(\Pi\mathcal{AL}\), this task was achieved by the axioms in (5.3).

The Transfer Property (P\textsubscript{\leftarrow})

The logic \(\otimes\mathcal{L}\) has the *transfer property* if for all concepts \(C, D\) and role names \(r\) there is an ontology \(\mathcal{O}_{C^\perp D}\) such that for every model \(I\) of \(\mathcal{O}_{C^\perp D}\) and every \(x, y \in \Delta^I\), if \(C^I(x) \in \text{Enc}(u)\) for some \(u \in \Sigma_0^+\) and \(r^I(x, y) = 1\), then \(D^I(y) \in \text{Enc}(u)\).

To ensure that encodings of \(u_p \cdot u_i^\perp, u_+, v_{pi},\) and \(v_j, j \in N\), are transferred from \(x\) to the \(r_i\)-successor \(y_i\) for every \(i \in N\), we use the ontology

\[
\mathcal{O}_{P_{\leftarrow}} := \bigcup_{i \in N} \mathcal{O}_{D_{M_{ou}}^\perp} \cup \mathcal{O}_{M_{+}}^\perp \cup \mathcal{O}_{D_{V^Iu}^\perp} \cup \mathcal{O}_{D_{W^Iw}^\perp} \cup \bigcup_{i,j \in N} \mathcal{O}_{V^I_j^\perp V^I_j^\perp} \cup \mathcal{O}_{W^I_j^\perp W^I_j^\perp}.
\]

This was accomplished by the \(\Pi\mathcal{AL}\)-axioms in (5.4).

As argued before, if we combine these four properties, then we obtain the canonical model property.
Theorem 5.3 Let $\text{Enc}$ be a valid encoding function for $\otimes$. If the logic $\otimes\mathcal{L}$ satisfies $P_{\text{ini}}$, $P_\otimes$, $P_\rightarrow$, and $P_\leftarrow$, then it also satisfies $P_\triangle$.

Proof. We show that the ontology $O_P := O_{P,\text{ini}} \cup O_{P,\otimes} \cup O_{P,\rightarrow} \cup O_{P,\leftarrow}$ satisfies the conditions from the definition of $P_\triangle$. We prove this only for the concept names $V$ and $V_i$, as the claim for $W$, $W_i$ and $M$, $M+$ follows from analogous arguments. For a model $\mathcal{I}$ of $O_P$, we construct the function $g : N^* \rightarrow \Delta^\mathcal{I}$ inductively as follows.

We set $g(\varepsilon) := c^\mathcal{I}$. Since $\mathcal{I}$ is a model of $O_{P,\text{ini}}$, we have $V^\mathcal{I}(g(\varepsilon)) = V^\mathcal{I}(c^\mathcal{I}) \in \text{Enc}(v_1)$, and thus $V^\mathcal{I}(g(\varepsilon)) \sim \text{enc}(v_1) = V^\mathcal{I}_P(\varepsilon)$.

Let now $\nu$ be a node of the search tree for which $g(\nu)$ has already been defined, and assume that $V^\mathcal{I}(g(\nu)) \sim \text{enc}(v_\nu)$ and $V^\mathcal{I}_P(g(\nu)) \sim \text{enc}(v_\nu)$. Since $\text{Enc}$ is a valid encoding function and by the definition of $\sim$, we know that $V^\mathcal{I}(g(\nu)) \in \text{Enc}(v_\nu)$ and $V^\mathcal{I}_P(g(\nu)) \in \text{Enc}(v_\nu)$ hold.

From the fact that $\mathcal{I}$ is a model of $O_{P,\otimes}$ we can infer that $D^\mathcal{I}_{\text{enc}}(g(\nu)) \in \text{Enc}(v_{\nu})$.

Since $\mathcal{I}$ satisfies $O_{P,\rightarrow}$, for each $i \in \{1, \ldots, n\}$ there must be an element $y_i \in \Delta^\mathcal{I}$ with $r_i^\mathcal{I}(g(\nu), y_i) = 1$. We define $g(v_\nu) := y_i$.

The ontology $O_{P,\leftarrow}$ now ensures that $V^\mathcal{I}(g(v_\nu)) \sim D^\mathcal{I}_{\text{enc}}(g(\nu)) \sim \text{enc}(v_{\nu}) = V^\mathcal{I}_P(v_\nu)$ and $V^\mathcal{I}_P(g(v_\nu)) \sim \text{enc}(v_\nu) = V^\mathcal{I}_P(v_\nu)$ hold for all $i \in N$. \hfill \Box

We now describe how the property $P_\triangle$ can be used to prove undecidability of $\otimes\mathcal{L}$. Recall that the idea is to add a set $O_{V \neq W}$ of axioms (as in (5.5)) to $O_P$ so that every model $\mathcal{I}$ is restricted to satisfy $V^\mathcal{I}(g(\nu)) \neq W^\mathcal{I}(g(\nu))$ for every $\nu \in N^*$, thus obtaining an ontology that is consistent if and only if $P$ has no solution.

More formally, we have to show that (i) every model of $O_P \cup O_{V \neq W}$ witnesses the non-existence of a solution for $P$, and (ii) if $P$ has no solution, then we can find a model of $O_P \cup O_{V \neq W}$. Part (i) uses the fact that every model of $O_P$ encodes the canonical model by $P_\triangle$. For part (ii), the idea is to show that $\mathcal{I}_P$ can be extended to a model of $O_P \cup O_{V \neq W}$. Formally, an interpretation $\mathcal{I}'$ is an extension of $\mathcal{I}_P$ if it agrees with $\mathcal{I}_P$ on the interpretation of all role names and the concept names relevant for $\mathcal{I}_P$, i.e. $V, W, M, M+, V_i, W_i$, but defines interpretations for additional (auxiliary) concept names. However, for this to work, $\mathcal{I}_P$ has to be a model of $O_P$ in the first place.

For the remainder of this section, we thus assume that $\mathcal{I}_P$ can actually be extended to a model of $O_P$—while $O_P$ might use additional concept names, it should not contradict the information about $V$, $W$, $M$, $V_i$, $W_i$, and $M+$ represented by $\mathcal{I}_P$. It is important to keep in mind for the subsequent sections that this constitutes an additional condition that has to be verified before we can show undecidability of a concrete fuzzy DL $\otimes\mathcal{L}$. We also assume that $\otimes\mathcal{L}$ satisfies $P_\triangle$. In Section 5.1.3, we will show that these assumptions actually hold for a variety of fuzzy description logics.

Recall that the key to showing undecidability of $\otimes\mathcal{L}$ is to be able to express the restriction that $V$ and $W$ encode different words at every node $\nu \in N^*$ of the search tree. Since $\text{Enc}$ is a valid encoding function and the concept name $M$ encodes the word $u_{\nu} \cdot u_{\nu}^{\square \nu}$ at every $\nu \in N^*$, it suffices to check whether, for all $\nu \in N^*$, either $(V \rightarrow W)^{\mathcal{I}_P}(\nu) \leq M^{\mathcal{I}_P}(\nu)$ or $(W \rightarrow V)^{\mathcal{I}_P}(\nu) \leq M^{\mathcal{I}_P}(\nu)$ holds (see Condition c) of Definition 5.2).
The Solution Property ($P_\neq$)

If the logic $\otimes\mathcal{L}$ satisfies $P_\triangleleft$ with $\mathcal{O}_P$ and $\mathcal{I}_P$ can be extended to a model of $\mathcal{O}_P$, then $\otimes\mathcal{L}$ has the solution property if there is an ontology $\mathcal{O}_{\neq W}$ such that the following conditions are satisfied:

1. For every model $\mathcal{I}$ of $\mathcal{O}_P \cup \mathcal{O}_{\neq W}$ and every $\nu \in \mathbb{N}^*$, we have
   \[
   \min\{V^\mathcal{I}(g(\nu)) \Rightarrow W^\mathcal{I}(g(\nu)), W^\mathcal{I}(g(\nu)) \Rightarrow V^\mathcal{I}(g(\nu))\} \leq M^\mathcal{I}(g(\nu)),
   \]
   where $g$ is the mapping obtained from $P_\triangleleft$ for $\mathcal{I}$.

2. If for every $\nu \in \mathbb{N}^*$ we have
   \[
   \min\{V^{P_\triangleleft}(\nu) \Rightarrow W^{P_\triangleleft}(\nu), W^{P_\triangleleft}(\nu) \Rightarrow V^{P_\triangleleft}(\nu)\} \leq M^{P_\triangleleft}(\nu),
   \]
   then $\mathcal{I}_P$ can be extended to a model of $\mathcal{O}_P \cup \mathcal{O}_{\neq W}$.

If a fuzzy DL satisfies this property, then consistency of ontologies is undecidable.

**Theorem 5.4** If $\otimes\mathcal{L}$ satisfies $P_\neq$, then $\mathcal{P}$ has a solution iff $\mathcal{O}_P \cup \mathcal{O}_{\neq W}$ is inconsistent.

**Proof.** If $\mathcal{O}_P \cup \mathcal{O}_{\neq W}$ is inconsistent, then in particular no extension of $\mathcal{I}_P$ can satisfy this ontology. By $P_\neq$, there is a $\nu \in \mathbb{N}^*$ such that
\[
V^{P_\triangleleft}(\nu) \Rightarrow W^{P_\triangleleft}(\nu) > M^{P_\triangleleft}(\nu) \quad \text{and} \quad W^{P_\triangleleft}(\nu) \Rightarrow V^{P_\triangleleft}(\nu) > M^{P_\triangleleft}(\nu).
\]
By the definition of $\mathcal{I}_P$ and Condition c) of Definition 5.2, we have $v_\nu = w_\nu$, and thus $\mathcal{P}$ has a solution.

For the converse direction, assume that $\mathcal{O}_P \cup \mathcal{O}_{\neq W}$ has a model $\mathcal{I}$ and let $g$ be the function given by $P_\triangleleft$. By $P_\neq$, for every $\nu \in \mathbb{N}^*$ we have
\[
V^{P_\triangleleft}(g(\nu)) \Rightarrow W^{P_\triangleleft}(g(\nu)) \leq M^{P_\triangleleft}(g(\nu)) \quad \text{or} \quad W^{P_\triangleleft}(g(\nu)) \Rightarrow V^{P_\triangleleft}(g(\nu)) \leq M^{P_\triangleleft}(g(\nu)).
\]
By $P_\triangleleft$, the definition of $\mathcal{I}_P$, and Condition c) of Definition 5.2, it follows that $v_\nu \neq w_\nu$. Since this holds for all $\nu \in \mathbb{N}^*$, we know that $\mathcal{P}$ has no solution.

Figure 5.3 informally depicts the relationships between all notions introduced in this section. The existence of a valid encoding function is a basic precondition for all our properties. The canonical model property is implied by the conjunction of the smaller properties. Finally, the solution property depends on the canonical model property and guarantees undecidability of consistency in the given logic $\otimes\mathcal{L}$.

Although we will consider several variants of this approach throughout this chapter, the basic structure consisting of the valid encoding function, the canonical model property, and the solution property remains the same. The first version presented in this section already suffices to show undecidability for many fuzzy description logics (see Theorem 5.11).

**5.1.3 First Results**

The first step in proving undecidability for concrete fuzzy DLs is to find a valid encoding function for our continuous t-norm $\otimes$. We assume in the following that $\otimes$ is not the
5.1 Consistency under Witnessed Model Semantics

Figure 5.3: A framework for showing undecidability of consistency in a fuzzy DL

Gödel t-norm. The reason for this is that our encoding function and the subsequent constructions depend on the choice of one component \( ((a,b), \otimes') \) of \( \otimes \) where \( \otimes' \) is either \( \mathfrak{t} \) or \( \Pi \). If \( \otimes \) is different from the Gödel t-norm, such a component must exist by Proposition 2.3. It is important that the component that we choose remains fixed throughout the whole construction. In the case that \( \otimes' = \mathfrak{t} \), we denote our choice by \( \mathfrak{t}^{(a,b)} \), and similarly for \( \otimes' = \Pi \). Correspondingly, we denote the fuzzy description logic by \( \mathfrak{t}^{(a,b)}-\mathcal{L} \) or \( \Pi^{(a,b)}-\mathcal{L} \).

For the case of the logic \( \mathfrak{t}^{(0,b)}-\mathcal{EL} \), i.e. the t-norm starts with \( \mathfrak{t} \) (or equivalently, it has zero divisors), recall that for every \( x \in (0,b] \) we have that \( x \Rightarrow 0 = b - x \); that is, the residual negation yields a “local involutive negation” over the interval \( (0,b] \) (see Section 2.1.3). Thus, the concept \( \exists C \) will be interpreted as the local involutive negation of the interpretation of \( C \), whenever the latter is in this interval. In this logic, we use the short-hand \( C \rightarrow D \) for \( \exists (C \cap \exists D) \) to express a function similar to the residuum. In fact, for all \( x,y \in [0,1] \), we have

\[
(x \otimes (y \Rightarrow 0)) \Rightarrow 0 = \begin{cases} y & \text{if } y < b \leq x \\ b - x + y & \text{if } y < x < b \\ 1 & \text{otherwise} \end{cases}
\]

In particular, \( (C \rightarrow D)^{T}(x) = (C \rightarrow D)^{T}(x) \) holds whenever \( D^{T}(x) < b \) for an interpretation \( T \) and \( x \in \Delta^{T} \).

Regardless of the logic, the following constructions often use the abbreviation \( C^{n} \) for the \( n \)-ary conjunction of a concept \( C \) with itself, i.e. \( C^{0} := \top \), and \( C^{n} := C \cap C^{n-1} \) for all \( n \geq 1 \).

We now use the chosen component to encode the words from \( \Sigma_{0}^{*} \). For \( u \in \{0\}^{*} \) (in particular for \( u = \varepsilon \)) we always use the encoding \( \text{Enc}(u) := [b,1] \), i.e. all values between the upper bound of our component and 1 are valid encodings for \( \varepsilon \). For these words, we define \( \text{enc}(u) := b \). For the remaining words \( u \in \Sigma_{0}^{*} \setminus \{0\}^{*} \), we use only a singleton
set Enc(\(u\)) := \{enc(\(u\))\}, where enc(\(u\)) depends on the chosen component. For the case of \(\Pi^{(a,b)}\), we define
\[
\text{enc}(\(u\)) := \sigma_{a,b}(2^{-u}) \in (a, b),
\]
and for \(\mathcal{L}^{(a,b)}\) we use
\[
\text{enc}(\(u\)) := \sigma_{a,b}(1 - 0.\overline{w}) \in (a, b),
\]
where \(\sigma_{a,b}(x) := a + (b - a)x\) for all \(x \in [0, 1]\). Observe that \(\sigma_{a,b}\) is a strictly monotone, bijective mapping from \([0, 1]\) to \([a, b]\) with the inverse defined by \(\sigma_{a,b}^{-1}(x) := \frac{x-a}{b-a}\) for all \(x \in [a, b]\).

These encodings are illustrated in Figures 5.4 and 5.5.

**Lemma 5.5** The functions Enc defined above are valid encoding functions for t-norms of the form \(\Pi^{(a,b)}\) or \(\mathcal{L}^{(a,b)}\).

*Proof.* In both cases, the encodings of different words \(u_1, u_2 \in \Sigma\Sigma_0^*\) are different, and in particular smaller than \(b\), and thus are not included in Enc(\(\varepsilon\)). Furthermore, the encodings do not depend on the number of leading zeros. Thus, the first two conditions of Definition 5.2 are satisfied. For Condition c), we analyze the two cases of \(\Pi^{(a,b)}\) and \(\mathcal{L}^{(a,b)}\) separately.

For \(\Pi^{(a,b)}\), consider two different words \(v, w \in \Sigma^*\) and assume w.l.o.g. that \(v < w\). Then \(v + 1 \leq w\) and hence \(2^{-w} \leq 2^{-(v+1)} = 2^{-v}/2\). If \(v \neq \varepsilon\), this implies that
\[
\text{enc}(v) \Rightarrow \text{enc}(w) = \sigma_{a,b}(2^{-w}/2^{-v}) \leq \sigma_{a,b}(1/2) = \text{enc}(1) < 1.
\]

For \(v = \varepsilon\), we have \(p \Rightarrow \text{enc}(w) = \text{enc}(w) \leq \text{enc}(1) < 1\) for any \(p \in \text{Enc}(\varepsilon) = [b, 1]\). Conversely, if \(v = w\), then \(\text{enc}(v) \Rightarrow \text{enc}(w) = 1 = \text{enc}(w) \Rightarrow \text{enc}(v)\). Thus, the words \(u_\varepsilon := 1\) and \(u_+ := \varepsilon\) satisfy Condition c) of Definition 5.2.

For the case of \(\mathcal{L}^{(a,b)}\), let \(k = \max\{|v_i|, |w_i| \mid i \in N\}\) be the maximal length of a word occurring in \(\mathcal{P}\). Then, for every \(\nu \in \mathcal{N}^*\), we have \(|v_\nu| \leq (|\nu| + 1)k\) and \(|w_\nu| \leq (|\nu| + 1)k\).

If \(v_\nu \neq w_\nu\), these words must differ in one of the first \(\ell := (|\nu| + 1)k\) letters. Thus, if \(v_\nu \neq \varepsilon\) and \(w_\nu \neq \varepsilon\), then either \(\text{enc}(v_\nu) > \text{enc}(w_\nu)\), and thus
\[
\text{enc}(v_\nu) \Rightarrow \text{enc}(w_\nu) = \sigma_{a,b}(\min\{1, 1 + 0.\overline{v_\nu} - 0.\overline{w_\nu}\})
\]
\[
= \min\{b, \sigma_{a,b}(1 + 0.\overline{v_\nu} - 0.\overline{w_\nu})\}
\]
\[
\leq \sigma_{a,b}(1 - (s + 1)^{-\ell+1})
\]
\[
= \sigma_{a,b}(1 - 0.\overline{\ell^f})
\]
\[
= \text{enc}(1 \cdot 0^f) < 1,
\]
or, similarly, \(\text{enc}(v_\nu) < \text{enc}(w_\nu)\) and \(\text{enc}(w_\nu) \Rightarrow \text{enc}(v_\nu) \leq \text{enc}(1 \cdot 0^f) < 1\). Note that again this also holds if \(v_\nu = \varepsilon\), since \(w_\nu\) also differs from \(\overline{0^f}\) in one of the first \(\ell\) letters,
and similarly if \( w_\nu = \epsilon \). Conversely, if \( v_\nu = w_\nu \), then both residua yield 1 as result, which is greater than \( \text{enc}(1 \cdot 0^k) \). Thus, setting \( u_\epsilon := 1 \cdot 0^k \) and \( u_+ := 0^k \) satisfies Condition c) of Definition 5.2.

Variants of the above encoding functions and words \( u_\epsilon, u_+ \) have been used before to show undecidability of fuzzy description logics based on the Product and Łukasiewicz t-norms (Baader and Peñaloza 2011b; Cerami and Straccia 2013).

We will now present several instances of \( \otimes \cdot \mathcal{L} \) that satisfy the properties of the previous section. Recall that one preconditions for \( P_{\text{ini}} \) is that \( I_P \) can be extended to a model of \( O_P \). Thus, in the following constructions of \( O_{C(c)=u}, O_{\exists r}, O_{C_{\text{out}}}, \) and \( O_{C \cdot \ominus D} \), it is important keep in mind that the resulting ontology \( O_P \) (as defined in the previous section) should not contradict information in \( I_P \). However, we are allowed to define values for auxiliary concept names like \( D_{V_{\text{out}}} \).

We now present several cases for \( \otimes \cdot \mathcal{L} \) in which the initialization property holds.

**Lemma 5.6** For every continuous t-norm \( \otimes \) except the Gödel t-norm, the following logics satisfy \( P_{\text{ini}} \):

- \( \otimes \cdot \mathcal{EL} \) with equality assertions,
- \( \otimes \cdot \mathcal{ELC} \) with inequality assertions, and
- \( \mathbb{L}^{(a,b)} \cdot \mathcal{NEL} \).

**Proof.** If we are allowed to use equality assertions, we can use the simple ontology

\[
O_{C(c)=u} := \{ \langle c: C = \text{enc}(u) \rangle \}
\]

to enforce that \( C^I(c^I) = \text{enc}(u) \in \text{Enc}(u) \) is satisfied by every model \( I \).

In \( \otimes \cdot \mathcal{ELC} \), the two inequality assertions

\[
\langle c: C \geq \text{enc}(u) \rangle, \quad \langle c: \neg C \geq 1 - \text{enc}(u) \rangle
\]

express the same restriction. The first axiom ensures that \( C^I(c^I) \geq \text{enc}(u) \), while the second requires that \( 1 - C^I(c^I) \geq 1 - \text{enc}(u) \), i.e. \( C^I(c^I) \leq \text{enc}(u) \), holds.

For the logic \( \mathbb{L}^{(a,b)} \cdot \mathcal{NEL} \), a more involved construction is necessary. We first ensure that a fresh auxiliary concept name \( A \) has a value from \( \text{Enc}(u) \) at all domain elements, and then require that \( C \) and \( A \) have the same value at \( c \). For the first part, we use the two axioms

\[
\langle H^{(s+1)|u} \equiv \square H^{(s+1)|u} \rangle, \quad \langle A \equiv H^{2|u} \rangle
\]

Observe that, whenever \( H^I(x) \in [0, b] \) for some interpretation \( I \) and \( x \in \Delta^I \), then for every \( m \in \mathbb{N} \) we have by linearity of \( \sigma_{0,b} \) that

\[
(H^m)^I(x) = \sigma_{0,b}(\max\{0, m(\sigma_{0,b}(H^I(x)) - 1) + 1\}) = \max\{0, m(H^I(x) - b) + b\}.
\]
Let now $\mathcal{I}$ be an interpretation that satisfies these axioms and $x \in \Delta^\mathcal{I}$. If $u \in \{0\}^*$, then the second axiom enforces that $A^\mathcal{I}(x) = \top^\mathcal{I}(x) = 1 \in \text{Enc}(u)$ holds. If $u \notin \{0\}^*$, then by the first axiom we have

$$\max\{0, (s+1)^{|u|}(H^\mathcal{I}(x) - b) + b\} = b - \max\{0, (s+1)^{|u|}(H^\mathcal{I}(x) - b) + b\}. $$

This shows that $-b = 2(s+1)^{|u|}(H^\mathcal{I}(x) - b)$, and thus $H^\mathcal{I}(x) = b - \frac{b}{2(s+1)^{|u|}}$. From the second axiom, it follows that

$$A^\mathcal{I}(x) = \max\{0, 2\hat{u}(-\frac{b}{2(s+1)^{|u|}}) + b\}. $$

Since $\frac{\hat{u}}{(s+1)^{|u|}} = 0.\hat{u} < 1$, we obtain $A^\mathcal{I}(x) = b - b(0.\hat{u}) = \sigma_{0,b}(1 - 0.\hat{u}) = \text{enc}(u)$.

For the second part, we use the axiom

$$(c:(C \rightarrow A) \cap (A \rightarrow C)).$$

If $u \in \{0\}^*$, then the semantics of $\rightarrow$ and the fact that $A^\mathcal{I}(c^\mathcal{I}) \in \text{Enc}(u) = \text{Enc}(\varepsilon) = [b, 1]$ imply that also $C^\mathcal{I}(c^\mathcal{I}) \in [b, 1] = \text{Enc}(u)$. If $u \notin \{0\}^*$, then $A^\mathcal{I}(c^\mathcal{I}) = \text{enc}(u) < b$, which implies that $C^\mathcal{I}(c^\mathcal{I}) < b$, and thus $C^\mathcal{I}(c^\mathcal{I}) = A^\mathcal{I}(c^\mathcal{I}) = \text{enc}(u)$. \qed

We now analyze the successor and concatenation properties. It turns out that they hold for all logics $\otimes$-$\mathcal{L}$ that we consider. In particular, the successor property only needs the constructors $\top$ and $\exists$ and the restriction to witnessed models, whereas the concatenation property only requires the constructors $\top$ and $\land$.

**Lemma 5.7** For every continuous t-norm $\otimes$, the logic $\otimes$-$\mathcal{L}$ satisfies $P_{\rightarrow}$.

**Proof.** Consider the ontology $O_{\exists r} := \{(\top \sqsubseteq \exists r. \top)\}$. Any model $\mathcal{I}$ of this axiom satisfies $\langle \exists r. \top \rangle^\mathcal{I}(x) = 1$ for every $x \in \Delta^\mathcal{I}$. Since reasoning is restricted to witnessed models, there must exist a $y \in \Delta^\mathcal{I}$ with $r^\mathcal{I}(x, y) = 1$. \qed

We mention here that the successor property can easily be obtained even without the restriction to witnessed models if the logic allows axioms of the form $\text{crisp}(r)$ that state that a role should be crisp, similar to the restriction in Section 3.1.5. In this case, the following undecidability results hold even for general model semantics (Borgwardt and Peñaloza 2012c).

**Lemma 5.8** For every continuous t-norm $\otimes$ except the Gödel t-norm, the logic $\otimes$-$\mathcal{L}$ satisfies $P_{\circ}$.

**Proof.** By assumption, $\otimes$ must contain either the Product or the Łukasiewicz t-norm in some interval. We divide the proof depending on the representative chosen for the encoding function.

For the case of $\Pi^{(a,b)}$-$\mathcal{L}$, observe that for every $u \in \Sigma^*_0$ and $u' \in \Sigma^*_0$, it holds that $u'(s+1)^{|u|} + u = u'u$. Given $u \in \Sigma^*_0$, we define the ontology

$$O_{C_{ou}} := \{\langle D_{C_{ou}} \equiv C^{(s+1)^{|u|}} \cap C_u\rangle\}. $$

Observe that for every interpretation $\mathcal{I}$ and $x \in \Delta^\mathcal{I}$, if $C^\mathcal{I}(x)$ is of the form $\sigma_{a,b}(p)$ for some $p \in [0,1]$, then for all $m \in \mathbb{N}$, we have

$$(C^m)^\mathcal{I}(x) = \sigma_{a,b}(p^m).$$
Let now $\mathcal{I}$ be a model of $\mathcal{O}_{\text{Cou}}$, $x \in \Delta^I$, and $u' \in \{\varepsilon\} \cup \Sigma^*_0$ such that $C^I_u(x) \in \text{Enc}(u)$ and $C^I(x) \in \text{Enc}(u')$. If $u \notin \{0\}^*$ and $u' \neq \varepsilon$, then we have

$$D^I_{\text{Cou}}(x) = \sigma_{a,b}(2^{-(u'(s+1)^{|u|}+u)}) = \text{enc}(u'u).$$

If $u \in \{0\}^*$ and $u' \neq \varepsilon$, we have $C^I_u(x) \in [b, 1]$, and thus

$$D^I_{\text{Cou}}(x) = (C(x)^{|u|}I(x) = \sigma_{a,b}(2^{-(u'(s+1)^{|u|}+u)}) = \text{enc}(u'u).$$

Similarly, for $u \notin \{0\}^*$ and $u' = \varepsilon$ we get $(C(x)^{|u|}I(x) \in [b, 1]$, which implies that

$$D^I_{\text{Cou}}(x) = C^I_u(x) = \text{enc}(\varepsilon u).$$

Finally, if $u \in \{0\}^*$ and $u' = \varepsilon$, then $D^I(x) = (C(x)^{|u|} \cap C_u)^{I(x)} \in [b, 1] = \text{Enc}(\varepsilon u)$.

For the case of $\mathcal{L}^{(a,b)}_\mathcal{E}$, we define the ontology

$$\mathcal{O}_{\text{Cou}} := \{\langle C^{|u|} \equiv C \rangle, \langle D^I_{\text{Cou}} \equiv C' \cap C_u \rangle \}.$$

Let $\mathcal{I}$ be a model of $\mathcal{O}_{\text{Cou}}$, $x \in \Delta^I$, and assume that $C^I_u(x) \in \text{Enc}(u)$ and $C^I(x) \in \text{Enc}(u')$ for some $u' \in \{\varepsilon\} \cup \Sigma^*_0$. If $u' \neq \varepsilon$, then from the first axiom it follows that

$$(C(x)^{|u|}I(x) = C^I(x) = \sigma_{a,b}(1 - \alpha_\varepsilon u') \in (a, b).$$

Since $\otimes (a, b)$-contains Łukasiewicz, this implies that $C(x) \in (a, b)$. Thus,

$$\sigma_{a,b}(\max\{0, (s + 1)^{|u|}(\sigma_{a,b}(C^I(x)) - 1)\}) = C^I(x) = \sigma_{a,b}(1 - \alpha_\varepsilon u'),$$

which shows that

$$C^I(x) = \sigma_{a,b}(1 - (s + 1)^{-|u|} \alpha_\varepsilon u').$$

If $u \notin \{0\}^*$, then it follows that

$$D^I_{\text{Cou}}(x) = \sigma_{a,b}(\max\{0, (1 - 0, \varepsilon u') + (1 - (s + 1)^{-|u|} \alpha_\varepsilon u')\})$$

$$= \sigma_{a,b}(1 - 0, \varepsilon u' - (s + 1)^{-|u|} \alpha_\varepsilon u')$$

$$= \sigma_{a,b}(1 - 0, \varepsilon u')$$

$$= \text{enc}(u'u).$$

If $u \in \{0\}^*$, then $C^I_u(x) \in [b, 1]$, and thus

$$D^I_{\text{Cou}}(x) = C^I(x) = \sigma_{a,b}(1 - (s + 1)^{-|u|} \alpha_\varepsilon u') = \text{enc}(u'u).$$

It remains to consider the case that $u'$ is the empty word, and thus $C^I(x) \in [b, 1]$. By the first axiom, we also have $C^I(x) \in [b, 1]$. If $u \notin \{0\}^*$, then

$$D^I_{\text{Cou}}(x) = C^I_u(x) = \text{enc}(u) = \text{enc}(\varepsilon u).$$

On the other hand, if $u \in \{0\}^*$, then we have $D^I_{\text{Cou}}(x) \in [b, 1] = \text{Enc}(\varepsilon u)$.

\[\square\]
So far, we have established three of the properties required for the canonical model property for the logics mentioned in Lemma 5.6: $\otimes\mathcal{FL}$ with equality assertions, $\otimes\mathcal{ELC}$ with inequality assertions, and $\mathcal{L}^{(0,b)}\mathcal{FL}$ with crisp ontologies. This leaves open only the transfer property, which holds for the latter two, but for $\otimes\mathcal{ELC}$ only if we additionally allow value restrictions.

**Lemma 5.9** For every continuous t-norm $\otimes$ except the G"{o}del t-norm, the logics $\otimes\mathcal{AL}$, $\otimes\mathcal{ELC}$, and $\mathcal{L}^{(0,b)}\mathcal{FL}$ satisfy $P_\otimes$.

*Proof.* Let $I$ be an interpretation and $x, y \in \Delta^I$ such that $C^I(x) \in \text{Enc}(u)$ for some $u \in \Sigma_0^I$ and $r^I(x, y) = 1$. Regardless of whether we have chosen $\mathcal{L}^{(a,b)}$ or $\mathcal{L}^{(u,b)}$, if $u \notin \{0\}^*$, then the goal is to ensure that $D^I(y) = C^I(x)$. On the other hand, if $u \in \{0\}^*$, then $C^I(x) \geq b$, and we only need to ensure that $D^I(x) \geq b$.

In all fuzzy DLs based on $\mathcal{EL}$, we can formulate the axiom $\langle \exists r.D \sqsubseteq C \rangle$. If $I$ satisfies this axiom, then

$$D^I(y) = r^I(x, y) \otimes D^I(y) \leq (\exists r.D)^I(x) \leq C^I(x).$$

We now add an axiom ensuring that also $D^I(y) \geq C^I(x)$ holds if $u \notin \{0\}^*$, and $D^I(y) \geq b$ holds if $u \in \{0\}^*$. The precise form of this axiom depends on the expressivity of the logic.

In $\otimes\mathcal{AL}$, we can use the axiom $\langle C \sqsubseteq \forall r.D \rangle$ to restrict $I$ to satisfy

$$C^I(x) \leq (\forall r.D)^I(x) \leq r^I(x, y) \Rightarrow D^I(y) = D^I(y),$$

and thus also $D^I(y) \geq C^I(x) \geq b$ if $u \in \{0\}^*$.

In the case of $\otimes\mathcal{ELC}$, if $I$ is a model of $\langle \exists r.\neg D \sqsubseteq \neg C \rangle$, then

$$1 - D^I(y) = r^I(x, y) \otimes (1 - D^I(y)) \leq (\exists r.\neg D)^I(x) \leq 1 - C^I(x),$$

and thus $C^I(x) \leq D^I(y)$ as in the previous case.

Finally, for $\mathcal{L}^{(0,b)}\mathcal{FL}$, we use the axiom $\langle \exists r.D \sqsubseteq \exists C \rangle$, similar to the one for $\otimes\mathcal{ELC}$. If $I$ satisfies this axiom, then

$$\otimes D^I(y) = r^I(x, y) \otimes (\otimes D^I(y)) \leq (\exists r.\exists D)^I(x) \leq \otimes C^I(x).$$

If $u \notin \{0\}^*$, then $D^I(y) \leq C^I(x) < b$, which shows that $b - D^I(y) \leq b - C^I(x)$, and thus $C^I(x) \leq D^I(y)$. If $u \in \{0\}^*$, then $\otimes D^I(y) \leq \otimes C^I(x) = 0$, and thus $D^I(y) \geq b$ as required. \hfill $\Box$

Together with Theorem 5.3, the previous lemmata show that the logics $\otimes\mathcal{AL}$ with equality assertions, $\otimes\mathcal{ELC}$ with inequality assertions, and $\mathcal{L}^{(0,b)}\mathcal{FL}$ have the canonical model property (if $\otimes$ is not the G"{o}del t-norm). We now show that the latter two logics also have the solution property, while for $\otimes\mathcal{AL}$ we additionally need the implication constructor.

Recall that a necessary condition for the solution property is that the canonical model $I_P$ can be extended to a model of the ontology $O_P$ constructed from the individual parts in Lemmata 5.6–5.9. It is a simple task to verify that this holds in all the cases described above. We only need to assume that a unique new concept name is used
5.1 Consistency under Witnessed Model Semantics

for any auxiliary concept name appearing in the different ontologies, e.g. $A$, $H$, or $D_{V_{O_{V}}}$. In fact, the values of these auxiliary concept names at each node $\nu$ are uniquely determined by the values of the concept names $V, W, V_{i}, W_{i}, M, M_{+}$ at $\nu$. Moreover, since every $\nu$ has exactly one $r_{i}$-successor with degree greater than 0 for every $i \in N$, it follows that $\mathcal{I}_{\mathcal{P}}$ can be extended to a finitely branching, and thus witnessed, model of $\mathcal{O}_{\mathcal{P}}$ (see Lemma 2.15).

Lemma 5.10 Let $\otimes$ be any continuous t-norm except the Gödel t-norm. If any logic based on $\otimes$-$\mathcal{ELC}$, $\otimes$-$\mathcal{ELL}$, or $\mathcal{L}_{\mathcal{ER}}^{(0, b)}$ satisfies $P_{\Delta}$ with $\mathcal{O}_{\mathcal{P}}$ and $\mathcal{I}_{\mathcal{P}}$ can be extended to a model of $\mathcal{O}_{\mathcal{P}}$, then this logic also satisfies $P_{\neq}$.

Proof. For $\otimes$-$\mathcal{ELC}$, we define the ontology $\mathcal{O}_{\mathcal{V}_{\neq W}} := \{(V \rightarrow W) \cap (W \rightarrow V) \subseteq M\}$. For every model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{\mathcal{V}_{\neq W}}$ and every $\nu \in N^{*}$, we have

$$(V^{\mathcal{I}} (g(\nu)) \Rightarrow W^{\mathcal{I}} (g(\nu))) \otimes (W^{\mathcal{I}} (g(\nu)) \Rightarrow V^{\mathcal{I}} (g(\nu))) \leq M^{\mathcal{I}} (g(\nu)),$$

where $g$ is the function given by $P_{\Delta}$ for $\mathcal{I}$. Since at least one of the two residua must be 1, this implies $\min\{V^{\mathcal{I}} (g(\nu)) \Rightarrow W^{\mathcal{I}} (g(\nu)), W^{\mathcal{I}} (g(\nu)) \Rightarrow V^{\mathcal{I}} (g(\nu))\} \leq M^{\mathcal{I}} (g(\nu))$.

For the second condition, assume that $\mathcal{I}_{\mathcal{P}}$ cannot be extended to a model of the combined ontology $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{\mathcal{V}_{\neq W}}$. Since there is an extension $\mathcal{I}$ of $\mathcal{I}_{\mathcal{P}}$ that satisfies $\mathcal{O}_{\mathcal{P}}$, we know that $\mathcal{I}$ must violate $\mathcal{O}_{\mathcal{V}_{\neq W}}$. This means that there is a $\nu \in N^{*}$ such that

$$M^{\mathcal{I}_{\mathcal{P}}} (\nu) < (V^{\mathcal{I}_{\mathcal{P}}} (\nu) \Rightarrow W^{\mathcal{I}_{\mathcal{P}}} (\nu)) \otimes (W^{\mathcal{I}_{\mathcal{P}}} (\nu) \Rightarrow V^{\mathcal{I}_{\mathcal{P}}} (\nu)) \leq \min\{V^{\mathcal{I}_{\mathcal{P}}} (\nu) \Rightarrow W^{\mathcal{I}_{\mathcal{P}}} (\nu), W^{\mathcal{I}_{\mathcal{P}}} (\nu) \Rightarrow V^{\mathcal{I}_{\mathcal{P}}} (\nu)\}.$$

For $\otimes$-$\mathcal{ELL}$, consider the ontology

$$\mathcal{O}_{\mathcal{V}_{\neq W}} := \{\langle X \subseteq X \cap X \rangle, \langle \top \subseteq \neg (X \cap \neg X) \rangle, \langle X \cap V \subseteq X \cap W \cap M \rangle, \langle \neg X \cap W \subseteq \neg X \cap V \cap M \rangle\}. \tag{5.6}$$

For any model $\mathcal{I}$ of the axioms (5.6) and all $x \in \Delta^{\mathcal{I}}$, we have $X^{\mathcal{I}} (x) \leq X^{\mathcal{I}} (x) \otimes X^{\mathcal{I}} (x)$, and hence $X^{\mathcal{I}} (x)$ must be an idempotent element w.r.t. $\otimes$. Recall that $X^{\mathcal{I}} (x)$ can thus not lie in any component of $\otimes$, which implies that $\otimes$ behaves like the Gödel t-norm for $X^{\mathcal{I}} (x)$. In particular, we get $0 \geq (X \cap \neg X)^{\mathcal{I}} (x) = \min\{X^{\mathcal{I}} (x), 1 - X^{\mathcal{I}} (x)\}$, and thus $X^{\mathcal{I}} (x) \in \{0, 1\}$.

Let now $\mathcal{I}$ be a model of $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{\mathcal{V}_{\neq W}}$, $g$ be the associated mapping from $\Delta^{\mathcal{I}_{\mathcal{P}}}$ to $\Delta^{\mathcal{I}}$, and $\nu \in N^{*}$. If $X^{\mathcal{I}} (g(\nu)) = 1$, then axiom (5.7) states that $V^{\mathcal{I}} (g(\nu)) \leq W^{\mathcal{I}} (g(\nu)) \otimes M^{\mathcal{I}} (g(\nu))$.

We consider which representative was chosen for the encoding function:

$\Pi_{(a,b)}$: Since $W^{\mathcal{I}} (g(\nu)) \in \text{Enc}(w_{\nu})$, we know in particular that $W^{\mathcal{I}} (g(\nu)) > a$. Furthermore, since $M^{\mathcal{I}} (g(\nu)) = \text{enc}(1) < b$ and $\Pi$ is a strict t-norm, for every $z > M^{\mathcal{I}} (g(\nu))$, we have $W^{\mathcal{I}} (g(\nu)) \otimes z > W^{\mathcal{I}} (g(\nu)) \otimes M^{\mathcal{I}} (g(\nu)) \geq V^{\mathcal{I}} (g(\nu))$.

$\mathcal{L}_{(a,b)}$: If $w_{\nu} \neq \varepsilon$, then since the length of $w_{\nu}$ is bounded by $\ell := (|\nu| + 1)k$ and

$$W^{\mathcal{I}} (g(\nu)) \otimes M^{\mathcal{I}} (g(\nu)) = \sigma_{a,b}(\max\{0, 1 - \frac{w_{\nu}}{\ell} - (0.9^{\ell} \cdot \frac{1}{\ell})\}),$$

$^{1}$A continuous t-norm is strict if it is strictly monotone (Klement, Mesiar, and Pap 2000).
we have

\[ W^I(g(\nu)) \otimes M^I(g(\nu)) = \sigma_{a,b}(1 - 0.\frac{w_\nu}{w_\nu} - (0.\frac{g}{g} \cdot \frac{1}{1})) \in (a, b). \]

For \( w_\nu = \varepsilon \), it follows that

\[ W^I(g(\nu)) \otimes M^I(g(\nu)) = M^I(g(\nu)) = \sigma_{a,b}(1 - (0.\frac{g}{g} \cdot \frac{1}{1})) \in (a, b). \]

Thus, by the properties of \( \ell \) we again have for any \( z > M^I(g(\nu)) \) that

\[ W^I(g(\nu)) \otimes z > W^I(g(\nu)) \otimes M^I(g(\nu)) \geq V^I(g(\nu)). \]

In both cases, we get

\[ W^I(g(\nu)) \Rightarrow V^I(g(\nu)) = \sup\{z \in [0, 1] \mid W^I(g(\nu)) \otimes z \leq V^I(g(\nu))\} \]

\[ = \inf\{z \in [0, 1] \mid W^I(g(\nu)) \otimes z > V^I(g(\nu))\} \]

\[ \leq \inf\{z \in [0, 1] \mid z > M^I(g(\nu))\} \]

\[ = M^I(g(\nu)). \]

If \( X^I(g(\nu)) = 0 \), then we know that \( V^I(g(\nu)) \Rightarrow W^I(g(\nu)) \leq M^I(g(\nu)) \) by similar arguments, using axiom (5.8) instead of (5.7). Thus, we always have

\[ \min\{V^I(g(\nu)) \Rightarrow W^I(g(\nu)), W^I(g(\nu)) \Rightarrow V^I(g(\nu))\} \leq M^I(g(\nu)). \]

To show the second point of \( P_{\neg} \), assume that

\[ \min\{V^I_P(\nu) \Rightarrow W^I_P(\nu), W^I_P(\nu) \Rightarrow V^I_P(\nu)\} \leq M^I_P(\nu) < 1 \]

and consider an extension \( I \) of \( \mathcal{I}_P \) that satisfies \( \mathcal{O}_P \), which exists by assumption. We show that \( I \) can be further extended to a model of \( \mathcal{O}_{V \neq W} \).

To find the values for \( X \), consider any \( \nu \in N^* \). By assumption, exactly one of the residua \( V^I_P(\nu) \Rightarrow W^I_P(\nu) \) and \( W^I_P(\nu) \Rightarrow V^I_P(\nu) \) is equal to 1. If it is the case that \( V^I_P(\nu) \Rightarrow W^I_P(\nu) = 1 \), we set \( X^I(\nu) := 1 \), which trivially satisfies axiom (5.8) at \( \nu \).

By assumption, we must then have \( W^I_P(\nu) \Rightarrow V^I_P(\nu) \leq M^I_P(\nu) \). By the definition of the residuum, we know that \( W^I_P(\nu) \otimes m' > V^I_P(\nu) \) for all \( m' > M^I_P(\nu) \). Since \( \otimes \) is continuous and monotone, this means that \( V^I_P(\nu) \leq W^I_P(\nu) \otimes M^I_P(\nu) \), i.e. axiom (5.7) is also satisfied at \( \nu \).

If the other residuum is equal to 1, we set \( X^I(\nu) := 0 \) and can use dual arguments to show that axioms (5.7) and (5.8) are satisfied at \( \nu \). We have thus constructed an extension of \( \mathcal{I}_P \) that satisfies both \( \mathcal{O}_P \) and \( \mathcal{O}_{V \neq W} \).

The last case is that of \( \ell^{0, b})-\mathcal{XL} \), for which we can use the ontology

\[ \mathcal{O}_{V \neq W} := \{(V \rightarrow W) \cap (W \rightarrow V) \subseteq M\}, \]

which is similar to the one for \( \otimes-\mathcal{XL} \). For every model \( I \) of \( \mathcal{O}_P \cup \mathcal{O}_{V \neq W} \) and every \( \nu \in N^* \), we have \( (V \rightarrow W) \cap (W \rightarrow V))^I(g(\nu)) \leq M^I(g(\nu)) \), where \( g \) is the mapping associated with \( I \) by \( P_\Delta \).
Table 5.1: The undecidability results of Theorem 5.11

<table>
<thead>
<tr>
<th>TBox</th>
<th>$\mathcal{NL}$</th>
<th>$\mathcal{LAC}$</th>
<th>$\mathcal{ECL}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>crisp assertions</td>
<td>$\mathcal{E}^{(0,b)}$</td>
<td>$\mathcal{E}^{(0,b)}$</td>
<td>$\mathcal{E}$</td>
</tr>
<tr>
<td>inequality assertions</td>
<td>$\mathcal{E}^{(0,b)}$</td>
<td>$\mathcal{E}^{(0,b)}$</td>
<td>$\otimes$</td>
</tr>
<tr>
<td>equality assertions</td>
<td>$\mathcal{E}^{(0,b)}$</td>
<td>$\otimes$</td>
<td>$\otimes$</td>
</tr>
</tbody>
</table>

If $V^I(g(\nu)) \leq W^I(g(\nu))$, then $(W \rightarrow V)^I(g(\nu)) \leq M^I(g(\nu)) = \text{enc}(1 \cdot 0^{[|\nu|+1)]) < b$ by the definition of $\rightarrow$ and construction of $O_P$. By the definition of $\rightarrow$, this implies $V^I(g(\nu)) < b$, and thus $W^I(g(\nu)) \Rightarrow V^I(g(\nu)) = (W \rightarrow V)^I(g(\nu)) \leq M^I(g(\nu))$.

Similarly, if $W^I(g(\nu)) \leq V^I(g(\nu))$, then $V^I(g(\nu)) \Rightarrow W^I(g(\nu)) \leq M^I(g(\nu))$. In both cases, we have $\min\{V^I(g(\nu)) \Rightarrow W^I(g(\nu)), W^I(g(\nu)) \Rightarrow V^I(g(\nu))\} \leq M^I(g(\nu))$.

To show the second condition of $P_\neq$, assume that $I_P$ cannot be extended to a model of $O_P \cup O_{V \neq W}$. Since there is an extension $I$ of $I_P$ that satisfies $O_P$, we know that $I$ violates $O_{V \neq W}$. This means that there is a $\nu \in N^*$ such that

$$(V \rightarrow W)^I_P(\nu) \otimes (W \rightarrow V)^I_P(\nu) > M^I_P(\nu).$$

As above, the value $(V \rightarrow W)^I_P(\nu) \otimes (W \rightarrow V)^I_P(\nu)$ is either $V^I_P(\nu) \Rightarrow W^I_P(\nu)$ or $W^I_P(\nu) \Rightarrow V^I_P(\nu)$, depending on which of the values $V^I_P(\nu)$ and $W^I_P(\nu)$ is greater. Thus, both $V^I_P(\nu) \Rightarrow W^I_P(\nu)$ and $W^I_P(\nu) \Rightarrow V^I_P(\nu)$ must be strictly greater than $M^I_P(\nu)$, showing that $\min\{V^I_P(\nu) \Rightarrow W^I_P(\nu), W^I_P(\nu) \Rightarrow V^I_P(\nu)\} > M^I_P(\nu)$. 

This concludes the first round of undecidability proofs using the framework presented in Section 5.1.2. Observe that all presented constructions use only a single individual name in the ABox, and therefore these undecidability results hold already for local consistency.

**Theorem 5.11** For every continuous t-norm $\otimes$ except the Gödel t-norm, (local) consistency w.r.t. witnessed models is undecidable in the following logics:

- $\otimes-\mathcal{LAC}$ with crisp general TBoxes and equality assertions;
- $\otimes-\mathcal{ECL}$ with crisp general TBoxes and inequality assertions; and
- $\mathcal{E}^{(0,b)}-\mathcal{NL}$ with crisp ontologies.

Table 5.1 summarizes these preliminary results and distinguishes between crisp, inequality, and equality assertions on the vertical axis, and different combinations of constructors on the horizontal axis. An entry “$\otimes$” stands for every continuous t-norm except the Gödel t-norm. Thus, for instance, the upper-left cell states that $\mathcal{E}^{(0,b)}-\mathcal{NL}$ is undecidable. Note that $\mathcal{L}-\mathcal{ECL}$ is equally expressive as $\mathcal{E}^{(0,1)}-\mathcal{NL}$, and thus consistency $\mathcal{L}-\mathcal{ECL}$ is also undecidable. This shows that the following fuzzy DLs for which undecidability of consistency w.r.t. witnessed models has been shown before are included in this result:

- $\Pi-\mathcal{AC}$ with fuzzy general TBoxes and inequality assertions if additionally strict GCIs of the form $(C \subseteq D > p)$ are allowed (Baader and Peñaloza 2011a);
- $\otimes-\mathcal{LAC}$ with fuzzy general TBoxes and equality assertions if $\otimes$ starts with $\Pi$ (Baader and Peñaloza 2011b); and
- $\mathcal{E}^{(0,b)}-\mathcal{ECL}$ with fuzzy GCIs and inequality assertions (Cerami and Straccia 2013).
We have shown in Section 4.1 that consistency w.r.t. witnessed models is decidable even in $\otimes$-$\mathcal{S}I\mathcal{H}O\mathcal{I}$ with fuzzy general TBoxes and inequality assertions if $\otimes$ has no zero divisors, i.e. it does not start with $\mathcal{L}$. Furthermore, consistency w.r.t. witnessed models in $G$-$\mathcal{E}L\mathcal{C}$ with fuzzy general TBoxes and equality assertions is also decidable (see Section 4.2). Together with the above theorem, this already covers many fuzzy description logics. However, two obvious gaps remain for which the decidability status of consistency is still open.

The first gap concerns the fuzzy DLs with equality assertions above $\otimes$-$\mathcal{R}\mathcal{E}L$, where $\otimes$ does not start with $\mathcal{L}$. For such t-norms, we show in Section 5.1.5 that consistency is undecidable for $\otimes$-$\mathcal{E}L$ with equality assertions. Unfortunately, we must leave open the decidability status of consistency in $\otimes$-$\mathcal{R}\mathcal{E}L$ and $\otimes$-$\mathcal{R}\mathcal{A}L$ with equality assertions.

The second gap is about fuzzy DLs $\otimes$-$\mathcal{E}L\mathcal{C}$ with involutive negation over crisp ontologies. In addition to the Łukasiewicz t-norm, in Section 5.1.4, we show that consistency is also undecidable for the Product t-norm. However, apart from the fundamental t-norms, not much is known about the decidability of consistency in $\otimes$-$\mathcal{E}L\mathcal{C}$ with crisp ontologies. Related to this is another gap hidden in Table 5.1, concerning $\otimes$-$\mathcal{E}L\mathcal{U}$. For this sublogic of $\otimes$-$\mathcal{E}L\mathcal{C}$, only the decidable cases of Chapter 4 are known.

5 Undecidable Fuzzy Description Logics over the Standard Interval

5.1.4 The Case of $\Pi$-$\mathcal{E}L\mathcal{C}$

To prove that consistency in $\Pi$-$\mathcal{E}L\mathcal{C}$ is also undecidable, we need to slightly modify the framework presented in Section 5.1.2. The most important change is that we consider a different version of the PCP in which the compared words do not start with $v_1/w_1$. More formally, in this section a solution to an instance $\mathcal{P} = \{(v_1, w_1), \ldots, (v_n, w_n)\}$ of the PCP is a non-empty sequence $\nu = i_1 \ldots i_k \in \{1, \ldots, n\}^+$ for which $v_{i_1} \ldots v_{i_k} = w_{i_1} \ldots w_{i_k}$ holds. Correspondingly, we redefine $w_\nu := v_{i_1} \ldots v_{i_k}$ and $w_{\nu'} := w_{i_1} \ldots w_{i_k}$. We call the canonical model resulting from these modified definitions $\mathcal{I}_{\mathcal{P}}$. It can be defined just as in Section 5.1.2, but the values it holds are now different. This also results in a modified canonical model property $P_{\Delta}^\nu$, which is defined exactly as before, except that $\mathcal{I}_{\mathcal{P}}$ is replaced by $\mathcal{I}_{\mathcal{P}}^\nu$. Observe that $\mathcal{Enc}$, as defined in Section 5.1.3 for $\Pi^{(a,b)}$, remains a valid encoding function, and we can use $u_\varepsilon = 1$ and $u_+ = \varepsilon$ as before.

The second difference is that we cannot show the initialization property, but it suffices to consider a weaker version of $P_{\ini}$, where only the two words $\varepsilon$ and $u_\varepsilon$ need to be initialized. Together with a property to express constant concepts, this weak initialization property also allows us to show the canonical model property $P_{\Delta}^\nu$.

The Weak Initialization Property ($P_{\ini}^w$)

The logic $\otimes$-$\mathcal{L}$ has the weak initialization property if for every concept $C$, individual name $c$, and $u \in \{\varepsilon, u_\varepsilon\}$ there is an ontology $\mathcal{O}_{C(c)=u}$ such that for every model $\mathcal{I}$ of $\mathcal{O}_{C(c)=u}$, it holds that $C^\mathcal{I}(c^\mathcal{I}) \in \mathcal{Enc}(u)$.

Given a logic $\otimes$-$\mathcal{L}$ that satisfies $P_{\ini}^w$, we can initialize the values of $V$, $W$, and $M$ at the root of the search tree by the ontology

$$\mathcal{O}_{P,\ini}^w := \mathcal{O}_{V(c_r)=\varepsilon} \cup \mathcal{O}_{W(c_r)=\varepsilon} \cup \mathcal{O}_{M(c_r)=u_\varepsilon}.$$
It remains to express the values of the concept names $V_i, W_i,$ and $M_+$. Note that they are constant throughout the canonical model, which was previously achieved by initializing them at the root and then transferring these values along all introduced role successors. In general, a constant interpretation of a concept can be enforced through the following property.

**Constant property** ($P_\Sigma$)

The logic $\otimes$-$L$ has the *constant property* if for every concept $C$ and word $u \in \Sigma_0^+$ there is an ontology $O_{C=u}$ such that for every model $I$ of $O_{C=u}$ and every $x \in \Delta^I$ it holds that $C^I(x) \in \text{Enc}(u)$.

The constant values of $V_i, W_i,$ and $M_+$ can then be ensured by the ontology

$$O_{P_\Sigma} := O_{M_+=u_+} \cup \bigcup_{i \in N} O_{V_i=v_i} \cup O_{W_i=w_i}.$$

**Theorem 5.12** Let $\text{Enc}$ be a valid encoding function for $\otimes$. If the logic $\otimes$-$L$ satisfies $P^w_{\text{ini}}, P_\Sigma, P_o, P_{\rightarrow},$ and $P_{\leftarrow},$ then it also satisfies $P_\Delta$.

**Proof.** One can show that $O'_P := O^w_{P,\text{ini}} \cup O_{P_\Sigma} \cup O_{P_o} \cup O_{P_{\rightarrow}} \cup O_{P_{\leftarrow}}$ yields the canonical model property, using arguments similar to those in the proof of Theorem 5.3. The only difference is that at the root node $g(\varepsilon) := c^I_\varepsilon$ of any model $I$ of this ontology, only the values of $V, W,$ and $M$ are enforced by $O^w_{P,\text{ini}},$ while the values of $V_i, W_i,$ and $M_+$ are given by $O_{P_\Sigma}.$

As before, it is necessary to ensure that $T'_P$ can be extended to a witnessed model of the above ontology $O'_P$. In light of the different version of the PCP we consider here, it is clear that we also need a different solution property. It has to be enforced that $V$ and $W$ encode different words at every node of the search tree except the root node, where they both encode $\varepsilon$. We will denote by $P_{\text{w}}$ the solution property in which $N^*$ has been replaced by $N^+$ to reflect this change, and by $O'_{V \neq W}$ the associated ontology. It is easy to see that Theorem 5.4 also holds under these changes.

**Theorem 5.13** If $\otimes$-$L$ satisfies $P_{\text{w}},$ then $P$ has a solution iff $O'_P \cup O'_{V \neq W}$ is inconsistent.

The resulting modified framework is depicted in Figure 5.6. We now verify that $\Pi$-$\mathcal{ELC}$ indeed satisfies the new properties. Note that it already satisfies $P_o, P_{\rightarrow},$ and $P_{\leftarrow}$ by Lemmata 5.7–5.9.

**Lemma 5.14** The logic $\Pi$-$\mathcal{ELC}$ satisfies $P^w_{\text{ini}}$ and $P_\Sigma$.

**Proof.** For $P^w_{\text{ini}},$ note that we have $\text{enc}(\varepsilon) = 1,$ and hence the crisp assertion $\langle c: C \geq 1 \rangle$ yields the desired condition for $\varepsilon$. For $u_\varepsilon = 1,$ we use the axiom $\langle C \equiv \lnot C \rangle,$ which in particular restricts $C^I(c^I) = 1 - C^I(c^I)$ to be $\text{enc}(1) = 1/2$.

For $P_\Sigma,$ consider the ontology $O_{C=\varepsilon} := \{ \langle H \equiv \lnot H \rangle, \langle C \equiv H^w \rangle \}$, where $H$ is a fresh concept name. From the first axiom, it follows that for every model $I$ of this ontology and every $x \in \Delta^I$ we have $H^I(x) = 1 - H^I(x),$ and thus $H^I(x) = 1/2 = 2^{-1}$. Hence, from the second axiom we obtain $C^I(x) = (2^{-1})^u = 2^{-u} = \text{enc}(u).$
undecidability of consistency in $\otimes\text{-}\mathcal{L}$

solution property $P'_\neq$

Figure 5.6: Showing undecidability with $P'_\text{wini}$ and $P_\text{=} = \otimes$ instead of $P_\text{ini}$

Note that $I'_P$ can only be extended to a model of the resulting ontology $O'_P$ since we know that $u_+ = \varepsilon$, i.e. the value of $M$ is constant. Otherwise, the axiom $\langle M \equiv \neg M \rangle$ would cause $O'_P$ to be inconsistent.

Now that we know that $\Pi\text{-}\mathcal{ELC}$ has the canonical model property, we can proceed to show the (modified) solution property. The proof of the following lemma is very similar to the one of Lemma 5.10; we only describe the differences here.

**Lemma 5.15** The logic $\Pi\text{-}\mathcal{ELC}$ satisfies $P'_\neq$.

**Proof.** The ontology $O'_{V \neq W}$ is similar to the one used for $\otimes\text{-}\mathcal{ELC}$ in the proof of Lemma 5.10, with the addition of a flag $Y$ to distinguish the root node $\varepsilon$ of $I'_P$. We define

$$O'_{V \neq W} := \{ \langle \exists r_i, \neg Y \subseteq \neg \top \rangle \mid 1 \leq i \leq n \} \cup$$

$$\{ \langle X \subseteq X \cap X \rangle, \langle \top \subseteq \neg (X \cap \neg X) \rangle, \langle c_i : \neg Y \rangle, \langle Y \cap X \cap V \subseteq Y \cap X \cap W \cap M \rangle, \langle Y \cap \neg X \cap W \subseteq Y \cap \neg X \cap V \cap M \rangle \}.$$  

(5.9)

Every model of the axioms in (5.9) has to satisfy that every $r_i$-successor with degree 1 must belong to $Y$ with degree 1, for every $i \in N$. In particular, because of the construction of $O_{P, \rightarrow}$ (see the proof of Lemma 5.7), this means that for every model $I$ of $O_P \cup O_{V \neq W}$ and every $\nu \in N^+$, we have $Y^I(g(\nu)) = 1$, where $g$ is the mapping associated with $I$ by $P'_{\Delta}$. On the other hand, $Y^I(g(\varepsilon))$ must be 0. The role of $X$ is the same as before. The remainder of the first condition of $P'_\neq$ can thus be shown as in the proof of Lemma 5.10, but using $N^+$ instead of $N^*$.

For the second condition of $P'_\neq$, consider an extension $I$ of $I'_P$ that satisfies $O'_P$. To extend $I$ to a model of $O'_{V \neq W}$, we first set $Y^I(\nu) := 1$ for every $\nu \in N^+$ and
\(X^I(\varepsilon) := Y^I(\varepsilon) := 0.\) The remaining values \(X^I(\nu)\) for \(\nu \in N^+\) can be chosen exactly as in the proof of Lemma 5.10. Again, the proof is the same as before, with \(N^+\) instead of \(N^*\).

We thus obtain the following result, which again also holds for local consistency since in our construction we used only one individual name.

**Theorem 5.16** (Local) consistency w.r.t. witnessed models in \(\Pi-\mathcal{ELC}\) with crisp ontologies is undecidable.

**5.1.5 The Case of \(\otimes-\mathcal{ELC}\) with Equality Assertions**

We have already shown that \(\otimes-\mathcal{ELC}\) with crisp general TBoxes and equality assertions satisfies the properties \(P_{\text{ini}}\), \(P_\circ\), and \(P_\rightarrow\), whenever \(\otimes\) is not the Gödel t-norm. By Theorem 5.3, it only remains to show the transfer property to obtain \(P_\Delta\) and, by Lemma 5.10, undecidability of consistency. Rather than showing that \(\otimes-\mathcal{ELC}\) satisfies \(P_\sim\), in this section we strengthen Theorem 5.3 by showing that a weaker property, which we call the *simultaneous transfer property*, together with the other properties, implies the canonical model property. We then show that, for every continuous t-norm except the Gödel t-norm, \(\otimes-\mathcal{ELC}\) satisfies the simultaneous transfer property.

Recall that the transfer property is used to transfer a membership degree from any domain element to all its \(r\)-successors. In the reduction from the PCP, this property is used to copy several degrees. It thus makes sense to allow for all these degrees to be transferred simultaneously.

**Simultaneous transfer property** (\(P_\sim\))

The logic \(\otimes-\mathcal{L}\) has the *simultaneous transfer property* if for all finite sequences \((C_1, D_1), \ldots, (C_k, D_k)\) of pairs of concepts there is an ontology \(O_{(C_j), \ldots, (D_j)}\) such that for every model \(I\) of \(O_{(C_j), \ldots, (D_j)}\) and every \(x \in \Delta^I\), if for every \(j, 1 \leq j \leq k\), there is an \(u_j \in \Sigma^I_0\) such that \(C_j^I(x) \in \text{Enc}(u_j)\) and \(u_1 \notin \{0\}^*\), then there is a \(y \in \Delta^I\) such that for all \(j, 1 \leq j \leq k\), it holds that \(D_j^I(y) \in \text{Enc}(u_j)\).

Note that this combines the successor and transfer properties into a single property, without explicitly requiring a role connection between \(x\) and \(y\). However, in the constructions of Lemmata 5.19 and 5.20 below, we of course use auxiliary role connections to relate the values \(C_j^I(x)\) and \(D_j^I(y)\). Given an instance \(P\) of the PCP with words \((v_1, w_1), \ldots, (v_n, w_n)\), we can assume without loss of generality that \(v_1 \neq \varepsilon\), and thus \(v_1 \notin \{0\}^*\). Thus, we can choose for every \(i, 1 \leq i \leq n\), the following sequence of pairs \((C_j^{(i)}, D_j^{(i)})\), \(1 \leq j \leq 2n + 4\), to ensure the existence of the \(i\)-th successor, representing the concatenation of \(v_i\) with \(v_1\):

\[(V_1, V_1), \ldots, (V_n, V_n), (W_1, W_1), \ldots, (W_n, W_n), (M_+, M_+), (D_{M_{\text{out}+}}, M), (D_{V_{v_1}}, V), (D_{W_{\text{out}+}}, W)\]

The last three pairs are used to transfer the computed concatenations to the \(i\)-th successors, while the remaining pairs ensure that all constants are available for the next
undecidability of consistency in $\otimes$-$\mathcal{L}$

solution property $P_{\neq}$

canonical model property $P_{\Delta}$

initialization property $P_{\text{ini}}$

concatenation property $P_{\circ}$

simultaneous transfer property $P_{\rightarrow}\Rightarrow$

valid encoding function $\text{Enc}$

Figure 5.7: Showing undecidability with $P_{\rightarrow}\Rightarrow$ instead of $P_{\rightarrow}$ and $P_{\Rightarrow}$.

round of concatenations (cf. Section 5.1.2). We then define $O_{\rightarrow}\Rightarrow$ as the union of the resulting ontologies $O_{(C^{(i)})\rightarrow(D^{(i)})}$ to generate all necessary successors.

It is easy to see that any logic that satisfies $P_{\rightarrow}$ and $P_{\Rightarrow}$ must also satisfy $P_{\rightarrow}\Rightarrow$. Indeed, $P_{\rightarrow}$ ensures that there is an $r$-successor with degree 1, and $P_{\Rightarrow}$ states that encoding $C_I^{(i)}(x)$ can be copied to $D_I^{(i)}(y)$ if $r^I(x,y) = 1$. On the other hand, we can already obtain $P_{\Delta}$ using $P_{\rightarrow}\Rightarrow$ instead of $P_{\rightarrow}$ and $P_{\Rightarrow}$.

**Theorem 5.17** Let $\text{Enc}$ be a valid encoding function for $\otimes$. If the logic $\otimes$-$\mathcal{L}$ satisfies $P_{\text{ini}}$, $P_{\circ}$, and $P_{\rightarrow}\Rightarrow$, then it also satisfies $P_{\Delta}$.

**Proof.** The ontology $O''_{\rightarrow} := O_{\rightarrow}\Rightarrow \cup O_{\rightarrow}\text{ini} \cup O_{\rightarrow}\circ$ satisfies the conditions of $P_{\Delta}$. The function $g$ for a model $I$ of this ontology can be constructed as in the proof of Theorem 5.3, with the exception that we define as $g(\nu_i)$ that element $y \in \Delta^I$ whose existence is guaranteed by $O_{\rightarrow}\Rightarrow$ if we consider $x = g(\nu)$. □

Figure 5.7 depicts the modified framework for showing undecidability using the simultaneous transfer property instead of the successor and transfer properties.

We now prove an auxiliary result that has a similar function as the successor property and will be useful for the subsequent proofs of the simultaneous transfer property.

**Lemma 5.18** Consider the logic $\boxtimes$-$\mathcal{L}$, where $\otimes$ is a continuous t-norm of the form $\Pi^{(a,b)}$ or $\bigwedge^{(a,b)}$. For every role name $r$ and all concept names $C,D$, there is a crisp ontology $O_{C\rightarrow D}$ such that, for every model $I$ of this ontology and every $x \in \Delta^I$ with $C^I(x) \otimes C^I(x) \in (a,b)$, there is a $y \in \Delta^I$ such that $r^I(x,y) \geq b$ and $D^I(y) = C^I(x)$.

**Proof.** We can use the ontology $O_{C\rightarrow D} := \{(C \sqsubseteq \exists r.D), (\exists r.(D \sqcap D) \sqsubseteq C \sqcap C)\}$ to achieve this behavior. To see this, consider a model $I$ of this ontology and some $x \in \Delta^I$ with $C^I(x) \otimes C^I(x) \in (a,b)$. Since $I$ is witnessed, the first axiom ensures that there is an element $y \in \Delta^I$ such that

$$C^I(x) \leq \sup_{z \in \Delta^I} r^I(x,z) \otimes D^I(z) = r^I(x,y) \otimes D^I(y),$$

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while the second axiom implies that
\[ r^I(x, y) \otimes D^I(y) \otimes D^I(y) \leq \sup_{z \in \Delta^I} r^I(x, z) \otimes D^I(z) \otimes D^I(z) \leq C^I(x) \otimes C^I(x). \]

From these two inequalities and the monotonicity of \( \otimes \), we get
\[ r^I(x, y) \otimes D^I(y) \otimes D^I(y) \leq C^I(x) \otimes C^I(x) \leq r^I(x, y) \otimes r^I(x, y) \otimes D^I(y) \otimes D^I(y). \] (5.10)

Since \( C^I(x) \otimes C^I(x) \in (a, b) \), from this it follows that \( r^I(x, y) \otimes D^I(y) \otimes D^I(y) \) is also in \( (a, b) \). This means that \( r^I(x, y) \) must be greater than or equal to \( b \) since otherwise we would have
\[ r^I(x, y) \otimes (r^I(x, y) \otimes D^I(y) \otimes D^I(y)) < r^I(x, y) \otimes D^I(y) \otimes D^I(y), \]

by the definitions of ordinal sums and the Product and the Łukasiewicz t-norms, in contradiction to (5.10). This implies that \( D^I(y) \otimes D^I(y) \in (a, b) \), and thus (5.10) can be simplified to \( D^I(y) \otimes D^I(y) = C^I(x) \otimes C^I(x) \).

If \( \otimes \) contains the Product t-norm in \((a, b)\), then we obtain \((D^I(y))^2 = (C^I(x))^2\), i.e. \( D^I(y) = C^I(x) \). On the other hand, if \( \otimes \) contains the Łukasiewicz t-norm in \((a, b)\), then the fact that \( D^I(x) \otimes C^I(x) > a \) implies that \( C^I(x) \) must be strictly greater than \( \frac{a+b}{2} \), and similarly for \( D^I(y) \). We obtain \( 2 \cdot D^I(y) - b = 2 \cdot C^I(x) - b \), which again shows that
\[ D^I(y) = C^I(x). \]

We now show that \( \otimes - \mathfrak{E} \mathfrak{L} \) satisfies \( P_{\otimes} \) whenever \( \otimes \) is not the Gödel t-norm. We divide the proof in two parts, depending on whether \( \otimes \) contains the Product or the Łukasiewicz t-norm.

Lemma 5.19 The logic \( \Pi^{(a,b)}_{\mathfrak{E} \mathfrak{L}} \) satisfies \( P_{\otimes} \).

Proof. For every word \( u \in \Sigma_0^* \), we know that \( \text{enc}(u) = \sigma_{a,b}(2^{-u}) > a \). In particular, for every interpretation \( I \), \( x \in \Delta^I \), and \( j, 1 \leq j \leq k \), we have \( C^I_j(x) > a \). We now define the ontology \( O_{(C_j) \rightarrow (D_j)} \) as follows:
\[ O_{(C_j) \rightarrow (D_j)} := O_{H \leftarrow H'} \cup \{ \langle H \equiv C^I_j \cap \cdots \cap C^I_k \rangle \} \cup \{ \langle \exists r. D_j \subseteq C_j \rangle, \langle \exists r. (D_j \rightarrow H') \subseteq C_j \rightarrow H \rangle \mid 1 \leq j \leq k \}, \]

(5.11)
(5.12)

where \( r \) is a fresh role name, \( H \) and \( H' \) are fresh concept names, and \( O_{H \leftarrow H'} \) is the ontology given by Lemma 5.18.

We show that this ontology satisfies the conditions for the simultaneous transfer property. Let \( I \) be a model of this ontology and \( x \in \Delta^I \) such that there exists a word \( u \in \Sigma_0^* \setminus \{0\}^* \) with \( C^I_j(x) = \text{enc}(u) \in (a, b) \), and furthermore, \( C^I_j(x) \in (a, 1] \) for all \( j, 2 \leq j \leq k \). Using the axiom from (5.11), it follows that \( H^I(x) \in (a, b) \). Since \( \otimes \) behaves as the Product t-norm in \((a, b)\), this implies \( H^I(x) \otimes H^I(x) \in (a, b) \), and thus by Lemma 5.18 there exists an element \( y \in \Delta^I \) with \( r^I(x, y) \geq b \) and \( H^I(y) = H^I(x) \).

For \( P_{\otimes} \), we still need to show that the following holds for every \( j, 1 \leq j \leq k \):

- if \( u_j \notin \{0\}^* \), then \( D^I_j(y) = C^I_j(x) = \text{enc}(u_j) \), and
• if $C_j^2(x) \geq b$, then $D_j^2(y) \geq b$.

Consider any $j$, $1 \leq j \leq k$, and suppose first that $C_j^2(x) \geq b$ holds. Since $H^\|_2(x) < b$, it follows that $C_j^2(x) \Rightarrow H^\|_2(x) = H^\|_2(x) < b$. The second axiom from (5.12) ensures that

$$r^\|_2(x, y) \otimes (D_j^2(y) \Rightarrow H^\|_2(y)) \leq (\exists r. (D_j \rightarrow H'))^\|_2(x) \leq C_j^2(x) \Rightarrow H^\|_2(x) = H^\|_2(x) < b.$$ 

Since $r^\|_2(x, y) \geq b$ and $H^\|_2(y) = H^\|_2(x)$, this implies $a < D_j^2(y) \Rightarrow H^\|_2(x) \leq H^\|_2(x) < b$, and thus by the definition of the residuum $\Rightarrow$ of an ordinal sum, it must be the case that $D_j^2(y) \geq b$.

For the other case, suppose now that $C_j^2(x) = \text{enc}(u_j) < b$ for some $u_j \in \Sigma_0 \setminus \{0\}^*$. We show that the two axioms from (5.12) ensure that $D_j^2(y) = C_j^2(x)$. The first axiom restricts $\mathcal{I}$ to satisfy

$$r^\|_2(x, y) \otimes D_j^2(y) \leq (\exists r. D_j)^\|_2(x) \leq C_j^2(x) < b,$$

and since $r^\|_2(x, y) \geq b$, it follows that $D_j^2(y) \leq C_j^2(x)$. Analogously, from the second axiom, we derive that $D_j^2(y) \Rightarrow H^\|_2(y) \leq C_j^2(x) \Rightarrow H^\|_2(x)$. Recall that we have $a < H^\|_2(y) = H^\|_2(x) < b$, and thus by the axiom in (5.11) it follows that $C_j^2(x) > H^\|_2(x)$. We can infer that $D_j^2(y) \Rightarrow H^\|_2(x) \leq C_j^2(x) \Rightarrow H^\|_2(x) < b$, which implies $D_j^2(x) > H^\|_2(x)$. From the definition of the residuum of $\otimes$, we obtain

$$\frac{\sigma_{a,b}^{-1}(H^\|_2(x))}{\sigma_{a,b}^{-1}(D_j^2(y))} \leq \frac{\sigma_{a,b}^{-1}(H^\|_2(x))}{\sigma_{a,b}^{-1}(C_j^2(x))},$$

and since $H^\|_2(x) > a$ and $\sigma_{a,b}$ is a strictly monotone bijection between $[0, 1]$ and $[a, b]$, we get $\sigma_{a,b}^{-1}(H^\|_2(x)) > 0$ and $D_j^2(y) \geq C_j^2(x)$. As this holds for every $j$, it is possible to transfer all the values simultaneously. \hfill $\square$

The novel idea in this construction is to exploit the fact that the residuum is antitone in its first argument to provide a lower bound for $D_j^2(y)$. For this construction to work, it is necessary that $a < H^\|_2(x) < C_j^2(y)$ since otherwise the implication $C_j^2(x) \Rightarrow H^\|_2(x)$ will simply be $a$ or $1$. This restriction is ensured by the axiom in (5.11).

For the case in which $\otimes$ contains the Łukasiewicz t-norm in the interval $(a, b)$, we use the same idea for showing that the simultaneous transfer property holds. However, in this case we cannot ensure that $H$, which is interpreted as the conjunction of all the concepts $C_j^2$, has a degree strictly greater than $a$. Thus, we need to add some additional restrictions to handle the case where $H^\|_2(x) = a$.

**Lemma 5.20** The logic $\mathcal{L}^{(a,b)}$ satisfies $P\Rightarrow$.

**Proof.** We define the ontology $\mathcal{O}_{(C_j) \rightarrow (D_j)}$ as follows:

$$\mathcal{O}_{(C_j) \rightarrow (D_j)} := \mathcal{O}_{G \supseteq G'} \cup \mathcal{O}_{E \supseteq E'} \cup \{ (H \equiv C_j^2 \cap \cdots \cap C_k^2), (H \equiv G \cap G'), (H' \equiv G' \cap G'), (C_1 \equiv E \cap E'), (\exists r. H' \subseteq H'), (\exists r. ((E' \rightarrow H') \rightarrow H') \subseteq E) \} \cup \{ (\exists r. D_j \subseteq C_j), (\exists r. (D_j \rightarrow H') \subseteq C_j \rightarrow H) \mid 1 \leq j \leq k \},$$

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where \( r \) is a fresh role name, \( H, H', G, G', E, \) and \( E' \) are fresh concept names, and \( \mathcal{O}_{G,G'} \), \( \mathcal{O}_{E,E'} \) are the ontologies given by Lemma 5.18.

Let \( \mathcal{I} \) be a model of this ontology and \( x \in \Delta^\mathcal{I} \). It is easy to see that \( H^\mathcal{I}(x) \leq C^\mathcal{I}_j(x) \) holds for all \( j, 1 \leq j \leq k \). Additionally, we know that \( H^\mathcal{I}(x) \in [a, b) \). Using Lemma 5.18, we first show that there exists a \( y \in \Delta^\mathcal{I} \) such that \( r^\mathcal{I}(x, y) \geq b \) and \( G^\mathcal{I}(y) = H^\mathcal{I}(x) \). For this, we make a case distinction on whether \( H^\mathcal{I}(x) > a \) holds.

If \( H^\mathcal{I}(x) > a \), the second axiom in (5.13) implies \( G^\mathcal{I}(x) \otimes G^\mathcal{I}(x) = H^\mathcal{I}(x) \in (a, b) \). Thus, Lemma 5.18 yields the existence of an element \( y \in \Delta^\mathcal{I} \) with \( r^\mathcal{I}(x, y) \geq b \) and \( G^\mathcal{I}(y) = G^\mathcal{I}(x) \). The third axiom in (5.13) now yields

\[
H^\mathcal{I}(x) = G^\mathcal{I}(y) \otimes G^\mathcal{I}(y) = G^\mathcal{I}(x) \otimes G^\mathcal{I}(x) = H^\mathcal{I}(x).
\]

If \( H^\mathcal{I}(x) = a \), then we use the axioms from (5.14). From the first axiom and by our assumption on \( u_1 \), we have \( E^\mathcal{I}(x) \otimes E^\mathcal{I}(x) = C^\mathcal{I}_1(x) \in (a, b) \), and hence by Lemma 5.18 there is an element \( y \in \Delta^\mathcal{I} \) such that \( r^\mathcal{I}(x, y) \geq b \) and \( E^\mathcal{I}(y) = E^\mathcal{I}(y) \). The second axiom in (5.14) states that \( r^\mathcal{I}(x, y) \otimes H^\mathcal{I}(y) \leq H^\mathcal{I}(x) = a \). Since \( r^\mathcal{I}(x, y) \geq b \), it follows that \( H^\mathcal{I}(y) \leq a \). From the third axiom, we get \( (E^\mathcal{I}(y) \Rightarrow H^\mathcal{I}(y)) \Rightarrow H^\mathcal{I}(y) \leq E^\mathcal{I}(y) < b \).

In particular, this means that \( E^\mathcal{I}(y) \Rightarrow H^\mathcal{I}(y) > H^\mathcal{I}(y) \) since otherwise the residuum would be \( 1 \geq b \). But since \( E^\mathcal{I}(y) > a \) and by the definition of the residuum of \( \otimes \), this can only be the case if \( H^\mathcal{I}(y) = a = H^\mathcal{I}(x) \).

As in Lemma 5.19, we need to show that, whenever \( C^\mathcal{I}_j(x) \geq b \), then also \( D^\mathcal{I}_j(y) \geq b \), and if \( u_j \notin \{0\}^* \), then \( D^\mathcal{I}_j(y) = C^\mathcal{I}_j(x) = \text{enc}(u_j) \). The former case can be shown as in the proof of Lemma 5.19. In the latter case, the first axiom from (5.15) again ensures that \( D^\mathcal{I}_j(y) \leq C^\mathcal{I}_j(x) \leq b \) and \( r^\mathcal{I}(x, y) \geq b \). From the second axiom and the fact that \( H^\mathcal{I}(x) = H^\mathcal{I}(x) \), it similarly follows that \( D^\mathcal{I}_j(y) \Rightarrow H^\mathcal{I}(x) \leq C^\mathcal{I}_j(x) \Rightarrow H^\mathcal{I}(x) < b \).

We now know that \( H^\mathcal{I}(x) < C^\mathcal{I}_j(x) < b \) and \( H^\mathcal{I}(x) < D^\mathcal{I}_j(y) < b \), and therefore

\[
1 - \sigma_{-\frac{1}{a}}(D^\mathcal{I}_j(y)) + \sigma_{-\frac{1}{b}}(H^\mathcal{I}(x)) \leq 1 - \sigma_{a\theta}(C^\mathcal{I}_j(x)) + \sigma_{a\theta}(H^\mathcal{I}(x)).
\]

Thus, we have \( D^\mathcal{I}_j(y) \geq C^\mathcal{I}_j(x) \), which finishes the proof. \( \square \)

Together with Theorem 5.17, we obtain that \( \otimes-\mathcal{EL} \) with equality assertions satisfies the canonical model property whenever \( \otimes \) is not the Gödel t-norm.

It is also easy to see that \( \mathcal{I}_\mathcal{P} \) can be extended to a model of the ontology \( \mathcal{O}_{\mathcal{P}}' \) constructed from the ontologies provided by the initialization, concatenation, and simultaneous transfer properties. In fact, the values of the auxiliary concept names \( H \) and \( H' \) are uniquely determined by the values of \( V, W, V, W, \) at each node \( \nu \).\(^2\) For the case of \( \mathcal{P}(a,b) \), we can additionally define \( (G')^\mathcal{I}_\mathcal{P}(\nu) := G^\mathcal{I}_\mathcal{P}(\nu) := \frac{(H^\mathcal{I}_\mathcal{P}(\nu)+b)}{2} \) and \( (E')^\mathcal{I}_\mathcal{P}(\nu) := E^\mathcal{I}_\mathcal{P}(\nu) := \frac{(V^\mathcal{I}_\mathcal{P}(\nu)+b)}{2} \) for the \( \mathcal{P}(i) \)th successor of a node \( \nu \in N^* \). The values of \( G' \) and \( E' \) at the root node \( \nu \) are unconstrained and can be fixed arbitrarily. By Lemma 5.10, this implies that \( \otimes-\mathcal{EL} \) with crisp general TBoxes and equality assertions has the solution property, which yields the following result.

**Theorem 5.21** For every continuous t-norm except the Gödel t-norm, (local) consistency w.r.t. witnessed models in \( \otimes-\mathcal{EL} \) with crisp general TBoxes and equality assertions is undecidable.

\( \square \)

\(^2\)Note that there are actually \( i \) copies of \( H \) and \( H' \) in \( \mathcal{O}_{\mathcal{P}}' \) with slightly different definitions, and similarly for \( G, G', E, \) and \( E' \).
5 Undecidable Fuzzy Description Logics over the Standard Interval

Table 5.2: Undecidability of consistency in fuzzy DLs over the standard interval with witnessed model semantics and crisp and fuzzy general TBoxes

<table>
<thead>
<tr>
<th></th>
<th>NEŁ</th>
<th>NL Ł</th>
<th>EL</th>
<th>SUHOI</th>
<th>EL Ł</th>
<th>ALC</th>
<th>A LC</th>
<th>SCHOI</th>
</tr>
</thead>
<tbody>
<tr>
<td>crisp assertions</td>
<td>Ł(0, b)</td>
<td>Ł(0, b)</td>
<td>Ł(0, b)</td>
<td>Ł(0, b)</td>
<td>Ł(0, b)</td>
<td>Ł(0, b)</td>
<td>Ł(0, b)</td>
<td>Ł(0, b)</td>
</tr>
<tr>
<td>inequality assertions</td>
<td>Ł(0, b)</td>
<td>Ł(0, b)</td>
<td>Ł(0, b)</td>
<td>Ł(0, b)</td>
<td>Ł(0, b)</td>
<td>⊗</td>
<td>⊗</td>
<td>⊗</td>
</tr>
<tr>
<td>equality assertions</td>
<td>Ł(0, b)</td>
<td>Ł(0, b)</td>
<td>⊗</td>
<td>⊗</td>
<td>⊗</td>
<td>⊗</td>
<td>⊗</td>
<td></td>
</tr>
</tbody>
</table>

We have now completed our analysis of fuzzy description logics over the standard interval with witnessed model semantics. Section 5.2 contains a few additional undecidability proofs for general model semantics. The results obtained in Chapter 4 and in this chapter so far are summarized in Table 5.2. As in Table 5.1, the columns describe the logical constructors allowed in the logic, while the rows denote the types of assertions. The content of a cell then shows the class of continuous t-norms for which consistency has been shown to be undecidable, where ⊗ stands for any continuous t-norm except the Gödel t-norm. Cells with gray background mark logics for which the decidability of consistency has been fully characterized, either between t-norms with/without zero divisors, or between the Gödel t-norm and all other t-norms. For the other logics, only the stated undecidability results are known. Notice that we have shown undecidability using only crisp general TBoxes, while the decidability results were proven also in the presence of fuzzy GCIs. Thus, the results depicted in Table 5.2 hold independently of whether we use crisp or fuzzy GCIs.

Regarding the Gödel t-norm, it is still open whether G-SCHOI with fuzzy general TBoxes and equality assertion has a decidable consistency problem. However, the results of Section 4.2 indicate that this question can be answered positively. The main insight—that only the order between membership degrees matters—should not be affected by the additional expressivity of the inverse roles, role axioms, and nominals. The algorithms described in Chapter 3 already provide some ideas on how to deal with the former two. Moreover, the technique of dealing with nominals described in Appendix A should also be applicable for the Gödel semantics.

For the case of fuzzy DLs with involutive negation allowing only crisp assertions, the undecidability results for the Łukasiewicz and Product t-norms suggest that consistency is also undecidable for all other continuous t-norms (except the Gödel t-norm). However, no proof of this exists so far. The main problem is that the involutive negation cannot be localized to a certain component interval [a, b] as it always acts globally on [0, 1].

The last obvious gap concerns the fuzzy DLs ⊗-NEŁ and ⊗-NL Ł with equality assertions if ⊗ has no zero divisors. Since the residual negation is very inexpressive in this case (see Section 2.1.3), it seems unlikely that one can show undecidability of consistency. However, no decidability results are known either.

5.2 Consistency under General Model Semantics

Relaxing the restriction to witnessed models affects the behavior of the existential restrictions used in the successor or the simultaneous transfer properties. Instead of
guaranteeing the existence of a role successor with degree 1, in a general model we have
to deal with the possibility of infinitely many role successors with increasing degrees, the
supremum of which is 1. The main idea exploited in this section is that it is sometimes
enough to have a successor with a large enough role degree. This has repercussions for
the transfer property, which now needs to account for an error in the transfer of an
encoding.

For this reason, the main change in this section is to relax the encodings $Enc(u)$ of
words $u \in \Sigma^*$ to allow for a safety margin around $enc(u)$. Furthermore, we view words
$u \in \Sigma^*$ again as numbers, but not in base $s + 1$, but now $\beta := s + 2$. Correspondingly,
while $\Sigma$ is still the original alphabet $\{1, \ldots, s\}$ of the instance $\mathcal{P}$ of the PCP, we now
consider $\Sigma_+ := \{1, \ldots, s + 1\}$ instead of $\Sigma_0$. For technical reasons, we assume without
loss of generality that $s \geq 4$.

We use an adaptation of the framework introduced in Section 5.1.4 (see Figure 5.6)
to show undecidability of consistency in $\mathcal{L}^{(0,0)}(\mathcal{E}\mathcal{L}$ with crisp ontologies and $\Pi_{\mathcal{E}\mathcal{L}}$
with inequality assertions under general model semantics. However, we do not, as in
Section 5.1.4, assume that the search tree of the PCP starts with $(v_\ell, v_\ell) = (\varepsilon, \varepsilon)$
but rather $(v_1, w_1)$, as usual. To distinguish the parts of the two different frameworks, we
annotate the new properties, ontologies, and canonical model with a small $g$.

### 5.2 Consistency under General Model Semantics

#### 5.2.1 A Modified Framework

Under general model semantics, we employ a stricter definition of validity for encoding
functions that introduces two numbers to accurately distinguish encodings of different
words. In contrast to Definition 5.2, however, we do not need to deal with leading zeros
in our words.

**Definition 5.22 (valid encoding function)** A function $Enc^g : \Sigma_+^* \rightarrow \mathbb{Z}_{\geq 0}$ is called a
valid encoding function for $\otimes$ if

a) the sets $Enc^g(u_1)$ and $Enc^g(u_2)$ are nonempty and disjoint for any two different
words $u_1, u_2 \in \Sigma_+^*$, and

b) there exist two words $u_\ell, u_+ \in \Sigma_+^*$ such that for every $\nu \in N^*$, $p \in Enc^g(v_\nu)$,
$q \in Enc^g(w_\nu)$, $m_\nu \in Enc^g(u_\ell \cdot u_+ |^{[\nu]}|)$, and $m_w \in Enc^g(u_\ell \cdot u_+|^{[\nu]}|)$ it holds that
$v_\nu \neq w_\nu$ iff $\min\{p \Rightarrow q, q \Rightarrow p\} \leq \min\{m_\nu, m_w\}$. \(\square\)

We again assume that there is a function $enc^g : \Sigma_+^* \rightarrow [0, 1]$ that chooses a representative
d value $enc^g(u) \in Enc^g(u)$ for each $u \in \Sigma_+^*$.

In accordance with Condition b) of the above definition, we introduce several new
concept names, taking the place of $M$ and $M_+$ in the previous constructions. The
concept name $M_V$ will store an encoding of $u_\ell \cdot u_+|^{[\nu]}|$, while $M_V, i \in N$, represent the
words $u_\ell|^{[\nu]}|$ that have to be concatenated in each step; likewise for $M_W$ and $M_W$. We
gain the following canonical model $\mathcal{T}_P^g = (N^*, \mathcal{T}_P)$ that augments the search tree for $\mathcal{P}$
by interpretations of these new concept names for all $\nu \in N^*$ and $i \in N$:

- $\ell^{(g)} := \varepsilon$,
- $V^{(g)}(\nu) := enc^g(v_\nu)$, \quad $W^{(g)}(\nu) := enc^g(w_\nu)$,
- $V_i^{(g)}(\nu) := enc^g(v_i)$, \quad $W_i^{(g)}(\nu) := enc^g(w_i)$,
We now describe the modifications to the smaller properties that are necessary to deal with additional errors introduced by the encodings of the constant words with general model semantics. The main approach is that depicted in Figure 5.6, but we use the initialization property instead of the weak initialization property. The second change is that in the constant and concatenation properties the constant word should be encoded precisely as $\text{enc}(u)$. This is because the new framework is not able to deal with additional errors introduced by the encodings of the constant words $v_1$, $w_2$, $u_+^{[|w_1|]}$, and $u_+^{[|v_1|]}$. Moreover, we cannot guarantee a role successor with degree exactly 1 anymore, and thus the successor and transfer properties must be changed accordingly.

The initialization property is the only property that remains essentially unchanged, except for the alphabet and the encoding function.

The Canonical Model Property ($\mathcal{P}_\cap^*$)

The logic $\otimes\mathcal{L}$ has the canonical model property if there is an ontology $\mathcal{O}_p^\cap$, such that for every model $\mathcal{I}$ of $\mathcal{O}_p^\cap$, there is a mapping $g: \Delta_p^\cap \rightarrow \Delta^\mathcal{I}$ with

$$A^p_\cap(\nu) \sim A^\mathcal{I}(g(\nu))$$

for every $A \in \{V, W, M_V, M_W\} \cup \bigcup_{i \in N} \{V_i, W_i, M_{V_i}, M_{W_i}\}$ and $\nu \in N^*$.

The Initialization Property ($\mathcal{P}_{\text{ini}}^\cap$)

The logic $\otimes\mathcal{L}$ has the initialization property if for every concept $C$, individual name $c$, and $u \in \Sigma_+^*$ there is an ontology $\mathcal{O}_{C(c)=u}^\cap$ such that for every model $\mathcal{I}$ of $\mathcal{O}_{C(c)=u}^\cap$, it holds that $\Delta^\mathcal{I}(c^\mathcal{I}) \in \text{Enc}(u)$.

However, we now use this property in a slightly different way, which is similar to the construction in Section 5.1.4. The ontology

$$\mathcal{O}_{p,\text{ini}}^\cap := \mathcal{O}_{V(c_r)=v_1}^\cap \cup \mathcal{O}_{W(c_r)=w_1}^\cap \cup \mathcal{O}_{M_V(c_r)=u_+^{[|v_1|]}}^\cap \cup \mathcal{O}_{M_W(c_r)=u_+^{[|w_1|]}}^\cap$$

initializes $V$, $W$, $M_V$, and $M_W$ to the appropriate values. A modified constant property is used to deal with the remaining concept names.

The Constant Property ($\mathcal{P}_\cap^c$)

The logic $\otimes\mathcal{L}$ has the constant property if for every concept $C$ and word $u \in \Sigma_+^*$ there is an ontology $\mathcal{O}_{C=u}^\cap$ such that for every model $\mathcal{I}$ of $\mathcal{O}_{C=u}^\cap$, and every $x \in \Delta^\mathcal{I}$ it holds that $\Delta^\mathcal{I}(x) = \text{Enc}(u)$. 

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Using this property, we encode all needed constants via

\[ O^g_{P,-} := \bigcup_{i \in N} O^g_{V_i=v_i} \cup O^g_{W_i=w_i} \cup O^g_{M_{V_i=v_i}^{u_u}|v_i|} \cup O^g_{M_{W_i=w_i}^{u_u}|w_i|}. \]

As mentioned above, the concatenation property remains more or less unchanged, apart from the requirement that the constant word \( u \) that is to be concatenated should be encoded without an error term.

**The Concatenation Property (P\(_g\))**

The logic \( \otimes \mathcal{L} \) has the **concatenation property** if for all words \( u \in \Sigma^*_+ \) and concepts \( C \) and \( C_u \) there is an ontology \( O^g_{\mathcal{C}_u} \) and a concept name \( D_{\mathcal{C}_u} \) such that for every model \( I \) of \( O^g_{\mathcal{C}_u} \) and every \( x \in \Delta^I \), if \( C^I_u(x) = \text{enc}^g(u) \) and \( C^I(x) \in \text{Enc}^g(u') \) for some \( u' \in \Sigma^*_+ \), then \( D^I_{\mathcal{C}_u}(x) \in \text{Enc}^g(u'u) \).

We can compute all necessary concatenations with

\[ O^g_{P,\rightarrow} := \bigcup_{i \in N} O^g_{V_{\otimes v_i}} \cup O^g_{W_{\otimes w_i}} \cup O^g_{M_{V_{\otimes v_i}}^{u_u}|v_i|} \cup O^g_{M_{W_{\otimes w_i}}^{u_u}|w_i|}. \]

We now describe the main change in the framework, namely that \( r \)-successors with degree exactly 1 cannot be guaranteed anymore. Instead, the modified successor property ensures the existence of \( r \)-successors with degrees arbitrarily close to 1.

**The Successor Property (P\(_g\))**

The logic \( \otimes \mathcal{L} \) has the **successor property** if for all role names \( r \) there is an ontology \( O^g_{\mathcal{R}_r} \) such that for every model \( I \) of \( O^g_{\mathcal{R}_r} \), every \( x \in \Delta^I \), and every \( p \in [0,1) \) there is an element \( y \in \Delta^I \) with \( r^I(x,y) > p \).

As before, we only need to ensure the existence of appropriate \( r_i \)-successors for all \( i \in N \) via

\[ O^g_{P,-} := \bigcup_{i \in N} O^g_{\mathcal{R}_{r_i}}. \]

The following modified transfer property takes this change in the \( r_i \)-connections into account by providing lower bounds on the role degrees that suffice to transfer the encoding of a word without incurring a too large additional error.

**The Transfer Property (P\(_g\))**

The logic \( \otimes \mathcal{L} \) has the **transfer property** if there is a function \( f : [0,1] \rightarrow [0,1] \) such that for all concepts \( C, D \) and role names \( r \) there is an ontology \( O^g_{\mathcal{C}_r \otimes D} \) such that for every model \( I \) of \( O^g_{\mathcal{C}_r \otimes D} \) and every \( x, y \in \Delta^I \), if \( C^I(x) \in \text{Enc}^g(u) \) for some \( u \in \Sigma^*_+ \) and \( r^I(x,y) = f(C^I(x)) \), then \( D^I(y) \in \text{Enc}^g(u) \).

Using the ontology

\[ O^g_{P,-} := \bigcup_{i \in N} O^g_{D_{\otimes v_i}^{x_i}V_{\otimes v_i}} \cup O^g_{D_{\otimes w_i}^{x_i}W_{\otimes w_i}} \cup O^g_{D_{M_{\otimes v_i}^{u_u}|v_i|}^{x_i}M_{V_{\otimes v_i}}^{u_u}} \cup O^g_{D_{M_{\otimes w_i}^{u_u}|w_i|}^{x_i}M_{W_{\otimes w_i}}^{u_u}}, \]

we can transfer all relevant encodings by choosing \( r_i \)-successors with degrees above all bounds given by \( f \). Such successors always exist by the successor property.
Theorem 5.23  Let $\text{Enc}^g$ be a valid encoding function for $\otimes$. If the logic $\otimes\mathcal{L}$ satisfies $P^g_{\Delta\otimes}$, $P^g_{\oplus}$, $P^g_{\otimes}$, and $P^g_{\otimes\otimes}$, then it also satisfies $P^g_{\Delta}$.

Proof. The ontology $O^g_{\Delta\otimes} := O^g_{\Delta\otimes\oplus} \cup O^g_{\Delta\otimes\otimes}$ satisfies the conditions of $P^g_{\Delta\otimes\otimes}$. Indeed, it is easy to see that $O^g_{\Delta\otimes\oplus}$ and $O^g_{\Delta\otimes\otimes}$ work as in the proofs of Theorems 5.3 and 5.12 to ensure that all relevant concepts have the correct values at the root node $g(\varepsilon) := c^g_i$ in any model $I$ of $O^g_{\Delta\otimes\oplus}$, with the difference that now the constant words $v_i$, $w_i$, $u_i$, and $u_i[w_i]$ are encoded exactly.

Let now $\nu$ be a node of the search tree for which $g(\nu)$ has already been defined and consider any $i \in N$. Since the constant words are encoded exactly, $O^g_{\Delta\otimes\oplus}$ ensures that $D^g_{\nu} \in \text{Enc}^g(\nu_i)$, and similarly for the other concatenations. We now define

$$p := \max\{f(D^g_{\nu v_1}(g(\nu))), f(D^g_{\nu w_1}(g(\nu))), f(D^g_{\nu v_1 \mid v_1}(g(\nu))), f(D^g_{\nu w_1 \mid w_1}(g(\nu)))\},$$

where $f$ is the function from the transfer property. Since $I$ satisfies $O^g_{\Delta\otimes\otimes}$, there must be an element $y_i \in \Delta^I$ with $r^I(\nu, y_i) > p$. We set $g(v_i) := y_i$. The ontology $O^g_{\Delta\otimes\otimes}$ ensures that the value of $D^g_{\nu v_1}(g(\nu))$ is transferred correctly to $V^I(g(v_i))$, and similarly for $W$, $M_V$, and $M_W$, while the values of the constant concepts are fixed by $O^g_{\Delta\otimes\oplus}$.

This leaves only one crucial property to show undecidability. To check whether the constructed search tree contains a solution of $\mathcal{P}$, we have to compare the values of $V$, $W$, $M_V$, and $M_W$ at each node according to Definition 5.22.

The Solution Property (P$^g_{\neq}$)

<table>
<thead>
<tr>
<th>If the logic $\otimes\mathcal{L}$ satisfies $P^g_{\Delta}$ with $O^g_{\Delta}$ and $T^g_{\Delta}$ can be extended to a model of $O^g_{\Delta}$, then $\otimes\mathcal{L}$ has the solution property if there is an ontology $O^g_{\neq W}$ such that the following conditions are satisfied:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. For every model $I$ of $O^g_{\Delta \cup O^g_{V \neq W}}$ and every $\nu \in N^*$, we have</td>
</tr>
<tr>
<td>$$\min{V^I(g(\nu)), W^I(g(\nu)) \Rightarrow W^I(g(\nu)) \Rightarrow V^I(\nu)} \leq \min{M_V^I(g(\nu)), M_W^I(g(\nu))},$$</td>
</tr>
<tr>
<td>where $g$ is the mapping obtained from $P^g_{\Delta}$ for $I$.</td>
</tr>
<tr>
<td>2. If for every $\nu \in N^*$ we have</td>
</tr>
<tr>
<td>$$\min{V^T_{\otimes\otimes}(\nu) \Rightarrow W^T_{\otimes\otimes}(\nu), W^T_{\otimes\otimes}(\nu) \Rightarrow V^T_{\otimes\otimes}(\nu)} \leq \min{M_V^T(\nu), M_W^T(\nu)},$$</td>
</tr>
<tr>
<td>then $T^g_{\Delta}$ can be extended to a model of $O^g_{\Delta \cup O^g_{V \neq W}}$.</td>
</tr>
</tbody>
</table>

The proof of the following theorem is very similar to that of Theorem 5.4.

Theorem 5.24  If $\otimes\mathcal{L}$ satisfies $P^g_{\neq}$, then $\mathcal{P}$ has a solution iff $O^g_{\Delta \cup O^g_{V \neq W}}$ is inconsistent.

It only remains to apply the new framework to the two logics mentioned in the beginning of this section.
We now describe the new encoding required for the Łukasiewicz t-norm. Mirroring the encoding used in Section 5.1.3, we define $\text{enc}^8(u) := b \cdot \overline{u}$ for all words $u \in \Sigma^*_+$. Likewise, the encoding of $\varepsilon$ is now 0 instead of $b$. For ease of presentation, we use the notation $0, \overline{\varepsilon} := 0$ also for $\varepsilon$, and therefore have $\text{enc}^8(\varepsilon) = b \cdot 0, \overline{\varepsilon}$. Note that we have $\text{enc}^8(u) < b$ for all $u \in \Sigma^*_+$.

But the most important change lies in the introduction of an error term into the definition of $\text{Enc}^8$.

**Definition 5.25 (bounded-error encoding)** The set $\text{Enc}^8(u)$ of bounded-error encodings of a word $u \in \Sigma^*_+$ contains all real numbers of the form $b(0, \overline{u} + e) \in [0, b)$ with an error term $e$ that satisfies $|e| < \beta^{-(|u|+2)}$.

This definition keeps the error terms small enough to avoid overlapping between the encodings of adjacent words (see Figure 5.8).

**Lemma 5.26** The function $\text{Enc}^8$ defined above is a valid encoding function for a t-norm of the form $\xi^{(0,b)}$.

**Proof.** Note that $\text{Enc}^8(u)$ is the open interval $(b(0, \overline{u} - \beta^{-(|u|+2)}), b(0, \overline{u} + \beta^{-(|u|+2)})$ for every $u \in \Sigma^*_+$, while $\text{Enc}^8(\varepsilon) = [0, b\beta^{-2})$.

Consider now two different words $u_1, u_2 \in \Sigma^*_+$ and assume without loss of generality that $|u_1| \leq |u_2|$. Since they are different, the numbers $0, \overline{u_1}$ and $0, \overline{u_2}$ must differ at least in the $(|u_1| + 1)$-th digit after the decimal point, and thus we have $|0, \overline{u_1} - 0, \overline{u_2}| \geq \beta^{-(|u_1|+1)}$. But since the error terms of encodings of $u_1$ and $u_2$ are bounded by $\beta^{-(|u_1|+2)}$ and $\beta^{-(|u_2|+2)}$, respectively, they can never sum up to bridge the gap of $\beta^{-(|u_1|+1)}$ between $0, \overline{u_1}$ and $0, \overline{u_2}$. This shows that Condition a) of Definition 5.22 is satisfied by $\text{Enc}^8$.

For Condition b), we choose the words $u_\nu := s - 1 \cdot s + 1 \in \Sigma^*_+$ and $u_+ := s + 1 \in \Sigma^*_+$. Consider now any $\nu \in N^*$ and assume without loss of generality that $|v_\nu| \leq |w_\nu|$. We thus have

$$\min\{m_{v_\nu}, m_{w_\nu}\} = (b \cdot 0, s + 1 \cdot s + 1 + e) = b(1 - 3\beta^{-(|v_\nu|+2)} + e)$$

with $|e| < \beta^{-(|v_\nu|+4)}$. Additionally, we have $p = b(0, \overline{v_\nu} + e_1)$ and $q = b(0, \overline{w_\nu} + e_2)$, where $|e_1| < \beta^{-(|v_\nu|+2)}$ and $|e_2| < \beta^{-(|w_\nu|+2)} \leq \beta^{-2(|v_\nu|+2)}$, and thus $|e_1 - e_2| < 2\beta^{-(|v_\nu|+2)}$.

If $v_\nu = w_\nu$, then we make a case distinction on the order between $p$ and $q$. If $p = q$, then $\min\{p \Rightarrow q, q \Rightarrow p\} = 1 \geq \min\{m_{v_\nu}, m_{w_\nu}\}$. Otherwise, we can assume without loss of generality that $p < q$, and thus $e_1 < e_2$. We infer that

$$\min\{p \Rightarrow q, q \Rightarrow p\} = q \Rightarrow p = b - be_2 + be_1 = b - b(e_1 - e_2) > b(1 - 2\beta^{-(|v_\nu|+2)})$$

Since $2 < 3 - \beta^{-2}$ and $e \leq |e| < \beta^{-(|v_\nu|+4)}$, we obtain

$$\min\{p \Rightarrow q, q \Rightarrow p\} > b(1 - 3\beta^{-(|v_\nu|+2)} + \beta^{-(|v_\nu|+4)}) = \min\{m_{v_\nu}, m_{w_\nu}\}.$$

Figure 5.8: The bounded-error encodings for $\xi^{(0,b)} \cdot NEL (s = 4)$
Conversely, if \( v_{\nu} \neq w_{\nu} \), then \( 0.\overline{v}_{\nu} \) and \( 0.\overline{w}_{\nu} \) must differ at least in the \((|v_{\nu}| + 1)\)-th digit after the decimal point, and thus \(|0.\overline{v}_{\nu} - 0.\overline{w}_{\nu}| \geq \beta^{-|v_{\nu}|+1}\). We assume without loss of generality that \( 0.\overline{v}_{\nu} < 0.\overline{w}_{\nu} \) and obtain

\[
\min\{p \Rightarrow q, q \Rightarrow p\} = b - b(0.\overline{v}_{\nu} + e_2) + b(0.\overline{w}_{\nu} + e_1) \\
\leq b(1 - |0.\overline{v}_{\nu} - 0.\overline{w}_{\nu}| + |e_1 - e_2|) \\
< b(1 - \beta^{-|v_{\nu}|+1} + 2\beta^{-|v_{\nu}|+2}).
\]

Since \( \beta - 2 = s \geq 4 > 3 + \beta^{-2} \) and \( e \geq -|e| > -\beta^{-|v_{\nu}|+4} \), this implies that

\[
\min\{p \Rightarrow q, q \Rightarrow p\} < b(1 - 3\beta^{-|v_{\nu}|+2} - \beta^{-|v_{\nu}|+4}) < \min\{m_v, m_w\}. \tag{☐}
\]

For the following constructions, recall that we use \( C \rightarrow D \) as abbreviation for the concept \( \sqcap(C \sqcap \Box D) \), and that \( (C \rightarrow D)^\mathcal{I}(x) = C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x) \) whenever \( D^\mathcal{I}(x) < b \) holds for an interpretation \( \mathcal{I} \) and \( x \in \Delta^\mathcal{I} \).

**Lemma 5.27** The logic \( \mathcal{L}^{(0,b)} \mathcal{RELC} \) satisfies \( \mathbb{P}^g_{ini} \) and \( \mathbb{P}^g_{=} \).

**Proof.** The constructions are similar to the one used for \( \mathbb{P}_{ini} \) in Lemma 5.6, but have to be adapted to the new encoding. We first consider the constant property and define

\[
\mathcal{O}^g_{C=0} := \{ \langle H^{\beta_{|u|}} \equiv \Box H^{\beta_{|u|}} \rangle, \langle \Box C \equiv H^{2\nu} \rangle \},
\]

where \( H \) is an auxiliary concept name. Let \( \mathcal{I} \) be a model of this ontology and \( x \in \Delta^\mathcal{I} \). If \( u = \varepsilon \), then the second axiom implies that \( \forall C^\mathcal{I}(x) = I^\mathcal{I}(x) = 1 \), and thus \( C^\mathcal{I}(x) = 0 = b \cdot 0.\overline{\nu} = \text{enc}(\varepsilon) \). If \( u \neq \varepsilon \), then by the first axiom we have

\[
\max\{0, \beta_{|u|}(H^\mathcal{I}(x) - b) + b\} = b - \max\{0, \beta_{|u|}(H^\mathcal{I}(x) - b) + b\},
\]

and thus \( H^\mathcal{I}(x) = b - \frac{b}{2\beta_{|u|}} \). Using the second axiom, we get \( \forall C^\mathcal{I}(x) = \max\{0, b - \frac{b}{2\beta_{|u|}}\} \).

Since \( \frac{b}{2\beta_{|u|}} < 1 \), we conclude that \( C^\mathcal{I}(x) = b \cdot 0.\overline{\nu} = \text{enc}^g(u) \).

For the initialization property, we reuse the previous construction and set

\[
\mathcal{O}^g_{C=0} := \mathcal{O}^g_{A=0} \cup \{ \langle C \rightarrow A \rangle \cap (A \rightarrow C) \},
\]

for another auxiliary concept name \( A \). Since \( A^\mathcal{I}(x) = \text{enc}^g(u) < b \) holds for all \( x \in \Delta^\mathcal{I} \) of any model \( \mathcal{I} \) of this ontology, we in particular have \( C^\mathcal{I}(c^\mathcal{I}) \Rightarrow A^\mathcal{I}(c^\mathcal{I}) = 1 \), i.e. \( C^\mathcal{I}(c^\mathcal{I}) \leq \text{enc}^g(u) < b \). This in turn implies that \( A^\mathcal{I}(c^\mathcal{I}) \Rightarrow C^\mathcal{I}(c^\mathcal{I}) = 1 \), and thus also \( \text{enc}^g(u) \leq C^\mathcal{I}(c^\mathcal{I}) \). \( \tag{☐} \)

The concatenation property can be shown by similar arguments as in the proof of Lemma 5.8, with small adaptations for the new encoding.

**Lemma 5.28** The logic \( \mathcal{L}^{(0,b)} \mathcal{RELC} \) satisfies \( \mathbb{P}^g_{=} \).

**Proof.** We define

\[
\mathcal{O}^g_{\text{Cou}} := \{ (\langle \exists\exists C' \rangle)^{\beta_{|u|}} \equiv \exists C', \langle \exists D_{\text{Cou}} \equiv (\exists C') \cap (\exists C_u) \rangle \},
\]

\[118\]
where $C'$ is an auxiliary concept name, and consider a model $I$ of $\mathcal{O}^g_{\mathcal{C}u}$ and any $x \in \Delta^I$ with $C'_T(x) = \text{enc}^g(u)$ and $C^I(x) \in \text{Enc}^g(u')$ for some $u' \in \Sigma^*_+$. We thus have $C^I(x) = b(0, \overrightarrow{u'} + e)$ with $|e| < \beta^{-|u'|+2}$.

If $u' = \varepsilon$, then $C^I(x) = 0$, and thus $C'^I(x) = 0 = \beta^{-|u|} C^I(x)$. If $u' \neq \varepsilon$, then $\ominus C'^I(x) \in (0, b)$, which implies that $\ominus C'^I(x) \in (0, b)$ and $b - C^I(x) = \beta^{-|u|}(-C'^I(x)) + b$, and hence $C'^I(x) = \beta^{-|u|} C^I(x)$. In both cases, $C'^I(x) = b(0, \overrightarrow{u'} + e)$, shifted $|u|$ digits to the right.

If either $u$ or $u'$ is $\varepsilon$, then $D^e_{\mathcal{C}ou}(x) = C^I(x)$ or $C^I_{u}(x)$, respectively. In both cases, this encodes the concatenation $\overrightarrow{u'u}$. If both $u$ and $u'$ are non-empty words over $\Sigma^*_+$, then

$$\ominus D_{\mathcal{C}ou}(x) = b \max\{0, 1 - \beta^{-|u|}(0, \overrightarrow{u'} + e) - 0, \overrightarrow{u'}\} = b \max\{0, 1 - (0, \overrightarrow{u'} + e)\},$$

where $|e' | = \beta^{-|u|} |e| < \beta^{-|u'|+2}$. This implies that $0 < 0, u'u + e' < 1$, and thus $D^e_{\mathcal{C}ou}(x) \in \text{Enc}^g(u')$ as desired. $\square$

We now come to the last two properties required for the canonical model property.

**Lemma 5.29** The logic $\mathfrak{L}^{(0,b)}\cdot\mathfrak{REL}$ satisfies $P^g_\rightarrow$ and $P^g_\leftarrow$.

**Proof.** Using $\mathcal{O}^g_\exists := \{\langle \top \sqsubseteq \exists \top, \top \rangle\}$, we know that $\sup_{y \in \Delta^I} r^T(x, y) = 1$ must hold in every model $I$ of $\mathcal{O}_\exists$ and for every $x \in \Delta^I$. From page 116, this immediately implies the existence of a $y \in \Delta^I$ with $r^T(x, y) > p$.

For the transfer property, we define $\mathcal{O}^g_{\exists, \sqsubseteq, D} := \{\langle \exists D \sqsubseteq C \rangle, \langle \exists \exists, \exists D \sqsubseteq \exists C \rangle\}$, as in Lemma 5.9.

In the case that $b < 1$, we can simply choose the function $f$ defined by $f(p) := b$ for all $p \in [0, 1]$. Indeed, if $r^T(x, y) > b$, then it behaves similarly to 1 as far as values from $[0, b]$ are concerned. Under the assumptions of $P_\rightarrow$, by the above two axioms we have

$$r^T(x, y) \otimes D^T(y) \leq C^I(x) \text{ and } r^T(x, y) \otimes D^T(y) \leq \om D^I(x).$$

If $C^I(x) = 0$, then $r^T(x, y) \otimes D^T(y) = 0$, and thus $D^T(y) = 0$. Consider now the case that $C^I(x) \in (0, b)$. If $D^T(y) \geq b$, then $C^I(x) \geq r^T(x, y) \otimes D^T(y) \geq b$, contradicting the assumption. Likewise, $D^T(y) = 0$ implies $b - C^I(x) \geq r^T(x, y) > b$, which is also impossible. Thus, we must have $D^T(y) \in (0, b)$, which implies $b - D^T(y) \leq b - C^I(x)$ and hence $D^T(y) \leq C^I(x)$, i.e., we can transfer all encodings exactly.

If $b = 1$, then consider any $u \in \Sigma^*_+$ and $0, \overrightarrow{u} + e \in \text{Enc}^g(u)$ with $|e| < \beta^{-|u|+2}$. We define $f(0, \overrightarrow{u} + e) := 1 - (\beta^{-|u|+2} - |e|) \in (1 - \beta^{-2}, 1)$. This value is well-defined since $\text{Enc}^g$ satisfies the disjointness condition of Definition 5.22. For all other values from $[0, 1]$, $f$ can be fixed arbitrarily. Assume now that $C^I(x) = 0, \overrightarrow{u} + e$ and $r^T(x, y) > f(C^I(x))$.

From the axioms in $\mathcal{O}^g_{\exists, \sqsubseteq, D}$, we obtain

$$r^T(x, y) + D^T(y) - 1 \leq \max\{0, r^T(x, y) + D^T(y) - 1\} = r^T(x, y) \otimes D^T(y) \leq C^I(x)$$

and

$$r^T(x, y) - D^T(y) \leq \max\{0, r^T(x, y) - D^T(y)\} = r^T(x, y) \otimes D^T(y) \leq 1 - C^I(x).$$
From the first inequality, it follows that \( D^\ell(y) - C^\ell(x) \leq 1 - r^\ell(x, y) < 1 - f(C^\ell(x)) \), and the second one yields \( C^\ell(x) - D^\ell(y) \leq 1 - r^\ell(x, y) < 1 - f(C^\ell(x)) \). This shows that the absolute difference between \( C^\ell(x) \) and \( D^\ell(y) \) is strictly smaller than \( \beta^-(|u| + 2) - |e| \), which implies that \( D^\ell(y) \) is also a bounded-error encoding of \( u \).

Together with Theorem 5.23, the previous lemmata prove the canonical model property for \( \mathcal{L}^{(0,b)}_{\mathcal{IMEL}} \) under general model semantics. Furthermore, it is easily verified that \( \mathcal{I}^g_\mathcal{P} \) can be extended to the auxiliary concept names in such a way that it satisfies \( O^g_\mathcal{P} \). This leaves us with the task of verifying the solution property.

**Lemma 5.30** The logic \( \mathcal{L}^{(0,b)}_{\mathcal{IMEL}} \) satisfies \( P^g_\# \).

**Proof.** We define \( O^g_{\mathcal{V} \neq W} := \{ ((V \rightarrow W) \cap (W \rightarrow V)) \subseteq M_V \cap (M_V \rightarrow M_W) \} \). For any interpretation \( \mathcal{I} \) and any \( x \in \Delta^\ell \), if the values \( M^\mathcal{I}_V(x) \) and \( M^\mathcal{I}_W(x) \) are in the interval \([0, b] \), then either \( M^\mathcal{I}_V(x) \leq M^\mathcal{I}_W(x) \), and thus \( (M_V \cap (M_V \rightarrow M_W))^\mathcal{I}(x) = M^\mathcal{I}_V(x) \otimes 1 = M^\mathcal{I}_W(x) \), or \( M^\mathcal{I}_V(x) > M^\mathcal{I}_W(x) \), and then

\[
(M_V \cap (M_V \rightarrow M_W))^\mathcal{I}(x) = \max\{0, M^\mathcal{I}_V(x) + (b - M^\mathcal{I}_W(x) + M^\mathcal{I}_W(x) - b) \} = M^\mathcal{I}_W(x).
\]

Let now \( \mathcal{I} \) be a model of \( O^g_{\mathcal{V} \neq W} \), \( \nu \in N^* \), and \( g \) be the function given by \( P^g_\# \) for \( \mathcal{I} \). We know by \( P^g_\# \) that \( V^\mathcal{I}(g(\nu)) \) is a bounded-error encoding of a word from \( \Sigma^*_+ \), and thus strictly smaller than \( b \), and similarly for \( W \), \( M_V \), and \( M_W \). Thus, we have

\[
(M_V \cap (M_V \rightarrow M_W))^\mathcal{I}(g(\nu)) = \min\{M^\mathcal{I}_V(g(\nu)), M^\mathcal{I}_W(g(\nu))\}.
\]

Likewise, by the definition of \( \rightarrow \), we get

\[
((V \rightarrow W) \cap (W \rightarrow V))^\mathcal{I}(g(\nu)) = \min\{V^\mathcal{I}(g(\nu)) \Rightarrow W^\mathcal{I}(g(\nu)), W^\mathcal{I}(g(\nu)) \Rightarrow V^\mathcal{I}(g(\nu))\}.
\]

The first condition of \( P^g_\# \) immediately follows from these two equations. Assume now that \( \mathcal{I}^g_{\mathcal{P}} \) cannot be extended to a model of \( O^g_{\mathcal{V} \neq W} \cup O^g_{\mathcal{V} = W} \). Since there is a model \( \mathcal{I} \) of \( O^g_{\mathcal{P}} \) that extends \( \mathcal{I}^g_{\mathcal{P}} \), we know that \( \mathcal{I} \) violates \( O^g_{\mathcal{V} \neq W} \), and thus

\[
\min\{V^\mathcal{I}_{\mathcal{P}}(\nu) \Rightarrow W^\mathcal{I}_{\mathcal{P}}(\nu), W^\mathcal{I}_{\mathcal{P}}(\nu) \Rightarrow V^\mathcal{I}_{\mathcal{P}}(\nu)\} > \min\{M^\mathcal{I}_V(\nu), M^\mathcal{I}_W(\nu)\}
\]

by the arguments above (for \( \mathcal{I}^g_{\mathcal{P}} \), we can set \( g(\nu) := \nu \) for all \( \nu \in N^* \)).

We thus obtain the following result from Theorem 5.24.

**Theorem 5.31** (Local) consistency w.r.t. general models in \( \mathcal{L}^{(0,b)}_{\mathcal{IMEL}} \) with crisp ontologies is undecidable.

Since \( \mathcal{I}^g_{\mathcal{P}} \) is actually a witnessed model, this again shows undecidability of consistency in this logic w.r.t. witnessed models (cf. Theorem 5.11). However, for this new proof we had to verify different properties of \( \mathcal{L}^{(0,b)}_{\mathcal{IMEL}} \). Formally, the framework of Section 5.2.1 is incomparable to the ones of Sections 5.1.2 and 5.1.4 since it uses a different alphabet and requires the stronger constant property \( P^g_\# \).
5.2 Consistency under General Model Semantics

5.2.3 The Case of $\Pi\text{-ELCC}$ with Inequality Assertions

The final undecidability proof of this chapter is very similar to the one from the previous section. In particular, the encoding is also based on the numbers $0, \overline{u}$. More precisely, we define $\text{enc}_b(u) := 2^{-0, \overline{u}, v}$ for every $u \in \Sigma^*_e$. For $\varepsilon$, this means that $\text{enc}_b(\varepsilon) = 1$. Similarly, we now define the set $\text{Enc}_b(u)$ of all bounded-error encodings of $u \in \Sigma^*_e$ to contain all real numbers of the form $2^{-0, \overline{u}, v} \in (0,1]$ with an error term $e$ satisfying $|e| < \beta^{-[(u)v] + 2}$. This is the same encoding as for $\text{-EL}$ from Definition 5.25, translated via the strictly antitone mapping $x \mapsto 2^{-x}$ from $[0,1)$ to $(0,1]$ (see Figure 5.9).

Since the original encoding satisfies the disjointness condition of Definition 5.22 and $x \mapsto 2^{-x}$ is a bijection, these encodings are still disjoint. However, the antitonicity of this mapping also means that the condition to distinguish encodings of different words is now different. For this, consider $u_* := s - 1 \cdot s + 1$ and $u_{+} := s + 1$ as before and any $u \in \Sigma^*$, $p = 2^{-0, \overline{u}, v} + e_1$, $q = 2^{-0, \overline{u}, v} + e_2$, $m_v = 2^{3\beta^{-[(u)v] + 2}} - 1 + e_3$, and $m_w = 2^{3\beta^{-[(w)v] + 2}} - 1 + e_4$ with appropriate error terms. If we assume that $|v_v| \leq |w_v|$, then by Lemma 5.26 we have $v_v \neq w_v$ iff

$$\min\{1 + \log_2(p) - \log_2(q), 1 + \log_2(q) - \log_2(p)\} \leq \min\{-\log_2(m_v), -\log_2(m_w)\},$$

which is equivalent to

$$\max\left\{\frac{q}{2p}, \frac{p}{2q}\right\} \geq \max\{m_v, m_w\}. \quad (5.16)$$

Correspondingly, for the solution property we have to check a different inequality than before. However, first we have to verify the canonical model property.

Lemma 5.32 The logic $\Pi\text{-ELCC}$ satisfies $P^g_\Delta$.

Proof. For the constant property, we define $O_{C=\text{u}}^g := \{H \equiv \neg H, (C^{|u|}) \equiv H^\overline{u}\}$, to ensure that $H^X(x)$ is always 0.5, and thus $C^X(x) = ((2^{-1})^\overline{u})^{\beta^{-|u|}} = 2^{-0, \overline{u}} = \text{enc}_b(u)$.

We reuse $O_{C(e)=\text{u}}^g := \langle c : C \geq \text{enc}_b(u) \rangle, (c : \neg C \geq 1 - \text{enc}_b(u) \rangle$ from Lemma 5.6 for the initialization property.

The concatenation property can be obtained as in Lemma 5.8 by using the ontology $O_{C \sqcup u}^g := \langle (C^{|u|}) \equiv C, (D_{C \sqcup u} \equiv C' \sqcup C_u) \rangle$. Indeed, if we have $C^X(x) = 2^{-0, \overline{u}}$ and $C^X(x) = 2^{-0, \overline{u}} + e'$ with $|e| < \beta^{-[(u)v] + 2}$, then $D_{C \sqcup u}^X(x) = 2^{-0, \overline{u}} + e' + e''$ with $|e''| = |e| < \beta^{-[(u)v] + 2}$.

The successor property was already verified in the proof of Lemma 5.29 using the axiom $\langle T \equiv \exists \text{r} \cdot T \rangle$.

For the transfer property, we use the axioms $O_{C \sqcup D}^g := \langle (\exists \text{r}. D \sqsubseteq C), (\exists \text{r}. \neg D \sqsubseteq \neg C) \rangle$. Consider any $u \in \Sigma^*_e$ and $p := 2^{-0, \overline{u}, v} \in \text{Enc}_b(u)$ with $|e| < \beta^{-[(u)v] + 2}$. We set $f(p) := \max\{d, \frac{1-p}{1-d}\} \in [0,1)$, where $d := 2|e| < \beta^{-[(u)v] + 2}$. This value is well-defined since the encodings of different words are disjoint. Assume now that $C^X(x) = p$ and
As usual, checked whether as required by (5.16). A similar argument can be made for $122 \in {\mathcal{O}}_{{\mathcal{P}}^f}$ that satisfies extended to a model of axioms in (5.17) and (5.18) additionally enforce that for all $x$.

Furthermore, we have potent truth degrees. Since we are dealing with $\Pi$, the only idempotent values are 0 and 1. 

As usual, $\mathcal{I}^f_\Pi$ can be extended to a model of the resulting ontology $\mathcal{O}^f_\Pi$, and it remains to be checked whether $V$ and $W$ encode different words at each node $\nu \in N^*$. Following (5.16), this can be achieved by the ontology

$$\mathcal{O}^f_{V \not= W} := \{ \langle X \subseteq X \cap X \rangle, \langle Y \subseteq Y \cap Y \rangle, \langle H \equiv \neg H \rangle, \langle \neg X \cap M \subseteq \neg X \cap M_V \rangle, \langle M_V \subseteq M \rangle, \langle X \cap M \subseteq X \cap M_W \rangle, \langle M_W \subseteq M \rangle, \langle Y \cap M \cap V \subseteq Y \cap H \cap W \rangle, \langle \neg Y \cap M \cap W \subseteq \neg Y \cap H \cap V \rangle \} \quad (5.19)$$

As in the proof of Lemma 5.10, the first two axioms ensure that $X$ and $Y$ only take idempotent truth degrees. Since we are dealing with $\Pi$, the only idempotent values are 0 and 1.

Finally, consider the axioms in (5.19), any model $\mathcal{I}$ of $\mathcal{O}^f_{V \not= W}$, the corresponding mapping $g$ given by $\mathcal{P}^f_\Delta$, and any $\nu \in N^*$. If $Y^f(\varphi(\nu)) = 0$, then we have

$$M^f(\varphi(\nu)) \otimes W^f(\varphi(\nu)) \leq H^f(\varphi(\nu)) \otimes V^f(\varphi(\nu)),$$

and thus $M^f(\varphi(\nu)) \leq W^f(\varphi(\nu)) \Rightarrow (0.5 \cdot V^f(\varphi(\nu)))$. Since $W^f(\varphi(\nu)) \in \text{Enc}^f(w_\nu)$, we know that $W^f(\varphi(\nu)) > 0.5$, and hence

$$\max \left\{ \frac{W^f(\varphi(\nu))}{2 \cdot V^f(\varphi(\nu))}, \frac{V^f(\varphi(\nu))}{2 \cdot W^f(\varphi(\nu))} \right\} \geq \frac{V^f(\varphi(\nu))}{2 \cdot W^f(\varphi(\nu))} \geq M^f(\varphi(\nu)) = \max\{M_V^f(\varphi(\nu)), M_W^f(\varphi(\nu))\},$$

as required by (5.16).
5.3 Satisfiability and Entailment

to satisfy the axioms in (5.17) and (5.18). To find a value for $Y^T(\nu)$, we consider the order relation between the values $\frac{W^T(\nu)}{2V^T(\nu)}$ and $\frac{V^T(\nu)}{2W^T(\nu)}$. If the latter is greater than the former, by assumption we know that it is also greater than or equal to $M^T(\nu)$. Since $W^T(\nu) > 0.5$, this is equivalent to $M^T(\nu) \otimes W^T(\nu) \leq H^T(\nu) \otimes V^T(\nu)$. Thus, in this case we can define $Y^T(\nu) := 0$ to satisfy both axioms in (5.19), and dually for the other case.

This shows that $O_{V \neq W}^\otimes$ realizes the inequality test from (5.16). We thus conclude as in Theorem 5.4 that $\mathcal{P}$ has a solution iff $O_P^\otimes \cup O_{V \neq W}^\otimes$ is inconsistent.

**Theorem 5.33** (Local) consistency w.r.t. general models in $\Pi\text{-}\mathcal{ELC}$ with crisp general TBoxes and inequality assertions is undecidable.

5.3 Satisfiability and Entailment

So far, we have only shown undecidability of (local) consistency for fuzzy DLs over the standard interval. We now briefly comment on how to adapt the techniques of the preceding sections for the satisfiability and subsumption problems.

Recall that satisfiability of a concept w.r.t. an ontology is at least as hard as deciding the consistency of that ontology (see Section 2.2.4). Thus, the undecidability results of Chapter 5 also apply to concept satisfiability. However, Lemma 2.22 illustrates that, if the ontology is consistent, then satisfiability of concepts is not affected by the ABox, and thus it is still interesting to analyze the concept satisfiability problem without an ABox.

The undecidability results for local consistency of purely crisp ontologies immediately carry over to this problem since an ontology with a crisp local ABox $\{\langle c:C_1 \rangle, \ldots, \langle c:C_n \rangle \}$ is consistent iff $C_1 \cap \cdots \cap C_n$ is 1-satisfiable w.r.t. this ontology. However, in general local consistency is harder than pure concept satisfiability since a local ABox can assert different values for several concepts. But if we additionally allow fuzzy GCIs, then at least the consistency of local inequality assertions $\langle c:C_1 \geq p_1 \rangle, \ldots, \langle c:C_n \geq p_n \rangle$ can be reduced to the 1-satisfiability of a new concept name $A$ w.r.t. the original ontology and the fuzzy GCIs $\langle A \sqsubseteq C_1 \geq p_1 \rangle, \ldots, \langle A \sqsubseteq C_n \geq p_n \rangle$. Together with the considerations of Section 4.3, this shows that all results in the first row of Table 5.2 also apply to concept satisfiability w.r.t. witnessed models and crisp general TBoxes. Similarly, the second row can be used to determine the decidability of concept satisfiability in the presence of fuzzy GCIs.

We can similarly consider the subsumption problem for two concepts w.r.t. an ontology with empty ABox. One possibility to adapt the undecidability proofs would be to show that any counter-model for the given subsumption must contain an element that acts as the root of the search tree for a PCP instance. However, for this one can only use the knowledge that a certain implication has a value smaller than a given constant. To deal with this, it may be necessary to adapt the encoding to be more flexible, e.g. consider an arbitrary value instead of $1/2$ as the base for the encoding for the Product t-norm.

However, a more rigorous analysis of undecidability proofs for the satisfiability and subsumption problems remains open.
6 Fuzzy Description Logics over Infinite Lattices

We now investigate how the constructions from Chapters 4 and 5 for the standard interval can be transferred to other classes of infinite complete residuated De Morgan lattices. As a first result, in Section 6.1 we extend the undecidability result for $L$-$\LELC$ with crisp general TBoxes and inequality assertions to the case where $L$ has a particularly simple structure.

Afterwards, we analyze the influence of the presence of zero divisors on the decidability of consistency in $L$-$\ISUHOI$. Recall that over the standard interval the decidability of consistency is characterized by the absence of zero divisors (see Table 5.2). We show that, although consistency in $L$-$\ISUHOI$ is decidable if $L$ has no zero divisors (see Theorem 4.4), the converse does not hold in general.

To demonstrate this, we construct two particular infinite families of infinite lattices with exactly one zero divisor. For the lattices of the first family, consistency in $L$-$\ISUHOI$ with fuzzy general TBoxes and inequality assertions is undecidable (see Section 6.2.1), while for those in the second family it is decidable (see Section 6.2.2).

6.1 An Infinite Lattice with Two Limit Points

Recall that consistency in $\otimes$-$\LELC$ with crisp general TBoxes and inequality assertions is undecidable for all continuous t-norms $\otimes$ except the Gödel t-norm (see Theorem 5.11). The main reasons are that the interval $[0,1]$ is dense, i.e. between any two values we can always find another one, and the t-norms are sufficiently expressive to manipulate encodings of arbitrary words.

As a strengthening of this result, we now construct a countable total order $L_Z$ for which consistency in $L_Z$-$\LELC$ with crisp general TBoxes and inequality assertions is undecidable, and which has only two limit points. A limit point is an element $x \in L_Z$ for which every open set containing $x$ contains at least one other element of $L_Z$, where we consider the order topology on $L_Z$, whose open sets are exactly the intervals of the form \{$x \in L_Z \mid a < x < b$\}, \{$x \in L_Z \mid a < x$\}, and \{$x \in L_Z \mid x < b$\} with $a,b \in L_Z$. In other words, a limit point $x$ can be approximated to arbitrary precision by elements from $L_Z \setminus \{x\}$, or $L_Z$ is “dense” in the vicinity of $x$.

We define the total order $L_Z := \mathbb{Z} \cup \{-\infty, \infty\}$ with the usual ordering over the integers and $-\infty$ and $\infty$ as the minimal and maximal element, respectively. Its De Morgan negation defined by $\sim x := x$ for $x \in \mathbb{Z}$, $\sim -\infty := -\infty$, and $\sim (\infty) := \infty$. The t-norm $\otimes$ is defined as follows for all $x,y \in L_Z$:

$$x \otimes y := \begin{cases} x + y & \text{if } x,y \in \mathbb{Z} \text{ and } x,y \leq 0 \\ \min\{x,y\} & \text{otherwise.} \end{cases}$$
Note that $-\infty$ and $\infty$ are the only two limit points of $L[Z]$ and there are no zero divisors. Furthermore, $\otimes$ is join-preserving and meet-preserving, and thus we obtain a complete residuated lattice with the following residuum:

$$x \Rightarrow y = \begin{cases} \infty & \text{if } x \leq y \\ y & \text{if } x > y \text{ and } x \geq 0 \\ y - x & \text{if } x > y \text{ and } x < 0. \end{cases}$$

We now show that (local) consistency in $L[Z]$-$\mathcal{ELC}$ with crisp general TBoxes and inequality assertions is undecidable under both witnessed and general model semantics. As in Chapter 5, given an instance $P$ of the PCP, we construct an ontology $O_P$ that is consistent iff $P$ has no solution. As before, the alphabet of $P$ is $\Sigma = \{1, \ldots, s\}$ with $s \geq 2$, and we read words in $\Sigma_0^s$ as integers in base $s + 1$. In particular, the empty word $\varepsilon$ is regarded as 0.

Although the framework of Section 5.1.2 was developed for continuous t-norms over $[0,1]$, it can also be considered in the context of an arbitrary complete residuated De Morgan lattice. In the following construction, we mainly follow the proof of undecidability for the case of $\Pi$-$\mathcal{ELC}$ with inequality assertions from Section 5.1.3.

We define the encoding of a word $u \in \Sigma_0^s$ as $\text{enc}(u) := -u$ and set $\text{Enc}(u) := \{\text{enc}(u)\}$. Thus, we always have $-\infty < \text{enc}(u) \leq 0$. This defines a valid encoding function in the sense of Definition 5.2 with the words $u_\varepsilon := 1$ and $u_+ := \varepsilon$. This can be shown in the same way as for $\Pi^{(a,b)}-\mathcal{L}$ in Lemma 5.5, based on the observation that the encodings of two different words have a difference of at least 1.

The canonical model $I_P$ can be defined as usual, but using $-\infty$ and $\infty$ instead of 0 and 1, respectively. We now construct the ontology $O_P$ for the canonical model property as in Section 5.1.2, using the following ontologies:

$$O_{C(c)=u} := \{\langle c:C \geq -u \rangle, \langle \neg c: -C \geq u \rangle\},$$

$$O_{C^+} := \{\langle D_{C^+} \equiv (s+1)^{|u|} \cap C_u \rangle\},$$

$$O_{\exists r} := \{\langle \top \subseteq \exists r.\top \rangle\},$$

$$O_{C^+ \sqsubseteq D} := \{\langle \exists r. D \sqsubseteq C \rangle, \langle \exists r. \neg D \sqsubseteq -C \rangle\}.$$ 

These provide exactly the properties needed to satisfy $P_\Delta$. The proofs are easy and very similar to the ones for $\Pi$-$\mathcal{ELC}$ with inequality assertions in Section 5.1.3. For example, if we have $C^\triangle(x) = -u'$ and $C^\triangle(x) = -u$ for any model $I$ of $O_{C^+}$, $u \in \Sigma_q^s$, and $u' \in \{\varepsilon\} \cup \Sigma q_s^\varepsilon$, then $D^\triangle_{C^+}(x) = -((s+1)^{|u|}u' + u) = -u'u$. We note only a slight difference in the successor property, which is closer to $P_{\varphi^g}$ introduced in Section 5.2.1 in that the generated $r$-successor need only have a degree strictly greater than $-C^\triangle(x)$ (assuming that $C^\triangle(x) = \text{enc}(u) = -u$ for some $u \in \Sigma_0^s$). Every such $r$-successor $y$ can be used to exactly transfer $\text{enc}(u)$ to $D^\triangle(y)$.

**Lemma 6.1** The logic $L[Z]$-$\mathcal{ELC}$ with inequality assertions satisfies $P_\Delta$. 

As before, it can be verified that $I_P$ can be extended to a witnessed model of $O_P$. 

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Checking whether $V$ and $W$ always encode different words also works as in the proof for $\PiELC$ (see Lemma 5.10), using

$$\mathcal{O}_{V\neq W} := \{ \langle X \subseteq X \cap X \rangle, \langle \top \subseteq \neg(X \cap \neg X) \rangle, \langle X \cap V \subseteq X \cap W \cap M \rangle, \langle \neg X \cap W \subseteq \neg X \cap V \cap M \rangle \}.$$ 

The first two axioms ensure that $X^T(x)$ is always idempotent and $X^T(x) \otimes \sim X^T(x) = -\infty$, i.e. $X^T(x) \in \{-\infty, \infty\}$. If we have $X^T(g(\nu)) = \infty$ for a node $\nu \in N^*$, then we obtain

$$V^T(g(\nu)) \leq W^T(g(\nu)) \otimes M^T(g(\nu)) = W^T(g(\nu)) - 1 < W^T(g(\nu)) = W^T(g(\nu)) \otimes 0,$$

and thus

$$W^T(g(\nu)) \Rightarrow V^T(g(\nu)) = \bigvee \{ x \in L_Z \mid W^T(g(\nu)) \otimes x \leq V^T(g(\nu)) \} < 0.$$ 

This yields $W^T(g(\nu)) \Rightarrow V^T(g(\nu)) \leq -1 = M^T(g(\nu))$. Similarly, if $X^T(g(\nu)) = -\infty$, then $V^T(g(\nu)) \Rightarrow W^T(g(\nu)) \leq M^T(g(\nu)).$

Conversely, if

$$\min \{ V^{\mathcal{P}}(\nu) \Rightarrow W^{\mathcal{P}}(\nu), W^{\mathcal{P}}(\nu) \Rightarrow V^{\mathcal{P}}(\nu) \} \leq M^{\mathcal{P}}(\nu),$$

then $\mathcal{P}$ can be extended to a model of $\mathcal{O}_P \cup \mathcal{O}_{V\neq W}$ by defining the interpretation of $X$ depending on the order between $V$ and $W$ at each node (see the proof of Lemma 5.10). This shows that $\mathcal{P}$ has a solution iff $\mathcal{O}_P \cup \mathcal{O}_{V\neq W}$ is inconsistent. Note that this also holds for witnessed model semantics since $\mathcal{I}_P$ is witnessed.

**Theorem 6.2** (Local) consistency w.r.t. witnessed or general models in $L_Z\cdot\ELC$ with crisp general TBoxes and inequality assertions is undecidable. 

As illustrated in Section 5.3, this also proves undecidability of $\infty$-satisfiability in $L_Z$ with fuzzy general TBoxes (and without ABox).

### 6.2 Infinite Lattices with One Zero Divisor

In this section, we investigate whether the dichotomy for $\otimes\text{-SUHOI}$ with inequality assertions between t-norms starting with $\otimes$ and those without zero divisors extends to all infinite lattices. We construct a family of infinite complete residuated De Morgan lattices to illustrate two facts. First, while a continuous t-norm $\otimes$ over $[0, 1]$ either has no zero divisors or infinitely many, there are infinite total orders with only one zero divisor for which consistency in $L\text{-SUHOI}$ with inequality assertions is already undecidable. Second, surprisingly there also are infinite total orders with one zero divisor for which this problem remains decidable.

**Definition 6.3** ($L_\infty$) Let $\otimes$ be a continuous t-norm over the standard interval $[0, 1]$. The complete De Morgan lattice $L_\infty$ is given by $L_\infty := [0, 1] \cup \{-\infty, -2, 2, \infty\}$ with the usual order (i.e. we have $0 = -\infty$ and $1 = \infty$) and De Morgan negation

$$\sim x := \begin{cases} 1 - x & \text{if } x \in [0, 1], \\ -x & \text{if } x \in \{-\infty, -2, 2, \infty\}. \end{cases}$$
6 Fuzzy Description Logics over Infinite Lattices

The t-norm $\otimes_\infty$ over $L_\infty$ is defined as follows for all $x, y \in L_\infty$:

$$x \otimes_\infty y := \begin{cases} 
  x \otimes y & \text{if } x, y \in [0, 1] \\
  1 & \text{if } x = y = 2 \\
  -\infty & \text{if } x = y = -2 \\
  \min\{x, y\} & \text{otherwise.}
\end{cases}$$

A simple consequence of the continuity of $\otimes$ is that $\otimes_\infty$ is join-preserving, and hence has a unique residuum $\Rightarrow_\infty$, where, for all $x, y \in L_\infty$,

$$x \Rightarrow_\infty y = \begin{cases} 
  \infty & \text{if } x \leq y \\
  x \Rightarrow y & \text{if } 0 \leq y < x \leq 1 \\
  2 & \text{if } x = 2, y = 1 \\
  -2 & \text{if } x = -2, y = -\infty \\
  y & \text{otherwise.}
\end{cases}$$

The t-conorm $\oplus_\infty$ defined by $\otimes_\infty$ is given, for all $x, y \in L_\infty$, by

$$x \oplus_\infty y = \begin{cases} 
  x \oplus y & \text{if } x, y \in [0, 1] \\
  \infty & \text{if } x = y = 2 \\
  0 & \text{if } x = y = -2 \\
  \max\{x, y\} & \text{otherwise.}
\end{cases}$$

Moreover, note that $-2$ is the only zero divisor w.r.t. $\otimes_\infty$.

6.2.1 An Undecidable Family

We now show that whenever $\otimes$ has zero divisors, then ontology consistency in the fuzzy DL $L_\infty$-$\mathcal{ELU}$ is undecidable. We prove this by a reduction from ontology consistency in $\otimes$-$\mathcal{EL}$, which is undecidable even for crisp ontologies (see Theorems 5.11 and 5.31).

For a given crisp $\otimes$-$\mathcal{EL}$ ontology $O = (A, T, \emptyset)$, we build an $L_\infty$-$\mathcal{ELU}$ ontology $O_\infty$ that simulates the semantics of the axioms in $O$ under $\otimes$. Let Bot be a concept name not appearing in $O$. We first recursively define the function $\varrho$ that maps $\mathcal{EL}$-concepts to $\mathcal{ELU}$-concepts as follows:

- $\varrho(A) := A$ for all $A \in N_C \cup \{\top\}$,
- $\varrho(C \sqcap D) := \varrho(C) \sqcap \varrho(D)$,
- $\varrho(\exists r.C) := \exists r.\varrho(C)$, and
- $\varrho(\Box C) := \varrho(C) \rightarrow \text{Bot}$.

The ontology $O_\infty := (A_\infty, T_\infty, \emptyset)$ is then given by

$$A_\infty := \{\{x: \varrho(C) \geq 1\} | \{x: C \geq 1\} \in A\} \cup \{\{(c, d): r \geq 1\} | \{(c, d): r \geq 1\} \in A\}$$

$$T_\infty := \{(\varrho(C) \sqsubseteq \varrho(D) \geq 1) | (C \sqsubseteq D \geq 1) \in T\} \cup \{\{\top \sqsubseteq \text{Nil} \geq -2\}, \{\top \sqsubseteq (\text{Nil} \sqcap \text{Nil}) \rightarrow (\bot \geq \infty)\} \cup \{\{\text{Bot} \sqsubseteq \text{Nil} \sqcup \text{Nil} \geq \infty\}, \{\text{Nil} \sqcup \text{Nil} \sqsubseteq \text{Bot} \geq \infty\}\}.$$ (6.1)
where \( \text{Nil} \) is a new concept name not appearing in \( \mathcal{O} \) and different from \( \text{Bot} \). Note that neither \( \mathcal{A}_\infty \) nor \( \mathcal{T}_\infty \) are crisp since 1 is not the greatest element of \( L_\infty \).

The first axiom in (6.1) requires the interpretation of \( \text{Nil} \) to be always greater or equal to \(-2\). The second axiom expresses that for every model \( I \) and every \( x \in \Delta^I \) it holds that \( \text{Nil}^I(x) \leq \bot^I(x) = -\infty \). Thus, together these two axioms restrict every model of \( \mathcal{O}_\infty \) to interpret the concept name \( \text{Nil} \) as the constant \(-2\). Consider now the axioms in (6.2). They state that

\[
\text{Bot}^J(x) = \text{Nil}^J(x) \oplus \infty \text{Nil}^J(x) = -2 \oplus \infty -2 = 0
\]

for every model \( I \) of \( \mathcal{O}_\infty \) and every \( x \in \Delta^I \). The idea behind this restriction is that \( \text{Bot} \) will be used to simulate the bottom concept \( \bot \) from the original ontology \( \mathcal{O} \), as suggested by the transformation \( \varrho \). We now show that \( \mathcal{O} \) is consistent iff \( \mathcal{O}_\infty \) is consistent.

Let \( I \) be a model of \( \mathcal{O} \), and let \( J = (\Delta^J, \cdot^J) \) be the interpretation where \( \Delta^J := \Delta^I \), for every role name \( r \) and individual name \( c \) we have \( r^J := r^I \) and \( c^J := c^I \), and for every \( x \in \Delta^J \) and concept name \( A \),

\[
A^J(x) := \begin{cases} 
0 & \text{if } A = \text{Bot}, \\
-2 & \text{if } A = \text{Nil}, \\
A^I(x) & \text{otherwise}.
\end{cases}
\]

Observe that \( J \) is witnessed whenever \( I \) is.

**Lemma 6.4** \( J \) is a model of \( \mathcal{O}_\infty \).

**Proof.** As demonstrated above, \( J \) satisfies the axioms from (6.1) and (6.2). We now show by induction that for every \( \mathfrak{R}_\mathcal{L} \)-concept \( C \) that does not contain \( \text{Bot} \) or \( \text{Nil} \) and every \( x \in \Delta^J \) it holds that \( C^J(x) = \min\{ (\varrho(C))^J(x), 1 \} \). In particular, this means that \( (\varrho(C))^J(x) \geq 0 \). For concept names and \( \top \), the claim holds by definition of \( \varrho \).

- Consider a concept of the form \( C \cap D \). If \( (\varrho(C \cap D))^J(x) > 1 \), then \( (\varrho(C))^J(x) > 1 \) and \( (\varrho(D))^J(x) > 1 \). By induction, \( C^J(x) = D^J(x) = 1 \), and hence \( (C \cap D)^J(x) = 1 \). If \( (\varrho(C \cap D))^J(x) \leq 1 \), then (i) \( (\varrho(C))^J(x) \leq 1 \) or (ii) \( (\varrho(D))^J(x) \leq 1 \). Thus,

\[
(\varrho(C \cap D))^J(x) = \min\{ (\varrho(C))^J(x), 1 \} \otimes \min\{ (\varrho(D))^J(x), 1 \}
\]

\[
= C^J(x) \otimes D^J(x)
\]

\[
= (C \cap D)^J(x).
\]

- For a concept of the form \( \exists r.C \), we get

\[
\min\{ (\varrho(\exists r.C))^J(x), 1 \} = \min\left\{ \bigvee_{y \in \Delta^J} r^J(x, y) \otimes \infty (\varrho(C))^J(y), 1 \right\}
\]

\[
= \bigvee_{y \in \Delta^J} r^J(x, y) \otimes \infty \min\{ (\varrho(C))^J(y), 1 \}
\]

\[
= \bigvee_{y \in \Delta^J} r^I(x, y) \otimes \infty C^J(y)
\]

\[
= \bigvee_{y \in \Delta^I} r^I(x, y) \otimes C^J(y)
\]

\[
= (\exists r.C)^J(x).
\]
Finally, for concepts of the form $\exists C$,
$$\min\{(\varrho(C))^{J}(x), 1\} = \min\{(\varrho(C))^{J}(x) \Rightarrow_{\infty} 0, 1\}$$
$$\begin{cases} 1 & \text{if } (\varrho(C))^{J}(x) = 0 \\ \min\{(\varrho(C))^{J}(x), 1\} \Rightarrow 0 & \text{otherwise} \end{cases}$$
$$= C^{J}(x) \Rightarrow 0$$
$$= (\exists C)^{J}(x).$$

Suppose now that $J$ is not a model of $T_{\infty}$, i.e. there is a GCI $\langle C \sqsubseteq D \geq 1 \rangle \in T$ such that $(\varrho(C))^{J}(x) \Rightarrow_{\infty} (\varrho(D))^{J}(x) < 1$ for some $x \in \Delta^{J}$. Hence, $(\varrho(D))^{J}(x) < 1$, which implies that $D^{J}(x) = (\varrho(D))^{J}(x) \in [0,1]$. If $(\varrho(C))^{J}(x) < 1$, then we have $C^{J}(x) = (\varrho(C))^{J}(x) \in [0,1]$ and $C^{J}(x) \Rightarrow D^{J}(x) = (\varrho(C))^{J}(x) \Rightarrow_{\infty} (\varrho(D))^{J}(x) < 1$: if $(\varrho(C))^{J}(x) \geq 1$, then $C^{J}(x) = 1$ and $C^{J}(x) \Rightarrow D^{J}(x) = D^{J}(x) < 1$. Both cases violate the assumption that $I$ is a model of $(C \sqsubseteq D \geq 1)$. A similar argument shows that $J$ satisfies $A_{\infty}$.

For the converse direction, let now $J$ be a model of $O_{\infty}$. The interpretation $I = (\Delta^{J}, r^{J})$ over $[0,1]$ uses the same domain as $J$, i.e. $\Delta^{J} := \Delta^{J}$. Furthermore, for every individual name $c$ we set $c^{J} := c^{\varrho}$, and for every role name $r$, every concept name $A$, and all $x, y \in \Delta^{J}$,
$$r^{J}(x, y) := \begin{cases} 0 & \text{if } r^{J}(x, y) \leq 0 \\ 1 & \text{if } r^{J}(x, y) \geq 1 \end{cases}$$
$$A^{J}(x) := \begin{cases} 0 & \text{if } A^{J}(x) \leq 0 \\ 1 & \text{if } A^{J}(x) \geq 1 \end{cases}$$

The interpretation $I$ can be seen as an approximation of $J$ to the interval $[0,1]$ by mapping all values outside this interval to the closest element. Again, $I$ is witnessed if $J$ is witnessed.

Lemma 6.5 $I$ is a model of $O$.

Proof. We first show that the transformation $\varrho$ is compatible with the approximation $I$ of $J$. Formally, for every $\mathcal{LE}$-concept $C$ and every $x \in \Delta^{J}$, it holds that
$$C^{J}(x) = \begin{cases} 0 & \text{if } (\varrho(C))^{J}(x) \leq 0 \\ 1 & \text{if } (\varrho(C))^{J}(x) \geq 1 \end{cases} (\varrho(C))^{J}(x) \text{ otherwise.}$$

The proof is by induction on the structure of $C$. For all concept names and for $\top$, the claim holds by construction.

• Consider a concept of the form $C \sqcap D$. If $(\varrho(C \sqcap D))^{J}(x) < 0$, then we must have $(\varrho(C))^{J}(x) < 0$ or $(\varrho(D))^{J}(x) < 0$. By induction, $C^{J}(x) = 0$ or $D^{J}(x) = 0$, and hence we have $(C \sqcap D)^{J}(x) = C^{J}(x) \otimes D^{J}(x) = 0$.

If $(\varrho(C \sqcap D))^{J}(x) \geq 1$, then $(\varrho(C))^{J}(x) \geq 1$ and $(\varrho(D))^{J}(x) \geq 1$. By induction, $C^{J}(x) = 1 = D^{J}(x)$, and hence $(C \sqcap D)^{J}(x) = C^{J}(x) \otimes D^{J}(x) = 1$.

Finally, if $(\varrho(C \sqcap D))^{J}(x) \in [0,1)$, then $(\varrho(C))^{J}(x), (\varrho(D))^{J}(x) \in [0,1)$, and thus $(\varrho(C \sqcap D))^{J}(x) = (\varrho(C))^{J}(x) \otimes_{\infty} (\varrho(D))^{J}(x) = C^{J}(x) \otimes D^{J}(x) = (C \sqcap D)^{J}(x)$. 

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• Consider now a concept of the form $\exists r.C$.
  
  If $(\rho(\exists r.C))^J(x) < 0$, then for every $y \in \Delta^J$, we have $r^J(x, y) \otimes_{\infty} C^J(y) < 0$.
  
  By induction, this implies that $r^I(x, y) \otimes C^I(y) = 0$ for every $y \in \Delta^I$, and hence $(\exists r.C)^I(x) = 0$.
  
  If $(\rho(\exists r.C))^J(x) > 1$, then there exists a $y \in \Delta^J$ with $r^J(x, y) \otimes_{\infty} C^J(y) > 1$.
  
  This implies $r^I(x, y) = 1 = C^I(y)$ and $1 \geq (\exists r.C)^I(x) \geq r^I(x, y) \otimes C^I(y) = 1$.
  
  Otherwise, $\rho(\exists r.C)^J(x) = \bigvee_{y \in \Delta^J} r^J(x, y) \otimes_{\infty} C^J(y) = \bigvee_{y \in \Delta^I} r^I(x, y) \otimes C^I(y) = (\exists r.C)^I(x)$.
  
  • For a concept of the form $\exists C$, we have $(\rho(\exists C))^J(x) = (\rho(C))^J(x) \Rightarrow_{\infty} 0 \geq 0$.
  
  If $(\rho(\exists C))^J(x) > 1$, then $(\rho(C))^J(x) \leq 0$ and hence $C^I(x) = 0$, which yields $(\exists C)^I(x) = 1$.
  
  Otherwise, we must have $(\rho(C))^J(x) > 0$. If $(\rho(C))^J(x) \geq 1$, then $C^I(x) = 1$ and $(\exists C)^I(x) = 0 = (\rho(\exists C))^J(x)$; otherwise, we have $C^I(x) = (\rho(C))^J(x)$, which yields the result.

Suppose now that there are $(C \subseteq D \geq 1) \in \mathcal{T}$ and $x \in \Delta^I$ with $C^I(x) \Rightarrow D^I(x) < 1$.

This means that $0 \leq D^I(x) < 1$ and $C^I(x) > D^I(x)$. But then, $(\rho(D))^J(x) < 1$ and $(\rho(C))^J(x) > (\rho(D))^J(x)$. This implies $\rho(C)^J(x) \Rightarrow_{\infty} (\rho(D))^J(x) < 1$, which violates the assumption that $\mathcal{J}$ is a model of $\mathcal{T}_{\infty}$. Again, a similar argument shows that $\mathcal{J}$ must satisfy $\mathcal{A}$.

Lemmata 6.4 and 6.5 and Theorems 5.11 and 5.31 now yield the following result.

**Theorem 6.6** If $\otimes$ is a continuous t-norm over $[0, 1]$ with zero divisors, then consistency w.r.t. witnessed or general models in $L_{\infty} \Box \mathcal{EU}$ with fuzzy general TBoxes and inequality assertions is undecidable.

It may be possible to use similar reductions to lift other undecidability results from Chapter 5 to $L_{\infty}$. In particular, in the presence of fuzzy GCIs the logics $L_{\infty} \Box \mathcal{EU}$ with inequality assertions and $L_{\infty} \Box \mathcal{ALU}$ with equality assertions are probably undecidable regardless of $\otimes$ (unless it is the Gödel t-norm, in which case one can easily extend the constructions of Section 4.2 to show decidability in ExpTime).

Theorem 6.6 suggests that a similar dichotomy as for $[0, 1]$ holds for infinite lattices: ontology consistency in $L_{\infty} \Box \mathcal{SUHOL}$ is decidable if and only if the t-norm has no zero divisors. However, as we show next, this is not the case.

### 6.2.2 A Decidable Family

Complementing Theorem 6.6, we show that if $\otimes$ has no zero divisors, then ontology consistency in $L_{\infty} \Box \mathcal{SUHOL}$ is decidable in exponential time, even though $L_{\infty}$ has a zero divisor, namely $-2$. The idea for proving this is similar to the one used in Section 4.1, but more cases need to be distinguished.

Instead of $2$, we consider here the sublattice $4 := \{-\infty, -2, 0, \infty\}$ of $L_{\infty}$. This set is closed under the operations $\land, \lor, \otimes_{\infty}, \oplus_{\infty}$, and $\Rightarrow_{\infty}$. Furthermore, the restriction of $\Rightarrow_{\infty}$
to 4 is the residuum of the restriction of \( \otimes_{\infty} \) to 4. Thus, 4 is a residuated sublattice of \( L_\infty \). However, the restriction of \( \oplus_{\infty} \) to 4 cannot be represented as a t-conorm of this lattice. We nevertheless consider the fuzzy DL 4-\( \mathbb{S}UHOI \) that results from interpreting \( \sqcup \) by this operator and observe that is it easy to extend the constructions of Chapter 3 by the constructor \( \sqcup \) interpreted in this way (recall in particular Definition 3.2 and Table 3.1 and the proofs of Lemmata 3.6 and 3.25). Thus, consistency w.r.t. general (and witnessed) models in this logic with fuzzy general TBoxes and inequality assertions is EXPTime-complete (cf. Theorem 3.30).

Similar to Section 4.1, we reduce consistency w.r.t. witnessed models in \( L_\infty \)-\( \mathbb{S}UHOI \) to consistency in 4-\( \mathbb{S}UHOI \). For this, we extend the function 2 to 4: \( L_\infty \rightarrow 4 \), where

\[
4(p) := \begin{cases} 
  p & \text{if } p \leq 0 \\
  \infty & \text{otherwise.}
\end{cases}
\]

As before, the key property of this function is its compatibility with the relevant lattice operations (cf. Lemma 4.1).

**Lemma 6.7** For all \( x, y \in L_\infty \) and all non-empty sets \( X \subseteq L_\infty \), it holds that

- \( 4(x \otimes y) = 4(x) \otimes 4(y) \),
- \( 4(x \oplus y) = 4(x) \oplus 4(y) \),
- \( 4(x \Rightarrow y) = 4(x) \Rightarrow 4(y) \),
- \( 4(\forall_{x \in X}) = \bigvee_{x \in X} 4(x) \), and
- if \( X \) has a least element, i.e. \( \bigwedge_{x \in X} x \in X \), then \( 4(\bigwedge_{x \in X} x) = \bigwedge_{x \in X} 4(x) \).

**Proof.** Since \( \otimes \) has no zero divisors, if \( x > 0 \) and \( y > 0 \), then \( x \otimes y > 0 \). This implies that \( 4(x) \otimes_{\infty} 4(y) = \infty = 4(x \otimes_{\infty} y) \). Otherwise, we have \( x \otimes_{\infty} y \leq 0 \), and thus \( 4(x \otimes_{\infty} y) = x \otimes_{\infty} y \). Since either \( x \leq 0 \) or \( y \leq 0 \), this is in turn equal to \( 4(x) \otimes_{\infty} 4(y) \).

Similarly, we have \( x \oplus_{\infty} y > 0 \) whenever \( x > 0 \) or \( y > 0 \), and thus in this case it holds that \( 4(x) \oplus_{\infty} 4(y) = \infty = 4(x) \oplus_{\infty} 4(y) \). If both \( x \leq 0 \) and \( y \leq 0 \), then we also have \( 4(x) \oplus_{\infty} 4(y) = x \oplus_{\infty} y = 4(x) \oplus_{\infty} 4(y) \).

Consider now the residuum. If \( x \geq 0 \) and \( y \geq 0 \), then both \( 4(x) \Rightarrow_{\infty} 4(y) \) and \( 4(x \Rightarrow y) \) are either 0 or \( \infty \). Furthermore, by Proposition 2.9 we have \( 4(x \Rightarrow_{\infty} y) = \infty \) if \( x \Rightarrow_{\infty} y > 0 \) iff \( x = 0 \) or \( y > 0 \) iff \( 4(x) = 0 \) or \( 4(y) = \infty \) iff \( 4(x) \Rightarrow_{\infty} 4(y) = \infty \).

For the remaining cases, we show that \( x \Rightarrow_{\infty} y = 4(x) \Rightarrow_{\infty} 4(y) \) holds, from which we obtain \( x \Rightarrow_{\infty} y \in 4 \), and therefore \( 4(x \Rightarrow_{\infty} y) = x \Rightarrow_{\infty} y = 4(x) \Rightarrow_{\infty} 4(y) \). If \( x < 0 \) and \( y \geq 0 \), then we get \( 4(x) \Rightarrow_{\infty} 4(y) = \infty = x \Rightarrow_{\infty} y \). If \( x \geq 0 \) and \( y > 0 \), we have \( 4(x) \Rightarrow_{\infty} 4(y) = 4(y) = y = x \Rightarrow_{\infty} y \). Finally, if both \( x < 0 \) and \( y < 0 \), then \( 4(x) = x \) and \( 4(y) = y \), which implies that \( 4(x) \Rightarrow_{\infty} 4(y) = x \Rightarrow_{\infty} y \).

Consider now a non-empty set \( X \subseteq L_\infty \). If we have \( x \leq 0 \) for all \( x \in X \), then \( \forall_{x \in X} x \leq 0 \), and thus \( 4(\forall_{x \in X} x) = \bigvee_{x \in X} x = \bigvee_{x \in X} 4(x) \). Otherwise, there is an element \( x_0 \in X \) with \( x_0 > 0 \), and thus \( \forall_{x \in X} x > 0 \) and \( 4(\forall_{x \in X} x) = \infty = 4(x_0) = \bigvee_{x \in X} 4(x) \).

Assume now that \( X \) has a least element \( x_0 \). By monotonicity of 4, \( 4(x_0) \) is the least element of \( \{4(x) \mid x \in X\} \), and thus \( 4(\bigwedge_{x \in X} x) = 4(x_0) = \bigwedge_{x \in X} 4(x) \).

\[ \square \]
6.2 Infinite Lattices with One Zero Divisor

Given an ontology $O = (A, T, R)$ in $L_{\infty}$-$\mathfrak{SHOIL}$, we now construct the ontology $O' = (A', T', R)$ in $4$-$\mathfrak{SHOIL}$, where

$$A' := \{ \langle \alpha \geq 4(p) \rangle | \langle \alpha \geq p \rangle \in A \}, \text{ and}$$

$$T' := \{ \langle C \subseteq D \geq 4(p) \rangle | \langle C \subseteq D \geq p \rangle \in T \}.$$

**Lemma 6.8**: $O$ is consistent in $L_{\infty}$-$\mathfrak{SHOIL}$ iff $O'$ is consistent in $4$-$\mathfrak{SHOIL}$.

**Proof**. Let $J$ be a model of $O'$ in $4$-$\mathfrak{SHOIL}$. Since $4$ is closed under the relevant lattice operations of $L_{\infty}$, all concepts get the same interpretation in $L_{\infty}$ as in $4$. As all the axioms in $O'$ are stronger than those in $O$, $J$ is also a model of $O$.

To prove the converse, for an arbitrary model $I$ of $O$, we define the $4$-interpretation $J = (\Delta^{J}, \cdot^{J})$ with $\Delta^{J} := \Delta^{I}$ such that for every individual name $c$ we have $c^{J} := c^{I}$, and for every role name $r$, concept name $A$, and $x, y \in \Delta^{J}$,

$$A^{J}(x) := 4(A^{I}(x)) \text{ and } r^{J}(x, y) := 4(r^{I}(x, y)).$$

It is easy to see that $r^{J}(x, y) = 4(r^{I}(x, y))$ also holds for every complex role $r$, and if $r^{I}$ is transitive, then monotonicity of $4$ implies that $r^{J}$ is also transitive. Similarly, all role inclusions in $R$ remain satisfied by $J$ (see the proof of Lemma 4.2).

Observe now that for every concept $C$ and $x \in \Delta^{J}$, it holds that $C^{J}(x) = 4(C^{I}(x))$.

For concept names, $\top$, and $\bot$, this holds by definition of $J$ and 4. The arguments for concepts of the form $C \sqcap D$, $C \sqcup D$, $C \rightarrow D$, $\exists r.C$, and $\forall r.C$ are straightforward due to fact that $I$ is witnessed and by Lemma 6.7 (see the proof of Lemma 4.2).

We now show that $J$ is a model of $O'$. Given an assertion $\langle a : C \geq \ell \rangle \in A$, we know that $C^{I}(a^{I}) \geq \ell$, thus $J$ is a model of $O$. By monotonicity of $4$ and the above claim, we get $C^{J}(a^{J}) \geq 4(\ell)$, and hence $J$ satisfies the assertion $\langle a : C \geq 4(\ell) \rangle \in A'$. The claim for role assertions follows from a similar argument.

For every GCI $\langle C \subseteq D \geq \ell \rangle \in T$, we know that $C^{I}(x) \Rightarrow_{\infty} D^{I}(x) \geq \ell$ for all $x \in \Delta^{I}$.

By Lemma 6.7, we obtain $C^{J}(x) \Rightarrow_{\infty} D^{J}(x) = 4(C^{I}(x)) \Rightarrow_{\infty} 4(D^{I}(x)) \geq 4(\ell)$ for every $x \in \Delta^{J} = \Delta^{I}$. Thus, $J$ satisfies the axiom $\langle C \subseteq D \geq 4(\ell) \rangle \in T'$.

This yields the claimed result.

**Theorem 6.9**: If $\otimes$ is a continuous t-norm over $[0, 1]$ without zero divisors, then consistency w.r.t. witnessed models in $L_{\infty}$-$\mathfrak{SHOIL}$ with fuzzy general TBoxes and inequality assertions is decidable in EXPTime.

The constructed lattice $L_{\infty}$ has exactly one zero divisor, regardless of which continuous t-norm $\otimes$ it is based upon. If we include additional values $\pm 3, \pm 4, \ldots, \pm (n + 1)$, it is possible to extend the t-norm $\otimes_{\infty}$ in such a way that it has exactly $n$ zero divisors, simply by setting $x \otimes_{\infty} x = -\infty$ for every $x \leq -2$. Arguments analogous to the ones used for Theorems 6.6 and 6.9 can be used to prove that for any natural number $n$ there is an infinite family of residuated lattices with exactly $n$ zero divisors for which ontology consistency in $L_{\infty}$-$\mathfrak{SHOIL}$ is undecidable, and another infinite family for which this problem is decidable in exponential time. In other words, the decidability of ontology consistency in $L_{\infty}$-$\mathfrak{SHOIL}$ cannot be determined by the number of zero divisors that the t-norm has, in contrast to the case of the interval $[0, 1]$.  

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One possible reason for this is that over $[0,1]$ the presence of finitely many zero divisors already implies that $\otimes$ has \textit{infinitely} many zero divisors. It may be the case that any lattice $L$ with infinitely many zero divisors causes consistency in $L$-$\text{ISWIOI}$ (with fuzzy general TBoxes and inequality assertions) to become undecidable. On the other end of the expressivity spectrum, for t-norms that have only finitely many non-idempotent elements, it is possible that the ideas of Section 4.2 for the Gödel t-norm can be adapted to show decidability of consistency.

However, we have to leave open a precise characterization of the decidability of consistency in fuzzy description logics over infinite residuated De Morgan lattices.
7 Conclusions and Future Work

As a conclusion to this thesis, we summarize the obtained results about reasoning in fuzzy description logics with general concept inclusions. We then identify the gaps left open in this work and mention some directions for future research on fuzzy description logics.

7.1 Summary

The study of fuzzy description logics has started by applying the basic semantics proposed by Zadeh (1965) to $\mathcal{FL}_0$ and $\mathcal{ALC}$ with unfoldable TBoxes, and generalizing classical decision procedures to reason in these new logics. Since Hájek (2005b) repaired the first tableau algorithms by introducing the restriction to witnessed models and proposed general t-norm-based semantics for fuzzy DLs, more and more decision procedures for consistency in t-norm-based fuzzy DLs have appeared. These are either reductions to classical reasoning or fuzzy tableau algorithms enhanced with systems of polynomial inequations. Soon, proofs for termination of classical tableau algorithms were adapted for fuzzy DLs with fuzzy general TBoxes by introducing naive blocking conditions. In 2011, these algorithms were shown to be unsound and the first undecidability results for consistency were published.

The main contribution of this work is the extension of these results to large classes of fuzzy description logics and the proof of decidability of most of the rest (see Table 5.2). Additionally, we analyzed the complexity of reasoning in fuzzy description logics with semantics based on finite lattices. For most decidability results, tight complexity bounds have been obtained.

The decidability of fuzzy DLs over the standard interval $[0,1]$ is very sensitive to the choice of constructors and t-norm. On the one hand, there are fuzzy description logics that allow many constructors, but are decidable due to the properties of the t-norm, such as $\otimes\text{-SUHOI}$ with fuzzy general TBoxes and inequality assertions where $\otimes$ has no zero divisors. On the other hand, extensions of $\mathcal{EL}$ with any kind of negation constructor easily become undecidable, e.g. $\otimes\text{-}\mathcal{ILC}$ with crisp ontologies if $\otimes$ has zero divisors or $\otimes\text{-}\mathcal{ELC}$ with crisp general TBoxes and inequality assertions for any t-norm $\otimes$ except the Gödel t-norm.

To prove the latter results, in Chapter 5 a framework was developed that describes several properties that a fuzzy DL has to satisfy in order for consistency to be undecidable. Each of these properties corresponds to a small step in the reduction from the Post correspondence problem. This general idea allows to prove undecidability of consistency in many fuzzy description logics under witnessed and general model semantics.

Both the undecidability results of Chapter 5 and the decidability results of Section 4.1 put a damper on any kind of application of these fuzzy description logics. In the former case, no sound, complete, and terminating implementation for fuzzy reasoning can
exist, while in the latter case, one can simply ignore all fuzzy values and apply classical reasoning algorithms. This leaves few options that are worth pursuing when considering a knowledge representation system based on fuzzy DLs: either we give up soundness and/or completeness, we do not allow general concept inclusions in the knowledge base, or we use only finite-valued semantics (see Chapter 3), Zadeh semantics over $[0, 1]$ (Bobillo, Delgado, and Gómez-Romero 2008), or Gödel semantics (see Section 4.2).

Regarding concrete implementations for these special cases, there are several options. A direct reduction to crisp reasoning as the one in (Bobillo, Delgado, and Gómez-Romero 2008) can take advantage of existing optimized reasoners for classical DLs, provided that the resulting ontology is not too large. One can also implement a fuzzy tableau algorithm from scratch, as was done for the fuzzyDL reasoner (Bobillo and Straccia 2008a), or adapt existing implementations for classical DLs to deal with fuzzy values. The latter approach has the advantage that in theory all developed optimizations can be reused, but in practice this requires an intimate knowledge of the system one wants to extend. If one does not care about correctness, one can also use a translation of finite-valued fuzzy DLs into fuzzy first-order languages and employ existing reasoners for these more expressive formalisms, e.g., $\mathcal{F^{LP}}$ (Beckert, Hähnle, Oel, and Sulzmann 1996).

### 7.2 Open Problems

This work leaves some gaps in the decidability analysis and opens up new topics for future research on fuzzy description logics.

- The precise complexity of reasoning in $L$-$\mathcal{SCI}$ with fuzzy roles and acyclic TBoxes over a finite lattice $L$ is still open. It may be possible to adapt one of the PSPACE-results for classical DLs obtained via tableau algorithms (Baader and Sattler 2001; Horrocks, Sattler, and Tobies 2000; Lutz 1999), but such an approach likely faces similar problems as the automata-based one from Section 3.1.5.

- The decidability results of Section 4.1 can also be shown in the presence of fuzzy role inclusions similar to fuzzy GCIs (Borgwardt, Distel, and Peñaloza 2012h). It seems possible to extend these and also the proofs of Chapter 3 to fuzzy role inclusions and fuzzy transitivity axioms $\langle \text{trans}(r) \geq p \rangle$ requiring that $p \otimes r^T(x, y) \otimes r^T(y, z) \leq r^T(x, z)$ holds for all domain elements $x, y, z$ of a model of this axiom. However, the proofs of Chapter 3 would have to be enhanced with more involved constructions to account for the degrees of the role axioms.

- The results of Section 4.2 suggest a combination with the techniques of Chapter 3 in order to prove ExpTime-completeness of witnessed reasoning in G-$\mathcal{SCCHI}$ with fuzzy general TBoxes and equality assertions. Likewise, PSPACE upper bounds can probably be obtained for G-$\mathcal{ALCHI}$ and G-$\mathcal{SCI}_{(c)}$.

- To reason in the presence of nominals, a common approach is to keep some global information about the behavior of the named individuals (Baader, Lutz, et al. 2005; Horrocks and Sattler 2005; Sattler and Vardi 2001). This has been used in Appendix A for classical DLs, and recently in (Borgwardt 2014) for fuzzy DLs over finite residuated De Morgan lattices, extending the results of Chapter 3. It
remains open to apply the same idea to the automata-based algorithm for $\mathbb{G}\mathcal{ALC}$ described in Section 4.2.

- One could add any constructor to a decidable fuzzy DL, e.g. fuzzy modifiers or number restrictions, and try to stay decidable. In the case of finite lattice semantics, they will probably add no more problems to the decision procedures than in the classical case. However, we want to mention that the semantics of fuzzy number restrictions is somewhat controversial, especially in the absence of the restriction to witnessed models. If an existential restriction $\exists r.C$ can only be satisfied by using three successors, then what is the meaning of an expression of the form $\leq 2 r.C$? Following the semantics proposed in (Bobillo and Straccia 2011) for finite-valued Łukasiewicz semantics w.r.t. witnessed models, this concept could never have a value of 1. However, it might be desirable to evaluate it “modulo” the number of witnesses needed for the value of $\exists r.C$.

- The decidability of consistency of $\otimes\mathcal{RLC}$ and $\otimes\mathcal{IALC}$ with equality assertions is unknown if $\otimes$ does not start with $\mathcal{L}$ (see Table 5.2). Likewise, for $\otimes\mathcal{ELC}$ the only known results concern the three fundamental continuous t-norms.

Regarding the second gap, observe that the proofs of undecidability for both $\mathcal{ELC}$ and $\Pi\mathcal{ELC}$ use the fact that one can construct the constant 1/2 using the axiom $\langle H \equiv \neg H \rangle$. We conjecture that these proofs can be lifted to $\otimes\mathcal{ELC}$, where $\otimes$ is any continuous t-norm for which 1/2 is not an idempotent element. This condition ensures that 1/2 lies in a component of norm that uses either the Łukasiewicz or the Product t-norm. Starting from this value, one can construct encodings of the words $v_i$ and $w_i$. However, the encoding has to be adapted since 1/2 need not lie in the exact center of the component interval.

- For sublogics of $\otimes\mathcal{EUL}$, consistency is trivial since every ontology has a model. However, except for partial results, the precise complexities of subsumption and instance checking in $\otimes\mathcal{EL}$ and $\otimes\mathcal{EUL}$ with an arbitrary continuous t-norm $\otimes$ remain open (Borgwardt and Peñaloza 2013a,b; Mailis et al. 2012).

- The precise complexity of reasoning with acyclic TBoxes in fuzzy DLs over the standard interval is still open since previous decidability results for $\otimes\mathcal{IALC}$ do not yield tight complexity bounds (cf. Section 2.4). For Zadeh semantics, the algorithm in (Straccia 2001) shows an ExpSpace upper bound, which, however, should be easy to reduce to PSPACE by including lazy unfolding in the tableaux rules. For the Łukasiewicz t-norm, (Bobillo and Straccia 2009) proves inclusion in NExpTime; for the Product t-norm, the upper bound is ExpSpace. It seems unlikely, however, that the methods used in this thesis can help to lower these bounds, apart from the cases in Section 4.1.

- If any of the decidable fuzzy DLs are in fact used for an application, it is desirable to extend the decidability results to non-standard reasoning tasks that aid in ontology design and maintenance, e.g. finding least common subsumers, matching, unification, axiom pinpointing, or conjunctive query answering.
Appendix
A Classical Description Logics with Nominals and Acyclic TBoxes

The complexity of the standard reasoning problems in classical $ALCHO$ and $SO$ with acyclic TBoxes is believed to be $PSPACE$-complete. We use this appendix to provide, to the best of our knowledge, the first proof of this fact. It combines the automata-based proof technique for $SI$ from (Baader, Hladik, and Peñaloza 2008) with an approach to deal with nominals described in the context of $ALCOQ$ in (Baader, Lutz, et al. 2005) and rules for role inclusions and transitivity axioms in $SHI$ from (Horrocks, Sattler, and Tobies 2000). Thus, the following constructions are very similar to the (ordered) Hintikka trees of Sections 3.1 and 4.2.

We describe the decision procedure in terms of the logic $SHO$ ($2$-$SCHO$), and then provide faithful families of functions to obtain $PSPACE$ on-the-fly constructions for $ALCHO$ and $SO$ (see Section 3.1.3). We consider the syntax of these logics to allow the concept constructors $\top$, $\bot$, $\sqcap$, $\sqcup$, $\neg$, nominals, and existential and value restrictions. Implications and residual negations are not relevant here since they can already be expressed by the above constructors. The axioms are restricted to (crisp) assertions, (acyclic) concept definitions, role inclusions, and transitivity axioms.

We also assume that all concepts we deal with are in negation normal form, i.e. negation occurs only directly in front of concept names. Any concept can be transformed in linear time to an equivalent one in negation normal form using the De Morgan laws, the duality of existential and value restrictions, and elimination of double negations. For a concept $C$, we denote the negation normal form of $\neg C$ by $\neg neg C$.

It suffices to analyze only the complexity of consistency, as the other reasoning problems can be reduced to an (in-)consistency check. In the presence of nominals, any consistency problem can further be expressed as a local consistency problem as follows: An ontology $\mathcal{O} = (A, T, R)$ is consistent iff $(\{c_0 : C_A\}, T, R)$ is locally consistent, where $c_0$ is a fresh individual name and

$$C_A := \bigcap_{(c: C) \in A} \exists r. \{(c) \cap C\} \cap \bigcap_{(c,d : s) \in A} \exists r. \{(c) \cap \exists s. \{d\}\},$$

where $r$ is a fresh role name; see (Baader, Lutz, et al. 2005).

Thus, in the following we want to decide local consistency of an ontology $\mathcal{O} = (A, T, R)$, where $A$ is a local ABox and $T$ is an acyclic TBox. We denote by $\text{Ind}$ the set of all individual names occurring in $\mathcal{O}$. We extend the definition of $\text{sub}_R(C)$ from Definition 3.1 to nominals and disjunctions by setting $\text{sub}_R(C \sqcup D) := \{C \sqcup D\} \cup \text{sub}_R(C) \cup \text{sub}_R(D)$ and $\text{sub}_R(\{c\}) := \{\{c\}\}$. We further consider the negation closure of $\text{sub}(\mathcal{O})$ similarly as in Section 4.2, keeping in mind that we want all concepts to be in negation normal form:

$$\text{cl}(\mathcal{O}) := \{C, \neg C \mid C \in \text{sub}(\mathcal{O})\}.$$
In the crisp case, instead of Hintikka functions we only need to consider Hintikka sets (cf. Baader, Hladik, and Peñaloza (2008)). As in Section 3.1, the goal is to describe a model using a tree consisting of such Hintikka sets. Due to the presence of nominals, we actually need to consider several trees, each rooted in a different individual. We again use a special element \( \varrho \) in the Hintikka sets, which is used to indicate whether the role connection to the parent node in the Hintikka tree is present.

**Definition A.1 (Hintikka set)** A set \( H \subseteq \text{cl}(O) \cup \{ \varrho \} \) is called a *Hintikka set* for \( O \) if

1. \( \bot \not\in H \);
2. if \( C \sqcap D \in H \), then \( \{ C, D \} \subseteq H \); and
3. if \( C \sqcup D \in H \), then \( \{ C, D \} \cap H \neq \emptyset \);
4. there is no concept name \( A \) with \( \{ A, \neg A \} \subseteq H \).

A Hintikka set \( H \) is *compatible* with \( T \) if for every \( \langle A \equiv C \rangle \in T \), it holds that \( A \in H \) implies \( C \in H \) and \( \neg A \in H \) implies \( \neg C \in H \). It is *compatible* with \( A \) if for every \( \langle c : C \rangle \in A \) we have \( C \in H \).

For a Hintikka set \( H \), we define the set \( \text{Ind}(H) := \{ c \in \text{Ind} | \{ c \} \in H \} \) that contains all individual names represented by \( H \).

To deal with the individual names, our algorithm starts by guessing an equivalence relation \( \approx \) on \( \text{Ind} \) that specifies which individual names should be interpreted as the same domain elements. We denote by \( [c]_\approx \) the equivalence class of \( \approx \) that contains \( c \in \text{Ind} \), and by \( \text{Ind}/_\approx \) the set of all resulting equivalence classes. Subsequently, we guess a family \( \{ H_X \}_{X \in \text{Ind}/_\approx} \) of subsets of \( \text{cl}(O) \). These sets represent the concepts the named domain elements should satisfy. This is similar to the approach used in (Baader, Lutz, et al. 2005) to decide concept satisfiability in \( \text{ALCOQ} \) with acyclic TBoxes.

We then need to ensure that these guesses satisfy some basic conditions, namely that they are Hintikka sets containing the correct nominals and that the local ABox \( A \) is satisfied:

A.1 For every \( X \in \text{Ind}/_\approx \), the set \( H_X \) is a Hintikka set for \( O \) that is compatible with \( T \) such that \( \text{Ind}(H_X) = X \).

A.2 \( H_{[c_0]}_\approx \) is compatible with \( A \), where \( c_0 \) is the unique individual name used in the assertions of \( A \).

Since we are aiming for a complexity of \( \text{PSPACE} \) and \( \text{NPSPACE} \) equals \( \text{PSPACE} \) by a result from Savitch (1970), this initial polynomial guess and the subsequent tests do not affect our result.

In the following, we call a Hintikka set \( H \) simply *compatible* if it is compatible with \( T \) and it is compatible with the family \( \{ H_X \}_{X \in \text{Ind}/_\approx} \) in the following sense: For every \( c \in \text{Ind}(H) \), we have \( H \subseteq H_{[c]}_\approx \). This means that we require all Hintikka sets to respect our initial guess about the behavior of the named individuals. By (A.1), each \( H_X \), \( X \in \text{Ind}/_\approx \), is compatible since it is compatible with \( T \) and for every \( c \in \text{Ind}(H_X) = X \) it holds that \( [c]_\approx = X \).

Another difference to Section 3.1 is that here we consider the arity \( k \) of our Hintikka trees to be only the number of *existential* restrictions in \( \text{cl}(O) \). Again, we fix an arbitrary bijection

\[ \varphi : \{ \exists r.C | \exists r.C \in \text{cl}(O) \} \rightarrow K, \]
where $K := \{1, \ldots, k\}$. For a fixed role name $r$, we denote by $\varphi_r(O)$ the set of all indices $i \in K$ such that $i = \varphi(\exists r.C)$ for some $\exists r.C \in cl(O)$.

**Definition A.2 (Hintikka condition)** A tuple $(H_0, H_1, \ldots, H_k)$ of Hintikka sets for $O$ satisfies the Hintikka condition if the following hold for every $\exists r.C \in cl(O)$ with $i = \varphi(\exists r.C)$:

- if $\exists r.C \in H_0$, then $\{\varnothing, C\} \subseteq H_i$; and
- if $\varnothing \in H_i$, then for every $\forall r'.D \in H_0$ with $r \subseteq_R r'$, we have $D \in H_i$, and additionally $\forall s.D \in H_i$ for every transitive role name $s$ with $r \subseteq_R s \subseteq_R r'$.

Since we need to find several Hintikka trees, each representing a different set of individuals $X \in \text{ind}/_\approx$ for which we already know the Hintikka set, we introduce the notion of Hintikka trees starting with a given Hintikka set.

**Definition A.3 (Hintikka tree)** A Hintikka tree for $O$ starting with a Hintikka set $H_X$ is a mapping $T$ that assigns to each node $u \in K^+$ a compatible Hintikka set for $O$ such that $T(\epsilon) = H_X$ and, for every $u \in K^+$, the tuple $(T(u), T(u1), \ldots, T(uk))$ satisfies the Hintikka condition.

In the proof of the following lemma, it is convenient to view $C^I$ for an interpretation $I$ as a subset of $\Delta^I$ rather than the characteristic function $C^I: \Delta^I \to \{0, 1\}$ introduced in Section 2.2, and similarly for role names.

**Lemma A.4** $O$ is locally consistent iff there exist a family of Hintikka sets $(H_X)_{X \in \text{ind}/_\approx}$ for $O$ that are compatible with $T$ and satisfy (A.1) and (A.2), and for each $X \in \text{ind}/_\approx$ a Hintikka tree for $O$ starting with $H_X$.

**Proof.** Assume first that there are such a family $(H_X)_{X \in \text{ind}/_\approx}$ and Hintikka trees $T_X$ starting with $H_X$ for each $X \in \text{ind}/_\approx$. The first step in the construction of a model $T$ of $O$ is to get rid of irrelevant nodes in these Hintikka trees. We say that a node is irrelevant in $T_X$ if it is (or has an ancestor) of the form $ui$ with $u \in K^+$ and $i \in K$ such that either

a) $\text{ind}(T_X(ui)) \neq \emptyset$ or

b) $\varnothing \notin T_X(ui)$.

All other nodes are called relevant in $T_X$. Note that $\epsilon$ is relevant in all Hintikka trees. For a node satisfying a), we have $c \in \text{ind}(T_X(ui))$ for some $c \in \text{ind}$, and thus the compatible Hintikka set $T_X(ui)$ can be replaced by $T_{[c]_\approx}(\epsilon)$ since $T_X(ui) \subseteq H_{[c]_\approx} = T_{[c]_\approx}(\epsilon)$. Similarly, a node satisfying b) is irrelevant since the corresponding existential restriction is not present in the parent node. We now define the domain of $T$ as

$$\Delta^T := \{(X, u) \in (\text{ind}/_\approx) \times K^+ | u \text{ is relevant in } T_X\}.$$  

We set $c^T := ([c]_\approx, \epsilon)$ for each $c \in \text{ind}$ and define, for each role name $r$, a binary relation $r^T$ on $\Delta^T$ as follows: $((X, u), (Y, v)) \in r^T$ iff there is an index $i \in \varphi_r(O)$ with $r' \subseteq_R r$ such that $\varnothing \notin T_X(ui)$ and either (i) $X = Y$ and $v = ui$, or (ii) $v = \epsilon$ and $\text{ind}(T_X(ui)) \cap Y \neq \emptyset$. To obtain a model of $R$, we then set

$$r^T := r^T \cup \bigcup_{s \subseteq_R r} (s^T)^+. $$
Thus, in particular all transitive roles are interpreted by transitive binary relations. From the transitivity of $\subseteq_R$, we also obtain $r^s \subseteq s^t$ for every $r \subseteq s \in R$.

It remains to define the interpretations of the concept names in such a way that $\mathcal{I}$ becomes a model of $A$ and $\mathcal{T}$. For every primitive concept name $A$, we simply set

$$A^\mathcal{I} := \{(X, u) \in \Delta^\mathcal{I} \mid A \in T_X(u)\}.$$ 

We now extend $\mathcal{I}$ to the defined concept names, while showing the following claim: for every $C \in \text{cl}(O)$ and every $(X, u) \in \Delta^\mathcal{I}$ with $C \in T_X(u)$, we have $(X, u) \in C^\mathcal{I}$. We prove this by induction on the weight function $\omega$ defined in the proof of Lemma 3.6, extended to nominals and disjunctions as follows: $\omega(\{c\}) := 0$ and $\omega(C \cup D) := \max\{\omega(C), \omega(D)\} + 1$.

For $\top$, the claim is trivial, and for $\bot$ it follows from the fact that $T_X(u)$ is a Hintikka set. The primitive concept names $A$ satisfy the claim by the definition of $A^\mathcal{I}$ above.

Consider now a defined concept name $A$ with the (crisp) definition $\langle A \equiv C \rangle \in \mathcal{T}$. If $A \in T_X(u)$, then by the compatibility of $T_X(u)$ with $\mathcal{T}$, we also have $C \in T_X(u)$, and thus $(X, u) \in C^\mathcal{I}$ by induction. Hence, by setting $A^\mathcal{I} := C^\mathcal{I}$ we ensure that $\mathcal{I}$ satisfies the concept definition and we have $(X, u) \in A^\mathcal{I}$.

If $\{d\} \in T_X(u)$ for a $d \in \text{Ind}$, this means that $d \in \text{Ind}(T_X(u))$, and thus we must have $u = \varepsilon$ since otherwise $u$ would be irrelevant in $T_X$. Hence, we have $d \in \text{Ind}(H_X) = X$ by (A.1), which shows that $d^\mathcal{I} = ([d]_\mathcal{I}, \varepsilon) = (X, u)$, i.e. $(X, u) \in \{d\}^\mathcal{I}$.

Since $\neg A \notin T_X(u)$ for a (primitive or defined) concept name $A$, then we have $A \notin T_X(u)$ since $T_X(u)$ is a Hintikka set. If $A$ is primitive, then we have $(X, u) \notin A^\mathcal{I}$ by definition. If $A$ is defined by $\langle A \equiv C \rangle \in \mathcal{T}$, then $\neg C \in T_X(u)$ since $T_X(u)$ is compatible with $\mathcal{T}$. Since $\omega(\neg C) \leq \omega(C) + 1 = \omega(A) < \omega(\neg A)$, by induction we get $(X, u) \in (\neg C)^\mathcal{I}$, and thus again $(X, u) \notin C^\mathcal{I} = A^\mathcal{I}$.

If $C \cap D \in T_X(u)$, then $(C, D) \subseteq T_X(u)$, and thus by induction it follows that $(X, u) \in C^\mathcal{I}$ and $(X, u) \in D^\mathcal{I}$, i.e. $(X, u) \in (C \cap D)^\mathcal{I}$. The claim for $C \cup D$ follows similarly.

If $\exists r.C \in T_X(u)$, then since $T_X$ is a Hintikka tree, we have $\{r, C\} \subseteq T_X(u)$, where $i = \varphi(\exists r.C)$. Since $u$ is relevant in $T_X$, $ui$ can only be irrelevant in $T_X$ if $\text{Ind}(T_X(ui)) \neq \emptyset$. If indeed we have $c \in \text{Ind}(T_X(ui))$ for some $c \in \text{Ind}$, then the compatibility of $T_X(ui)$ ensures that $C \in T_X(ui) \subseteq H_{[c]_\mathcal{I}} = T_X([c]_\mathcal{I}, \varepsilon)$. Since $\varepsilon$ is relevant, by induction we obtain $([c]_\mathcal{I}, \varepsilon) \in C^\mathcal{I}$. By the definition of $r^\mathcal{I}$ and reflexivity of $\subseteq_R$, we also get $((X, u), ([c]_\mathcal{I}, \varepsilon)) \in r^\mathcal{I} \subseteq r^\mathcal{T}$, and thus $(X, u) \in (\exists r.C)^\mathcal{I}$. Otherwise, we have $\text{Ind}(T_X(ui)) = \emptyset$ and $(X, ui) \in \Delta^\mathcal{I}$, and thus induction yields $(X, ui) \in C^\mathcal{I}$. Since we also have $(X, u), (X, ui) \in r^\mathcal{T} \subseteq r^\mathcal{I}$, this again implies $(X, u) \in (\exists r.C)^\mathcal{I}$.

If $\forall r.C \in T_X(u)$, then consider any $(Y, v) \in \Delta^\mathcal{I}$ with $((X, u), (Y, v)) \in r^\mathcal{T}$. If the pair $((X, u), (Y, v))$ is already in $r^\mathcal{I}$, then there is an index $i \in \varphi(\forall r.C')$ with $r' \subseteq_R r$ such that $g \in T_X(xi)$ and either (i) $X = Y$ and $v = ui$, or (ii) $v = \varepsilon$ and $\text{Ind}(T_X(xi)) \cap Y \neq \emptyset$. In both cases, the Hintikka condition yields $C \in T_X(xi)$. In case (i), we obtain $(Y, v) = (X, ui) \in C^\mathcal{I}$ by induction, which shows that the value restriction is satisfied for this $r$-successor of $(X, u)$. If (ii) holds, then $C \in T_X(xi) \subseteq H_Y = T_Y(\varepsilon)$ by the compatibility of $T_X(xi)$, which again implies by induction that $(Y, v) = (Y, \varepsilon) \in C^\mathcal{I}$.

Consider now the remaining case that $((X, u), (Y, v)) \in (s^\mathcal{T})^+$ for some transitive role $s$ with $s \subseteq_R r$. This means that there are domain elements $(Z_i, w_i)$, $1 \leq i \leq n$, such that the pairs $((X, u), (Z_1, w_1)), \ldots, ((Z_n, w_n), (Y, v))$ are all contained in $s^\mathcal{T}$. 

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For the first pair, we again know by the Hintikka condition that there is an index \( i \in \varphi_r(O) \) with \( r^i \sqsubseteq s \) and \( \forall s.C \in T_X(u_i) \). If (a) \( Z_1 = X \) and \( w_1 = u_i \), we thus have \( \forall s.C \in T_{Z_1}(w_1) \). Otherwise, we know that (b) \( w_1 = \varepsilon \) and \( \text{Ind}(T_X(u_i)) \cap Z_1 \neq \emptyset \), which also yields \( \forall s.C \in T_X(u_i) \subseteq H_{Z_1} = T_{Z_1}(\varepsilon) = T_{Z_1}(w_1) \). Analogously, one can show that \( \forall s.C \in T_{Z_1}(w_i) \) holds for all \( i, 2 \leq i \leq n \). Finally, as in the case for \( r^T \) above, we obtain \( C \in T_Y(v) \), and thus \((Y,v) \in C^\varepsilon \) by induction.

This in particular shows that \( \mathcal{I} \) satisfies all concept definitions in \( T \). Since \( H_{[c_0]_\approx} \) is compatible with \( \mathcal{A} \), for every \( \langle c_0,C \rangle \in \mathcal{A} \) we have \( C \in H_{[c_0]_\approx} = T_{[c_0]_\approx}(\varepsilon) \), and thus \( c_0^\varepsilon = ([c_0]_\approx,\varepsilon) \in C^\varepsilon \) by the above claim, i.e. \( \mathcal{I} \) also satisfies \( \mathcal{A} \).

Conversely, assume that there is a model \( \mathcal{I} \) of \( O \). We use this model to define the required equivalence relation \( \approx \), Hintikka sets \( H_X \), and Hintikka trees \( T_X \). For all \( c,d \in \text{Ind} \), we set \( c \approx d \) iff \( c^\varepsilon = d^\varepsilon \). For each \( X \in \text{Ind}_\approx \), we then define \( H_X := H(c^\varepsilon) \), where \( c \) is any element of \( X \) and \( H(x) \) is defined for any \( x \in \Delta^\varepsilon \) by

\[
H(x) := \{ C \in \text{cl}(O) \mid x \in C^\varepsilon \}.
\]

Since \( \mathcal{I} \) satisfies \( T \), this obviously defines a Hintikka set for \( O \) that is compatible with \( T \), and we additionally have \( \text{Ind}(H_X) = X \), i.e. (A.1) is satisfied. Furthermore, for every \( \langle c_0,C \rangle \in \mathcal{A} \), we have \( c_0^\varepsilon \in C^\varepsilon \), and thus \( C \in H_{[c_0]_\approx} \), which shows that (A.2) is also satisfied.

Note that, for every \( x \in \Delta \), the Hintikka set \( H(x) \) is compatible with the resulting family \( \{ H_X \}_{X \in \text{Ind}_\approx} \) since \( c \in \text{Ind}(H(x)) \) implies \( x \in \{ c \}^\varepsilon \), i.e. \( x = c^\varepsilon \), and thus \( H(x) = H_{[c]_\approx} \).

For a given \( X \in \text{Ind}_\approx \), we now define the Hintikka tree \( T_X \) starting with \( H_X \) by inductively constructing a mapping \( g: K^* \to \Delta^\varepsilon \) that specifies which elements of \( \Delta^\varepsilon \) represent which nodes of \( T_X \). In this construction, we ensure that \( T_X(u) \cap \text{cl}(O) = H(g(u)) \) holds for all \( u \in K^* \). This in particular ensures that all Hintikka sets we use in the construction are compatible. We start by setting \( g(\varepsilon) := c^\varepsilon \) and \( T_X(\varepsilon) := H_X \), where \( c \) is any element of \( X \). Thus, \( T_X \) obviously starts with \( H_X \) and \( T_X(\varepsilon) \) is of the required form since \( H_X = H(c^\varepsilon) = H(g(\varepsilon)) \).

Let now \( u \in K^* \) be any node for which \( g(u) \) and \( T_X(u) \) have already been defined, and consider any \( i \in K \) and the associated existential restriction \( \exists r.C = \varphi^{-1}(i) \). If \( \exists r.C \notin H(g(u)) \), then \( T_X(u_i) \) is irrelevant and we can set, e.g. \( g(u)i := g(\varepsilon) \) and \( T_X(u_i) := H_X \). Since \( g \notin T_X(\varepsilon) \), the Hintikka condition for \( \exists r.C \) is satisfied.

Otherwise, we know that \( g(u) \in (\exists r.C)^\varepsilon \), and thus there must exist a \( y_i \in \Delta^\varepsilon \) such that \( (g(u), y_i) \in r^\varepsilon \) and \( y_i \in C^\varepsilon \). In this case, we set \( g(u)i := y_i \) and \( T_X(u_i) := H(y_i) \cup \{ g \} \).

To verify the Hintikka condition, observe first that \( C \in H(y_i) \), since \( y_i \in C^\varepsilon \). Consider now any \( \forall r'.D \in H(g(u)) \) with \( r \sqsubseteq r' \). By the definition of \( H \), we obtain \( g(u) \in (\forall r'.D)^\varepsilon \). Since \( \mathcal{I} \) satisfies \( R \), we have \( (g(u), y_i) \in r^\varepsilon \subseteq r'^\varepsilon \), and thus \( y_i \in D^\varepsilon \). This shows that \( D \in H(y_i) \). If additionally there is a transitive role \( s \) with \( r \sqsubseteq s \sqsubseteq r' \), then we have to show that \( y_i \in (\forall s.D)^\varepsilon \). Assume to the contrary that there is a \( y' \in \Delta^\varepsilon \) such that \( (y_i, y') \in s^\varepsilon \), but \( y' \notin D^\varepsilon \). Since \( (g(u), y_i) \in r^\varepsilon \subseteq s^\varepsilon \) and \( s \) is transitive, we also have \( (g(u), y') \in s^\varepsilon \subseteq r^\varepsilon \). But now \( y' \notin D^\varepsilon \) stands in contradiction to our assumption that \( g(u) \in (\forall r'.D)^\varepsilon \).

As usual, we use looping tree automata to decide the existence of Hintikka trees (see Section 3.1.2). We again augment each Hintikka set in these trees by an index specifying its position relative to its siblings. We assume here without loss of generality that \( k \geq 1 \).
since otherwise any Hintikka tree starting with $H_X$ would simply consist of $H_X$ and not contain other meaningful information.

**Definition A.5 (Hintikka automaton)** Given a compatible Hintikka set $H_X$ for $O$, the Hintikka automaton $A_{O,H_X}$ for $O$ and $H_X$ is the looping automaton $(Q_O, I_{O,H_X}, \Delta_O)$, where

- $Q_O$ is the set of all pairs $(H, i)$, where $H$ is a compatible Hintikka set for $O$ and $i \in K$;
- $I_{O,H_X} := \{(H_X, i) \in Q_O \mid i \in K\}$; and
- $\Delta_O$ is the set of all tuples $((H_0, i_0), (H_1, 1), \ldots, (H_k, k))$ such that $(H_0, H_1, \ldots, H_k)$ satisfies the Hintikka condition.

Obviously, $A_{O,H_X}$ is non-empty iff there is a Hintikka tree for $O$ starting with $H_X$. Our local consistency test for $O$ thus consists of guessing an equivalence relation $\approx$ on $\text{Ind}/_\approx$ and a family $(H_X)_{X \in \text{Ind}/_\approx}$ satisfying (A.1) and (A.2) and checking non-emptiness of the automaton $A_{O,H_X}$ for each $X \in \text{Ind}/_\approx$. As detailed before, the first steps can be executed in nondeterministic polynomial space, and thus it suffices to show that emptiness of the (linearly many) looping automata $A_{O,H_X}$ can be checked in PSPACE.

As in Section 3.1, we employ faithful families of functions to obtain PSPACE on-the-fly constructions of subautomata $A_{O,H_X}^S$. For $\text{ACCHO}$, this is easy since we can ensure that the maximal role depth of concepts appearing in the Hintikka sets decreases linearly with the depth of the Hintikka tree. We extend the definition of the role depth $\text{rd}_T$ of concepts w.r.t. $T$ from Section 3.1.4 to disjunctions and nominals as follows: $\text{rd}_T(\{c\}) := 0$ and $\text{rd}_T(C \cup D) := \max\{\text{rd}_T(C), \text{rd}_T(D)\}$. Given a Hintikka set $H$ for $O$, we denote by $\text{rd}_T(H)$ the maximal role depth $\text{rd}_T(C)$ of all concepts $C \in H$. For $n \geq 0$, we denote by $\text{cl}^{\leq n}(O)$ the set of all concepts of $\text{cl}(O)$ with role depth less than or equal to $n$. Note that for all concepts $C$ it holds that $\text{rd}_T(C) = \text{rd}_T(\text{\text{\#}}C)$.

The following definition is similar to Definition 3.15 and the faithful functions employed in (Baader, Hladik, and Peñaloza 2008).

**Definition A.6 (functions $f_{(H,i)}$)** Let $(H, i)$ and $(H', i')$ be two states of $A_{O,H_X}$ and $n := \text{rd}_T(H)$. We define the function $f_{(H,i)}(H', i') := (H'', i'')$, where

$$H'' := H' \cap (\text{cl}^{\leq n-1}(O) \cup \{\varnothing\}).$$

Note that the resulting pairs are again elements of $Q_O$.

**Lemma A.7** In $\text{ACCHO}$, the family $f_{(H,i)}$ is faithful w.r.t. $A_{O,H_X}$.

**Proof.** Consider states $q = (H, i)$, $q_0 = (H_0, i_0)$, and $q_j = (H_j, j)$, $1 \leq j \leq k$, and define $n := \text{rd}_T(H)$, $q'_0 := (H'_0, i_0) := f_q(q_0)$, and $q'_j := (H'_j, j) := f_{q_j}(q_j)$ for each $j$, $1 \leq j \leq k$. For the first condition of Definition 3.10, we assume that $(H, H_1, \ldots, H_k)$ satisfies the Hintikka condition and verify it for $(H, H'_1, \ldots, H'_k)$ by considering each $\exists r.C \in \text{cl}(O)$ and $j := \varphi(\exists r.C)$.

If $\exists r.C \in H$, then $\{\varnothing, C\} \subseteq H_j$. Since $\text{rd}_T(C) < \text{rd}_T(\exists r.C) \leq \text{rd}_T(H) = n$, this implies that $\{\varnothing, C\} \subseteq H_j \cap (\text{cl}^{\leq n-1}(O) \cup \{\varnothing\}) = H'_j$. Assume now that $\varnothing \in H'_j$, and thus $\varnothing \in H_j$. Then for any $\forall r'.D \in H$ with $r \sqsubseteq R'$ we have $\text{rd}_T(D) < \text{rd}_T(\forall r'.D) \leq \text{rd}_T(H) = n$ and...
$D \in H_j$, and thus $D \in H'_j$. This already finished the first part of this proof since in AC0 there are no transitive roles.

The second condition of Definition 3.10 requires us to show that $(H'_0, H'_1, \ldots, H'_k)$ satisfies the Hintikka condition if $(H_0, H_1, \ldots, H_k)$ does. We consider again any $\exists r.C \in \mathsf{cl}(O)$ and $j := \varphi(\exists r.C)$.

If $\exists r.C \in H'_0 = H_0 \cap (\mathsf{cl}^{\leq n-1}(O) \cup \{q\})$, then $\mathsf{rd}_T(C) < \mathsf{rd}_T(\exists r.C) \leq n - 1$ and $\exists r.C \in H'_0$, which implies that $\{q, C\} \subseteq H_j \cap (\mathsf{cl}^{\leq n-1}(O) \cup \{g\}) = H'_j$. Assume now that $q \in H'_j$, i.e. $q \in H_j$, and consider any $\forall r'.D \in H'_0 = H_0 \cap (\mathsf{cl}^{\leq n-1}(O) \cup \{g\})$ with $r \subseteq \mathcal{R} r'$. This means that we also have $\forall r'.D \in H_0$, and thus $D \in H_j$. Since $\mathsf{rd}_T(D) < \mathsf{rd}_T(\forall r'.D) \leq n - 1$, we infer that $D \in H'_j$. □

By Proposition 3.11, the emptiness test of $A_{O,H_X}$ can thus be reduced to the one of $A_{O,H_X}^S$, which can be done in PSPACE by the following lemma and Proposition 3.13.

**Lemma A.8** The construction of $A_{O,H_X}^S$ from $O$ and $H_X$ is a PSPACE on-the-fly construction.

**Proof.** As in Sections 3.1.4 and 3.1.5, it only remains to verify that these automata are polynomially blocking (cf. Definition 3.9). We use here the equality on $Q_O$ as blocking relation. Consider any path in a run of $A_{O,H_X}^S$. After the first $m := \mathsf{rd}_T(H_X) + 1$ transitions on this path, we must have reached a state $(H,i)$ with $H = \emptyset$. Afterwards, all states have an empty first component. Thus, after $m + k + 2$ nodes, we have seen at least one state twice. This number is clearly linear in the size of $O$. □

For $SO$, the construction is again a little more involved (cf. Definition 3.18 and (Baader, Hladik, and Peñaloza 2008)). For a Hintikka set $H$ and a role name $r$, we define the sets

$$H|_r := \{C \in H \mid C = \forall r.D\} \text{ and } H^{-r} := \{C \in H \mid \forall r.C \in H|_r\}.$$

**Definition A.9** (functions $g_{(H,i)}$) Let $(H,i)$ and $(H',i')$ be two states of $A_{O,H_X}$ and $n := \mathsf{rd}_T(H)$. We define the function $g_{(H,i)}(H',i') := (H'',i'')$, where

$$P := \begin{cases} \mathsf{cl}^{\leq n}(O) \cap H'|_r & \text{if } i' \in \varphi_r(O) \text{ for a transitive } r \in \mathcal{N}_R, \\ \emptyset & \text{otherwise,} \end{cases}$$

$$H'':= H' \cap (\mathsf{cl}^{\leq n-1}(O) \cup \{g\} \cup P).$$

Again, these functions map elements of $Q_O$ to elements of $Q_O$ and constitute a faithful family.

**Lemma A.10** In $SO$, the family $g_{(H,i)}$ is faithful w.r.t. $A_{O,H_X}$.

**Proof.** Let $q = (H,i)$, $q_0 = (H_0,i_0)$, $q_j = (H_j,j)$, $q'_j := (H'_j,j) := g_{(H,i)}(q_j)$, and $q_j := (H'_j,j)$ := $g_{(H,i)}(q_j)$ be states of $A_{O,H_X}$ for all $j$, $1 \leq j \leq k$. We further define $n := \mathsf{rd}_T(H)$, assume that $(q,q_1,\ldots,q_k) \in \Delta_O$, and verify that then we have $(q,q'_1,\ldots,q'_k) \in \Delta_O$. For this purpose, consider any $\exists r.C \in \mathsf{cl}(O)$ and $j := \varphi(\exists r.C)$.

As in Lemma A.7, if $\exists r.C \in H$, then we have $\mathsf{rd}_T(C) < n$, and thus $\{q,C\} \subseteq H'_j$. If $g \in H'_j$, i.e. $g \in H_j$, then consider any value restriction $\forall r.D \in H$. Since $\mathsf{rd}_T(D) < n$, we have $D \in H_j$. Furthermore, if $r$ is transitive, then $\mathsf{rd}_T(\forall r.D) \leq n$ and $\forall r.D \in H_j \cap r$ imply that also $\forall r.D \in H'_j$. □
For the second condition of Definition 3.10, assume that \((H_0, H_1, \ldots, H_k)\) satisfies the Hintikka condition, and consider any \(\exists r.C \in \text{cl}(O)\) and \(j := \varphi(\exists r.C)\).

If \(\exists r.C \in H'_0\), then \(\exists r.C \in H_0\). Since \(r_d^T(C) < n - 1\), we obtain \(\{\varrho, C\} \subseteq H'_j\). If \(\varrho \in H'_j\) and \(\forall r.D \in H'_0\), then also \(\varrho \in H_j\) and \(\forall r.D \in H_0\), and thus \(D \in H_j\) by the Hintikka condition. Since \(r_d^T(D) < n - 1\), we also have \(D \in H'_j\). If \(r\) is transitive, then \(\forall r.D \in H_j\) and \(r_d^T(\forall r.D) \leq n - 1\), and thus \(\forall r.D \in H'_j\), as required.

We define the blocking relation \(\sim_{SO}\) on \(Q_O\) by setting \((H, i) \sim_{SO} (H', i')\) iff there is a \(\exists r.C \in \text{cl}(O)\) such that \(i = i' = \varphi(\exists r.C)\), and either

- \(\varrho \notin H \cup H'\); or
- \(r\) is transitive, \(\varrho \in H \cap H'\), \(C \in H\) iff \(C \in H'\), \(H^{-r} = H'^{-r}\), and \(H|_r = H'|_r\).

We now verify that the automaton \(A^S_{O,H_X}\) is \(\sim_{SO}\)-invariant. If \(\varrho\) is neither in \(H\) nor in \(H'\), then \(\exists r.C\) is not present in the parent node and nothing else is required by the Hintikka condition. Let now \(((H_0, i_0), (H_1, 1), \ldots, (H_k, k))\) be a valid transition of \(A^S_{O,H_X}\), \(i \in K\), and \(H'\) be a compatible Hintikka set for \(O\) such that \(i = \varphi(\exists r.C)\), \(r\) is transitive, \(\varrho \in H_i \cap H'\), \(C \in H_i\) iff \(C \in H'\), \(H_i^{-r} = H'^{-r}\), and \(H|_r = H'|_r\). We have to show that the Hintikka condition still holds with \((H', i)\) instead of \((H_i, i)\). If \(\exists r.C \in H_0\), then \(C \in H_i\), and thus also \(\{\varrho, C\} \subseteq H'\) by assumption. Furthermore, for every \(\forall r.D \in H_0\), we have \(\{D, \forall r.D\} \subseteq H_i\), and thus \(\forall r.D \in H_i|_r\) and \(D \in H_i^{-r}\). Since \(H_i^{-r} = H'^{-r}\), this implies that \(D \in H'\) as well as \(\forall r.D \in H'\).

The final lemma of this thesis shows that this blocking relation allows the depth-first emptiness test for the automata \(A^S_{O,H_X}\) to stop after polynomially many consecutive transitions.

**Lemma A.11** The construction of \(A^S_{O,H_X}\) from \(O\) and \(H_X\) is a PSPACE on-the-fly construction.

**Proof.** It remains to show that the automata are polynomially blocking w.r.t. \(\sim_{SO}\). Consider any path in a run of \(A^S_{O,H_X}\) and three consecutive nodes, labeled by \((H_0, i_0)\), \((H_1, i_1)\), and \((H_2, i_2)\), on this path. Further assume that \(r_0, r_1, r_2\) are the roles of the existential restrictions associated with \(i_0, i_1, i_2\) via \(\varphi\), respectively. If \(r_2\) is not transitive, then \(r_d^T(H_2)\) is strictly smaller than \(r_d^T(H_1)\). If \(r_2\) is transitive but different from \(r_1\), then it is at least smaller than \(r_d^T(H_0)\). Thus, there can be at most \(r_d^T(H_X) + 1\) transitions with different role names or the same non-transitive role name before a state \((H, i)\) with \(H = \emptyset\) is reached.

Consider now a path \((H_0, i_0), \ldots, (H_n, i_n)\) in the run, where each \(i_j, 1 \leq j \leq n,\) is associated to the same transitive role name \(r\). It is easy to see that there can be at most \(k\) such transitions with \(\varrho \notin H_j\) before the blocking relation triggers. Assume therefore that we have \(\varrho \in H_j\) for all \(j, 1 \leq j \leq n\). The Hintikka condition then implies that \(H_j|_r \subseteq H_{j+1}|_r\) holds for all \(j, 1 \leq j \leq n - 1\). Thus, after at most \(|\text{cl}(O)|\) transitions, the sets \(H_j|_r\) do not change anymore. But after this point, we also have \(H_j^{-r} = \{C \in \text{cl}(O) \mid \forall r.C \in H_j|_r\}\) by the Hintikka condition, and thus the sets \(H_j^{-r}\) also stay the same. From this point on, we can make at most \(2k\) transitions without triggering the remaining conditions of \(\sim_{SO}\), namely that the \(i_j = i_{j'}\) and \(C \in H_j\) iff \(C \in H_{j'}\) for two indices \(1 \leq j < j' \leq n\).
To summarize, every path of length \((\text{rd}_T(H_X)+1) \cdot k \cdot (|\text{cl}(O)| + 2k) + 1\) in the run must contain at least two states that satisfy the blocking relation. This number is polynomial in the size of \(O\).

As described before, this not only provides a \(\text{PSPACE}\) decision procedure for local consistency and satisfiability, but also for consistency. Furthermore, in classical DLs with negation, a (1-)subsumption between two concepts \(C\) and \(D\) w.r.t. \(O\) can be checked by testing whether \(C \sqcap \neg D\) is not (1-)satisfiable w.r.t. \(O\). Hardness follows as usual from \(\text{PSPACE}\)-hardness of reasoning in \(\text{ALC}\) w.r.t. the empty TBox (Schmidt-Schauß and Smolka 1991).

**Theorem A.12** In classical \(\text{ALCHO}\) and \(\mathcal{S}\) with acyclic TBoxes, consistency, satisfiability, and subsumption are \(\text{PSPACE}\)-complete.
List of Results

Theorem 3.8 ................................................................. 41
Let $L$ be a finite residuated De Morgan lattice. Then local consistency w.r.t. general models in $L$-SCHI with fuzzy general TBoxes is decidable in ExpTime. It is ExpTime-hard already in $2$-$\mathcal{NL}$ and $2$-$\mathcal{ELC}$.

Theorem 3.22 ................................................................. 49
Let $L$ be a finite residuated De Morgan lattice. Then local consistency w.r.t. general models in $L$-ISCHI and $L$-ALCHI with acyclic TBoxes is decidable in PSpace. It is PSpace-hard already in $2$-$\mathcal{NL}$ and $2$-$\mathcal{ELC}$.

Theorem 3.30 ................................................................. 58
Let $L$ be a finite residuated De Morgan lattice. Then consistency w.r.t. general models in $L$-SCHI with fuzzy general TBoxes and equality assertions is decidable in ExpTime. When restricted to either $L$-ALCHI or $L$-SCIc and acyclic TBoxes, the problem is in PSpace. Corresponding hardness results hold already in $2$-$\mathcal{NL}$ and $2$-$\mathcal{ELC}$.

Theorem 3.31 ................................................................. 58
Let $L$ be a finite residuated De Morgan lattice. Then satisfiability, subsumption, and instance checking w.r.t. general models in $L$-SCHI with fuzzy general TBoxes and equality assertions are decidable in ExpTime. When restricted to $L$-ALCHI or $L$-SCIc and acyclic TBoxes, these problems are in PSpace. Corresponding hardness results hold already in $2$-$\mathcal{NL}$ and $2$-$\mathcal{ELC}$.

Theorem 4.4 ................................................................. 64
Let $L$ be a complete residuated De Morgan lattice without zero divisors. Then consistency w.r.t. witnessed models in $L$-SUI$^{OH}$ with fuzzy general TBoxes and inequality assertions is decidable in ExpTime. It is ExpTime-hard already in $2$-$\mathcal{NL}$.

Theorem 4.5 ................................................................. 65
Let $L$ be a complete residuated De Morgan lattice without zero divisors. Then consistency w.r.t. witnessed models in $L$-ALCHI, $L$-SUI, $L$-ALCHI$^{OH}$, and $L$-SUI$^{O}$ with acyclic TBoxes and inequality assertions is decidable in PSpace. It is PSpace-hard already in $2$-$\mathcal{NL}$.

Theorem 4.16 ................................................................. 76
In $G$-$\mathcal{ALC}$ with fuzzy general TBoxes and order assertions, local consistency w.r.t. witnessed models is decidable in ExpTime. It is ExpTime-hard already in $G$-$\mathcal{NL}$ with inequality assertions.

Theorem 4.19 ................................................................. 79
In $G$-$\mathcal{ALC}$ with fuzzy general TBoxes and order assertions, consistency w.r.t. witnessed models is decidable in ExpTime. It is ExpTime-hard already in $G$-$\mathcal{NL}$ with inequality assertions.
Theorem 4.20  
In G-$\mathcal{IALC}$ with fuzzy general TBoxes and order assertions, satisfiability, subsumption, and instance checking w.r.t. witnessed models are decidable in ExpTime. They are ExpTime-hard already in G-$\mathcal{IREL}$ and G-$\mathcal{ECL}$.

Theorem 4.22  
In G-$\mathcal{IALC}$ with fuzzy general TBoxes and order assertions, the best satisfiability, subsumption, and instance degrees w.r.t. witnessed models can be computed in exponential time.

Theorem 5.11  
For every continuous t-norm $\otimes$ except the Gödel t-norm, (local) consistency w.r.t. witnessed models is undecidable in the following logics:
- $\otimes$-$\mathcal{IAL}$ with crisp general TBoxes and equality assertions;
- $\otimes$-$\mathcal{ELC}$ with crisp general TBoxes and inequality assertions; and
- $\mathcal{L}^{(0,b)}$-$\mathcal{IEL}$ with crisp ontologies.

Theorem 5.16  
(Local) consistency w.r.t. witnessed models in $\Pi$-$\mathcal{ECL}$ with crisp ontologies is undecidable.

Theorem 5.21  
For every continuous t-norm except the Gödel t-norm, (local) consistency w.r.t. witnessed models in $\otimes$-$\mathcal{IEL}$ with crisp general TBoxes and equality assertions is undecidable.

Theorem 5.31  
(Local) consistency w.r.t. general models in $\mathcal{L}^{(0,b)}$-$\mathcal{IE}$ with crisp ontologies is undecidable.

Theorem 5.33  
(Local) consistency w.r.t. general models in $\Pi$-$\mathcal{ECL}$ with crisp general TBoxes and inequality assertions is undecidable.

Theorem 6.2  
(Local) consistency w.r.t. witnessed or general models in $L_2$-$\mathcal{ECL}$ with crisp general TBoxes and inequality assertions is undecidable.

Theorem 6.6  
If $\otimes$ is a continuous t-norm over $[0,1]$ with zero divisors, then consistency w.r.t. witnessed or general models in $L_\infty$-$\mathcal{IELU}$ with fuzzy general TBoxes and inequality assertions is undecidable.

Theorem 6.9  
If $\otimes$ is a continuous t-norm over $[0,1]$ without zero divisors, then consistency w.r.t. witnessed models in $L_\infty$-$\mathcal{ISUHOr}$ with fuzzy general TBoxes and inequality assertions is decidable in ExpTime.

Theorem A.12  
In classical $\mathcal{ALCHO}$ and $\mathcal{SO}$ with acyclic TBoxes, consistency, satisfiability, and subsumption are PSPACE-complete.
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