Exchange Graphs via Quiver Mutation

DISSERTATION

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Dipl.-Math. Matthias Warkentin
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Supervisors: Prof. Dr. Dieter Happel †
Prof. Dr. Jan Schröer, Universität Bonn
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Introduction

The starting point of this thesis was a question by Dieter Happel with the following background. Let $Q$ be a finite acyclic quiver (another word for directed graph) and $K$ an algebraically closed field. The set of tilting modules over the path algebra $KQ$ carries the structure of an abstract simplicial complex $\Sigma(Q)$ and is partially ordered; the quiver of tilting modules is the associated Hasse diagram and an orientation of the dual graph $K_{KQ}$ of $\Sigma(Q)$. Happel and Unger examined the possibility of reconstructing $Q$ from its partially ordered set of tilting modules in their paper [HU09] and gave a positive answer “up to multiplicity of arrows” as follows. It is possible to determine the number of vertices of $Q$ as well as the information whether there are no arrows, precisely one arrow, or several arrows between two vertices, but in the last case not the precise number of arrows. An example for this phenomenon is the generalised Kronecker quiver with two vertices and $m$ arrows; for all $m \geq 2$ the associated partially ordered sets of tilting modules are isomorphic.

If we call quivers that are isomorphic “up to multiplicity of arrows” basically equal, then Happel’s question can be phrased as follows.

**Question** (Happel). Is it true that the quivers (partially ordered sets, simplicial complexes) of tilting modules of finite basically equal acyclic quivers are isomorphic?

In this thesis we give partial positive answers to this question for the following classes of quivers. We call a quiver almost abundant if any two vertices are connected and no vertex is incident to more than one simple arrow. We show in particular the following results (Corollaries 10.7 and 13.26):

**Theorem.** Let $Q$ and $Q'$ be basically equal acyclic $n$-point-quivers. If $n = 3$ or one of the quivers (and then also the other) is almost abundant, then $K_{KQ} \cong K_{KQ'}$ and $\Sigma(Q) \cong \Sigma(Q')$. Notably there is a canonical bijection between the sets of isomorphism classes of exceptional modules over $KQ$ and $KQ'$. 
INTRODUCTION

Our approach exploits a deep relation between the combinatorics of tilting modules and the combinatorics of cluster algebras introduced by Fomin and Zelevinsky in [FZ02]. Their quiver mutation rule allows to define the so-called exchange graph \( \Gamma(Q) \) in a purely combinatorial way, and it turns out that \( \mathcal{K}_{KQ} \) can be seen as a full subgraph of \( \Gamma(Q) \). The latter is by construction \( n \)-regular (where \( n \) is the number of vertices of \( Q \)). Moreover, for each vertex of \( \Gamma(Q) \) there is an associated quiver, and any neighbouring quiver is obtained via one of the \( n \) possible quiver mutations. Therefore elementary estimates, based on the mutation rule only, yield surprisingly strong structural results.

Specifically, we define a class of quivers called forks (see Definition 2.1) with the following property: Every fork has a distinguished vertex such that mutation in any other vertex yields a fork with more arrows (compare Lemma 2.5). This generalises results of Assem, Blais, Brüstle and Samson ([ABBS08]) for the case \( n = 3 \). Hence, if we have a fork in \( \Gamma(Q) \) and delete the edge corresponding to the distinguished mutation, the connected component containing the fork will be a complete infinite \( (n - 1) \)-ary tree consisting of forks only (see Lemma 2.8), which is the reason for the name. This implies in particular that a fork is mutation-infinite, i.e. iterated mutations yield infinitely many isomorphism classes of quivers. On the other hand we can show that the converse also holds, i.e. any connected mutation-infinite quiver \( Q \) can be mutated to a fork (Theorem 3.2). Moreover, we get the following result on the global structure of \( \Gamma(Q) \) (Theorem 5.1):

**Theorem.** Let \( \Gamma \) be the exchange graph of a connected mutation-infinite \( n \)-point-quiver. Then there is a bound \( D(n) \) depending only on \( n \) such that any vertex of \( \Gamma \) has at most distance \( D(n) \) to a fork.

Another consequence concerns the conjecture of Unger that \( \mathcal{K}_{KQ} \) has infinitely many connected components when \( Q \) is wild with at least three vertices. Combined with results of Felikson, Shapiro and Tumarkin from [FST12] we obtain the following “almost complete” confirmation (see Theorem 11.3):

**Theorem.** Unger’s conjecture holds in all but finitely many cases, specifically for all acyclic quivers with at least 11 vertices or with arrows of multiplicity greater than two.

The structural role of forks in the exchange graph in combination with basic results from Auslander-Reiten theory and cluster theory allows a classification of the exchange graphs for certain classes of quivers which finally yields the partial positive answers to Happel’s question given above. Notably we show for quivers that have only multiple arrows (and are hence called abundant) in Theorem 9.4 the following result:
Theorem. The exchange graph of an abundant acyclic $n$-point-quiver is an $n$-regular tree.

Furthermore, we get a complete classification of exchange graphs for quivers with three vertices (see Theorems 10.1 and 10.3).

To transfer the classification results from the exchange graph to the cluster complex – this is the associated extension of the simplicial complex of tilting modules – we use a fundamental result by Hubery ([Hub10]) that guarantees that the cluster complex is a simplicial abstract polytope and a method following Fomin and Zelevinsky ([FZ03a]) that requires a certain property of the fundamental group of $\Gamma(Q)$, namely that it is generated by certain cycles obtained by alternating mutations in two vertices, and those cycles have a certain length.

In this context the knowledge about forks is again quite useful as the tree parts of $\Gamma(Q)$ do not contribute to the fundamental group, and the necessary property is therefore easy to check for abundant quivers (because the fundamental group of a tree is trivial) and those with three vertices by the given classification. Moreover, the length of the said cycles is determined by the arrow number between the two mutation vertices such that two quivers will satisfy the necessary condition for identifying their cluster complexes if they are basically equal. So it seems justified to conjecture that one of the reasons for the partial positive answers to Happel’s question, namely the property that basically equal acyclic quivers remain basically equal under mutation, holds in general (Conjecture 8.16).

In the case $n = 3$ a possible explanation of this phenomenon can be observed. It turns out (again using the properties of forks) that one can perform mutations without specifying the precise multiplicity of multiple arrows; so one actually calculates with parameters instead of numbers. This inspires the idea of “polynomial quivers” with arrow numbers replaced by “arrow polynomials”. It is an interesting question when iterated mutations of such a polynomial quiver are well-defined; we suggest in Conjecture 12.28 that this holds for a certain class of “basic” acyclic polynomial quivers (see Definition 12.26) and show it for the class of almost abundant basic acyclic polynomial quivers (Theorem 13.23). Moreover, this would indeed provide an explanation for the observation that basically equal acyclic quivers remain basically equal under mutation; namely two such quivers can be seen as “evaluations” of the same basic acyclic polynomial quiver; and if the latter can be mutated without restrictions, the two corresponding evaluations of the parameters for an arbitrary mutation yield again basically equal quivers as shown in Theorem 12.29.

The organisation is as follows. In Chapter 1 we collect the basic definitions
for the purely combinatorial part. In Chapter 2 we introduce forks and show their characteristic properties. Chapter 3 relates forks and mutation-infinite quivers. Chapter 4 contains the definition of exchange graphs as well as further structural results. The bounded distance to forks in exchange graphs is explained in Chapter 5. In Chapter 6 we treat a slightly larger class of quivers than forks with similar behaviour. We introduce the cluster complex and discuss its relation to the exchange graph in Chapter 7. After explaining the connection to tilting theory and presenting Happel’s question in Chapter 8, we give first partial answers and a classification for 3 vertices in Chapters 9 and 10. Unger’s conjecture is the topic of Chapter 11. In Chapter 12 we introduce the concept of polynomial quivers and give several examples. Finally Chapter 13 is devoted to a class of polynomial quivers that considerably generalises the notion of forks and leads to our most general partial answer for Happel’s question.

Acknowledgement

At this point my first thought goes to Dieter Happel who initiated this dissertation project but sadly died too early to see it finished. His work and his insight in mathematics deeply influenced my own, and I dedicate this thesis to his memory. I am grateful to Jan Schröer for his kind willingness to continue the supervision of this project. I am indebted to many colleagues and friends for their support in so many ways: for technical advice, for helpful discussions, for persevering proofreading . . . . Thanks also go to my family for providing a home in the best sense. Finally and most of all I want to thank God because I regard every gift I have as coming from Him.
Chapter 1

Quivers and quiver mutation

We start by fixing notation and giving the basic definitions.

Definition 1.1. A quiver $Q = (Q_0, Q_1, s, t)$ is given by a set of vertices or points $Q_0$ and a set of arrows $Q_1$ together with the maps $s, t : Q_1 \to Q_0$ associating with every arrow $a \in Q_1$ its source $s(a)$ and its target $t(a)$: $s(a) \stackrel{a}{\longrightarrow} t(a)$. The quiver $Q$ is called finite if the sets $Q_0$ and $Q_1$ are both finite. A subquiver $Q' = (Q'_0, Q'_1, s', t')$ is given by subsets $Q'_0 \subseteq Q_0, Q'_1 \subseteq Q_1$ and the restrictions $s' = s|_{Q'_1}$ and $t' = t|_{Q'_1}$ such that $s(a), t(a) \in Q'_0$ for all $a \in Q'_1$. The subquiver is called full, if $s(a), t(a) \in Q'_0$ implies $a \in Q'_1$. Note that a full subquiver is already determined by a subset $Q'_0$ of the vertices of $Q$. If $Q'$ is a subquiver of $Q$ (or a subset of $Q_0$), we denote by $Q - Q'$ the quiver obtained by deleting all vertices of $Q'$ together with the incident arrows. We often write $i \in Q$ instead of $i \in Q_0$. The notation $\{i \neq j\} \subset Q_0$ means $\{i, j\} \subset Q_0$ with $i \neq j$.

A quiver morphism $f : Q \to Q'$ between two quivers $Q = (Q_0, Q_1, s, t)$ and $Q' = (Q'_0, Q'_1, s', t')$ is given by maps $f_0 : Q_0 \to Q'_0$ and $f_1 : Q_1 \to Q'_1$ such that $s'(f_1(a)) = f_0(s(a))$ and $t'(f_1(a)) = f_0(t(a))$ for all arrows $a \in Q_1$. As usual $f$ is an isomorphism (and then $Q$ isomorphic to $Q'$) if there is a morphism $g : Q' \to Q$ such that both compositions give the respective identity. Often we only talk of the map $f_0$, which can (or cannot) be extended to a quiver morphism in an obvious way. The opposite quiver $Q^{op}$ is obtained from $Q$ by reversing all arrows. This operation is often useful for additional assumptions concerning the orientation of a given arrow because it is compatible with all our notions in the sense that everything has a dual version and the argument for the opposite quiver would be the same up to using the dual notions. We refer to this idea with the phrase “up to duality” or similarly.

A path of length $m \in \mathbb{N}$ in $Q$ is a sequence of arrows $a_1, a_2, \ldots, a_m$ with
CHAPTER 1. QUIVERS AND QUIVER MUTATION

$s(a_{i+1}) = t(a_i)$ for all $i = 1, 2, \ldots, m - 1$, it is denoted by $a_m \ldots a_2 a_1$. The maps $s$ and $t$ are canonically extended to paths via $s(a_m \ldots a_2 a_1) = s(a_1)$ and $t(a_m \ldots a_2 a_1) = t(a_m)$. A path $p$ of length $m \in \mathbb{N}$ with $s(p) = t(p)$ is called closed or an (oriented) $m$-cycle. 1-cycles are also called loops. If $Q$ contains no oriented cycles, it is called acyclic. Additionally we want to have a path of length 0 or trivial path $e_i$ for every vertex $i \in Q_0$. We define $s(e_i) = i = t(e_i)$. Paths of arbitrary length $m \in \mathbb{N}_0$ can be concatenated in the obvious way.

Given a vertex $k \in Q_0$, the set of vertices that are reachable from $k$ (or successors of $k$) consists of all vertices of the form $t(p)$ for some path $p$ in $Q$ with $s(p) = k$. Further we define $Q^+(k)$ to be the full subquiver of $Q$ induced by all direct successors of $k$, i.e. vertices $i$ with an arrow $k \rightarrow i$. A vertex $k$ is a sink if $Q^+(k)$ is empty. (Direct) predecessors, $Q^-(k)$ and sources are defined dually. We say that $Q_0$ is admissibly numbered (usually by the numbers $1, \ldots, n$) if there are no arrows $i \rightarrow j$ whenever $i < j$. Clearly every (finite) acyclic quiver has a sink and hence, by induction, an admissible numbering.

Let us also fix some notation for graphs. Similar to a quiver, a graph $\Gamma$ is given by a set of vertices and a set of edges with endpoints in the vertices. A graph can be realised as a topological space in a standard way, so we can use the results of [Mas77], which is also the source for our notation and the topological details. Any edge can be oriented in two ways with an induced notion of source and target. Then the definition of paths is the same as for quivers, but note that each edge may occur several times with changing orientation; we call a path reduced if this does not happen in direct succession. As above we have the notion of closed paths, but as common in graph theory we reserve the term cycle for a (w. r. t. inclusion) minimal set of edges allowing a non-trivial closed reduced path. A graph is connected if any two vertices can be joined by a path. An edge is a bridge if its deletion increases the number of connected components. A tree is a connected graph in which any closed reduced path starting in a specific (equivalently, any) vertex is trivial. This is equivalent to ask that any vertex can be joined to a specific (or again any) vertex by a unique reduced path. We call a quiver $Q$ connected if the underlying unoriented graph is connected.

For a quiver $Q$ with vertices $i, j$ we set

$$q_{ij} := \# \{\text{arrows from } i \text{ to } j\} - \# \{\text{arrows from } j \text{ to } i\}.$$ 

Unless stated otherwise, we assume without mentioning that a quiver has no loops or 2-cycles, so quivers are determined (up to isomorphism) by these numbers, which form a skew-symmetric $Q_0 \times Q_0$ matrix. Instead of drawing
\( q_{ij} \geq 2 \) arrows from \( i \) to \( j \) we usually draw just one arrow together with its \textit{multiplicity} \( (i \overset{q_{ij}}{\rightarrow} j) \) and speak of a multiple arrow. We call a vertex \( k \in Q_0 \) \textbf{abundant} if \( |q_{jk}| \geq 2 \) for all \( j \neq k \); a quiver is \textbf{abundant} if all its vertices are abundant. By slight abuse of notation a 3-cycle means not only a quiver of the form \( \overbrace{i \overset{q_{ik}}{\rightarrow} k \overset{q_{kj}}{\rightarrow} j}^{3} \), but the full subquiver \( i \overset{q_{ik}}{\rightarrow} k \overset{q_{kj}}{\rightarrow} j \).

**Definition 1.2** ([FZ02]). Given a quiver \( Q \), for any vertex \( k \in Q_0 \) we define a new quiver \( \tilde{Q} = \mu_k(Q) \) with the same vertices, the \textbf{mutation} of \( Q \) in \( k \), via:

\[
\tilde{q}_{ij} = \begin{cases} 
-q_{ij} & \text{if } i = k \text{ or } j = k; \\
q_{ij} + \frac{1}{2}(q_{ik}|q_{kj}| + |q_{ik}|q_{kj}) & \text{else.}
\end{cases}
\]

Concretely, \( \mu_k(Q) \) is constructed as follows. For all \( i \in Q^-(k), j \in Q^+(k) \), we add \( q_{ik}q_{kj} \) arrows from \( i \) to \( j \). If this creates any 2-cycles, we remove them by deleting \( \min\{q_{ik}q_{kj}, q_{ji}\} \) arrows in both directions. Finally, we change the orientation of all arrows incident to \( k \). It is easy to see that quiver mutations are involutive in the sense that \( \mu_k(\mu_k(Q)) \cong Q \). We will only apply mutation to finite quivers, so this is an implicit assumption from now on.

**Definition 1.3.** Two quivers are called \textbf{mutation-equivalent} if one can be transformed into (a quiver isomorphic to) the other by a sequence of mutations. Since mutations are involutive, this indeed defines an equivalence relation on the set of all finite quivers. A quiver is called \textbf{mutation-acyclic} if it is mutation-equivalent to an acyclic quiver, else it is called \textbf{mutation-cyclic}. Given a quiver \( Q \), its \textbf{mutation class} consists of the isomorphism classes of all quivers mutation-equivalent to \( Q \). It builds the vertex set of the \textbf{mutation graph} of \( Q \); two vertices are joined by an edge if corresponding representatives are related by a mutation. Note that loops can also occur.

We note that quiver mutation is a local operation, which has the following consequence.

**Lemma 1.4.** Let \( Q \) be a quiver and \( R \subset Q_0 \). Then mutations in vertices of \( Q - R \) commute with deleting the vertices in \( R \), i.e. we obtain the same quiver when we first perform a sequence of mutations of \( Q \), but only in vertices of \( Q - R \), and then delete \( R \) as when we first delete \( R \) and perform the sequence of mutations afterwards.

**Proof.** It is clearly enough to prove this for a single mutation in a vertex \( k \in Q - R \). So let \( i, j \in Q - R \). By Definition 1.2 the way the arrow number between \( i \) and \( j \) changes depends only on the full subquiver induced by \( \{i, j, k\} \subset Q_0 - R \). \( \square \)
CHAPTER 1. QUIVERS AND QUIVER MUTATION
Chapter 2

Forks in the mutation graph

In this chapter we introduce the crucial notion of a fork. This will allow us to describe certain parts of mutation graphs as trees. Note that for an abundant quiver $Q$ and a vertex $k$ we get a decomposition $Q_0 = Q^-(k)_0 \cup \{k\} \cup Q^+(k)_0$.

**Definition 2.1.** We call an abundant quiver $Q$ a fork if it is not acyclic and there is a vertex $r$ called the point of return such that

(F1) For all $i \in Q^-(r)$ and $j \in Q^+(r)$ we have $q_{ji} > q_{ir}, q_{rj}$.

(F2) $Q^-(r)$ and $Q^+(r)$ are acyclic.

**Remark 2.2.** It is worth mentioning that any mutation-acyclic quiver $Q$ with an arbitrary vertex $r$ satisfies (F2). The reason is the following: By [BMR08, Corollary 5.3] full subquivers of mutation-acyclic quivers are again mutation-acyclic, so in particular the full subquiver induced by $r$ and a shortest oriented cycle in $Q^-(r)$ would be mutation-acyclic. Now any such $m$-cycle with $m \geq 4$ could be "shortened": mutation at one of its vertices would yield a quiver $Q'$ with $Q'^-(r)$ containing an $(m - 1)$-cycle. Hence it is enough to show the claim for $|Q_0| = 4$. On the other hand there is a result by Seven [Seva, Corollary 1.5] (based on work of Speyer and Thomas), which implies that each mutation-acyclic quiver contains a so-called "admissible cut"; this is a subset $C$ of its (multiple) arrows such that each oriented cycle shares precisely one (multiple) arrow with $C$ and each non-oriented cycle...
shares an even number of (multiple) arrows with $C$. It is easy to see that this excludes the possibility that $Q^{-}(r)$ (or dually $Q^{+}(r)$) is a 3-cycle. Seven’s result could also be applied without shortening the cycle, but Seva uses the same technique.

We collect some simple observations about forks in the following

**Lemma 2.3.** Let $Q$ be a fork with point of return $r$.

a. $Q - \{r\}$ is acyclic.

b. $Q - Q^{-}(r)$ and $Q - Q^{+}(r)$ are acyclic.

c. Each 3-cycle in $Q$ is of the form $i \rightarrow r \rightarrow j$ with $q_{ji} > q_{ir}, q_{rj}$.

d. For all vertices $i \in Q_{0}$ we have $Q^{-}(i), Q^{+}(i) \neq \emptyset$.

e. The point of return of a fork is uniquely determined.

**Proof.** By (F2) a cycle in $Q - \{r\}$ would have to contain vertices from both $Q^{-}(r)$ and $Q^{+}(r)$ and thus also arrows from $Q^{-}(r)$ to $Q^{+}(r)$, in contradiction to (F1). This shows a. Hence each cycle in $Q$ contains $r$ and thus a subquiver $i \rightarrow r \rightarrow j$ with $i \in Q^{-}(r)$ and $j \in Q^{+}(r)$. This already implies

b. Moreover, by (F1) $Q$ contains the 3-cycle $i \overset{q_{ir}}{\rightarrow} r \overset{q_{rj}}{\rightarrow} j$ with $q_{ji} > q_{ir}, q_{rj}$.

In particular any 3-cycle in $Q$ is of this form, which is c. As we have at least one cycle in $Q$ by definition, d follows for $i = r$. For $i \neq r$ assume (up to duality) $r \in Q^{+}(i)$ and choose some $j \in Q^{+}(r)$; then $j \in Q^{-}(i)$ by (F1). e follows from c since $r$ is the vertex opposite the unique arrow with maximal multiplicity in an arbitrary 3-cycle.

We also note that forks are in a sense “hereditary”:

**Lemma 2.4.** A full subquiver $\tilde{Q}$ of a fork $Q$ with point of return $r$ is either abundant acyclic or again a fork with point of return $r$.

**Proof.** $\tilde{Q}$ is clearly abundant. If it is not acyclic, it must contain $r$ by Lemma 2.3.a, but then (F1) and (F2) are obvious.

It is even useful to think of an abundant acyclic quiver as a “degenerate fork” in the sense that one of $Q^{-}(r)$ and $Q^{+}(r)$ is empty and the point of return $r$ corresponds to the source resp. the sink. This explains the assumptions of the following
Lemma 2.5. Let $Q$ be either a fork and $k \in Q_0$ not the point of return, or abundant acyclic and $k$ neither source nor sink. Then $\tilde{Q} := \mu_k(Q)$ is a fork with point of return $k$ and $|\tilde{Q}_1| > |Q_1|$.

Proof. Assume (up to duality) $k \in Q^-(r)$ where $r$ is the point of return of $Q$ in the first case and the sink in the second case. We will first show that for any two vertices $i, j$ the number of arrows $|q_{ij}|$ cannot decrease through the mutation. Indeed, the only case in which this could happen is that $i, k$ and $j$ form a 3-cycle, say $i \xrightarrow{q_{ik}} k \xrightarrow{q_{kj}} j$ in $Q$. But then Lemma 2.3.c and $k \not\in Q^-(r)$ yield $q_{ij} = q_{ik}q_{kj} > 2q_{ik} - q_{ji} > q_{ik} > 0$. So in $\tilde{Q}$ we get $\tilde{q}_{ij} = q_{ik}q_{kj} - q_{ji} > q_{ik} > q_{ji}$ as claimed. Note that also $\tilde{q}_{ij} > q_{ik}, q_{kj}$ as required for $[F1]$. 

Now let $j \in \tilde{Q}^-(k) = Q^+(k)$ and $i \in \tilde{Q}^+(k) = Q^-(k)$, so in $Q$ we have a subquiver $i \xrightarrow{q_{ik}} k \xrightarrow{q_{kj}} j$. If $q_{ji} > 0$, we are in the case just discussed; otherwise we get $\tilde{q}_{ij} = q_{ik}q_{kj} + q_{ji} > q_{ik}, q_{kj}$ arrows from $i$ to $j$ in $\tilde{Q}$ and $[F1]$ follows. Since $Q^+(k), Q^-(k) \neq \emptyset$ by Lemma 2.3.d resp. the assumption, the number of arrows strictly increases for at least one pair of vertices $i, j$ and we get a 3-cycle in $\tilde{Q}$. Finally note that $\tilde{Q}^-(k) = Q^+(k) \subset Q - Q^+(r)$ and $\tilde{Q}^+(k) = Q^-(k) \subset Q - \{r\}$ – see the following sketch – are both acyclic by Lemma 2.3.b and 2.3.a resp. the assumption, so $\tilde{Q}$ satisfies $[F2]$. 

Next we want to show that we get non-isomorphic forks for different choices of $k$ in the last lemma. For this we first need

Lemma 2.6. Let $Q$ be a fork or abundant acyclic. Then any quiver automorphism $\varphi \in \text{Aut}(Q)$ is the identity on $Q_0$.

Proof. If $Q$ is abundant acyclic, there is a unique admissible numbering of the vertices, which notably implies that $\varphi$ must fix the vertices since it respects such a numbering. If $Q$ is a fork, let $r$ be the point of return. By Lemma 2.3.e we certainly know $\varphi(r) = r$. But $Q - \{r\}$ is abundant acyclic by Lemma 2.3.a, so the same argument as before shows that $\varphi|_{Q - \{r\}}$ must fix the vertices. 

Lemma 2.7. Let $Q, k$ be as in Lemma 2.5 and $k' \neq k$. Then $\mu_k(Q) \not\cong \mu_{k'}(Q)$.
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Proof. Suppose we have an isomorphism $\psi : \mu_k(Q) \to \mu_{k'}(Q)$. By Lemma 2.5, $\mu_k(Q)$ and hence also $\mu_{k'}(Q)$ are forks with more arrows than $Q$. In particular, $k'$ is the point of return of $\mu_{k'}(Q)$ (again by Lemma 2.5) because $\mu_{k'}$ – applied to $\mu_{k'}(Q)$ – reduces the numbers of arrows. Hence $\psi(k) = k'$ by Lemma 2.3.e. $\psi$ induces an isomorphism $\varphi$ between the quivers obtained by mutation at the respective points of return, which of course yields just $Q$ in both cases, so we get an automorphism of $Q$ with $\varphi(k) = k'$. Hence $k' = k$ by Lemma 2.6. ☐

Lemma 2.8 (Tree Lemma). Let $\Gamma$ be the mutation graph of a quiver with $n$ vertices and assume that one vertex of $\Gamma$ is a fork $Q$ with point of return $r$. When we delete the edge $e$ between $Q$ and $\mu_r(Q)$, the connected component $\Gamma'$ of $\Gamma - e$ containing $Q$ is an infinite complete $(n - 1)$-ary tree rooted in $Q$.

Proof. Consider an arbitrary reduced path in $\Gamma$ (!) starting in $Q$, but not with $e$. Such a path is given by a sequence $(k_1, k_2, \ldots, k_m)$ of vertices in $Q_0$ with $k_1 \neq r$, $k_{l+1} \neq k_l$ for all $l \geq 1$. We set $Q^{(l)} := \mu_{k_l}(Q^{(l-1)})$ with $Q^{(0)} := Q$. Lemma 2.5 inductively implies that every $Q^{(l)}$ is a fork with point of return $k_l$ and $|Q_1^{(0)}| < |Q_1^{(1)}| < \ldots < |Q_1^{(m)}|$, so this path cannot be closed (nor leave $\Gamma'$) if $m \geq 1$. Hence $\Gamma'$ is a tree whose vertices are all forks. By Lemma 2.7 any fork has $n - 1$ “greater” neighbours, so it is a complete $(n - 1)$-ary tree. ☐

Remark 2.9. This lemma motivates the names fork (as “fork in a tree”) and point of return (the only possible direction for returning).

Remark 2.10. In the case $n = 3$ the considerations of this chapter essentially specialise to work of Assem, Blais, Brüstle and Samson [ABBS08], see also [BBH11] and [FST12, Section 8]. Unger used a module-theoretic version of the idea (see [Ung96a, Lemma 3.1]) to obtain related classification results in the same case.
Chapter 3

Mutation-infinite quivers and forks

It is a natural and important question to ask which quivers are mutation-finite, i.e. have finite mutation class. After contributions of several authors had formed a list of candidates, Felikson, Shapiro and Tumarkin could show with considerable combinatorial effort that this list is complete, see [FST12] for details. A by-product of their analysis is the following very useful

**Theorem 3.1** ([FST12], Lemma 7.3). Any mutation-infinite quiver with at least 11 vertices already has a mutation-infinite full proper subquiver.

This implies that it suffices to check all subquivers with at most 10 vertices for deciding whether a quiver is mutation-infinite or not. The results of this chapter relate this question to our notion of forks, namely we have the following

**Theorem 3.2.** A connected quiver is mutation-infinite if and only if it is mutation-equivalent to a fork.

It is clear that a fork is mutation-infinite by the Tree Lemma. The other implication will follow from Proposition 3.4 for which we need the following

**Definition 3.3.** Let $Q$ and $Q'$ be quivers and $k \in Q_0$. Then $Q$ is $\hat{k}$-mutation-equivalent to $Q'$ if $Q$ can be mutated to (a quiver isomorphic to) $Q'$ without mutating at $k$.

**Proposition 3.4.** Let $Q$ be a connected quiver with $|Q_0| = n \geq 3$ and vertices $i, j$ with $|q_{ij}| > 2$. Then for any vertex $k \notin \{i, j\}$ such that $Q - \{k\}$ is still connected, $Q$ is $\hat{k}$-mutation-equivalent to a fork.
Note that any quiver meeting the assumptions of Proposition 3.4 has a suitable vertex \( k \); take for example any vertex \( k \neq i \) with maximal distance from \( j \) (with respect to the natural metric on the underlying graph). Notably any such quiver is mutation-equivalent to a fork. As there are only finitely many quivers with \( n \) vertices and all arrow multiplicities at most two, any mutation-infinite connected quiver is mutation-equivalent to one as above and hence to a fork. So Theorem 3.2 indeed follows from Proposition 3.4.

We will prove the latter by induction on \( n \). The idea for the inductive step is the following: Use the induction hypothesis to first mutate the subquiver \( Q - \{k\} \) to a fork \( Q' - \{k\} \), then “\( \hat{k} \)-mutate” another subquiver \( Q' - \{k'\} \) to a fork (while \( Q' - \{k\} \) remains a fork). Finally “\( \hat{k} \)-mutate” the whole quiver to a fork. To show (in Lemma 3.6) that this last step is possible, we need Lemma 3.5.

**Lemma 3.5.** Let \( Q \) be a quiver with \( k' \neq k \) and \( h \in Q_0 \) as follows:

(I) For \( \hat{k} \in \{k',k\} \), \( Q - \{\hat{k}\} \) is either abundant acyclic or a fork with point of return not equal to \( h \).

(II) \( h \in Q^+(k') \cap Q^-(k) \) and \( q_{k'h}q_{hk} + q_{k'k} > q_{kh}, q_{hk} \).

Then \( \hat{Q} := \mu_h(Q) \) is a fork (with point of return \( h \)).

**Proof.** First note that \( \hat{Q} - \{k\} \) is a fork with point of return \( h \) by Lemma 2.5 or still abundant acyclic. In particular \( \hat{Q}^+(h) \) (\( \hat{Q} - \{k\} \)) is acyclic by \([\text{F2}]\). Similarly \( \hat{Q}^-(h) \) is acyclic, so \([\text{F2}]\) holds for \( \hat{Q} \) and \( h \).

Next we show \([\text{F1}]\). For all \( i \in \hat{Q}^-(h) \) and \( j \in \hat{Q}^+(h) \) with \( (i,j) \neq (k,k') \) this is clear by \([\text{I}]\) and Lemma 2.5. For \( (i,j) = (k,k') \) we get \( \hat{q}_{k'k} = q_{k'h}q_{hk} + q_{k'k} > \hat{q}_{kh}, \hat{q}_{hk} \) by \([\text{II}]\), and hence \([\text{F1}]\) as well as abundance by \([\text{I}]\). This last case also shows that \( Q \) is not acyclic. \( \square \)

**Lemma 3.6.** Let \( Q \) be a quiver and \( k' \neq k \in Q_0 \) such that \( Q - \{k'\} \) and \( Q - \{k\} \) are both forks with a common point of return \( r \). Then \( Q \) is \( k \)-mutation-equivalent to a fork, and \( n = |Q_0| \) mutations are sufficient.
For the proof we introduce some notions that will – as a by-product – lead to classification results similar to the Tree Lemma for forks.

**Definition 3.7.** For a quiver \( Q \) and \( k', k \in Q_0 \) we define the set of “stopovers” 
\[
k^k Q^k := (Q^+(k')_0 \cap Q^-(k)_0) \cup (Q^-(k')_0 \cap Q^+(k)_0)
\]
between \( k' \) and \( k \). \( Q \) as in Lemma 3.6 will be called a **pre-fork** with point of return \( r \) if \( k^k Q^k = \emptyset \).

The following lemma implies that Lemma 3.6 holds for pre-forks; after that we show a reduction step.

**Lemma 3.8.** Let \( Q \) as in Lemma 3.6 be a pre-fork and \( \tilde{Q} := \mu_{k'}(Q) \). Then 
\[
|\tilde{Q}_1| > |Q_1|, \tilde{Q} - \{k\} \text{ is a fork with point of return } k', \tilde{Q} - \{k'\} \text{ is abundant acyclic, } k^k Q^k = \tilde{Q}_0 - \{k', k\} \text{ and } \mu_h(\tilde{Q}) \text{ is a fork for any } h \notin \{k', k\}.
\]

**Proof.** \( \tilde{Q} - \{k\} \) is a fork with point of return \( k' \) and more arrows than \( Q - \{k\} \) by assumption and Lemma 2.5. In particular

1. \( \tilde{Q} - \{k', k\} \) is acyclic (by Lemma 2.3.a) and
2. \( \tilde{q}_{ij} > 0 \) for \( i \in \tilde{Q}^+(k') - \{k\} \) and \( j \in \tilde{Q}^-(k') - \{k\} \) (by (F1)).

Next we claim that for all vertices \( i \notin \{k', k\} \) we get at least as many arrows between \( i \) and \( k \) in the same direction after the mutation as before (which immediately implies abundance of \( \tilde{Q} - \{k'\} \)). Otherwise we would have to have added arrows in the opposite direction – but this is only possible if \( i, k' \) and \( k \) form a 3-cycle in \( Q \) contradicting \( k^k Q^k = \emptyset \). In particular \( |\tilde{Q}_1| > |Q_1| \) and \( k^k Q^k = \tilde{Q}_0 - \{k', k\} \) since all arrows incident to \( k' \) are reversed.

Now suppose there is an oriented cycle \( C \) in \( \tilde{Q} - \{k'\} \). (1) implies that \( C \) must contain \( k \) and hence also vertices from both \( \tilde{Q}^-(k) - \{k'\} \) and \( \tilde{Q}^+(k) - \{k'\} \). It follows that there must be a pair of vertices \( i \in \tilde{Q}^-(k) - \{k'\} \) and \( j \in \tilde{Q}^+(k) - \{k'\} \) with arrows from \( j \) to \( i \). However, by \( k^k Q^k = \tilde{Q}_0 - \{k', k\} \) this is equivalent to \( i \in \tilde{Q}^+(k') - \{k\} \) and \( j \in \tilde{Q}^-(k') - \{k\} \), so \( \tilde{q}_{ij} > 0 \) by (2). Thus \( \tilde{Q} - \{k'\} \) is acyclic and \( \tilde{Q} \) satisfies (I) of Lemma 3.5 for \( h \notin \{k', k\} \).
Since any $h \notin \{k', k\}$ is in $k\hat{Q}^k$, we can assume $h \in \hat{Q}^+(k') \cap \hat{Q}^-(k)$ by duality. Then (11) clearly holds if $\tilde{q}_{k'k} \geq 0$. Otherwise $q_{k'k} = -\tilde{q}_{k'k} > 0$, and as $h \in Q^-(k')$ implies $q_{hk'} > 0$ (so also $q_{hk} > 0$ since $k\hat{Q}^k = \emptyset$) we obtain $q_{hk} = q_{hk}q_{k'k} + q_{hk} > q_{hk'}, q_{k'k}$. Hence $\tilde{q}_{k'k}q_{hk} + q_{k'k} \geq 2\tilde{q}_{hk} - q_{k'k} > \tilde{q}_{hk} > q_{hk'} = q_{k'k}$ and we can apply Lemma 3.5 to $Q$ and $\tilde{h}$.

**Lemma 3.9.** Let $Q$ be as in Lemma 3.6 and $\tilde{r} \notin \{r, k', k\}$. Then $\hat{Q} := \mu_\tilde{r}(Q)$ is again as in Lemma 3.6 (with $\tilde{r}$ instead of $r$) and $k\hat{Q}^k = kQ^k - \{r\}$.

**Proof.** $\hat{Q} - \{k'\}$ and $\hat{Q} - \{k\}$ are again both forks with point of return $\tilde{r}$ by assumption and Lemma 2.5. To see how $k\hat{Q}^k$ changes under mutation in $\tilde{r}$, we examine which of the multiple arrows incident to $k'$ or $k$ are reversed. Of course those incident to $\tilde{r}$ are reversed, and hence $\tilde{r} \in k\hat{Q}^k \iff \tilde{r} \in k\hat{Q}^k$. Any other such multiple arrow must be contained in a 3-cycle $C$ through $\tilde{r}$ in $Q$. By Lemma 2.3.a we know that $Q - \{k', r\}$ and $Q - \{k, r\}$ are acyclic, hence $C$ must contain, apart from $\tilde{r}$, either $r$ (and one of $k'$ and $k$) or both $k'$ and $k$.

The latter case is not important for $k\hat{Q}^k$. This leaves two potential 3-cycles $C_k$ through $\tilde{r}, r$ and $k$ with $k \in \{k', k\}$ and the corresponding multiple arrows $a(\tilde{k})$ between $r$ and $\tilde{k}$. Notably we have $i \in k\hat{Q}^k \iff i \in k\hat{Q}^k$ for $i \neq r$.

Assume (up to duality) $\tilde{r} \in Q^+(r)$, then, by (F1) $C_k$ occurs if and only if $k \in Q^-(r)$, and if so, $a(\tilde{k})$ is indeed reversed (using (F1) again). So we see that both or none of the $a(\tilde{k})$ are reversed precisely if $r \notin k\hat{Q}^k$, and in that case also $r \notin k\hat{Q}^k$. In the other case, $r \in k\hat{Q}^k$ implies the reversion of exactly one of the $a(\tilde{k})$ and we obtain $r \notin k\hat{Q}^k$ again, so in summary $k\hat{Q}^k = kQ^k - \{r\}$.

**Proof of Lemma 3.6.** By Lemma 3.9 we can use $|k' - k|$ as a decreasing semi-invariant for the quivers under consideration. If $kQ^k \neq \emptyset$, we can assume
$r \in kQ^k$ (else we can mutate in some $\tilde{r} \in k'Q^k$, and that will be the new common point of return; if we need this extra mutation, then $|k'Q^k| \leq n - 3$).

So mutation in some $\tilde{r} \notin \{r, k', k\}$ yields $Q$ with $|\hat{Q}| = |kQ^k| - 1$ by Lemma 3.9; hence we can strictly reduce $|k'Q^k|$ until $k' = k = \varnothing$ by always choosing $\tilde{r}$ in the set of stopovers to prepare the next reduction step. In this way we get a pre-fork after at most $n - 2$ mutations and a fork after two further mutations by Lemma 3.8.

Now we are ready for the

Proof of Proposition 3.4. Recall that $Q$ has two vertices $i, j$ with $|q_{ij}| > 2$ and a vertex $k \notin \{i, j\}$ such that $Q - \{k\}$ is still connected. First we treat the base case $n = 3$ in three steps (similarly also used in [FST12, Lemma 8.7]):

1. Assume that $Q$ is a 3-cycle, say $i \xrightarrow{q_{ik}} k \xrightarrow{q_{kj}} j$, and (up to duality) $q_{ik} \geq q_{kj}$. Then it is easily checked that $\mu_j(\mu_i(Q))$ is a fork.

2. Otherwise $Q$ is acyclic. If one of $i$ and $j$ is neither sink nor source, mutation at that vertex yields a 3-cycle and we can proceed with step 1.

3. It remains to consider the case where $i$ and $j$ are source and sink. We can assume (up to duality and interchanging $i$ and $j$) that $k$ is connected to the source $i$. Then mutation at $j$ produces a quiver as in step 2.

For the induction step let $n \geq 4$. We carry out the plan sketched above. By the induction hypothesis we can find a sequence of mutations that transforms the subquiver $Q - \{k\}$ into a fork, so we can as well assume already initially that it is a fork with point of return $k'$. We can moreover assume that $k$ is connected to at least one vertex besides $k'$. (If this is not the case, mutation at a vertex apart from $k'$ yields the desired because that vertex will be the new point of return and $k$ will still be connected to the old one.) This ensures that $Q - \{k'\}$ is connected. To see that the other conditions of the induction hypothesis are satisfied for $Q - \{k'\}$ (with the given $k$), note that in the fork $Q - \{k\}$ we have a 3-cycle, say $i \xleftarrow{q_{ik'}} k' \xrightarrow{q_{ki}} j$ with $q_{ki} > q_{ik'} \geq 2$ and thus the desired multiple arrow in $Q - \{k'\}$ with $k \notin \{i, j\}$; finally $Q - \{k', k\}$ is abundant and in particular connected. Hence the induction hypothesis gives a sequence of mutations (at vertices in $Q - \{k', k\}$) that transforms the subquiver $Q - \{k'\}$ into a fork. By Lemma 2.5 $Q - \{k\}$ “remains a fork” under these mutations, so we can apply Lemma 3.6 to the resulting quiver.

Corollary 3.10. A fork as in Proposition 3.4 can be found algorithmically with at most $\frac{1}{3}n^3$ mutations.
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Proof. Let us denote the necessary number of mutations for an $n$-point-quiver by $A(n)$. We proceed as in the proof above. There we see that $A(3) \leq 4$, but in fact $A(3) = 3$ since there is only one mutation necessary in step 1 if step 2 has been performed before. Now we analyse the induction step. Turning the subquiver $Q - \{k\}$ into a fork takes at most $A(n - 1)$ mutations. After at most one further mutation we can start to turn $Q - \{k'\}$ into a fork (for simplicity we keep the name), but we do this carefully as follows. First we choose a suitable multiple arrow and “$\hat{k}$-mutate” its vertices and $k$ to a fork in at most $A(3)$ mutations. Its point of return is then also the point of return of $Q - \{k\}$. Now suppose that $2 \leq m \leq n - 2$ vertices from $Q - \{k', k\}$ and $k$ form a fork with point of return $r$ equal to that of $Q - \{k\}$. Choose an $(m + 1)$st vertex from $Q - \{k\}$ such that the set $S$ of the $m$ vertices contains direct successors and predecessors of $r$ in $Q - \{k\}$. The full subquiver of $Q - \{k\}$ induced by $S$ is then a fork with point of return $r$ by Lemma 2.4. Hence we can apply Lemma 3.6 to the full subquiver induced by $S \cup \{k\}$ and obtain a fork as before (but now with $m + 1$ vertices from $Q - \{k\}$) in at most $m + 2$ mutations. For $m = n - 2$ this is just the final step. We get the following estimate for $A(n)$:

$$A(n) \leq A(n - 1) + 1 + A(3) + \sum_{m=2}^{n-2} (m + 2) = A(n - 1) + \binom{n + 1}{2} - 2$$

and hence recursively for $1 \leq l \leq n - 3$

$$A(n) \leq A(n - l) + \sum_{i=0}^{l-1} \binom{n + 1 - i}{2} - 2l.$$

Setting $l = n - 3$ and summing up the binomial coefficients yields

$$A(n) \leq A(3) + \binom{n + 2}{3} - \binom{5}{2} - 2(n - 3) = \binom{n + 2}{3} - 2n - 1$$

(with equality for $n = 3$), and the latter is easily seen to be less than $\frac{1}{2}n^3$. \qed

We want to finish this chapter with a stronger version of Theorem 3.2 for almost all quivers, which we will use in Chapter 11. We need the following variant of Proposition 3.4.

Corollary 3.11. Let $Q$ be connected and $k \in Q_0$ such that $Q - \{k\}$ is mutation-infinite and connected. Then $Q$ is $\hat{k}$-mutation-equivalent to a fork.

Proof. By assumption we can mutate within $Q - \{k\}$ to generate a multiple arrow with sufficient multiplicity for applying Proposition 3.4. \qed
Proposition 3.12. All but finitely many (namely those with at most ten vertices and without arrows of multiplicity greater than two) connected mutation-infinite quivers $Q$ are $\hat{k}$-mutation-equivalent to a fork for some $k \in Q_0$.

Proof. Clearly $|Q_0| \geq 3$. If $Q$ has arrows of multiplicity greater than two, the claim holds by Proposition 3.4. If $|Q_0| \geq 11$, there is some $k$ such that $Q - \{k\}$ is mutation-infinite by Theorem 3.1 and (w.l.o.g.) connected, so the claim holds by Corollary 3.11. This leaves only finitely many cases. \qed
Chapter 4

Seed mutation and exchange graphs

A slight modification of quiver mutation leads to an exchange process called seed mutation which turns out to be relevant for surprisingly many structures. No additional input is required; in fact the only difference is that variables (or rational functions) are associated with the vertices of the quiver, and along with mutation in a vertex the associated rational function is replaced by a new one determined by the old ones and the quiver. This “algebraic lift” is crucial for the link to representation theory and makes the mutation pattern much nicer by resolving undesired symmetry effects as described below. The precise definition is the following:

Definition 4.1. Let \( F = Q(x_1, x_2, \ldots, x_n) \). A seed is a quiver \( Q \) with the extra condition that \( Q_0 \) is a set of \( n \) algebraically independent elements of \( F \) (called a cluster). For any \( z \in Q_0 \) the (quiver) mutation of \( Q \) in \( z \) becomes again a seed \( \mu_z(Q) \) by replacing \( z \) with \( z^* \) given by the exchange relation

\[
z^*z = \prod_{y \in Q_0} y^{\# \{\text{arrows from } y \text{ to } z\}} + \prod_{y \in Q_0} y^{\# \{\text{arrows from } z \text{ to } y\}}.
\]

So \( \mu_z(Q)_0 = Q_0 \setminus \{z\} \cup \{z^*\} \) (which is indeed a new cluster). Two seeds are considered to be isomorphic if their clusters are equal and the identity map on the (therefore common) vertex set can be extended to a quiver isomorphism.

It is easily checked that seed mutations are also involutive in the sense that \( \mu_z(\mu_z(Q)) \cong Q \). As in the case of quivers, two seeds are called mutation-equivalent if one can be transformed into (a seed isomorphic to) the other by a sequence of mutations. Given a quiver \( Q \), the exchange graph \( \Gamma(Q) \) of \( Q \) has as vertices the isomorphism classes of all seeds mutation-equivalent to the so-called initial seed obtained by specifying \( Q_0 = \{x_1, \ldots, x_n\} \). (This is
There is an edge between two vertices if these are related by a mutation (in complete analogy to the mutation graph). The elements of the clusters of all these seeds are called the **cluster variables**.

**Remark 4.2.** In a series of papers starting with [FZ02], Fomin and Zelevinsky introduced the (now famous) **cluster algebra** $\mathcal{A}_Q$ as the subring of $\mathcal{F}$ generated by the cluster variables. More precisely, this is a skew-symmetric cluster algebra without coefficients, see [FZ02] for a more general definition.

It follows directly from the definitions that every exchange graph is connected and regular; indeed, the $n$ possible mutations of any seed and the seed itself are pairwise non-isomorphic since they differ in their clusters. However, it is in general not possible to label the edges of the exchange graph consistently with $n$ “global” vertex labels that are (as in Definition 1.2) not exchanged under mutation. The reason is that such labels are usually not fixed by a seed isomorphism, so a mutation indexed by such a label is not well-defined on the isomorphism class. Therefore, some care is necessary when using (iterated) quiver mutations as in Chapter 2 on the level of seeds without notational adaption. Despite this issue we chose to formulate and prove our results on the level of quivers (with the mutation graph as an auxiliary construction) because this simplified the notation. All our results about mutation graphs also hold for the corresponding exchange graphs; up to the exchange of vertices the proofs remain the same verbatim since the key idea is to show and use that certain quivers are not isomorphic (e.g. by comparing arrow numbers), which then also follows for corresponding seeds. So we will freely use all definitions and results of the previous chapters for seeds and exchange graphs, too.

As for the relation between exchange and mutation graph, the identification of non-isomorphic seeds that are isomorphic as quivers induces a “ramified covering map” from the former to the latter. This is usually not a “nice” graph covering since quivers with automorphisms not fixing the vertices have fewer neighbours in the mutation graph, so the “quotient” does not inherit regularity. A seed does not have such an automorphism by definition.

The regularity is useful for giving further structural characterisations such as identifying certain edges as bridges. When we delete an edge in the exchange graph, we have either still one connected component with two vertices not having $n$, but only $n - 1$ neighbours, or two components with only one such vertex each. Generalising the notion of an $(n-1)$-ary tree we call a component of the latter type $(n-1)$-ary.
Corollary 4.3. An edge in an exchange graph is a bridge if its deletion produces an \((n - 1)\)-ary component. This is in particular the case for an edge between a fork \(Q\) with point of return \(r\) and \(\mu_r(Q)\).

**Proof.** If one component is \((n - 1)\)-ary, it cannot contain both vertices that were incident with the deleted edge. So there have to be two components, which means that the edge is a bridge. The second claim follows from the Tree Lemma [2.8]

A further application of the Tree Lemma to exchange graphs gives a sufficient condition for the exchange graph to be a tree.

Corollary 4.4. Let \(Q\) be a fork with \(n\) vertices and point of return \(r\) such that \(q_{ir}q_{rj} - q_{ji}\) for all \(i \in Q^- (r)\) and \(j \in Q^+ (r)\). Then the exchange graph of \(Q\) is an \(n\)-regular tree.

**Proof.** The assumption implies that \(\tilde{Q} := \mu_r(Q)\) is again a fork with point of return \(r\), so using the Tree Lemma [2.8] twice yields the claim.

Note that also the mutation graph of \(Q\) is a tree if \(\tilde{Q} \not\cong Q\) as quivers, and an infinite \((n - 1)\)-ary tree rooted in \(Q\) with an additional loop in \(Q\) if \(\tilde{Q} \cong Q\). Thus Corollary [1.3] does not hold for mutation graphs.

The next result confirms the intuition that we can throw away all forks without losing information. Let \(Q\) be a quiver and \(\Gamma(Q)\) its exchange graph. Consider the fork-less part of \(\Gamma(Q)\) obtained by deleting all vertices that are forks. Recall that a subgraph of a graph is **convex** if any reduced path with endpoints in the subgraph is completely contained in the subgraph.

Corollary 4.5. The fork-less part of \(\Gamma(Q)\) is a convex subgraph and allows to reconstruct \(\Gamma(Q)\). Notably \(\Gamma(Q)\) is a tree if and only if the fork-less part is.

**Proof.** The reconstruction is simple because \(\Gamma(Q)\) is regular: Just add \(n - d\) edges with attached \((n - 1)\)-ary trees to any vertex of degree \(d < n\) in the fork-less part. For the convexity note that a reduced path starting in a non-fork and encountering a fork has no chance of leaving the tree of forks again by the Tree Lemma.

It is therefore enough to study this subgraph, which is often quite easy to determine. In concrete calculations we can even stop as soon as we reach a pre-fork as defined in Definition [3.7] by an analogue to the Tree Lemma which we will prove later in Lemma [6.7] and Lemma [13.22]. But note that for a reconstruction we have to keep in mind at least the types of the pre-forks to be reinserted.
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Chapter 5

Forks in abundance

In this chapter we show that our results give a fairly good picture of the global structure of exchange graphs. In particular we have the following

**Theorem 5.1.** Let $\Gamma$ be the exchange graph of a mutation-infinite connected $n$-point-quiver. Then there is a bound $D(n)$ depending only on $n$ such that any vertex of $\Gamma$ has at most distance $D(n)$ to a fork.

*Proof.* Let $Q$ be the quiver of an arbitrary vertex of $\Gamma$. By Theorem 3.1 the necessary multiple arrow for applying Corollary 3.10 can be generated within a subquiver of $Q$ with at most 10 vertices, though we may not know which (in fact this task is usually performed by random mutations). As there are only finitely many such quivers without multiple arrows, the number of necessary mutations is bounded by some constant, so the maximal distance to a fork is bounded by $\frac{1}{3}n^3 + \text{const}$.

This implies that “most” of the vertices in $\Gamma$ are forks. To make this precise we show that the ratio of non-forks to forks in balls of increasing radius tends to zero and that a simple random walk on $\Gamma$ will almost surely reach the “fork part” and stay there forever. We start with the latter.

Recall that a simple random walk on a graph starts in a vertex and chooses one of the incident edges with equal probability to move on to a neighbour, where the process is iterated. $\Gamma$ is $n$-regular, so the probabilities are $1/n$.

**Proposition 5.2.** Let $\Gamma$ be the exchange graph of a mutation-infinite quiver. A simple random walk on $\Gamma$ will almost surely leave the fork-less part and never come back.

The idea is the following. Since any vertex has bounded distance to a fork, the random walk will almost surely visit forks again and again, even if it returns to the fork-less part. If we can show that the probability for returning
is less than one, the probability of always returning will be zero. But each fork is the root of an \((n - 1)\)-ary tree, and the probability of leaving such a tree is less than one because such trees are \textit{transient}, i.e. a simple random walk will with positive probability never return to its starting point.

**Lemma 5.3.** A simple random walk on the complete infinite \((n - 1)\)-ary tree is transient for \(n \geq 3\).

**Proof.** We sketch the proof explained in [DS84], specifically Subsection 2.2.5 in the freely available version from 2006, using the relation between random walks on graphs and electric networks (see also [Kle08, Chapter 19] for a more rigorous treatment). Imagine the edges of a graph \(G\) as wires with unit resistance. Then certain probabilities concerning a simple random walk on \(G\) with starting point \(s\) can be expressed in terms of certain resistances in the resulting electric network. The idea is that the probabilities of reaching a set of vertices \(A_l \not\ni s\) with finite complement before returning to \(s\) satisfy the same equations as the voltages in the electric network obtained by shorting all vertices of \(A_l\) and establishing a voltage of 1 there while grounding \(s\); both yield a “harmonic” function (with the boundary values 1 on \(A_l\) and 0 in \(s\)), which turns out to be unique; so the values coincide. If \(G\) is infinite and we take a sequence of sets \((A_l)_{l \in \mathbb{N}}\) with \(\bigcap_{l \in \mathbb{N}} A_l = \emptyset\), a random walk that reaches each set \(A_l\) before it returns to \(s\) will of course never return, so if the limit of the probabilities of reaching \(A_l\) before returning to \(s\) is positive, the random walk is transient. (In the other direction, a transient random walk will almost surely reach each set \(A_l\).) In the interpretation as an electric network, we can express this limit using the “effective resistance to infinity” \(R_\infty\) from \(s\) to \(\infty\) defined as the limit of the effective resistances \(R_l\) from \(s\) to \(A_l\) for \(l \to \infty\); specifically transience is equivalent to \(R_\infty < \infty\).

In our case \(R_\infty\) is easily computed: We can assume that \(s\) is the root of the \((n - 1)\)-ary tree and take as \(A_l\) the set of all vertices with distance at least \(l\) to \(s\). By symmetry, vertices with the same distance \(l\) have the same voltage and can be shorted without changing anything. We get \((n - 1)^l\) parallel wires between level \(l - 1\) and level \(l\) and can replace them by one wire with resistance \((n - 1)^{-l}\). The effective resistance from \(s\) to level \(l\) is then the sum of all the resistances in the series connection, so \(R_\infty\) is given by the infinite geometric series \(\sum_{l \geq 1} (n - 1)^{-l}\), which converges since \(n \geq 3\).

In particular, the probability \(p\) of never reaching a non-fork from a fork is positive. To see this, pick an edge \(e\) between a fork \(f\) and a non-fork \(g\). \(f\) is the root of an \((n - 1)\)-ary tree \(\Gamma'\) contained in \(\Gamma\) and consisting only of forks by the Tree Lemma. If the random walk reaches \(f\) and does not move on to
in the next step (which happens with probability \((n - 1)/n\)), the transience of the tree shows that it will with positive probability never reach \(g\).

**Proof of Proposition 5.2.** If the considered random walk does not show the claimed behaviour, there are only two possibilities. Either it will eventually stay in the fork-less part forever, or it will enter and leave the fork part again and again. We will show that the probability is zero for both events, which shows our claim. First note that the probability for being in a fork after \(D(n)\) steps is at least \(\delta := (1/n)^{D(n)} > 0\), independent of the starting point. In particular the probability that the random walk stays in the fork-less part forever is at least \(\delta\). This tends to zero for increasing \(m\), so the probability of staying in the fork-less part forever must be zero.

On the other hand, the probability that the random walk enters and leaves the fork part \(m\) times is at most \((1 - p)^m\). This tends to zero again for increasing \(m\), so the random walk will almost surely stay in the fork part.

Now we estimate the limit of the ratio of non-forks to forks in balls of increasing radius around an arbitrary fixed vertex of the exchange graph \(\Gamma\). Let \(F\) and \(N\) be the sets of forks and non-forks, \(B_d\) the (open) ball of radius \(d\) around a fixed vertex, i.e. the set of vertices with distance less than \(d\) to the fixed vertex for \(d \geq 0\). We want to show

**Proposition 5.4.** Let \(\Gamma\) be the exchange graph of a mutation-infinite \(n\)-point-quiver. Then the ratio of non-forks to forks in balls of increasing radius around an arbitrary fixed vertex of \(\Gamma\) tends to zero, i.e. with the above notation

\[
\lim_{d \to \infty} \frac{|N \cap B_d|}{|F \cap B_d|} = 0.
\]

**Proof.** We set \(\partial B_d := B_{d+1} \setminus B_d\), \(f_d := |F \cap \partial B_d|\) and \(n_d := |N \cap \partial B_d|\), then clearly \(|N \cap B_d| = \sum_{l=0}^{d-1} n_l\) and similarly for \(F\). A fork contained in \(\partial B_d\) will have at least \(n - 1\) neighbours in \(\partial B_{d+1}\) since it would lie on a cycle otherwise in contradiction to the Tree Lemma. Moreover, these neighbours will all be forks for sufficiently large \(d\) since the fork-less part is convex by Corollary 4.5. This implies \(f_{d+1} \geq (n - 1)f_d\) and inductively \(f_{d+2D(n)} \geq (n - 1)^{2D(n)}f_d\) for \(d \gg 0\) where \(D(n)\) is the bound for the maximal distance to a fork from Theorem 5.1. If we abbreviate \(x := 2D(n)\) and \(c := n - 1\), this reads \(f_{d+x} \geq c^x f_d\). On the other hand, for non-forks we get the trivial estimate \(n_{d+1} \leq (n - 1)n_d\) for \(d \geq 1\) and hence inductively \(n_{d+x} \leq c^x n_d\). Now suppose that we had equality. This would imply that all non-forks in \(\partial B_d\) have \((n - 1)\) non-fork neighbours in \(\partial B_{d+1}\) each, which must be pairwise distinct; the same holds for those etc.
CHAPTER 5. FORKS IN ABUNDANCE

up to the $2D(n)^{th}$ generation. But then a non-fork of the $D(n)^{th}$ generation would not have a fork within distance $D(n)$; this contradicts Theorem 5.1. As this argument holds for all non-forks in $\partial B_d$, we get $n_{d+x} \leq n_d(c^x - 1)$. So we can estimate $n_{d+x}/f_{d+x} \leq (1 - c^{-x})n_d/f_d$, which inductively shows $n_{d+mx}/f_{d+mx} \leq (1 - c^{-x})^mn_d/f_d$ for $d \gg 0$ and hence $\lim_{d \to \infty} n_d/f_d = 0$.

Now the idea is obviously that the non-forks are eventually in an arbitrarily small minority compared to the forks among the new vertices that are added when we increase the radius. Since we clearly add more and more vertices (even more and more forks), this will eventually bring the overall ratio arbitrarily close to zero. Spelled out this idea gives the following estimate. Let $\varepsilon > 0$. Then $n_d < \varepsilon f_d$ for all $d > m$ with some sufficiently large $m$. Hence we also have $\sum_{l=m+1}^d n_l < \varepsilon \sum_{l=m+1}^d f_l$ and thus

$$\lim_{d \to \infty} \frac{|N \cap B_d|}{|F \cap B_d|} = \lim_{d \to \infty} \frac{\sum_{l=0}^{d-1} n_l}{\sum_{l=0}^{d-1} f_l} = \lim_{d \to \infty} \frac{\sum_{l=0}^{m} n_l}{\sum_{l=0}^{d-1} f_l} + \lim_{d \to \infty} \frac{\sum_{l=m+1}^{d-1} n_l}{\sum_{l=0}^{d-1} f_l} < \varepsilon$$

because the first summand is zero. Since $\varepsilon$ was arbitrary, we are done. $\square$
Chapter 6

Pre-forks

In this chapter we use the results and notions from Chapter 3, there provided for the proof of Proposition 3.4, to treat the slightly larger set of pre-forks. In particular we show an analogue to the Tree Lemma, which yields a “more interesting” part of the associated exchange graph.

Recall from Definition 3.7 that a pre-fork with point of return \( r \) is a quiver \( Q \) with two vertices \( k' \neq k \) such that \( Q - \{k'\} \) and \( Q - \{k\} \) are both forks with point of return \( r \) and \( k'Q^k = \emptyset \). Pre-forks allow classification results similar to those for forks because they are quite close to being forks; in fact, most pre-forks are forks as the following lemma shows.

**Lemma 6.1.** Let \( Q \) be a pre-fork as above. If \(|q_{k'k}| \geq 2\), then \( Q \) is a fork with point of return \( r \).

**Proof.** The assumptions imply immediately that \( Q \) is abundant and not acyclic. \([F1]\) also follows easily as \( r \notin k'Q^k = \emptyset \) and all other cases are covered by the assumptions. Similarly, any oriented cycle in \( Q - \{r\} \) must contain \( k' \) and \( k \) by Lemma 2.3.a \([F2]\) follows when we show that no such cycle exists. Assume (up to duality) \( q_{k'k} > 0 \) and suppose there is a shortest path \( a_m \ldots a_2a_1 \) from \( k \) to \( k' \) in \( Q - \{r\} \). Clearly \( m \geq 2 \) and \( i := t(a_1) \neq k' \).

Since \( k'Q^k = \emptyset \), there is an arrow \( a'_1 \) from \( k' \) to \( i \). But then \( a_m \ldots a_2a'_1 \) is a cycle in \( Q - \{r,k\} \), which is a contradiction. \( \square \)

So there remain essentially two “types” of pre-forks (\(|q_{k'k}| = 0 \) or \(|q_{k'k}| = 1\)) that are not forks and actually give rise to more interesting mutation patterns. Before we discuss the simpler one, we note the direct partial analogue to Lemma 2.5 which follows immediately from Lemma 3.9.

**Corollary 6.2.** Let \( Q \) be a pre-fork as above and \( \tilde{r} \notin \{r,k',k\} \). Then \( \tilde{Q} := \mu_{\tilde{r}}(Q) \) is a pre-fork with point of return \( \tilde{r} \), \(|\tilde{Q}_1| > |Q_1| \) and \( \tilde{q}_{k'k} = q_{k'k} \). \( \square \)
Definition 6.3. Let $Q$ be a quiver and $k' \neq k \in Q_0$ with $q_{k'k} = 0$. We call $Q$

a. a ♦-root with point of return $r$ if $Q$ is a pre-fork with point of return $r$.

b. a ♦-wing with point of return $k'$ if $k'Q^k = Q_0 - \{k', k\}$, $Q - \{k\}$ is a
fork with point of return $k'$ and $Q - \{k'\}$ is abundant acyclic.

c. a ♦-tip if $k'Q^k = \emptyset$, $Q - \{k\}$ is a fork with point of return $k'$ and vice versa.

Any such quiver is called a ♦-quiver. (Lemma 6.7 will explain the names, see also Figure 6.1.)

Lemma 6.4. Let $Q$ be a ♦-quiver as above.

(rr) If $Q$ is a ♦-root with point of return $r$ and $\tilde{r} \notin \{r, k', k\}$, then $\tilde{Q} := \mu_\varepsilon(Q)$
is a ♦-root with point of return $\tilde{r}$ and $|\tilde{Q}_1| > |Q_1|$.

(rw) If $Q$ is a ♦-root with point of return $r$ and $\tilde{r} \in \{k', k\}$, then $\tilde{Q} := \mu_\varepsilon(Q)$
is a ♦-wing with point of return $\tilde{r}$ and $|\tilde{Q}_1| > |Q_1|$.

(wt) If $Q$ is a ♦-wing with point of return $k'$, then $\tilde{Q} := \mu_k(Q)$ is a ♦-tip
and $|\tilde{Q}_1| > |Q_1|$.

(tr) If $Q$ is a ♦-tip and $\tilde{r} \notin \{k', k\}$, then $\tilde{Q} := \mu_\varepsilon(Q)$ is a ♦-root with point
of return $\tilde{r}$ and $|\tilde{Q}_1| > |Q_1|$.

Proof. (rr) is part of Corollary 6.2 (rw) of Lemma 3.8. For (wt) first recall
that $q_{k'k} = 0$, so the arrows incident to $k'$ are not changed. Since $Q - \{k'\}$
is acyclic, only the arrows incident to $k$ are reversed – so $k'\tilde{Q}^k = \emptyset$ – and
the multiplicities of all other arrows are not reduced, thus $\tilde{Q} - \{k\}$ is still a
fork with point of return $k'$ because also (F1) remains valid. To see that $k$
is neither source nor sink in $Q - \{k'\}$, note that $Q^\pm(k) = Q^\pm(k')$ and $Q^\pm(k') \neq \emptyset$
by Lemma 2.3.d. Hence the claim follows by Lemma 2.5 In (tr) the only
condition not following obviously from the assumptions with Lemma 2.5 is
that $k'\tilde{Q}^k$ remains empty. This is similar to the proof of Lemma 3.9. Again, a
multiple arrow between $k'$ and, say, $i \neq \tilde{r}$ is reversed if and only if $k'$, $i$ and $\tilde{r}$
form a 3-cycle in $Q - \{k\}$. But by $k'\tilde{Q}^k = \emptyset$ this is the case if and only if $k$, $i$ and $\tilde{r}$
form a 3-cycle in $Q - \{k'\}$, and then the multiple arrow between $k$ and $i$
is reversed, too. 

Next we need a special case of a result from cluster theory.

Lemma 6.5. Let $Q$ be a seed and $k' \neq k \in Q_0$ with $q_{k'k} = 0$. Then
$\mu_{k'}(\mu_k(Q)) \cong \mu_k(\mu_{k'}(Q))$, i.e. mutations in non-connected vertices commute.
Proof. This follows by a simple calculation.

This implies that there is a quadrilateral (which we shall call a diamond) in the exchange graph wherever we have a seed with non-connected vertices; for the other three seeds this is of course the very same quadrilateral coming from the corresponding exchanged vertices, which are still not connected. Note that this is the smallest possible cycle: After having exchanged two cluster variables we need at least two further mutations to retrieve them. (In fact it is not hard to see that every quadrilateral in an exchange graph is of this form, but we will not use this fact.) As a consequence, any ♦-root lies on a cycle, so we need a substitute for the tree structure. This inspires

Definition 6.6. We call a reduced path in an exchange graph ♦-reduced if no three consecutive edges belong to the same diamond.

Note that any shortest path in an exchange graph is necessarily ♦-reduced; if it were not, it could be shortened with the obvious short-cut. In particular, we can reach any vertex in an exchange graph by sequences of mutations corresponding to ♦-reduced paths. Now the promised statement is the following:

Lemma 6.7. Let Γ be the exchange graph of a quiver with n vertices, and assume that one vertex of Γ is a ♦-root Q with point of return r. Then the edge e between Q and µ_r(Q) is a bridge, and the connected component Γ' of Γ - e containing Q satisfies the following conditions: The distance d(−) to Q (in Γ', but also in Γ) is strictly increasing along every ♦-reduced path in Γ' starting in Q, notably each such closed path is trivial. Moreover, Γ' consists entirely of ♦-quivers and forks and allows a recursive description as follows: Q is the root of a diamond with two neighbouring wings and an opposite tip; the other n - 3 neighbours of Q in Γ' are again ♦-roots (and thus yield copies of Γ'). The ♦-wings have only two neighbours in the fork-less part of Γ' (namely the ♦-root and the ♦-tip). Finally the n - 2 remaining neighbours of the ♦-tip are again ♦-roots (see Figure 6.1 for an illustration).

Proof. Consider an arbitrary ♦-reduced path in Γ' starting in Q. For simplicity we will stick to the notation of quivers, so this path is encoded by a sequence of mutation indices (k_1, k_2, ..., k_m) with k_1 \neq r, k_{l+1} \neq k_l for all l = 1, ..., m - 1 and further k_{l+2} \neq k_l if k_{l+1} and k_l are not connected in Q^{l-1} for all l = 1, ..., m - 2, where we recursively set Q^{l+1} := µ_{k_l}(Q^{l-1}) with Q^{0} := Q. We can restrict further to paths not visiting forks, because the distance to Q is clearly increasing in a tree part. This means the following: If Q^{l} is a ♦-wing with point of return k_l for some l \in {1, ..., m - 1}, then k_{l+1} is the unique vertex not connected to k_l in Q^{l}, as all other mutations (except in the point
of return) yield forks by Lemma \ref{lem:forks}. But now Lemma \ref{lem:forks} inductively implies that every $Q^{(l)}$ is a ♦-quiver (with point of return $k_i$ if it is not a ♦-tip) and $|Q_1^{(0)}| < |Q_1^{(1)}| < \ldots < |Q_1^{(m)}|$, so this path cannot be closed if $m \geq 1$. It also follows that the fork-less part of $\Gamma'$ consists entirely of ♦-quivers. Next we show inductively that $d(Q^{(l)}) = l$ which is obvious for $l = 0$. Suppose that $d(Q^{(l+1)}) = l' \leq l$. So there is a ♦-reduced path in $\Gamma'$ from $Q$ to $Q^{(l+1)}$ of length $l'$ and we can consider its continuation to $Q^{(l)}$, which is clearly still reduced because otherwise $d(Q^{(l)}) = l' - 1 < l$ contradicting the induction hypothesis. If it is even ♦-reduced, we get the contradiction $|Q_1^{(l+1)}| < |Q_1^{(l)}|$. But if it is not, then it can be shortened by (at least) two edges and we get $d(Q^{(l)}) \leq l' + 1 - 2 < l$ again. Hence $d(Q^{(l+1)}) = l + 1$.

We conclude that Lemma \ref{lem:forks} induces the given description of $\Gamma'$. Notably, it is $(n - 1)$-ary, so $e$ is a bridge by Corollary \ref{cor:bridge} and $d(\cdot)$ the distance in $\Gamma$, too.

The given recursive description determines $\Gamma'$ up to isomorphism; accordingly we will call a graph isomorphic to $\Gamma'$ an $(n - 1)$-ary ♦-tree. Note that we do not necessarily get a ♦-tree in the mutation graph – there we would have to take care of non-trivial pre-fork automorphisms because there is no analogue to Lemma \ref{lem:auto}. If (in the usual notation) $q_{k'i} = q_{ki}$ for all $i$ in a pre-fork, then interchanging $k'$ and $k$ induces an automorphism.

Figure 6.1: A part of $\Gamma'$ in Lemma \ref{lem:forks}. Oriented edges indicate increasing arrow numbers; $F$, $R$, $W$ and $T$ stand for forks, ♦-roots, -wings, and -tips.
As in the Tree Lemma, the key point of the last proof was avoiding (or at least controlling) cycles. More generally, it is desirable to have a certain control over the fundamental group of the exchange graph, so we recall some relevant facts from [Mas77, Chapter VI]. If we have a tree $T$ as a subgraph of a graph $\Gamma$, $T$ is maximal (w.r.t. inclusion) among all such trees if and only if it contains all vertices of $\Gamma$, and such a tree always exists (assuming Zorn’s Lemma). For each edge $e$ not contained in a maximal tree $T$, there is a unique cycle $C_e$ in $T \cup \{e\}$. If we fix a vertex $v_0$ as base point and $e$ has the endpoints $v_1$ and $v_2$, there are unique reduced paths $p_i$ from $v_0$ to $v_i$ ($i = 1, 2$) in $T$. The path obtained by concatenating $p_1$, $e$ and $p_2^{-1}$ defines an element of the fundamental group $\pi(\Gamma, v_0)$; it can also be obtained by walking along $p_1$ until one reaches $C_e$, going around the cycle once and then returning along the same path. If, given a set of cycles, paths of the latter form define a generating set of the fundamental group, we say that the fundamental group is generated by these cycles. The following theorem describes a method for obtaining such a generating set.

**Theorem 6.8** ([Mas77], Theorem VI.5.2). Let $\Gamma$ be a graph with a maximal tree $T$. Then the fundamental group of $\Gamma$ is (freely) generated by $\{C_e \mid e \not\in T\}$.

**Corollary 6.9.** In the setting of Lemma 6.7, the fundamental group of $\Gamma'$ is freely generated by its diamonds.

**Proof.** For each diamond choose one of the $\Diamond$-wings and delete the edge leading to the $\Diamond$-tip. It is clear that this does not increase the distances to $Q$, because the $\Diamond$-tips can still be reached via the other $\Diamond$-wings with paths of the same length. But now every vertex in $\Gamma'$ has a unique neighbour with smaller distance and hence a unique reduced path leading to $Q$ as all other neighbours have greater distance. This implies that the remaining edges build a tree, which is obviously maximal. The result is now precisely Theorem 6.8. \(\square\)

These results show that the fundamental group of an exchange graph can be understood by restricting to the part without the $(n - 1)$-ary ($\Diamond$-)trees.

**Remark 6.10.** There are analogous results for pre-forks with $|q_{k,k'}| = 1$ which are slightly more complicated to prove. The gist is that pentagons appear instead of diamonds, so there are two vertices taking the role of one $\Diamond$-tip. This will be covered by a more general result in Chapter 13.

**Example 6.11.** An application of Lemma 6.7 yields a complete description of the exchange graph (partially depicted in [BS10, Fig. 1] as “a web of Calabi–Yau algebras” for the case $a = b = 2$) of the quiver $Q$ below, where $a, b \geq 2$. Any mutation of $Q$ yields a $\Diamond$-tip with quiver $\tilde{Q}$ whose only
neighbour except the two ♦-roots guaranteed by Lemma 6.4 is (as a quiver) again isomorphic to $Q$. It also follows that all these quivers have canonically isomorphic exchange graphs and that they are all mutation-cyclic.

\[
\begin{align*}
Q: & \quad \bullet \xrightarrow{a} \bullet \\
& \quad \uparrow \quad | \quad \downarrow \\
& \quad b \quad b \\
& \quad \bullet \xleftarrow{a} \bullet \\
\end{align*}
\quad \quad \quad \quad \quad \quad \quad
\begin{align*}
\tilde{Q}: & \quad \bullet \xrightarrow{a} \bullet \\
& \quad | \quad | \\
& \quad b \quad ab \quad b \\
& \quad \bullet \xleftarrow{a} \bullet
\end{align*}
\]
Chapter 7

The cluster complex

The exchange graph does not capture all combinatorial aspects of the seed mutation process because it ignores the cluster variables. Their distribution in the clusters is described by an abstract simplicial complex (i.e. a set of subsets called simplices of a certain ground set such that each subset of a simplex is again a simplex) which Fomin and Zelevinsky introduced in \cite{FZ03a}:

**Definition 7.1.** Let $Q$ be a quiver. The **cluster complex** $\Delta(Q)$ is the abstract simplicial complex on the ground set of all cluster variables associated with $Q$ (see Definition 4.1) whose maximal simplices are the clusters.

So the cluster complex is by definition pure of dimension $n - 1$, and we can consider its **dual graph** $\Gamma$, which has the clusters as vertices and an edge between two clusters $C$ and $C'$ whenever $|C \cap C'| = n - 1$. Mapping each seed to its cluster induces a morphism $\varphi$ from the exchange graph $\Gamma(Q)$ to $\Gamma$. It is in general not known whether $\varphi$ is an isomorphism (i.e. each seed is determined by its cluster, and two seeds sharing exactly $n - 1$ variables are joined in the exchange graph), but this is true for cluster algebras of geometric type (\cite[Theorems 4 and 5]{GSV08}), which includes our coefficient-free setting:

**Theorem 7.2.** Let $Q$ be a quiver. Then $\Gamma(Q)$ is the dual graph of $\Delta(Q)$.

For an acyclic initial seed this has been proven using the link to representation theory which we will explain in Chapter 8 see \cite{BMRT07}. In this case we also get another important result, namely that the seeds containing a specific cluster variable form a connected subgraph of the exchange graph. These properties play a crucial role in a method for identifying cluster complexes which is taken from \cite[Section 2]{FZ03a} and will lead to classification results depending on the exchange graph.

The idea is to take an abstract simplicial complex $\Delta$ that allows a “mutation” of its maximal simplices and to formulate conditions for the compatibility
of this mutation with the seed mutation for a specific initial seed. The former is guaranteed if $\Delta$ is a **pseudo-manifold** (without boundary), i.e. a pure -- say $(n-1)$-dimensional -- simplicial complex with connected dual graph $\Gamma$ such that each $(n-2)$-simplex is contained in precisely two $(n-1)$-simplices. We then define the mutation of an $(n-1)$-simplex $C = \{v_1, \ldots, v_n\}$ at $v_k$ to be the unique other $(n-1)$-simplex containing $C - \{v_k\}$. So each $(n-1)$-simplex has $n$ possible mutations, and $\Gamma$ is $n$-regular. Note that if we specify a bijection $\psi$ between an $(n-1)$-simplex and the vertices of a seed (this is called a seed attachment in [FZ03a]), then there is a unique canonical way of “transporting” this seed attachment to its neighbours via mutation. The goal is to extend $\psi$ in this manner to a well-defined map from the complete ground set of $\Delta$ to the cluster variables (which will then be surjective). The first condition is due to the fact that we cannot expect to encounter the same (group of) cluster variable(s) at different “unrelated” places in the exchange pattern -- it is in fact a conjecture that this will never happen. Recall that the **link** of a simplex $D \in \Delta$ is the simplicial complex $\Delta_D := \{D' \in \Delta \mid D' \cup D \in \Delta, D' \cap D = \emptyset\}$. The condition is that $\Delta_D$ is again a pseudo-manifold for each non-maximal simplex $D$. Since $\Delta_D$ is obviously pure and allows mutation, this is equivalent to the condition that the dual graph $\Gamma_D$ (which can be identified with the full subgraph of $\Gamma$ whose vertices contain $D$) is connected. A pseudo-manifold with this property is a simplicial **abstract polytope** as defined in [MS02].

It is clear that we can uniquely transport a seed attachment along paths in $\Gamma$ and hence get seed attachments for all $(n-1)$-simplices. It remains to ensure that the obtained seed attachment is independent of the chosen path. This is done by the assumption in the following lemma, which is essentially [FZ03a, Lemma 2.4], but replaces the seed attachment with a more general “simplex attachment”, which can be transported analogously. (According to Schulte [Sch13], this “extension method” is well-known and probably goes back at least to Seifert-Threlfall.)

**Lemma 7.3.** Let $\Delta$ be a simplicial abstract $n$-polytope with dual graph $\Gamma$ and $\Delta'$ an $(n - 1)$-dimensional pseudo-manifold. Fix a base point $C$ in $\Gamma$, a set of generators $\{g_i\}_{i \in I}$ of $\pi(\Gamma, C)$ and a bijection $\psi_C : C \to C'$ for some $(n - 1)$-simplex $C' \in \Delta'$. If the transport of $\psi_C$ along each $g_i$ is again $\psi_C$, then we have the following:

1. For each $(n-1)$-simplex $\bar{C} \in \Delta$ the transport of $\psi_C$ via mutation yields a well-defined bijection $\bar{\psi}_C : \bar{C} \to \bar{C}'$ with some $(n-1)$-simplex $\bar{C}' \in \Delta'$. The obtained bijections are compatible with mutations.

2. The map $\bar{C} \mapsto \bar{C}'$ is a surjection onto the set of $(n-1)$-simplices in $\Delta'$. 


Given any \((n - 1)\)-simplex \(\bar{C} \in \Delta\) and \(v \in \bar{C}\), the image \(\psi_{\bar{C}}(v)\) is independent of \(\bar{C}\) (and can hence be denoted by \(\psi(v)\)).

The map \(v \mapsto \psi(v)\) is a surjection from the ground set of \(\Delta\) onto the ground set of \(\Delta'\).

**Proof.** (1) Given any \((n - 1)\)-simplex \(\bar{C} \in \Delta\), we can transport \(\psi_{\bar{C}}\) along a path to \(\bar{C}\) since \(\Gamma\) is connected. Suppose we have two such paths \(p\) and \(q\). As the transport is clearly reversible, the transport along \(p\) gives the same result as that along \(qq^{-1}p\). But the loop \(q^{-1}p\) brings back \(\psi_{\bar{C}}\) by assumption, so the result is the same as that along \(q\). The compatibility with mutation is obvious from the construction.

(2) The dual graph \(\Gamma'\) of \(\Delta'\) is also connected, so for a given \((n - 1)\)-simplex \(\bar{C}' \in \Delta'\) there is a sequence of mutations giving a path from \(\bar{C}'\) to \(\bar{C}''\). The corresponding sequence of mutations in \(\Gamma\) leads to the desired pre-image \(\bar{C}\).

(3) \(\Gamma_{\{v\}}\) is connected because \(\Delta\) is an abstract polytope, so there is a path in \(\Gamma_{\{v\}}\) connecting \(\bar{C}\) with any other \((n - 1)\)-simplex containing \(v\). \(v\) is not exchanged during the transport along that path, so its image is the same under all bijections.

(4) \(\Delta'\) is pure, so each element of its ground set is contained in an \((n - 1)\)-simplex. Thus (2) implies the claim.

Lemma 2.4 in [FZ03a] is more general in the sense that the cluster complex, which implicitly takes the role of \(\Delta'\), is not a priori known to be a pseudomanifold in general, but as this is the case for the cluster complexes we are mainly interested in, we chose a formulation which is not restricted to the cluster setting. Another difference is that the assumption about the generators of the fundamental group is split into two more special assumptions, for which we need one further notion. If \(D \in \Delta\) satisfies \(|D| = n - 2\), then \(\Gamma_D\) is a 2-regular connected subgraph of \(\Gamma\), so either an infinite chain or a cycle, and in the latter case it is called a geodesic loop. (The diamonds of Chapter 6 are examples of geodesic loops.) The two conditions are that the fundamental group of \(\Gamma\) is generated by the geodesic loops and that the seed attachment can be transported around each geodesic loop consistently. In fact the latter condition is again split: The first assumption is that the necessary exchange data (which we encoded in a quiver) are already given for every \((n - 1)\)-simplex of \(\Delta\) and are “compatible with mutation” – we might call this consistent quiver attachments. In [FZ03a] this is achieved for a specific choice of \(\Delta\) by exploiting additional structure on the \((n - 1)\)-simplices to directly calculate the appropriate datum. It has the advantage that any geodesic loop \(\Gamma_D\) obtains a numerical invariant, which is in our setting the negative square of the number of arrows between \(v_i\) and \(v_j\), where \(D \cup \{v_i, v_j\}\) is some \((n - 1)\)-simplex on
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the loop. (This number is indeed independent of the choice of such a simplex since said arrows are only reversed by quiver mutations along $\Gamma_D$.) Now the cases in which the corresponding seed mutations yield a cycle in the exchange graph are classified in [FZ02] in terms of this invariant, which also determines the length of the associated cycle. We state the result for our setting.

**Lemma 7.4.** Let $Q$ be a quiver and $i$ and $j$ two vertices. Then alternating mutations in $i$ and $j$ yield a cycle in the exchange graph if and only if $i$ and $j$ are either unconnected or joined by a simple arrow; the length of the generated cycle is four in the former and five in the latter case.

The final condition is that any geodesic loop in $\Gamma$ falls under these cases and has the right length, which guarantees the consistent seed transport. Our reason for not separating the “quiver attachments” from the seed attachments is that we obtain both by mutation, so the separation seemed artificial. Nevertheless we will have to check that each geodesic loop has the right length, but this will be obvious from the construction. Specifically we use

**Corollary 7.5.** Let $\Delta$ and $\Delta'$ be simplicial abstract $n$-polytopes. Fix a bijection $\psi : C \to C'$ between some $(n-1)$-simplices $C \in \Delta$ and $C' \in \Delta'$. If mutation induces an isomorphism of the dual graphs $\Gamma$ and $\Gamma'$, then $\Delta \cong \Delta'$. This is in particular the case when the fundamental groups of $\Gamma$ and $\Gamma'$ are both generated by geodesic loops and any geodesic loop in $\Gamma$ corresponds (under mutation) to a geodesic loop of the same length in $\Gamma'$ and vice versa.

**Proof.** The assumption on the dual graphs ensures that we can use Lemma 7.3 to get a surjection between the ground sets. But by interchanging $\Delta$ and $\Delta'$ and using $\psi^{-1}$ we see that this surjection is in fact a bijection which is by construction an isomorphism of the simplicial complexes. \(\square\)

One might wonder whether the dual graph alone contains enough information to determine the polytope (compare [Ung07]). This is true for finite convex simplicial polytopes by a result of Blind and Mani ([BML87]), but we will see later (Example 10.10) that it fails for infinite simplicial abstract polytopes already in small dimensions.
Chapter 8
The link to tilting theory

In this chapter we describe the connection of the combinatorial structures explored in the previous chapters to representation theory. For unexplained terminology and results we refer to the literature (e.g. [ARS95,ASS06]).

Let $Q$ be a finite connected acyclic quiver and $K$ an algebraically closed field. As usual we set $n = |Q_0|$. Then the path algebra $KQ$ is the $K$-vector space spanned by all paths of length $m \in \mathbb{N}_0$ with the multiplication given by concatenation where possible and zero else. It is a finite-dimensional basic hereditary algebra, and we will consider finite-dimensional (left) $KQ$-modules. The corresponding category $\text{mod } KQ$ satisfies the Krull-Remak-Schmidt theorem (i.e. has unique indecomposable direct sum decompositions) and is equivalent to the category $\text{rep } Q$ of finite-dimensional (covariant) $K$-representations of $Q$, whose objects are given by collections of $K$-vector spaces $(V_i)_{i \in Q_0}$ and linear maps $(V_a : V_{s(a)} \to V_{t(a)})_{a \in Q_1}$. (There are different preferences concerning left or right modules or even the order of the multiplication in $KQ$. All these choices are not essential for our purpose because one can always apply a duality if necessary by reversing all arrows.) The support of a representation $V = (V_i, V_a)_{i \in Q_0, a \in Q_1}$ is the full subquiver $\text{supp } V$ of $Q$ generated by all vertices $i$ with $V_i \neq 0$. A $KQ$-module $V$ is called sincere if $\text{supp } V = Q$. Note that a representation $V$ can always be seen as a sincere representation of $\text{supp } V$.

We need a bit of Auslander-Reiten theory. Recall that $KQ$ has (up to isomorphism) $n$ indecomposable projective modules $P_i$ for $i \in Q_0$. Let $\tau$ denote the Auslander-Reiten translation. Then the indecomposable modules $\tau^{-s} P_j$ with $s \geq 0$ and $j \in Q_0$ are called postprojective. Dually we have the indecomposable preinjective modules $\tau^s I_j$ with $s \geq 0$ and $j \in Q_0$. If $Q$ is representation-infinite, this yields $2n$ infinite $\tau$-orbits. We also need the bounded derived category $\mathcal{D}^b(Q)$ of $\text{rep } Q$. It contains a copy $(\text{rep } Q)[s]$ of $\text{rep } Q$ for each integer $s$ and has an Auslander-Reiten translation which
extends that of rep $Q$ and is also denoted by $\tau$. Other than for rep $Q$, $\tau$ is an auto-equivalence of $D^b(Q)$, and we have $\tau P_i[s] \cong I_i[s-1]$ for $i \in Q_0$ and $s \in \mathbb{Z}$. We refer to Happel [Hap88] for details.

We are especially interested in the tilting modules over $KQ$; for more general notions and results we refer to [Ung07].

**Definition 8.1.** A $KQ$-module $T$ is called a **tilting module** if the following conditions are satisfied:

1. $\text{Ext}^1(T,T) = 0$  
2. There is a short exact sequence $0 \to {}_AA \to T_0 \to T_1 \to 0$ with $T_0, T_1 \in \text{add} T$.

Modules satisfying (T1) are called **rigid**. An indecomposable rigid module is called **exceptional**. Neither (T1) nor (T2) depend on the multiplicities of the direct summands of $T$, so we can assume that $T$ is **basic** (i.e. has pairwise non-isomorphic indecomposable direct summands). The additivity of Ext implies that the tilting modules induce a simplicial complex:

**Definition 8.2.** The simplicial complex $\Sigma(Q)$ of tilting modules over $KQ$ (or shorter: tilting complex) is the abstract simplicial complex on the ground set of (representatives of the isomorphism classes of) exceptional modules whose maximal simplices are the tilting modules.

In fact the simplices of $\Sigma(Q)$ are precisely the basic rigid modules (which are therefore also called partial tilting modules) by a result of Bongartz:

**Lemma 8.3 ([Bon81]).** Any rigid module $T''$ can be completed to a tilting module, i.e. there is a rigid module $T''$ such that $T' \oplus T''$ is a tilting module. (In this case $T''$ is called a complement to $T'$.)

It turns out that $\Sigma(Q)$ is an $(n-1)$-dimensional pseudo-manifold (but with boundary) due to classical results about tilting modules (see [HR82, Bon81, RS90, HU89]) collected in the following theorem. Let $n(M)$ be the number of pairwise non-isomorphic indecomposable summands of a module $M$, so e.g. $n(KQ) = n$. A partial tilting module $T'$ is called **almost complete** if it has an indecomposable complement. Then the following holds.

**Theorem 8.4.** a) Let $T$ be a rigid $KQ$-module. Then $n(T) \leq n$ with equality if and only if $T$ is a tilting module.

b) Let $T'$ be an almost complete tilting module. If $T'$ is sincere, then there are exactly two non-isomorphic complements $X$ and $Y$ to $T'$; if $T'$ is
not sincere, there is exactly one complement. Moreover, in the former case there is (after renaming if necessary) an essentially unique short exact sequence $0 \to X \to \tilde{T} \to Y \to 0$ with $\tilde{T} \in \text{add} T$ connecting the two complements.

Remark 8.5. By part a) of the theorem the almost complete tilting modules are precisely the $(n - 2)$-simplices of $\Sigma(Q)$; notably we get $n(T') = n - 1$ in part b). Hence the boundary of $\Sigma(Q)$ is formed by those almost complete tilting modules which are not sincere.

Ringel’s suggestion to study $\Sigma(Q)$ was taken up by Riedtmann and Schofield in ([RS91]). As one tool they introduced the quiver of tilting modules $\mathcal{K}_{KQ}$, which is in fact nothing but an orientation of the dual graph of $\Sigma(Q)$: If two tilting modules share an almost complete tilting module $T'$, the edge between them is oriented depending on the order in which the two complements occur in the exchange sequence of part b) of Theorem 8.4. It turns out that $\mathcal{K}_{KQ}$ is the Hasse diagram of a partial order on the set of tilting modules as shown by Happel and Unger, who studied both structures extensively (see e.g. [HU05a, HU05b, HU10]). We will approach some of their questions via the exchange graph $\Gamma(Q)$ of the cluster algebra $A_Q$, but with a certain restriction due to the combinatorial character of this approach: Though the orientation of $\mathcal{K}_{KQ}$ can be used to define an orientation for $\Gamma(Q)$ as well (see [IT09]), this does in general not induce a well-defined orientation for the mutation graph and is therefore not “governed by the combinatorics of quiver mutation”, so we will neglect the orientation and only deal with $\mathcal{K}_{KQ}$ as the exchange graph of tilting modules.

Some of the first examples for an exchange of tilting modules are provided by the so-called APR-tilting modules named after Auslander, Platzeck and Reiten [APR79] (see [ASS06]):

Example 8.6. Assume $n \geq 2$ and let $i$ be a sink in $Q$ (so the projective module $P_i$ is simple). Then $T := \bigoplus_{j \neq i} P_j \oplus \tau^{-1} P_i$ is a tilting module, hence there is an edge in $\mathcal{K}_{KQ}$ between $T$ and the projective tilting module.

In [HU09], Happel and Unger examined the possibility of reconstructing $Q$ from the partially ordered set of tilting modules and gave a positive answer “up to multiplicity” in the following sense: One can recover the number of vertices $n$ and determine for any two vertices $i$ and $j$ whether there is no arrow, exactly one arrow, or a multiple arrow from $i$ to $j$, but in the latter case not the exact multiplicity. In fact this can be observed already in the simplest case $n = 2$ (see [Ung96b, 2.2]):

Example 8.7. Consider the quiver $Q(m)$ with two vertices and $m$ arrows. Then $\mathcal{K}_{KQ(m)}$ looks as follows:
• for $m = 0$: $\circ$,
• for $m = 1$: $\circ \to \circ$,
• for $m \geq 2$: $\circ \to \circ \to \cdots \cdots \to \circ \to \circ$.

So we get isomorphic quivers (and in fact also isomorphic partially ordered sets and simplicial complexes) of tilting modules for all $m \geq 2$.

Happel asked whether the same phenomenon occurs in general, i.e. whether quivers that are “isomorphic up to multiplicity” have isomorphic quivers (partially ordered sets, simplicial complexes) of tilting modules, and this question was the starting point of our investigations. To make it precise, let us fix the following definitions.

**Definition 8.8.** For a quiver $Q$ we define the **basic quiver** $Q_{\text{bas}}$ as the quiver obtained by replacing every multiple arrow in $Q$ by a double arrow (with the same direction). We say that two quivers $Q$ and $Q'$ with $Q_0 = Q'_0$ are **basically equal** and write $Q \equiv Q'$ if $\text{id}_{Q_0}$ induces an isomorphism $Q_{\text{bas}} \cong Q'_{\text{bas}}$.

**Question 8.9** (Happel). Is it true that the quivers (partially ordered sets, simplicial complexes) of tilting modules of basically equal acyclic quivers are isomorphic?

Of course an isomorphism of the partially ordered sets would induce one for the quivers (being the corresponding Hasse diagrams) as well; similarly an isomorphism of the simplicial complexes would induce one at least for the unoriented quivers (being the corresponding dual graphs).

Example 8.7 shows that it might be hard to tackle this question with classical methods, because it allows to construct a well-behaved bijection between the sets of exceptional modules over very different algebras: The so-called Kronecker algebra for $m = 2$ is tame, whereas all the algebras for $m > 2$ are wild, so their module categories are dramatically different. Even in this case we have the feeling that the given bijection is rather implicit and relies more on combinatorial than on representation-theoretic aspects. Especially when the partial order of the tilting modules is concerned, one would like to have a representation-theoretic interpretation of this bijection, but this remains an open task. In a less conceptual direction one might try to find an inductive approach to lift the results for $n = 2$ to larger $n$, but we have not even seen a way to tackle the case $n = 3$ with this idea only.

Luckily the development of cluster theory in the last decade facilitates an elementary combinatorial approach we will exploit. Maybe the first
connection was drawn in [MRZ03], where the authors, inspired by certain polytopes arising in cluster theory, showed that in the Dynkin case the tilting complex has a natural (and more regular) completion with just \(n\) additional elements (which may be identified with \(Q_0\)) in the ground set.

Related to this idea and somehow parallel to the progress in understanding the connection, the concept of so-called support-tilting modules emerged (see [IT09, Hub]); these are rigid modules that are tilting modules when restricted to their support. Note that an insincere almost complete tilting module \(T'\) has only one complement, but there is a unique vertex \(i \notin \text{supp} T'\) because \(n - 1 = n(T') \leq |(\text{supp} T')_0| < n\), and \(T'\) is support-tilting by part a) of Theorem 8.4. Taking this vertex \(i\) as a “substitute complement” we can continue the exchange process across the boundary of \(\Sigma(Q)\) and (using this idea inductively) get a pseudo-manifold without boundary. The maximal simplices (and thus vertices of the dual graph) are now precisely the support-tilting modules. With the benefit of hindsight this extension of \(\Sigma(Q)\) to the simplicial complex of support-tilting modules does not seem very far-fetched, but with the focus on the exchange graph it is not at all obvious what one should do.

In fact a crucial step was the introduction of the cluster category \(\mathcal{C}_Q\) in [BMR +06]; this is a certain orbit category of \(D^b(Q)\). Ringel points out in [Rin07] that one can get rid of the boundary of \(\Sigma(Q)\) by passing to \(D^b(Q)\), but – with infinitely many copies (shifts) of each tilting module – the obtained simplicial complex is far “too large”. A way around this is factoring out an auto-equivalence of \(D^b(Q)\), i.e. taking the orbits as the objects of a new category. It turns out that \(\tau^{-1}[1]\) is the right equivalence (so in the orbit category \(D^b(Q)/\tau^{-1}[1]\) we have \([1] \cong \tau\) and avoid superfluous copies), which leaves – as desired – exactly \(n\) orbits of indecomposable objects that have no representative in \(\text{rep} Q\) (embedded via \((\text{rep} Q)[0] \hookrightarrow D^b(Q) \rightarrow \mathcal{C}_Q\)). As representatives of these orbits one can take the shifts \(P_i[1] \cong \tau P_i\) (\(i \in Q_0\)) of the \(n\) indecomposable projectives; so, together with the indecomposables of \(\text{rep} Q\), they form a complete list of representatives of the isomorphism classes of indecomposable objects of \(\mathcal{C}_Q\), and we will use them to denote the latter. In this setting the tilting modules are generalised to (cluster) tilting objects, which are by definition the maximal rigid objects in \(\mathcal{C}_Q\). Naturally there is an associated simplicial complex of cluster tilting objects on the ground set of exceptional objects of \(\mathcal{C}_Q\).

We summarise the connections between these structures ([IT09, CK06, BMRT07]):

**Theorem 8.10.** There are distinguished bijections between the following sets:

- the cluster variables of \(\mathcal{A}_Q\),
b. isomorphism classes of exceptional objects in $\mathcal{C}_Q$,

c. isomorphism classes of exceptional modules in $\text{rep} Q$ together with $Q_0$,

which induce bijections between

a. the clusters of $\mathcal{A}_Q$,

b. isomorphism classes of basic cluster tilting objects in $\mathcal{C}_Q$,

c. isomorphism classes of basic support-tilting modules over $KQ$,

and under which the initial cluster $\{x_1, \ldots, x_n\}$ corresponds to the cluster tilting object $\tau KQ$ and to the trivial support-tilting module $0$, which has empty support. Moreover, each seed is determined by its cluster via the corresponding cluster tilting object.

**Corollary 8.11.** The simplicial complexes of cluster tilting objects and support-tilting modules are both isomorphic to the cluster complex $\Delta(Q)$. Notably, the dual graphs are all isomorphic to the exchange graph $\Gamma(Q)$; the exchange graph of tilting modules $K_{KQ}$ is isomorphic to the full subgraph of $\Gamma(Q)$ induced by the clusters not containing an initial cluster variable, and the complex of tilting modules $\Sigma(Q)$ is isomorphic to the subcomplex of $\Delta(Q)$ formed by the non-initial cluster variables.

This result gives us “different descriptions of the same object”, which is very useful: Some properties are more obvious in one description than in another; e.g. we see that $\Delta(Q)$ and $\Gamma(Q)$ have a symmetry induced by the Auslander-Reiten translation $\tau$, which is an obvious symmetry on the level of cluster tilting objects. And of course it enables us to approach Happel’s question with our results about quiver mutations. To do this we first reformulate the general identification criterion from Chapter 7.

**Lemma 8.12.** Let $Q$ and $Q'$ be two quivers with common vertex set such that for any sequence $i$ of mutation indices the quivers $\mu_i(Q)$ and $\mu_i(Q')$ are basically equal. If both cluster complexes $\Delta(Q)$ and $\Delta(Q')$ are abstract polytopes, and the fundamental groups of both exchange graphs $\Gamma(Q)$ and $\Gamma(Q')$ are generated by geodesic loops, then $\Delta(Q) \cong \Delta(Q')$.

**Proof.** The given assumptions allow us to apply Corollary 7.5. Both cluster complexes are simplicial abstract $n$-polytopes; by Theorem 7.2 their dual graphs are the exchange graphs, whose fundamental groups are assumed to be generated by geodesic loops. We fix the canonical bijection between the initial clusters induced by the common vertex set. If one of them lies on a
geodesic loop obtained by alternating mutations in two vertices $i$ and $j$, then $i$ and $j$ are either unconnected or joined by a simple arrow due to Lemma 7.4—and that is the case in both quivers simultaneously because they are basically equal. Hence these mutations generate geodesic loops of the same length in both exchange graphs. Since the basic equality is by assumption invariant under mutation, the same argument holds for all geodesic loops.

In our acyclic setting we get one of the assumptions by the following result of Hubery, which he proved more generally for hereditary Artin algebras (see [Hub10, Theorem 19]):

**Theorem 8.13.** The complex of support-tilting modules over a basic hereditary Artin algebra is a simplicial abstract polytope.

**Corollary 8.14.** Let $Q$ and $Q'$ be two acyclic quivers with common vertex set such that for any sequence $i$ of mutation indices $\mu_i(Q) \equiv \mu_i(Q')$. If the fundamental groups of both exchange graphs $\Gamma(Q)$ and $\Gamma(Q')$ are generated by geodesic loops, then $\Delta(Q) \cong \Delta(Q')$. Moreover, restriction yields an isomorphism $\Sigma(Q) \cong \Sigma(Q')$ and hence also an isomorphism $K_{KQ} \cong K_{KQ'}$.

We will later provide instances of acyclic quivers which satisfy both other assumptions as well; we actually conjecture that all acyclic quivers do:

**Conjecture 8.15.** Let $Q$ be acyclic. Then the fundamental group of $\Gamma(Q)$ is generated by geodesic loops.

**Conjecture 8.16.** Let $Q \equiv Q'$ be acyclic. Then $\mu_\sharp(Q) \equiv \mu_\sharp(Q')$ for any sequence $\sharp$ of mutation indices.

Conjecture 8.15 is common (see e.g. [FZ02, after Theorem 7.7]), but Conjecture 8.16 seems to be new. Let us note that the former is wrong for general quivers according to [FST08, Remark 9.19]; a counterexample for the latter is given by two abundant 3-cycles of which one is mutation-acyclic and the other not. As a justification for Conjecture 8.16 we can offer the fact that computer experiments have not produced a counterexample and that we can prove it for special cases (see also Corollary 10.6):

**Theorem 8.17.** Let $Q \equiv Q'$ be either abundant acyclic or forks with common point of return $r$ and $k \neq r$. Then $\mu_k(Q) \equiv \mu_k(Q')$. Notably Conjecture 8.16 holds for abundant quivers.

**Proof.** The arrows incident with $k$ are only reversed, and by assumption $Q^-(k) \equiv (Q')^-(k)$, so for $\tilde{Q} := \mu_k(Q)$ and $\tilde{Q'} := \mu_k(Q')$ we get $\tilde{Q}^+(k) = Q^-(k) \equiv (Q')^-(k) = (\tilde{Q'})^+(k)$ and similarly $\tilde{Q}^-(k) \equiv (Q')^+(k)$. Furthermore,
for all $i \in \tilde{Q}^-(k)$ and $j \in \tilde{Q}^+(k)$ we have a multiple arrow from $j$ to $i$ in $\tilde{Q}$, and the same holds in $Q'$, because both are forks with point of return $k$ by Lemma 2.5 if the statement is not empty. So also $\tilde{Q} - \{k\} \equiv \tilde{Q}' - \{k\}$ and the quivers remain basically equal. Now Conjecture 8.16 follows inductively with Lemma 2.5.

Remark 8.18. In [Sevb, Theorem 1.3], Seven shows with quite different methods a result in the direction of Conjecture 8.16 that would also follow from it. With our notation it says the following: If an acyclic quiver $Q$ contains no simple arrows, the same holds for all quivers mutation-equivalent to $Q$. Now, if Conjecture 8.16 holds, we can construct $\tilde{Q} \equiv Q$ with all arrow numbers even. This property is clearly invariant under mutation (see [Sev13, Theorem 1.3] for a generalisation), so no simple arrows can occur in quivers which are mutation-equivalent to $\tilde{Q}$ and hence neither in those mutation-equivalent to $Q$ because each of the latter is basically equal to one of the former.

Remark 8.19. If Conjecture 8.15 holds, then any mutation at an abundant vertex corresponds to a bridge in the exchange graph because the corresponding edge is not part of a geodesic loop by Lemma 7.4. Our results show in particular that at least this (weaker) consequence holds in some special cases, namely for forks by the Tree Lemma, which covers all vertices except the point of return, and Corollary 4.3, which covers also the point of return; moreover for the point of return (and hence also all other abundant vertices) of $\diamond$-roots by Lemma 6.7. In Corollary 13.21 we will note an analogous result for a class of quivers we call semi-forks containing all pre-forks except $\diamond$-roots.
Chapter 9

Partial answers to Happel’s question

In Chapters 4 and 6 we have already seen some examples in which the exchange graph satisfies Conjecture partially (e.g. Corollary 6.9) or even completely (Corollary 4.4); but that the cluster complex is an abstract polytope we know only in the acyclic case, where we have no complete descriptions of exchange graphs yet – indeed, until now we have not considered mutations in sinks or sources as they lack the “key feature” of increasing the number of arrows. Luckily their role in the exchange graph can be easily described with the help of tilting theory. Let us nevertheless start with the quivers alone.

We can assume that $Q_0 = \{1, \ldots, n\}$ is admissibly numbered; in particular, 1 is a sink in $Q$. Mutation in this sink just reverses all arrows ending in 1, so 1 becomes a source in $Q^{(1)} := \mu_1(Q)$, whereas 2 is a sink after the mutation, in which we can mutate to make 3 a sink in $Q^{(2)} := \mu_2(Q^{(1)})$. By iterating these sink mutations until we have mutated once in each vertex, we obtain $Q^{(n)}$. During this process every arrow has been reversed precisely twice (namely when mutating in the two vertices it connects), so $Q^{(n)} \cong Q$ and we get a closed path in the mutation graph. By the discussion in Chapter 4 there is a subgraph of the exchange graph which covers this closed path.

What happens on the level of cluster tilting objects is essentially iterated APR-tilting (compare Example 8.6): We start with $\tau KQ$ and exchange the shifted simple projective $\tau P_1$, so we obtain $\bigoplus_{j>1} \tau P_j \oplus P_1$; continuing as above we obtain $\bigoplus_{j>m} \tau P_j \oplus \bigoplus_{j\leq m} P_j$ after $m \leq n$ and hence $KQ$ after $n$ mutations. Further (ordered) sink mutations lead to the cluster tilting objects $\bigoplus_{j>m} \tau^{1-s} P_j \oplus \bigoplus_{j\leq m} \tau^{-s} P_j$ also for $s > 0$, which therefore incorporate all indecomposable postprojectives. Dually source mutations will produce the sequence of cluster tilting objects with preinjective summands given by the same formula with $s < 0$. We call the corresponding vertices in the exchange...
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graph **canonical transjective.** (Note that they depend on the chosen admissible numbering which may not be unique!)

Now there are essentially two cases. If $Q$ is representation-finite (and hence contains no multiple arrows), the process must obviously become periodic, so we get a closed path in the exchange graph as well. It is also quite interesting to follow that path, which may cover the complete exchange graph, but since basically equal quivers without multiple arrows are already equal and Happel’s question is trivial in that case, we concentrate on the second case. If $Q$ is representation-infinite, there are infinitely many postprojective and preinjective modules, therefore the canonical transjective cluster tilting objects are pairwise non-isomorphic (this also holds when $Q$ is not connected) and we obtain the following

**Lemma 9.1.** Let $Q$ be a representation-infinite acyclic quiver with an arbitrary admissible numbering of its vertices. Then the canonical transjectives form an infinite chain in the exchange graph.

**Remark 9.2.** Since the exchange graph is connected, we can conclude from the above considerations that any two vertices in the same $\tau$-orbit have the same quiver. (This also follows directly from the way the cluster tilting objects determine the quivers.)

**Example 9.3.** Figure 9.1 shows – in the notation of support-tilting modules to include the initial quiver with the vertices 1, 2 and 3 – the first 3 steps of the described process for $n = 3$. The vertices of the exchange graph are the points where three regions labelled by the indecomposable summands meet. Exchanging one summand corresponds to moving along the edge between the remaining summands. These remaining summands are therefore part of two seeds with associated arrow numbers, so we get two arrows between them. As the mutations considered here are source- and sink-mutations, we get the same arrow number twice, but in general the arrow numbers will change according to the mutation rule. Note that $\tau$ acts as a glide reflection.

![Figure 9.1: Canonical transjectives for $n = 3$](image_url)
With Lemma 9.1 we can now prove the following

**Theorem 9.4.** The exchange graph of an abundant acyclic $n$-point-quiver is an $n$-regular tree.

*Proof.* Though we can interpret the result to hold also for $n = 1$, only $n \geq 2$ is of interest. Then we are in the representation-infinite case and have the described infinite chain of canonical transjectives with abundant acyclic quivers connected by source-/sink-mutations. If we apply to any of these quivers a mutation at one of the $n - 2$ vertices that are neither sources nor sinks, we get a fork by Lemma 2.5. Hence the fork-less part is just this infinite line and the claim follows with Corollary 4.5. ☐

It follows at once that Conjecture 8.15 holds (trivially) for abundant quivers, and in combination with Lemma 7.4 we get the following approximation.

**Corollary 9.5.** Let $Q$ be an acyclic quiver. If $\Gamma(Q)$ contains cycles, then $Q$ is not abundant, i.e. $\Gamma(Q)$ already contains geodesic loops. ☐

We also obtain the following partial answer to Happel’s question 8.9.

**Corollary 9.6.** Let $Q$ and $Q'$ be abundant acyclic (and w. l. o. g. basically equal) $n$-point-quivers. Then $\Delta(Q) \cong \Delta(Q')$, $\Sigma(Q) \cong \Sigma(Q')$ and $K_{KQ} \cong K_{KQ'}$. Notably there is a canonical bijection between the sets of isomorphism classes of exceptional modules over $KQ$ and $KQ'$.

*Proof.* This follows from Corollary 8.14 with Theorems 8.17 and 9.4. Note that by Theorem 8.13 we could also directly apply Corollary 7.5 to an arbitrary bijection between the initial clusters (or, in fact, arbitrary clusters) because both exchange graphs are $n$-regular trees and hence mutation cannot but induce an isomorphism between the dual graphs. Of course the bijection respecting the unique admissible order in both quivers is the canonical one. ☐

**Remark 9.7.** With the above results, the cluster complex of an abundant acyclic quiver is easily identified as the universal abstract polytope with Schläfli symbol \{3, 3, \ldots, \infty\}, see [MS02, Section 3D].
Chapter 10

Classification for 3 vertices

In this chapter we classify the exchange graphs \( \Gamma(Q) \) of arbitrary 3-point-quivers (without loops or 2-cycles). Let us stress again that the arguments we use about mutation of 3-cycles are essentially covered by the considerations in [BBH11] and [ABBS08]. (However, the implications for the exchange graphs, especially the classification in the mutation-acyclic case – which implicitly appears also in [FST12, Section 9] – are not worked out.) It should also be mentioned that, as pointed out in [BBH11], these considerations have a prominent ancestor in [Mar80], where the famous Markov equation \( x^2 + y^2 + z^2 = 3xyz \) is treated. Namely, there is the following relation to the mutation rule: Multiplication with 9 and setting \( X := 3x \) etc. yields the equation \( X^2 + Y^2 + Z^2 = XYZ \), which we can regard as being quadratic in, say, \( X \) and therefore having a second solution \( X' = YZ - X \) by Vieta. Now, if we mutate a 3-cycle with arrow numbers \( X, Y, Z \) opposite \( X \) and get a new 3-cycle, the new arrow numbers are \( X', Z, Y \). It follows that \( X^2 + Y^2 + Z^2 - XYZ \) is invariant under such mutations, see [ABBS08] for a detailed treatment. The positive integer solutions of the Markov equation are called Markov triples, and it follows easily from [Mar80] that their “mutation graph” is a 3-regular tree, the idea being again that every triple apart from \((1, 1, 1)\) (which has the three neighbours \((2, 1, 1)\), \((1, 2, 1)\) and \((1, 1, 2)\)) has precisely one “smaller” and two “greater” neighbours. Miraculously, this helps us even in the (single) case where our results about the mutation graphs cannot be applied.

First we treat the case that \( Q \) is mutation-cyclic. We should mention that there are also cluster category constructions for such quivers, but then \( \Gamma(Q) \) gives in general only one component of the exchange graph of cluster tilting objects, see [Pla13] for an example. Furthermore we cannot extend the classification to the cluster complex without additional information, but we consider the following result nevertheless worth mentioning.
CHAPTER 10. CLASSIFICATION FOR 3 VERTICES

Theorem 10.1. Let $Q$ be a mutation-cyclic 3-point-quiver. Then $\Gamma(Q)$ is a 3-regular tree.

Proof. As $Q$ is mutation-cyclic, all quivers in its mutation-class are 3-cycles, and we can assume $Q$ to be one with minimum total number of arrows. First note that $Q$ has to be abundant since a 3-cycle with a simple arrow is clearly mutation-acyclic. If it is even a fork (and $r$ its point of return), we are in a special case of Corollary 4.4, because the minimality assumption yields $q_{ir}q_{rj} - q_{ji} \geq q_{ji} > q_{ir}, q_{rj}$ for $\{i\} = Q^-(r)_0$ and $\{j\} = Q^+(r)_0$. It follows that $\Gamma(Q)$ is a 3-regular tree. So assume $Q$ is not a fork; this means that the arrow with maximum multiplicity cannot be unique, thus $Q$ is of the form

```
1 \xrightarrow{x} 2 \xrightarrow{y} 3
```

with $x \geq y \geq 2$. If $y \geq 3$, the mutation at an arbitrary vertex $r$ yields a fork with point of return $r$ (since then $x^2 - y \geq 3x - y > x$ resp. $xy - x \geq 2x > x, y$). Clearly, by the Tree Lemma, $\Gamma(Q)$ is again a 3-regular tree in that case. So it remains to consider $y = 2$. When $x \geq 3$, $\mu_2(Q)$ is a fork with point of return 2 (as above), but mutations at 1 and 3 yield quivers isomorphic to $Q$. So in the subgraph $\Gamma(\{2\})$ all quivers are isomorphic to $Q$, and this is the fork-less part of $\Gamma(Q)$. But $y = 2$ implies that $\Gamma(\{2\})$ is infinite by Lemma 7.4, so the claim follows again with Corollary 4.5. We are finally left with the case $x = y = 2$, in which $Q$ is invariant under mutation, so the mutation graph is trivial and does not help to understand the exchange graph. As announced above, the Markov equation saves us by coming into play a second time. The exchange relation for a cluster variable $y_1$ in an arbitrary cluster $\{y_1, y_2, y_3\}$ is $y_1^*y_1 = y_2^2 + y_3^2$. Following (a reference we owe Philipp Lampe) we observe that $y_1^* = (y_2^2 + y_3^2)/y_1$ corresponds (again) to the second solution of the Markov equation as a quadratic equation with a root $y_1$, this time by the other Vieta formula. Hence specialising the initial cluster variables to $(1, 1, 1)$, the smallest Markov triple, yields a surjective graph morphism from the exchange graph to the “mutation graph” of Markov triples. As this is already a 3-regular tree, the same is true for $\Gamma(Q)$. \[\Box\]

Remark 10.2. Schröer asked whether the exchange graphs of other mutation-finite mutation-cyclic quivers (besides the 3-cycle with double arrows) are also trees, but it turns out that this is not the case. Indeed, suppose we have such a quiver $Q$ with at least three vertices and an exchange graph without cycles. Then $Q$ is abundant by Lemma 7.4. On the other hand, $Q$ has no arrows of multiplicity greater than two as it is mutation-finite (see Proposition 3.4). So any two vertices are joined by a double arrow. Furthermore any full 3-point-subquiver has to be a 3-cycle because $Q$ would be mutation-infinite otherwise. But this is only possible if $Q$ has precisely three vertices.
Theorem 10.3. Let $Q$ be a mutation-acyclic 3-point-quiver. Then $\Gamma(Q)$ is one of the ten following 3-regular graphs:

- the edge graph of the cube, the pentagon prism or the 3-dimensional associahedron (if $Q$ is representation-finite);

- the doubly infinite ladder or the two-layer brick wall (if $Q$ is representation-infinite, but mutation-finite);

- one of the graphs partially sketched in Figures 10.1–10.5 (if $Q$ is mutation-infinite).

Proof. For a systematic study let us first determine a distinguished representative in the mutation class of an acyclic 3-point-quiver $Q = \bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \xrightarrow{c} \bullet$ (or its opposite quiver), where $a, b, c \geq 0$ denote the numbers of the arrows. In Figure 9.1 we observe that source- and sink-mutations cyclically permute $a$, $b$ and $c$, so after relabelling we can assume that $Q$ is a representative with $c \leq a, b$. As the opposite quiver has an isomorphic exchange graph, we can finally assume $a \geq b \geq c$.

Next we determine the exchange graph of $Q$. First, if $b = 0$ (which implies $c = 0$), then $Q$ is not connected, so $\Gamma(Q) \cong \Gamma(\bullet \xrightarrow{a} \bullet) \times \Gamma(\bullet)$, which yields the cube for $a = 0$, the pentagon prism for $a = 1$ and the doubly infinite ladder for $a > 1$ by Lemma 7.4. For $a = b = 1$ the mutation class is finite, and both possible cases are well known: $Q$ is representation-finite for $c = 0$, and we get the 3-dimensional associahedron (see [FZ03b]); for $c = 1$ we obtain the (infinite) two-layer brick wall (see [FZ02]).

$Q$ is mutation-infinite in the remaining cases with $a \geq 2$, $b \geq 1$ and $c \geq 0$, but luckily the fork-less part is easy to determine by the following procedure (which in principle works also in the mutation-finite case when there are no forks in $\Gamma(Q)$). We start with the infinite chain of canonical transjectives (see Lemma 9.1 and Figure 9.1) indicated by a thicker line in the figures. Next we construct geodesic loops of the right lengths according to Lemma 7.4 and the values of $a$, $b$ and $c$. Note that this is based essentially on the known case $n = 2$ and hence depends only on the basic quiver $Q_{bas}$; which yields five cases. $c \geq 2$ (see Figure 10.1) is a special case of Theorem 9.4; in the other cases we get a periodic pattern of quadrilaterals and/or pentagons as shown in Figures 10.2–10.5. It is easy to see that the added seeds are pairwise distinct.

The last step is to check in each case that any mutation leaving the so far constructed part gives a fork for all arrow multiplicities under consideration.
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Figure 10.1: Exchange graph and dual polytope for $a, b, c \geq 2$

Figure 10.2: Exchange graph and dual polytope for $a, b \geq 2$ and $c = 1$
Figure 10.3: Exchange graph and dual polytope for $a \geq 2$ and $b = c = 1$

Figure 10.4: Exchange graph and dual polytope for $a, b \geq 2$ and $c = 0$
CHAPTER 10. CLASSIFICATION FOR 3 VERTICES

Figure 10.5: Exchange graph and dual polytope for $a \geq 2$, $b = 1$ and $c = 0$

To facilitate this we have depicted the corresponding quivers for one representative in each $\tau$-orbit until we obtain forks, which is sufficient by Remark 9.2. Note that $\tau$ acts again as a glide reflection.

We conclude by noting that every possible exchange graph of an arbitrary mutation-acyclic 3-point-quiver occurs in the given list since every such graph contains by definition an acyclic seed and hence (after switching to the opposite quiver if necessary) a seed with quiver $Q$ as above.

Corollary 10.4. There is a bijection between the $\binom{3+2}{2}$ integer sets $\{a, b, c\}$ with $2 \geq a \geq b \geq c \geq 0$ and the isomorphism classes of exchange graphs of mutation-acyclic 3-point-quivers.

Proof. With each set $\{a, b, c\}$ we associate the isomorphism class of the exchange graph of $Q := \bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet$. It follows from the proof of Theorem 10.3 that this is a bijection: The ten given exchange graphs actually occur, are all non-isomorphic and cover all isomorphism classes.

Remark 10.5. The proof of Theorem 10.3 shows an interesting phenomenon related to Conjecture 8.16. It is evident from Figures 10.1-10.5 that we can
mutate the given quivers without specifying the exact values of the arrow numbers greater than 1. So, what we have actually done is the mutation of quivers with arrow parameters instead of arrow numbers. This inspires the notion of what we call “polynomial quivers” and the question when the mutation of these is well-defined; we address this issue in Chapter 12.

As a by-product of the classification in Theorem 10.3 we get

**Corollary 10.6.** Conjectures 8.15 and 8.16 hold for 3-point-quivers.

**Proof.** The former follows by direct inspection of the possible exchange graphs. The latter is easy to see in the mutation-finite cases. In the other cases we note that basically equal acyclic quivers are “evaluations of the same polynomial quiver”, which is (up to duality) among the quivers in Figures 10.1[10.5]. It is easily checked that different evaluations of its mutations are still basically equal. (Note that the forks in these cases are abundant 3-cycles.)

Thus Corollary 8.14 yields another partial answer to Question 8.9.

**Corollary 10.7.** Let \( Q \equiv Q' \) be acyclic 3-point-quivers. Then \( \Delta(Q) \cong \Delta(Q') \), \( \Sigma(Q) \cong \Sigma(Q') \) and \( \mathcal{K}_{KQ} \cong \mathcal{K}_{KQ'} \). Notably there is a canonical bijection between the sets of isomorphism classes of exceptional modules over \( KQ \) and \( KQ' \).

**Remark 10.8.** As explained earlier we do not consider the orientation of \( KQ \) in this thesis. Nevertheless we believe that the isomorphism \( \mathcal{K}_{KQ} \cong \mathcal{K}_{KQ'} \) in Corollary 10.7 induces also an isomorphism of the oriented graphs. The reason is that the orientation of the edges can be determined with representation-theoretic arguments about the direction of maps between the modules involved in the corresponding exchange sequence. But this procedure is quite cumbersome, so we have not checked all cases.

Moreover, the classification carries over to the cluster complexes:

**Theorem 10.9.** There are precisely ten isomorphism classes of cluster complexes of mutation-acyclic 3-point-quivers characterised by the fact that the dual graph of a representative is isomorphic to precisely one of the graphs listed in Theorem 10.3.

**Proof.** As in the proof of Theorem 10.3 we can assume that a cluster complex as considered is given as \( \Delta(Q) \) with \( Q = \bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \xrightarrow{c} \bullet \), \( a \geq b \geq c \). As we see in Corollary 10.4 there are precisely ten basic equality classes of such quivers, and quivers in different classes have non-isomorphic exchange graphs and hence also non-isomorphic cluster complexes. But the cluster complexes of basically equal quivers are isomorphic by Corollary 10.7.
One can get a rather concrete picture of the “abstract” cluster complexes, which we explain in the mutation-infinite case. The characteristic clippings of the exchange graphs in Figures 10.1–10.5 are drawn in a way that indicates how to define a (realisation of a) simplicial abstract 2-polytope $\Delta$ whose dual graph is the exchange graph. In fact, the dual polytope of $\Delta$ is plainly visible: Its vertices (i.e. 0-faces) and edges (i.e. 1-faces) are those of the exchange graph, and the 2-faces are the $\infty$-gons, pentagons and/or quadrilaterals bounded by the edges, all with the obvious incidences. (It covers an infinite open strip and is hence homeomorphic to the open disc.) Now we can apply Corollary 7.5 to identify $\Delta$ with the respective cluster complex because mutation induces an isomorphism of the dual graphs by construction.

**Example 10.10.** We can now give a counterexample to the theorem of Blind and Mani for infinite abstract polytopes. Consider the exchange graph in Figure 10.2. It is not hard to show that, if we “flip” one of the pentagons with the attached binary trees across a horizontal axis, the resulting embedding of the same graph defines a non-isomorphic 2-polytope. (Indeed, consider the $\infty$-gons that share an edge with exactly two pentagons and call them double agents. In Figure 10.2 every double agent shares an edge with two other double agents. But after the flip this will not be the case anymore.) Of course the induced graph isomorphism is then not compatible with mutation.
Chapter 11

Unger’s conjecture

In her habilitation thesis [Ung93], Unger stated the following conjecture and proved it for \( n = 3 \) (see also [Ung96a]):

**Conjecture 11.1.** If \( Q \) is wild with \( n \geq 3 \) vertices, then \( \mathcal{K}_{KQ} \) has infinitely many connected components.

Recall that \( \mathcal{K}_{KQ} \) can be seen as a full subgraph of \( \Gamma(Q) \), and the missing vertices correspond to those support-tilting modules without full support. The corresponding seeds can be easily characterised, namely the cluster variable \( x_k \) is contained in the cluster of a seed if and only if the associated vertex \( k \in Q_0 \) is not in the support of the corresponding support-tilting module. This allows us to apply our results about \( k\)-mutation-equivalence as follows.

**Lemma 11.2.** Let \( Q \) be acyclic and \( k\)-mutation-equivalent to a fork. Then \( \mathcal{K}_{KQ} \) has infinitely many connected components.

**Proof.** By assumption there is a sequence of mutations not involving \( \mu_k \) and turning \( Q \) into a fork. Applied to the trivial support-tilting module, this yields a support-tilting module \( T^{(0)} \) with \( k \notin \text{supp} T^{(0)} \) which is hence not a vertex of \( \mathcal{K}_{KQ} \). By further mutations that involve neither \( k \) nor the respective point of return (such mutations are always possible as we have enough vertices) we even get a sequence of support-tilting modules \( (T^{(m)})_{m \in \mathbb{N}} \) with \( k \notin \text{supp} T^{(m)} \) for all \( m \in \mathbb{N} \) whose quivers are all forks. Now mutation in \( k \) yields another sequence of forks, and by the Tree Lemma the deletion of the vertices corresponding to \( (T^{(m)})_{m \in \mathbb{N}} \) leaves (at least) a sequence \( (C^{(m)})_{m \in \mathbb{N}} \) of connected components of \( \Gamma(Q) - \{T^{(m)}| m \in \mathbb{N}\} \) which are all infinite \( (n - 1)\)-ary trees. It is easy to see that each of these components contains proper tilting modules, so we get infinitely many connected components for \( \mathcal{K}_{KQ} \), too. \( \square \)
As a corollary we get the following

**Theorem 11.3.** Conjecture 11.1 holds in all but finitely many cases, specifically for all acyclic quivers with at least 11 vertices or with arrows of multiplicity greater than two.

*Proof.* It follows from [BR06] that the acyclic mutation-infinite quivers are precisely the wild ones with \( n \geq 3 \), and by Proposition 3.12 at least the specified ones satisfy the assumption of Lemma 11.2.

**Remark 11.4.** Using the above method we get in fact infinitely many connected components of \( \mathcal{K}_{\mathcal{K}Q} \) isomorphic to infinite \( (n-1) \)-ary trees (since the link of \( x_j \), i.e. the subgraph of \( \Gamma(Q) \) induced by the seeds involving \( x_j \), is connected, every tree rooted in a proper tilting module consists entirely of proper tilting modules), but it is in general not true that all but finitely many of the connected components are of this form (other than for \( n = 3 \), see [Ung93]). A counterexample with infinitely many \( \diamond \)-trees can be easily constructed starting with the following quiver, from which \( \mu_3 \) yields a \( \diamond \)-root:

\[
\begin{array}{c}
1 & \leftarrow & 2 - 4 \\
\uparrow & & \downarrow \\
2 & & 2 \\
\downarrow & & \uparrow \\
2 - 2 & \rightarrow & 3
\end{array}
\]

**Remark 11.5.** There are many simple examples for mutation-infinite quivers that do not satisfy the assumption of Lemma 11.2 e.g. each proper subquiver of the following quiver is mutation-finite, whereas successive mutations in 2, 3 and 1 produce a triple arrow between 3 and 4:

\[
\begin{array}{c}
1 & \leftarrow & 4 \\
\uparrow & & \downarrow \\
2 & \rightarrow & 3
\end{array}
\]
Chapter 12

Polynomial quivers

In this chapter we will develop the idea of “polynomial quivers”. We have already noted in Remark 10.5 that we can sometimes mutate quivers with “arrow parameters” instead of arrow numbers – in fact doing this is very natural when working with paper and pencil; but of course this usually gets complicated quite quickly: The occurring expressions are not easily computed by hand (this is cumbersome enough for concrete numbers and usually avoided by using Keller’s great applet [Kel06]), moreover it depends in general on the parameters whether the value of the expression is positive or negative. This is crucial as we need to distinguish direct successors and predecessors of a vertex \( k \) as well as vertices not joined with \( k \) when we want to perform a mutation in \( k \). If we want to adapt the mutation concept for “polynomial quivers” without successive distinction of cases, the “polynomial arrows” should have a well-defined direction, equivalently the polynomial expressions replacing the arrow multiplicities should have a well-defined sign, i.e. one independent of concrete values we might want to substitute. This can be guaranteed if all coefficients have a common sign and we restrict the parameters appropriately. These considerations lead to the following definitions.

Definition 12.1. Let \( \mathcal{P} \) be some (multivariate) polynomial ring over an ordered ring \( (R, \geq) \). We introduce a partial order “\( \succ \)” on \( \mathcal{P} \) via \( f \succ g \) if no coefficient of \( f - g \) is negative. Note that this is compatible with \( \geq \) on any non-negative evaluation; in particular it restricts to \( \geq \) on constant polynomials. Moreover, we can calculate as usual, e.g. \( f \succ g \) and \( h \geq 0 \) imply \( fh \succ gh \). From now on, we assume that \( R = \mathbb{Z} \). We call a polynomial \( f \) positive (negative) if \( f \succ 1 \) (\( f \prec -1 \)) and write \( f \succ g \) if \( f - g \) is positive. We define \( |f| \) as \( f \) for \( f \succ 0 \) and \( -f \) for \( f \preceq 0 \). A polynomial is strictly signed if it is positive, negative or zero. Note that \( \succ \) is again compatible with \( > \).

In principle we could permit an arbitrary polynomial ring over an ordered
ring as $\mathbb{Z}$ or even $\mathbb{R}$, but for a start $\mathbb{Z}$ and one “arrow variable” $\alpha_{\{i,j\}} =: \alpha_{ij}$ for any two vertices $i \neq j$ should suffice. Hence we propose the following

**Definition 12.2.** Let $P_0$ be an ordered set (which will be finite for our purposes; hence we set as usual $|P_0| = n$). Then a polynomial quiver $P$ on $P_0$ is given by a tuple of strictly signed polynomials $p_{ij} \in \mathbb{Z}[\alpha_{rs} | r \neq s \text{ in } P_0]$ for all $i < j$ in $P_0$, or equivalently by a skew-symmetric $P_0 \times P_0$ matrix with strictly signed polynomial entries $p_{ij}$. A polynomial quiver is depicted in the obvious way; we draw an arrow $i \xrightarrow{p_{ij}} j$ whenever $p_{ij} \succ 0$. $P_0$ is the set of vertices of $P$; the $p_{ij}$ are the arrow polynomials. All notions of “ordinary quivers” are canonically adopted, e.g. (full) subquivers, successors, sinks, cycles, multiple arrows, abundance etc. (Note that $p_{ij} = \alpha_{ij} + 1$ encodes neither a simple nor a multiple arrow because it can evaluate to both.) Though we have not defined $P_1$ for a polynomial quiver $P$, it is useful to introduce $|P_1| := \sum_{i < j} |p_{ij}|$.

With these definitions, the mutation rule from Definition 1.2 can be applied to polynomial quivers literally, yet the result of a mutation $\mu_k$ will in general not be a polynomial quiver. Indeed, consider the crucial situation $i \in P^-(k)$ and $j \in P^+(k)$. If $p_{ij} \succeq 0$, then $\tilde{p}_{ij} = p_{ik}p_{kj} + p_{ij} \succeq 1$, but $\tilde{p}_{ij}$ is not necessarily strictly signed if $p_{ij} \preceq -1$. So we see that we get a new polynomial quiver (and then say that the mutation of $P$ in $k$ is admissible) if $p_{ik}p_{kj} - p_{ji}$ is strictly signed for all 3-cycles $i \xrightarrow{p_{ik}} k \xrightarrow{p_{kj}} j$.

**Definition 12.3.** We call a polynomial quiver completely mutable if arbitrary iterated mutations are admissible.

Note that the evaluation of a polynomial quiver at non-negative values yields an ordinary quiver with arrows in the same directions. We will call such an evaluation an arrow evaluation, and of course arrow evaluation and mutation commute; notably the constant part of a polynomial quiver (obtained by setting all arrow variables to zero) is mutated in the usual way.

**Remark 12.4.** For concrete examples we will allow ourselves a slight deviation from Definition 12.2 as follows. To keep the occurring expressions short we often use a variable, say $\alpha$, instead of a strictly signed polynomial as $\alpha + 2$. This also has the advantage of the original idea that several cases may be treated together by making further restrictions as $\alpha \geq 2$ only if necessary.

If admissible, $\mu_k$ is clearly again involutive, so we can speak of mutation-equivalence as soon as we have explained what an isomorphism of polynomial quivers should be. But this (let alone the notion of a general morphism) is not completely obvious. We will use the following conventions.
Definition 12.5. We say that polynomial quivers $P$ and $\tilde{P}$ are isomorphic ($P \sim \tilde{P}$) if there are bijections $\sigma : P_0 \to \tilde{P}_0$ and $\varphi : \{\alpha_{ij} \mid i \neq j \text{ in } P_0\} \to \{\alpha_{ij} \mid i \neq j \text{ in } \tilde{P}_0\}$ such that $\tilde{p}_{\sigma(i)\sigma(j)} = \varphi(p_{ij})$ for all $i < j$ in $P_0$, where $\varphi$ is the induced map on the polynomials. We write this as $P \varphi \sigma \sim \tilde{P}$. We speak of a canonical isomorphism and write $P \varphi \sigma \sim \tilde{P}$ if $\varphi$ is induced by $\sigma$, i.e. $\varphi(\alpha_{ij}) = \alpha_{\sigma(i)\sigma(j)}$. Furthermore, when $P_0 = \tilde{P}_0$, we say that $P$ and $\tilde{P}$ are strongly isomorphic ($P \varphi \sigma \sim \tilde{P}$) if $P \varphi \sigma \sim \tilde{P}$. It is easy to see that these are equivalence relations which are compatible with mutation.

Example 12.6. $1 \overset{\alpha_{12}}{\longrightarrow} 2 \overset{\alpha_{23}}{\longrightarrow} 3$ has a non-trivial canonical automorphism induced by the transposition $(1 \ 3)$, but no non-trivial strong automorphism.

Remark 12.7. Note that besides mutation also arrow evaluation is compatible with any isomorphism, yet for both we have to take into account the respective permutations. In particular, in the case $P_0 = \tilde{P}_0$ only a strong isomorphism induces a quiver isomorphism on a common arrow evaluation, but each isomorphism $P \varphi \sigma \sim \tilde{P}$ induces a sequence of isomorphic polynomial quivers $(P^{(m)})_{m \in \mathbb{N}}$ via $p^{(m)}_{\alpha_{ij}} = \varphi(p^{(m-1)}_{ij})$ for all $i < j$ in $P_0$ with $P^{(0)} := P$ (so $P^{(1)} = \tilde{P}$), which satisfies

$$p^{(m)}_{\alpha_{\sigma^m(i)\sigma^m(j)}} = \varphi(p^{(m-1)}_{\sigma^m(i)\sigma^m-1(j)}) = \ldots = \varphi^m(p_{ij}).$$

$\varphi$ has finite order as $P_0$ is finite, so there is some $m$ such that $P \varphi^m \sim P^{(m)}$.

Example 12.8. $1 \overset{\alpha_{12}}{\longrightarrow} 2 \overset{\alpha_{23}}{\longrightarrow} 3 \varphi \sim 1 \overset{\alpha_{13}}{\longrightarrow} 2 \overset{\alpha_{23}}{\longrightarrow} 3$ with $\sigma = (1 \ 3)$ and $\varphi$ induced by $(1 \ 2 \ 3)$ yields the strong isomorphism $1 \overset{\alpha_{12}}{\longrightarrow} 2 \overset{\alpha_{23}}{\longrightarrow} 3 \varphi \sim 1 \overset{\alpha_{23}}{\longrightarrow} 2 \overset{\alpha_{12}}{\longrightarrow} 3$.

Of course we have further analogous definitions:

Definition 12.9. Two polynomial quivers are called (strongly) mutation-equivalent if one can be transformed into a polynomial quiver (strongly) isomorphic to the other by a sequence of admissible mutations. This defines again equivalence relations on the set of all finite polynomial quivers. We have obvious notions of mutation-acyclic and mutation-cyclic polynomial quivers. Given a polynomial quiver $P$, its (strong) mutation class consists of the (strong) isomorphism classes of all quivers (strongly) mutation-equivalent to $P$. Again we get the (strong) mutation graph of $P$ by joining two (strong) isomorphism classes through an edge if corresponding representatives are related by a mutation.
Of course we get a map from the strong mutation graph to the mutation graph by identifying different strong isomorphism classes which build one isomorphism class. Similar to the map from the exchange graph to the mutation graph of a quiver this is not necessarily “nice” (compare the discussion in Chapter 4).

Example 12.10. \( P := \frac{1}{2^{\frac{\alpha_{12} - 2\alpha_{23}}{(1,3)}}} \) has only two neighbours in the mutation graph because \( \mu_1(P) \equiv \mu_3(P) \), but three in the strong mutation graph.

At least we have the following

Lemma 12.11. Any closed path in the mutation graph can be lifted to a closed path in the strong mutation graph.

Proof. Let \( P \stackrel{\sim}{\sim} \mu_k(P) =: P^{(1)} \) for some sequence \( \mu_k \) of admissible mutations. With the notation of Remark 12.7 this implies \( \mu_{\sigma^{-1}(1)}(P^{(m)}) = P^{(m+1)} \), so we get a path from \( P^{(m)} \) to \( P^{(m+1)} \) in the strong mutation graph, and for a suitable \( m \) with \( P \nsim P^{(m)} \) the combined path from \( P \) to \( P^{(m)} \) will be closed. \( \square \)

Furthermore we get a map from the strong mutation graph to the mutation graph of any arrow evaluation because a (global) arrow evaluation is well-defined on the strong isomorphism classes and commutes with mutation.

We will need the following lemma, which basically says that a mutation is admissible when the result coincides with a polynomial quiver on all arrow evaluations.

Lemma 12.12. Let \( P \) and \( \tilde{P} \) be polynomial quivers with \( P_0 = \tilde{P}_0 \) and \( k \in P_0 \). For an arrow evaluation we denote the resulting quivers by \( Q \) and \( \tilde{Q} \). If there is a permutation \( \sigma \) of \( P_0 \) such that for all arrow evaluations \( \mu_k(Q) \nsim \tilde{Q}, \) then the mutation of \( P \) in \( k \) is admissible and \( \mu_k(P) \nsim \tilde{P}. \)

Proof. We have to check that the polynomials \( p'_{ij} \), obtained by mutation in \( k \), are strictly signed. Since mutation and evaluation commute, we know by assumption that \( p'_{ij} \) takes the same values as \( \tilde{p}_{\sigma(i)\sigma(j)} \) for all arrow evaluations:

\[
\begin{array}{ccc}
P & \xrightarrow{\mu_k} & \mu_k(P) \\
\downarrow{ev} & & \downarrow{ev} \\
Q & \xrightarrow{\mu_k} & \mu_k(Q) \\
\end{array} \quad \begin{array}{ccc}
\tilde{P} & \nsim & \tilde{P} \\
\downarrow{ev} & & \downarrow{ev} \\
\tilde{Q} & \nsim & \tilde{Q} \\
\end{array}
\]

But since we have infinitely many positive choices for each arrow variable, both polynomials coincide (see e.g. [Lan65, V, §4]). In particular \( p'_{ij} \) is strictly signed, and the claim follows. \( \square \)
In particular we get the following

**Corollary 12.13.** Suppose we have a polynomial quiver \( P \) and \( \{i, j\} \subseteq P_0 \) such that \( |p_{ij}| = 1 \) and the mutations leading to \( \tilde{P} := \mu_j(\mu_i(P)) \) and \( \tilde{P'} := \mu_i(\mu_j(P)) \) are admissible. Then also \( \mu_i \) is admissible for \( \tilde{P} \) and \( \mu_i(\tilde{P}) \cong \tilde{P'} \).

**Proof.** It is a consequence of Lemma \[7.4\] that \( \mu_i(\tilde{Q}) \cong \tilde{Q'} \) for arbitrary arrow evaluations. Note that the transposition is necessary because the cycle in the exchange graph has odd length. The claim now follows with Lemma \[12.12\]. □

**Remark 12.14.** Of course there is an analogous statement for unconnected vertices, which can be proven in the same way.

Given a polynomial quiver, we can formally proceed as with ordinary quivers and try to construct a “polynomial cluster algebra” by replacing the vertices with variables and adding new “polynomial cluster variables” via the exchange relation whenever we perform an admissible mutation. Let us first consider an example where the mutation itself is trivial.

**Example 12.15.** Consider the polynomial quiver \( 1 \rightarrow \alpha \rightarrow 2 \), which is self-dual and invariant under mutation. Hence all exchange relations have the same form, and starting with the initial variables \( x_1 \) and \( x_2 \) we define \( x_n \) for \( n \geq 3 \) recursively by \( x_n = (x_{n-1}^\alpha + 1)/x_{n-2} \). We obtain

\[
x_3 = \frac{x_2^\alpha + 1}{x_1}, \quad x_4 = \frac{(x_2^\alpha + 1)^\alpha + x_1^\alpha}{x_1^\alpha x_2}, \quad \text{and} \quad x_5 = \frac{((x_2^\alpha + 1)^\alpha + x_1^\alpha)^\alpha + x_1^\alpha x_2^\alpha}{x_1^\alpha x_2^\alpha + 1}.
\]

where we have freely used the normal calculation rules that hold for any arrow evaluation. This presentation is not completely satisfying; we cannot observe the so-called Laurent phenomenon of Fomin and Zelevinsky, which guarantees that each cluster variable is in fact not only a rational function, but rather a Laurent polynomial in the initial cluster variables (see [FZ02]). The problem is that we cannot expand terms like \((x_2^\alpha + 1)^\alpha\) unless we allow ourselves formal sums with “\(\alpha\) summands”, for example

\[
((x_2^\alpha + 1)^\alpha + x_1^\alpha)^\alpha = \sum_{i=0}^{\alpha} \binom{\alpha}{i} (x_2^\alpha + 1)^{\alpha i} x_1^\alpha (\alpha - i)
\]

would allow us to rearrange terms, cancel \(x_2^\alpha + 1\) and write \( x_5 \) as

\[
\sum_{i=1}^{\alpha} \binom{\alpha}{i} (x_2^\alpha + 1)^{\alpha i - 1} x_1^{\alpha (\alpha - i)} + x_1^{\alpha 2}
\]

\[
x_1^\alpha x_2^\alpha + x_1^\alpha x_2.
\]

(It is a nice exercise to write \( x_6 \) as a Laurent polynomial in this way.)
Already this example shows that “polynomial cluster variables” are in general quite unusual expressions. Calculating more complex examples by hand becomes quickly cumbersome, so it would be nice to have a computer algebra system that is able to calculate with (or even simplify) the occurring expressions. It is also not trivial to construct an appropriate ambient ring for “polynomial cluster variables”, because one gets in general powers of sums of powers with polynomials in the exponents with higher and higher nesting levels. So we content ourselves with the following

**Definition 12.16.** Given a polynomial quiver, we build the initial seed by replacing the vertices with corresponding variables. Then we construct further **polynomial seeds** by applying iterated admissible mutations and replacing the mutation vertex with the formal expression obtained via the exchange relation (see Definition 4.1). In this way we get **polynomial clusters** consisting of \( n \) such expressions called **polynomial cluster variables**. We call two polynomial cluster variables equivalent if they represent the same rational function for all arrow evaluations; two polynomial clusters are equivalent if there is a bijection from one to the other such that each polynomial cluster variable is mapped to an equivalent one. (Note that such a bijection is unique because arrow evaluation yields ordinary clusters which consist of algebraically independent elements, see Definition 4.1.) Finally, two polynomial seeds are called isomorphic if they have equivalent polynomial clusters and the corresponding bijection induces a strong isomorphism between the polynomial quivers. Now we can define the **polynomial exchange graph** as in the quiver setting: The vertices are the isomorphism classes of polynomial seeds obtained by admissible mutations from the initial seed, and two vertices are joined by an edge if representatives are related by a mutation.

Again we get a map from the polynomial exchange graph to the strong mutation graph by just forgetting the polynomial clusters and another map from the polynomial exchange graph to the exchange graph of any arrow evaluation. It is probably reasonable to give a summary of the different graphs and the maps between them:

**Corollary 12.17.** Let \( P \) be a polynomial quiver and \( Q \) an arbitrary arrow evaluation of \( P \). By construction we have maps among the associated graphs as shown in the following diagram, where \( ? \) denotes forgetting the clusters:

\[
\begin{array}{c}
\text{polynomial exchange graph of } P \xrightarrow{\text{ev}} \text{graph of } P \xrightarrow{\text{ev}} \text{mutation graph of } P \\
\text{exchange graph of } Q \xrightarrow{?} \text{mutation graph of } Q
\end{array}
\]
Corollary 12.18. The polynomial exchange graph of a completely mutable polynomial quiver with \( n \) vertices is \( n \)-regular.

Example 12.19. Consider the polynomial quiver \( 1 \xrightarrow{\alpha+1} 2 \), which is clearly completely mutable. Its polynomial exchange graph covers the 2-regular tree (for \( \alpha \geq 1 \)) and the pentagon (for \( \alpha = 0 \)).

Also note that the polynomial exchange graph of a constant polynomial quiver is just the ordinary exchange graph. Our definition of the polynomial exchange graph has the advantage that we directly get the following analogue of Lemma 7.4:

Corollary 12.20. Let \( P \) be a polynomial quiver and \( \{i \neq j\} \subset P_0 \) such that all alternating mutations in \( i \) and \( j \) are admissible. Then these mutations yield a cycle in the polynomial exchange graph if and only if \( i \) and \( j \) are either unconnected or joined by a simple arrow; the length of the generated cycle is four in the former and five in the latter case.

Proof. If we first consider only the polynomial quivers, we see that we get isomorphic polynomial quivers after 5 (for \( |p_{ij}| = 1 \)) or 4 (for \( |p_{ij}| = 0 \)) mutations by Corollary 12.13 and Remark 12.14. Now the claim follows from Lemma 7.4 by our construction of the polynomial exchange graph: In the exchange graph of any arrow evaluation we get cycles of the given length in the described cases, so also the polynomial clusters are equivalent after 4 or 5 mutations in these cases and never else.

Remark 12.21. Let us just mention that it follows from the general theory of cluster algebras with coefficients in \([FZ02]\) that the polynomial clusters obtained by going once around one of the described cycles in the polynomial exchange graph are not only equivalent but even equal (for any sensible notion of equality of polynomial cluster variables).

It should not be surprising that we get a notion of a polynomial fork by just substituting in Definition 2.1 “\( > \)” for “\( > \)” (and adding “polynomial” where required) and that all results of Chapter 2 hold accordingly: Again, in the proofs we just have to replace all inequality signs with the corresponding polynomial version. Note that, given the admissibility, the results would already follow from the fact that the constant part of a polynomial fork is a fork. The crucial point is rather that the mutation in any vertex not equal to the point of return is not admissible \textit{a priori}, but that this also follows from the (polynomial version of the) proof of Lemma 2.5. Instead of going through this in detail we will show in Chapter 13 a generalisation which allows also several simple arrows. Let us first list some conclusions and examples (compare Corollary 4.4, Theorem 9.4 and Theorem 10.3).
Corollary 12.22. Let $P$ be a polynomial fork with point of return $r$ such that $\mu_r$ is admissible and $\mu_r(P)$ is again a polynomial fork with point of return $r$. Then $P$ is completely mutable. \hfill $\square$

Since source- and sink-mutations are always admissible, we also get

Corollary 12.23. Each abundant acyclic polynomial quiver is completely mutable. \hfill $\square$

For the next corollary we need a bit of preparation because acyclic polynomial quivers are in general not completely mutable.

Example 12.24. $\alpha^2 + 2\alpha$ is not strictly signed.

This counterexample occurs because the arrow evaluations for $\alpha = 0$ and $\alpha = 1$ are not basically equal; we do not have the trichotomy “zero, one, many” for the polynomial arrow multiplicities. In fact we can even show

Lemma 12.25. Let $P$ be a connected completely mutable polynomial quiver with $|P_0| \geq 3$. Then any two arrow evaluations of $P$ are basically equal.

Proof. Assume this does not hold. Since the arrow polynomials are strictly signed by assumption, the corresponding arrows in the evaluations have the same direction (or vanish if and only if the polynomial is zero). So the problem has to be that some polynomial arrow $1\overrightarrow{u}2$ evaluates to a simple arrow in one arrow evaluation and to a multiple arrow in another. This can only happen if its arrow polynomial $u$ is not constant, but then $u \succcurlyeq 1$ implies that its constant term $u(0)$ must be 1. We will show that this leads to a contradiction similar to Example 12.24.

Since $P$ is connected and $|P_0| \geq 3$, we find a vertex 3 that is joined to 1 or 2. First suppose we have a 3-cycle $1\overrightarrow{u}2\overleftarrow{v}3$. We can assume $v(0) \geq w(0)$ by duality. Mutation in 2 produces the polynomial $v - uw$ which is strictly signed by assumption and satisfies $(v - uw)(0) = v(0) - w(0) \geq 0$, so we get $v - uw \succcurlyeq 0$ and hence an acyclic subquiver containing $u$. By applying source- and sink-mutations if necessary we can assume (up to duality) that it has the form of the top left polynomial quiver in Figure 12.1. A dashed arrow means that the corresponding polynomial is zero or positive; in particular either $v$ or $w$ can be zero, but not both by the choice of vertex 3. We consider the sequence of alternating mutations in 2 and 1 as shown. On the right hand side we have drawn the arrow evaluations obtained by setting
all arrow variables to zero. As all occurring arrow polynomials are strictly signed by assumption, the dashed arrows on the left must have the same direction (or vanish accordingly) as those on the right. In particular we get \((2 - u^2)uw + (1 - u^2)v \geq 0\) in the bottom left polynomial quiver. But as \(u\) was assumed to be non-constant and at least one of \(v\) and \(w\) is positive, the coefficient in front of a term of highest degree in that polynomial is negative. So this polynomial is not strictly signed in contradiction to the assumption.

This result naturally leads to the following

**Definition 12.26.** Let \(P_0\) be a finite ordered set. We construct a basic acyclic polynomial quiver \(P\) as follows: For each pair \(i > j\) we set

\[
p_{ij} := \begin{cases} 
0 & \text{or} \\
1 & \text{or} \\
2 + \alpha_{ij} & 
\end{cases}
\]

Then we have the desired trichotomy, and the proof of Theorem 10.3 yields

**Corollary 12.27.** Each basic acyclic polynomial quiver with 3 vertices is completely mutable.

In fact we believe that Corollaries 12.27 and 12.23 are only special cases of a more general result, which also seems to be supported by computer experiments:

**Conjecture 12.28.** Basic acyclic polynomial quivers are completely mutable.

Note that Conjecture 12.28 is indeed more general than Corollary 12.23: Given an abundant acyclic polynomial quiver \(P\), we can form the corresponding abundant basic acyclic polynomial quiver. If the latter is completely mutable, this means that all polynomials in the arrow variables \(\alpha_{ij}\) occurring in the mutation process are strictly signed. But then the same holds for the polynomials obtained by substituting the non-negative polynomials \(p_{ij} - 2\) for \(\alpha_{ij}\).

We note also that Conjecture 12.28 is stronger than Conjecture 8.16: As there are polynomials with negative coefficients that nevertheless take only positive values, the condition that all occurring polynomials are strictly signed is maybe stronger than necessary for ensuring that the arrow directions and therefore the polynomial quiver mutations are well-defined. On the other hand Lemma 12.25 implies the following
Figure 12.1: The polynomial quiver mutations for Lemma 12.25
**Theorem 12.29.** Conjecture 12.28 implies Conjecture 8.16.

*Proof.* Both conjectures trivially hold for (polynomial) quivers with less than three vertices, so we assume that we deal with connected quivers with at least three vertices. If $Q \equiv Q'$ are acyclic, then they are both arrow evaluations of the same basic acyclic polynomial quiver $P$. But now Conjecture 12.28 would imply that $P$ – and therefore also every polynomial quiver mutation-equivalent to $P$ – satisfies the assumptions of Lemma 12.25 and the claim follows.

In the following example we present a completely mutable polynomial quiver that is neither a fork nor mutation-acyclic.

![Figure 12.2: Another completely mutable polynomial quiver](image)

**Example 12.30.** Consider the polynomial quiver $P$ on the left of Figure 12.2. First observe (ignoring the rest of the figure) that mutation in the sink 3 yields a polynomial quiver canonically isomorphic to the dual $P^{op}$ via the transposition $(1 2)$. In the figure we see that $\mu_4$ is admissible and does the same with the isomorphism induced by $(2 3)$ (a vertical “symmetry axis” might help to see this), and we get a “cycle of length 2” in the mutation graph as shown. Note that the two transpositions combine to the bijection given on the “upper mutation edge”. It is easily checked that the remaining mutations $\mu_1$ and $\mu_2$ are admissible and turn $P$ into a fork as indicated in the figure if $\alpha_{ij} \geq 2$ for all $i \neq j$. Of course this holds analogously for $P^{op}$, so $P$ is completely mutable with only two vertices in the fork-less part of the mutation graph; notably no polynomial quiver in its mutation class is acyclic. Using Lemma 12.11 we see that we get a (proper) cycle of length...
6 in the strong mutation graph because the permutation for the canonical isomorphism has order 3. Hence the strong mutation graph of $P$ (and thus also the mutation graph of any sufficiently general arrow evaluation) looks as in Figure 12.3. This raises at once the question whether this cycle corresponds to a cycle in the exchange graph of some arrow evaluation of $P$, which we answer in the following lemma.

**Lemma 12.31.** The exchange graph of any arrow evaluation of $P$ from Example 12.30 with $\alpha_{ij} \geq 2$ for all $i \neq j$ is a 4-regular tree. In particular the same holds for the polynomial exchange graph of $P$.

**Proof.** It is enough to show that the fork-less part cannot contain a cycle. Consider the initial seed with vertex $i$ replaced by the variable $x_i$ for all $i$ and perform two mutations $\mu_4$ and $\mu_2$ such that the polynomial quiver is again isomorphic to the initial one. This gives the new cluster variables $x_4^* = (x_1^{\alpha_{14}} + x_2^{\alpha_{24}}x_3^{\alpha_{34}})/x_4$ and $x_2^* = (x_1^{\alpha_{12}}x_3^{\alpha_{23}}(x_4^*)^{\alpha_{24}} + 1)/x_2$, and the isomorphism of the polynomial quivers is given by the following bijection of the vertices:

$$\left( \begin{array}{cccc} x_2 & x_1 & x_4 & x_3 \\ x_1 & x_3 & x_4^* & x_2^* \end{array} \right)$$

We will now show that this second cluster is in an appropriate sense “greater” than the initial cluster: If we evaluate the cluster variables such that $1 \leq x_2 < x_1 < x_4 < x_3$, we claim that also $1 \leq x_1 < x_3 < x_4^* < x_2^*$ and notably we have a strictly greater maximum. Indeed, as $\alpha_{34} \geq 2$ and $x_2^{\alpha_{24}} \geq 1$ we get $x_4^* > x_3^2/x_4 > x_3$ by assumption and then $x_2^* > (x_4^*)^2/x_2 > x_4^*$ because $\alpha_{24} \geq 2$ and $x_1^{\alpha_{12}}x_3^{\alpha_{23}} \geq 1$. So we can repeat this argument to show that
further mutations within the fork-less part produce again greater clusters, in particular the sequence of clusters cannot become periodic. (Note the weaker assumptions $\alpha_{ii} \geq 2$ and $\alpha_{ij} \geq 0$ for this.)
Chapter 13

Polynomial semi-forks

We have seen examples of completely mutable polynomial quivers, and the crucial point was to know that as soon as we reach a polynomial fork $P$ via its point of return we need not mutate further because mutations corresponding to a reduced path in the mutation graph are all admissible and never lead back to $P$. We want to generalise this idea, but we have to deal with the situation that a reduced path may be closed. This inspires the following

**Definition 13.1.** Let $P$ be a polynomial quiver. A sequence $(k_1, k_2, \ldots)$ of vertices of $P$ is called reduced if $k_l \neq k_{l+1}$ for all $l \in \mathbb{N}$. As long as the corresponding mutations are admissible, we recursively set $P^{(l)} := \mu_{k_l}(P^{(l-1)})$ with $P^{(0)} := P$. We call a sequence (finite or infinite) admissible if all corresponding mutations are admissible. Let $R \subseteq P_0$ be invariant under all automorphisms of $P$. We call a reduced sequence $(k_1, k_2, \ldots)$ $R$-avoiding if it satisfies the following condition: Whenever $P^{(l)} \sim P$ for some $l \geq 0$, then $\sigma(k_{l+1}) \notin R$. We say that $P$ is $R$-mutable if each $R$-avoiding sequence is admissible. We call $P$ almost completely mutable if there is a vertex $k_0 \in P_0$ fixed under all automorphisms of $P$ such that $P$ is $\{k_0\}$-mutable.

The condition that $R$ in Definition 13.1 is invariant under all automorphisms of $P$ is necessary to ensure that we get a notion for the complete isomorphism class that can therefore be applied to the (strong) mutation graph.

Of course polynomial forks are almost completely mutable. In general we try to avoid sequences of mutations that yield a cycle in the mutation graph, compare the $\bigtriangledown$-reduced paths of Chapter 6. Since we want to allow simple arrows, we have to consider an analogue for the pentagons induced by the latter in the polynomial exchange graph if the respective mutations are admissible (see Corollary 12.20). Let us first introduce a shorter notation for sequences of mutations and write $S_t$ for the sequence $(k_1, k_2, \ldots, k_t)$. $S_0$
denotes the empty and $S_\infty$ an infinite sequence. If we want to concatenate two sequences, we use the operator “◦”, for example $S_{l+1} = S_l \circ (k_{l+1})$. We have chosen to work with sequences of mutations rather than with the induced paths in one of the associated graphs because then the results will apply to all graphs. The reason not to do the same for the ♦-reduced paths lies in the concrete description of the ♦-trees (see Figure 6.1); the greater generality obtained by allowing several simple arrows (instead of just one pair of unconnected vertices) hinders us from giving such a concrete analogous description.

**Definition 13.2.** Given a polynomial quiver $P^{(0)}$, we recursively define ♦-reduced sequences as follows: All reduced sequences $S_l$ with $l \leq 2$ are ♦-reduced. A reduced sequence $S_{l+1}$ with $l \geq 2$ is ♦-reduced if $S_{l-2}$ is admissible, $S_l$ is ♦-reduced, and the following holds: If $|p_{k_{l-1}k_l}| = 1$, then $k_{l+1} \neq k_{l-1}$. In (other) words, whenever we would perform two alternating mutations in the endpoints of a simple arrow, the next mutation does not continue this. Note that we do not assume that the two mutations are admissible – but if they are, then $|p_{k_{l-1}k_l}| = |p_{k_{l-1}k_l}|$ and we can check the condition in $P^{(l-1)}$ or $P^{(l)}$.

This definition is of course motivated by the fact that a ♦-reduced sequence would not walk along three consecutive edges of a pentagon in the polynomial exchange graph generated by a simple arrow. We will show that ♦-reduced sequences will – under certain conditions – nevertheless suffice to reach any vertex in a part of the polynomial exchange graph. The idea is obvious; we take the other way around the pentagon and have a shorter sequence that will be ♦-reduced. But again we do not assume that the corresponding mutations are admissible; this will be a consequence of the conditions.

The other idea needed to ensure that a certain sequence of mutations yields no cycle in the mutation graph is of course again to “increase the number of arrows” – but this causes a problem in connection with simple arrows because the number of arrows may decrease already with the second mutation in the endpoint of a simple arrow; here is the prototypical example:

**Example 13.3.**

\[
\begin{array}{cccccc}
1 & \xrightarrow{a} & 2 & \xrightarrow{b} & 3 & \mu_1 \quad 1 & \xrightarrow{a} & 2 & \xrightarrow{a+b} & 3 & \mu_3 \quad 1 & \xleftarrow{b} & 2 & \xleftarrow{a+b} & 3
\end{array}
\]

In this case we want to compare the arrow numbers of the first and the third quiver. It is useful to do this “element-wise”, but for this we have to interchange some arrow numbers and compare the $a$ arrows from 1 to 2 on the left to the $a + b$ arrows from 2 to 3 on the right. This explains the following
Definition 13.4. Let $P$ and $\tilde{P}$ be polynomial quivers with $|P_0| = |\tilde{P}_0|$. We write $P \preceq \tilde{P}$ or more precisely $P \preceq \tilde{P}$ if there is a bijection $\varphi : \{\{i \neq j\} \subset P_0\} \to \{\{i \neq j\} \subset \tilde{P}_0\}$ such that $|p_{ij}| \preceq |\tilde{p}_{\varphi(ij)}|$. If for at least one polynomial we additionally have $|p_{ij}| \prec |\tilde{p}_{\varphi(ij)}|$, we write $P \prec \tilde{P}$. Clearly $P_1 \prec \tilde{P}_1$ implies $|P_1| \prec |\tilde{P}_1|$ and in particular $P \not\sim \tilde{P}$; furthermore this defines a partial order.

However, the suggested solution entails another problem: Now we want to compare polynomial quivers that are not directly related by a mutation. But how can we guarantee that the desired reference quiver occurs in our sequence? In the example above we might have another vertex $1'$ in the middle quiver playing a similar role as 1, so there would be no obvious “predecessor” or, in fact, no analogue to the point of return for forks. For this reason we exclude simple arrows sharing a common endpoint:

Definition 13.5. Let $P$ be a polynomial quiver. $P$ is almost abundant if $|p_{ij}| \neq 1$ implies $|p_{ij}| \geq 2$ (so we have only simple and multiple arrows) and moreover $|p_{ij}| |p_{jk}| \geq 2$ for all pairwise distinct vertices $i, j$ and $k$ (so no two simple arrows share a common endpoint).

This notion supplies us with the following useful lemma.

Lemma 13.6. Suppose we have a polynomial quiver $P^{(0)}$ and a $\circ$-reduced sequence $S_l$ with $l \geq 2$ and the following properties: $P^{(l-2)}$ is almost abundant and $|p_{k_l-2,k_l}| = 1$. Then also $S_{l-2} \circ (k_l, k_{l-1})$ is $\circ$-reduced.

Proof. Clearly $k_l \neq k_{l-1}$ by assumption, so for $l = 2$ there is nothing to show. Hence assume $l \geq 3$. $k_l = k_{l-2}$ contradicts the assumption that $S_l$ is $\circ$-reduced, so $S_{l-2} \circ (k_l, k_{l-1})$ is reduced. But otherwise $|p_{k_{l-3}}| \prec 1$ since $P^{(l-2)}$ is almost abundant, so $\mu_{k_{l-2}}$ and $\mu_{k_l}$ are not alternating mutations in the endpoints of a simple arrow, and thus $S_{l-2} \circ (k_l, k_{l-1})$ is $\circ$-reduced. \qed

We are now ready to formulate the basic idea. In analogy to the Tree Lemma we start with a suitable almost abundant polynomial quiver $P^{(0)}$ with a “point of return” $k_0$ (or two) and show by induction that all $\circ$-reduced sequences $S_{\infty}$ not starting with a “point of return” are admissible and the
obtained polynomial quivers $P^{(l)}$ with $l \geq 1$ have the following properties generalising those of polynomial forks:

A($l$) For all $i \in (P^{(l)})^-(k_l)$ and $j \in (P^{(l)})^+(k_l)$ we have the following:

A.1($l$) If $p_{ik_l}^{(l)} = 1$ and $p_{kj_l}^{(l)} \geq 2$, then either

A.1.a($l$) $p_{ji}^{(l)} \geq p_{kj_l}^{(l)} + 2$ or

A.1.b($l$) $i = k_{l-1}$ and $p_{kj_l}^{(l)} - 2 \geq p_{ji}^{(l)} \geq 2$

and dually.

A.2($l$) If $p_{ik_l}^{(l)}, p_{kj_l}^{(l)} \geq 2$, then $p_{ji}^{(l)} \succ p_{ik_l}^{(l)}, p_{kj_l}^{(l)}$.

B($l$) $(P^{(l)})^-(k_l)$ and $(P^{(l)})^+(k_l)$ are non-empty and acyclic.

C($l$) $P^{(l)}$ is almost abundant, and we have

C.1($l$) $P_1^{(l)} \succ P_1^{(l-1)}$ or

C.2($l$) $l \geq 2$, $|p_{k_l k_{l-1}}^{(l)}| = 1$ and $P_1^{(l)} \succ P_1^{(l-2)}$.

Conditions A.1.b($l$) and C.2($l$) deal with the situation in which the last mutation has been “the second one in a pentagon”. For the base case $l = 0$ this makes no sense; we could spare us any trouble by demanding that $k_0$ is abundant, but as we wanted to include simple arrows and it would be a bit artificial to make an exception just for $l = 0$, we allow another vertex $k'_0$ with $|p_{k_0 k'_0}| = 1$. This leads to the following

**Definition 13.7.** We call an almost abundant polynomial quiver $P$ a **polynomial semi-fork** if there is a vertex $r$ called a point of return such that

(S1) For all $i \in P^-(r)$ and $j \in P^+(r)$ we have $p_{ji} \geq 1$, and moreover, if $p_{ir}, p_{rj} \geq 2$, then $p_{ji} \succ p_{ir}, p_{rj}$.

(S2) $P^-(r)$ and $P^+(r)$ are non-empty and acyclic.

Let us first note the intended consequence,

**Corollary 13.8.** A polynomial quiver $P^{(l)}$ with a vertex $k_l$ satisfying A($l$), B($l$) and C($l$) is a polynomial semi-fork with point of return $k_l$.

**Proof.** (S1) follows from A($l$), because $P^{(l)}$ is almost abundant (by C($l$)) and hence all cases are covered. (S2) is precisely B($l$).

We also list the properties corresponding to Lemma 2.3.
Lemma 13.9. Let $P$ be a polynomial semi-fork with a point of return $r$.

a. $P - \{r\}$ is acyclic.

b. Both $P - P^-(r)$ and $P - P^+(r)$ are acyclic.

c. For all vertices $i \in P_0$ we have $P^-(i), P^+(i) \neq \emptyset$.

Proof. By (S2) a cycle in $P - \{r\}$ would contain vertices from both $P^-(r)$ and $P^+(r)$ and thus also arrows from $P^-(r)$ to $P^+(r)$ contradicting (S1), which implies a. So each cycle in $P$ contains $r$ and thus also a subquiver $i \rightarrow r \rightarrow j$ with $i \in P^-(r)$ and $j \in P^+(r)$; this shows b. Concerning c, we have $P^-(r), P^+(r) \neq \emptyset$ by (S2). For $i \neq r$ assume (up to duality) $r \in P^+(i)$ and choose some $j \in P^+(r)$; then $j \in P^-(i)$ by (S1).

But note that the point of return is not necessarily unique:

Example 13.10. Consider the polynomial quiver in Figure 13.1. If we assume that all variables are at least two, then it is a polynomial semi-fork with two points of return, namely 1 and 4. Note that both corresponding mutations could actually reduce the number of arrows for an appropriate arrow evaluation. (In fact this polynomial quiver is also completely mutable.)

![Figure 13.1](image.png)

Figure 13.1: A polynomial semi-fork with two points of return.

On the other hand we can show that there cannot be more than two points of return.

Lemma 13.11. Let $P$ be a polynomial semi-fork with point of return $r$, and $r'$ another point of return. Then $|p_{rr'}| = 1$, notably $r'$ is uniquely determined.

Proof. Suppose $|p_{rr'}| \neq 1$. Then (up to duality) $p_{rr'} \gg 2$ as $P$ is almost abundant. Choose a vertex $i \in P^-(r) \neq \emptyset$ by (S2). Then $p_{ri} \gg 1$ by (S1) and as $P$ is almost abundant, we get either $p_{ri} \gg 2$ or $p_{ir} \gg 2$. In both cases the third arrow in the 3-cycle would also have to be a multiple arrow, again by (S1). Now we get a contradiction because (S1) demands a unique maximal multiplicity opposite the point of return.
Moreover any automorphism of a polynomial semi-fork fixes the vertices.

**Lemma 13.12.** Let $P$ be a polynomial semi-fork. Then any polynomial quiver automorphism $\varphi \in \text{Aut}(P)$ is the identity on $P_{0}$.

**Proof.** Let $r$ be the point of return and, if $r$ is not abundant, $r'$ the unique vertex with $|p_{rr'}| = 1$. If there are vertices $i$ and $j$ with $p_{ir}, p_{rj} \geq 2$, then $r$ is uniquely determined by the fact that it lies on a 3-cycle opposite the arrow with maximal multiplicity by property [S1] (recall that $P - \{r\}$ is acyclic by Lemma 13.9.a). This also implies $\varphi(r) = r$. If there are no such vertices, then $r$ is not abundant and we can up to duality assume that $P^{-}(r) = \{r'\}$ since $P$ is almost abundant. Choose some $j \in P^{+}(r)$, then we get a 3-cycle with $p_{rj}, p_{jr'} \geq 2$, and each 3-cycle in $P$ is of that form (again by [S1] and Lemma 13.9.a). Since $\varphi$ preserves (up to a change of variables) the arrow multiplicities of such a 3-cycle and in particular the direction of the simple arrow, we obtain $\varphi(r) = r$ in this case, too. In both cases we can argue with the unique admissible numbering for the almost abundant acyclic polynomial quiver $P - \{r\}$, so $\varphi$ must fix all vertices. \qed

We also note the (almost literal) analogue of Lemma 2.4

**Lemma 13.13.** A full subquiver $\tilde{P}$ of a polynomial semi-fork $P$ with point of return $r$ is either almost abundant acyclic or again a polynomial semi-fork with point of return $r$.

**Proof.** $\tilde{P}$ is clearly almost abundant. If it is not acyclic, it must contain $r$ by Lemma 13.9.a and also some $i \in P^{-}(r)$ and $j \in P^{+}(r)$ by Lemma 13.9.b, but then [S1] and [S2] are obvious. \qed

We will now realise the described idea and prove

**Proposition 13.14.** Let $P^{(0)}$ be a polynomial semi-fork with point of return $k_{0}$ and, if $k_{0}$ is not abundant, $k'_{0}$ the unique vertex with $|p_{k_{0}k'_{0}}| = 1$. Then each $\circ$-reduced sequence $S_{l}$ not starting with $k_{0}$ or $k'_{0}$ is admissible, and the polynomial quivers $P^{(l)}$ with $l \geq 1$ satisfy the conditions [A(l), B(l)] and [C(l)]. In particular each $P^{(l)}$ is a polynomial semi-fork with point of return $k_{l}$.

**Proof.** We proceed by induction on the length $l$ of the sequence $S_{l}$. For $l = 0$ there is nothing to show, so suppose $S_{l}$ is admissible and the polynomial quiver $P^{(l)}$ satisfies the given conditions and is hence also a polynomial semi-fork with point of return $k_{l}$ by Corollary 13.8 if $l \geq 1$. We have to show that $\mu_{k_{l+1}}$ is admissible and that $[A(l + 1), B(l + 1)]$ and $[C(l + 1)]$ hold. We can (up to duality) assume $p_{k_{l}k_{l+1}}^{(l)} \geq 1$ (clearly $k_{l} \neq k_{l+1}$). First we show that $\mu_{k_{l+1}}$ is
admissible, so consider vertices \( i \in (P^{(l)})^-(k_{l+1}) \) and \( j \in (P^{(l)})^+(k_{l+1}) \). As this configuration corresponds precisely to the configuration \( j \in (P^{(l+1)})^-(k_{l+1}) \) and \( i \in (P^{(l+1)})^+(k_{l+1}) \) which we have to analyse for \( A(l + 1) \), we will show both claims together and also note that in all cases but one the “number of arrows strictly increases”. In order to make the notation a bit shorter we will in general use \( p_{jk_{l+1}}^{(i)} \) etc. instead of \( p_{jk_{l+1}}^{(i+1)} \) and not resubstitute it for checking \( A(l + 1) \).

1. If \( p_{ij}^{(l)} \gg 0 \), then \( p_{ij}^{(l+1)} = p_{ij}^{(l)} + p_{ik_{l+1}}^{(l)} p_{k_{l+1}j}^{(l)} \gg p_{ij}^{(l)}, p_{ik_{l+1}}^{(l)}, p_{k_{l+1}j}^{(l)} \) is in particular strictly signed. Also, if one of \( p_{ik_{l+1}}^{(l)} \) and \( p_{k_{l+1}j}^{(l)} \) equals 1, say the latter, then \( p_{ij}^{(l)} \gg 2 \) (as \( P^{(l)} \) is almost abundant) and \( p_{ij}^{(l+1)} = p_{ij}^{(l)} + p_{ik_{l+1}}^{(l)} \gg p_{ik_{l+1}}^{(l)} + 2 \) which is \( A.1.a(l + 1) \) as required by \( A.1(l + 1) \).

2. Otherwise we have a 3-cycle, so by Lemma 13.9.a \( k_l \in \{i, j\} \) and by assumption \( k_l = i \).

2.1. If \( p_{jk_l}^{(l)} = 1 \), then \( p_{k_{l+1}j}^{(l)} \gg 2 \) as \( P^{(l)} \) is almost abundant, and hence \( p_{k_{l+1}j}^{(l+1)} = p_{k_{l+1}j}^{(l)} p_{k_{l+1}j}^{(l)} - 1 \gg p_{k_{l+1}j}^{(l)} p_{k_{l+1}j}^{(l)} + 1 \) with \( A.2(l + 1) \) satisfied.

2.2. Otherwise \( p_{jk_l}^{(l)} \gg 2 \).

2.2.1. In case \( p_{k_{l+1}j}^{(l)} \gg 2 \) as well, (S1) implies \( p_{k_{l+1}j}^{(l)} \gg p_{jk_l}^{(l)}, p_{k_{l+1}j}^{(l)} \). (Notably the case \( p_{k_{l+1}j}^{(l)} = 1 \) for \( A.1(l + 1) \) cannot occur.) So we obtain \( p_{k_{l+1}j}^{(l+1)} = p_{k_{l+1}j}^{(l)} p_{k_{l+1}j}^{(l)} - p_{jk_l}^{(l)} \gg 2 p_{k_{l+1}j}^{(l)} - p_{jk_l}^{(l)} \gg p_{k_{l+1}j}^{(l)} p_{k_{l+1}j}^{(l)} + 2 \) holds again.

2.2.2. Else \( p_{k_{l+1}j}^{(l)} = 1 \), which implies \( l \geq 1 \). So we have either (the dual of) \( A.1.a(l) \) or \( A.1.b(l) \). In the latter case we only need the implication \( k_{l+1} = k_{l-1} \), which is a contradiction to the assumption that \( S_{l+1} \) is \( \sigma \)-reduced if \( l \geq 2 \), and to the assumption that \( k_1 \neq k_0 \) for \( l = 1 \) (because then \( p_{k_{l}k_{l}}^{(l)} = p_{k_{l}k_{l}}^{(l)} = 1 \). In the former case we get that \( p_{k_{l}j}^{(l+1)} = p_{k_{l}j}^{(l)} - p_{jk_l}^{(l)} \gg 2 \) is strictly signed and also \( p_{jk_l}^{(l+1) - 2} = p_{k_{l}j}^{(l)} - 2 \gg p_{k_{l}}^{(l)} \) as required by \( A.1.b(l + 1) \). Note that this is the only case in which \( |p_{ij}^{(l)}| \) may decrease, and it corresponds indeed to the second mutation in the endpoint of a simple arrow.

So \( \mu_{k_{l+1}} \) is admissible, and \( A(l + 1) \) holds. We proceed with \( B(l + 1) \). We see that \( (P^{(l+1)})^-{(k_{l+1})} = (P^{(l)})^+(k_{l+1}) \subset P^{(l)} - \{k_1\} \) is acyclic by Lemma 13.9.a. Moreover \( (P^{(l)})^+(k_{l+1}) \ni (P^{(l)})^-{(k_{l+1})} \neq \emptyset \). Similarly we have \( k_l \in (P^{(l+1)})^+(k_{l+1}) = (P^{(l)})^-{(k_{l+1})} \subset P^{(l)} - (P^{(l)})^-{(k_{l+1})} \), which is acyclic by Lemma 13.9.b.

It remains to show \( C(l + 1) \). Recall that \( |p_{ij}^{(l)}| \) may decrease only in the case 2.2.2, so if \( |p_{k_{l}k_{l+1}}^{(l)}| \neq 1 \), we get \( P_{i}^{(l)} \preceq P_{i}^{(l+1)} \) (see Definition 13.4), then
\(P^{(l+1)}\) is in particular almost abundant. By Lemma 13.9, we even get a strict increase for at least one pair of vertices, so \(P_{1}^{(l)} \prec P_{1}^{(l+1)}\) and hence \([l + 1]\). In the case \(p_{k_{l+1} l}^{(l)} = 1\) (so \(l \geq 1\)) we have to compare \(P^{(l-1)}\) and \(P^{(l+1)}\). First note that \(P_{1}^{(l-1)} \prec P_{1}^{(l)}\) since otherwise \(|p_{k_{l-1} l}^{(l)}| = 1\) by \([l + 1]\) and thus \(k_{l+1} = k_{l-1}\) (as \(P^{(l)}\) is almost abundant), which gives a contradiction (see 2.2.2). Next we analyse the crucial configuration \(j \leftarrow p_{k_{l}}^{(l)} k_{l} \rightarrow k_{l+1}\) of 2.2.2. By (the dual of) A.1.a we have \(p_{k_{l+1} l}^{(l)} \succ p_{j_{l}}^{(l)}\), so in \(P^{(l-1)}\) the situation looks as \(j \leftarrow p_{j_{l}}^{(l)} k_{l} \rightarrow k_{l+1}\) with \(p_{k_{l+1} l}^{(l-1)} = p_{k_{l+1} j_{l}}^{(l)} - p_{j_{l}}^{(l)}\) and \(p_{j_{l}}^{(l)}\). For all other pairs of vertices we get \(|p_{j_{l}}^{(l-1)}| \prec |p_{j_{l}}^{(l)}| \prec |p_{j_{l}}^{(l+1)}|\), so \(P_{1}^{(l-1)} \prec P_{1}^{(l+1)}\) as desired. Since \(|p_{j_{l}}^{(l-1)}| = 1\) implies \(\varphi\{i, j\} = \{i, j\}\) or, in other words, only multiple arrows are “switched”, \(P^{(l+1)}\) is also almost abundant, and the proof is finished.

The following lemma shows that considering only \(\circ\)-reduced sequences is no real restriction and also not essential for admissibility.

**Lemma 13.15.** Under the assumptions of Proposition 13.14 consider a \(\circ\)-reduced sequence \(S_{l}\) (\(l \geq 0\)) not starting with \(k_{0}\) or \(k_{0}'\) and a reduced continuation \(\tilde{S}_{l+1}\) that is not \(\circ\)-reduced. Then \(\tilde{S}_{l+1}\) is admissible, and \(P^{(l+1)}\) can be reached via the shorter sequence \(S_{l-2} \circ (k_{l}, k_{l-1})\), which is \(\circ\)-reduced and does not start with \(k_{0}\) or \(k_{0}'\), i.e. \(\mu_{k_{l-1}}(\mu_{k_{l}}(P^{(l-2)})) \cong P^{(l+1)}\).

\[
\begin{align*}
\mu_{k_{l-1}} & \quad P^{(l-2)} \quad \mu_{k_{l}}(P^{(l-2)}) \quad \mu_{k_{l-1}} \\
\mu_{k_{l}} & \quad P^{(l)} \quad P^{(l)} \quad P^{(l)} \quad P^{(l+1)}
\end{align*}
\]

**Proof.** Since \(\tilde{S}_{l+1}\) is reduced, but not \(\circ\)-reduced, we have \(l \geq 2\), \(k_{l+1} = k_{l-1}\) and \(|p_{k_{l+1} l}^{(l-2)}| = 1\). As \(S_{l}\) does not start with \(k_{0}\) or \(k_{0}'\), it is in particular admissible by Proposition 13.14. Now we can apply Lemma 13.6 and see that...
$S_{l-2} \circ (k_l, k_{l-1})$ is \( \circ \)-reduced. Note that this sequence does not start with $k_0$ or $k'_0$; this is obvious for $l \geq 3$, and if $l = 2$, then $k_2 \in \{k_0, k'_0\}$ would imply $k_1 \in \{k_0, k'_0\}$ by $|p_{k_1k_2}^{(0)}| = 1$, a contradiction. Hence $S_{l-2} \circ (k_l, k_{l-1})$ is also admissible by Proposition \ref{prop:adm}. Now Corollary \ref{cor:adm} shows that $\mu_{k_{l-1}}$ is admissible for $P^{(l)}$ and the obtained polynomial quiver $P^{(l+1)}$ satisfies $\mu_{k_{l-1}}(\mu_{k_l}(P^{(l-2)})) \simeq P^{(l+1)}$.

Next we want to show in analogy to Lemma \ref{lem:inc} that \( \circ \)-reduced sequences bring us further and further away from $P^{(0)}$. In fact this is true in any of the graphs associated with $P^{(0)}$ with respect to the appropriate notion of mutation distance: Whenever two polynomial quivers or polynomial seeds are (strongly) mutation-equivalent, we define their mutation distance to be the minimal number of mutations needed to mutate one to a polynomial quiver or polynomial seed (strongly) isomorphic to the other. (So if we say that we reach or obtain a polynomial quiver or polynomial seed via a sequence of mutations we actually mean its (strong) isomorphism class.) We will be interested in the mutation distance $d(-)$ to the fixed polynomial quiver $P^{(0)}$. Of course we have the triangle inequality, which we use for example in the form $d(\mu_k(P)) \leq d(P) + 1$ or equivalently $d(\mu_k(P)) \geq d(P) - 1$.

**Lemma 13.16.** Under the assumptions of Proposition \ref{prop:adm} the mutation distance $d(-)$ strictly increases along each \( \circ \)-reduced sequence $S_l$ not starting with $k_0$ or $k'_0$; moreover $P^{(l)} \not\simeq P^{(0)}$ for the corresponding polynomial quivers with $l \geq 1$.

*Proof.* Again we use induction on the length $l$ of the \( \circ \)-reduced sequence. The polynomial quivers $P^{(l)}$ satisfy $A(l)$, $B(l)$ and $C(l)$ by Proposition \ref{prop:adm}.

First note that $C(l)$ inductively implies $P^{(0)} \not\simeq P^{(l)}$ for all $l \geq 1$; in particular a \( \circ \)-reduced sequence can never return to $P^{(0)}$, which notably shows the claim for $l = 1$. So suppose that all \( \circ \)-reduced sequences of length $l \geq 1$ satisfy the claim, and consider a \( \circ \)-reduced sequence $S_{l+1}$. We get $d(P^{(m)}) = m$ for all $m \leq l$ and therefore $d(P^{(l+1) - 1}) \geq d(P^{(l)}) - 1 = l - 1$.

First we show $d(P^{(l+1)}) \geq l$. Suppose this is not the case, so $d(P^{(l+1)}) = l - 1$. This means that there are vertices $k_{l+2}, \ldots, k_{l+2}$ such that $P^{(2l)} \sim P^{(0)}$ and $d(P^{(l+m)}) = l - m$ for $1 \leq m \leq l$. The resulting sequence $S_{l+2}$ is therefore not \( \circ \)-reduced (in particular $l \geq 2$), but clearly reduced. Suppose $S_{l+2}$ is not \( \circ \)-reduced. Then we get $\mu_{k_l}(\mu_{k_{l+1}}(P^{(l-1)})) \simeq P^{(l+2)}$ by Lemma \ref{lem:admissible}.

$$
\begin{align*}
&\xymatrix{ & P^{(l-1)} \ar@{-}[dr]_{\mu_{k_l}} & \mu_{k_{l+1}}(P^{(l-1)}) \ar@{-}[dl]_{\mu_{k_l}} & P^{(l+1)} \ar@{-}[dl]_{\mu_{k_{l+1}}} & P^{(l+2)} \ar@{-}[dl]_{\mu_{k_{l+2}}} \\
\mu_{k_{l+1}} & P^{(l)} & \mu_{k_{l+1}}(P^{(l-1)}) & P^{(l+1)} & P^{(l+2)} \\
P^{(l-2)} & P^{(l-1)} & P^{(l)} & P^{(l+1)} & P^{(l+2)}}
\end{align*}
$$
But this implies \( d(\mu_{k+1}(P^{(l-1)})) \leq d(P^{(l+2)}) + 1 = l - 1 \), which contradicts the induction hypothesis as \( S_{l-1} \circ (k_{l+1}) \) is \( \bigcirc \)-reduced by Lemma 13.15.

So \( S_{l+2} \) is \( \bigcirc \)-reduced, and the problem has to occur later. At the first point where the sequence is not \( \bigcirc \)-reduced we can shorten it by Lemma 13.15 (as illustrated for \( S_{l+3} \)), and therefore \( d(P^{(l)}) \leq l - 1 \), which is again nonsense.

II Now suppose that \( d(P^{(l+1)}) = l \). Then we find vertices \( k_{l+2}, \ldots, k_{2l+1} \) such that \( P^{(2l+1)} \sim P^{(0)} \) and \( d(P^{(l+m)}) = l + 1 - m \) for \( 1 \leq m \leq l + 1 \). Again, \( S_{2l+1} \) cannot be \( \bigcirc \)-reduced, but is reduced. If \( S_{l+2} \) is not \( \bigcirc \)-reduced, Lemma 13.15 implies that \( \mu_k(\mu_{k+1}(P^{(l-1)})) \cong P^{(l+2)} \) and we can apply II to \( S_{l-1} \circ (k_{l+1}, k_l) \) to get the contradiction \( d(P^{(l+2)}) \geq l \).

If \( S_{l+3} \) is not \( \bigcirc \)-reduced, Lemma 13.15 shows \( \mu_{k_{l+2}}(\mu_{k_{l+3}}(P^{(l)})) \cong P^{(l+3)} \) and \( S_l \circ (k_{l+2}, k_{l+1}) \) is \( \bigcirc \)-reduced. This yields the estimate \( d(\mu_{k_{l+2}}(P^{(l)})) \leq d(P^{(l+3)}) + 1 = l - 1 \), which contradicts II again.

As in II we see that the problem cannot occur later because then \( d(P^{(l+1)}) \) would be smaller than assumed. So \( d(P^{(l+1)}) = l + 1 \).

We can now also describe the role of the “forbidden” mutations.

**Corollary 13.17.** Under the assumptions of Proposition 13.14 let \( S_l \) with \( l \geq 1 \) be a \( \bigcirc \)-reduced sequence not starting with \( k_0 \) or \( k'_0 \). Then an arbitrary continuation \( S_{l+1} \) is admissible, and \( P^{(l+1)} \) can be reached via a \( \bigcirc \)-reduced sequence. Moreover, precisely one of the possible continuations yields a
polynomial quiver with shorter distance to $P^{(0)}$ than $P^{(l)}$, at most one yields a polynomial quiver with the same distance to $P^{(0)}$, and at least $|P_0^{(0)}| - 2$ yield polynomial quivers with greater distance to $P^{(0)}$.

Proof. At least $|P_0^{(0)}| - 2$ of the possible continuations are $\mathcal{O}$-reduced; they are admissible by Proposition 13.14 and hence yield polynomial quivers with greater distance to $P^{(0)}$ by Lemma 13.16. Of the at most two remaining continuations, one is of course not reduced (hence trivially admissible) and yields the polynomial quiver $P^{(l-1)}$ reachable via the $\mathcal{O}$-reduced sequence $S_{l-1}$ with shorter distance to $P^{(0)}$ again by Lemma 13.16. If there remains another continuation $S_{l+1}$ that is not $\mathcal{O}$-reduced, Lemma 13.15 implies that it is admissible and, moreover, $S_{l-2} \circ (k_l, k_{l-1})$ of length $l$ is $\mathcal{O}$-reduced, does not start with $k_0$ or $k'_0$, and leads to $P^{(l+1)}$. So Lemma 13.16 gives $d(P^{(l+1)}) = l = d(P^{(l)})$.

This immediately implies that for each such polynomial quiver $P^{(l)}$ obtained from $P^{(0)}$ by a $\mathcal{O}$-reduced sequence $S_l$ we get a unique shortest mutation sequence leading back to $P^{(0)}$. However, a priori this does not imply that the $\mathcal{O}$-reduced sequence $S_l$ is also unique; already distinct mutations of $P^{(0)}$ might lead to the same (strong) isomorphism class in the (strong) mutation graph. Of course this cannot happen in the polynomial exchange graph, but we can use Lemma 13.12 to show this stronger statement for all settings.

Lemma 13.18. Under the assumptions of Proposition 13.14 let $P$ and $\tilde{P}$ be two polynomial quivers obtained by mutations along $\mathcal{O}$-reduced sequences not starting with $k_0$ or $k'_0$. Then $P \sim \tilde{P}$ implies that both sequences are equal.

Proof. First note that $P \sim \tilde{P}$ implies $d(P) = d(\tilde{P})$. Indeed, assume without loss of generality $d(P) \leq d(\tilde{P})$; if we had $d(P) < d(\tilde{P})$ we could apply the mutation sequence of length $d(P)$ via the isomorphism to $\tilde{P}$ and get a polynomial quiver isomorphic to $P^{(0)}$ in less than $d(\tilde{P})$ steps, which is nonsense. If $d(P) = 0$, the empty sequence is the only possible sequence as the distance strictly increases along the sequences under consideration by Lemma 13.16.

Now suppose we have a $\mathcal{O}$-reduced sequence $S_l$ ($l \geq 0$) not starting with $k_0$ or $k'_0$ and two continuations $S_l \circ (k_{l+1})$ and $S_l \circ (\tilde{k}_{l+1})$ with the same properties. We set as usual $P^{(l+1)} = \mu_{k_{l+1}}(P^{(0)})$ and $\tilde{P}^{(l+1)} = \mu_{\tilde{k}_{l+1}}(P^{(0)})$. We claim that $P^{(l+1)} \sim \tilde{P}^{(l+1)}$ implies $k_{l+1} = \tilde{k}_{l+1}$. Clearly $\tilde{k}_{l+1}$ is the unique vertex of $\tilde{P}^{(l+1)}$ that allows a distance-decreasing mutation (see Corollary 13.17), but $\sigma(k_{l+1})$ has the same property, hence $\sigma(k_{l+1}) = \tilde{k}_{l+1}$. Therefore we get an induced automorphism $P^{(0)} = \mu_{k_{l+1}}(P^{(l+1)}) \sim \mu_{\tilde{k}_{l+1}}(\tilde{P}^{(l+1)}) = \tilde{P}^{(0)}$, which fixes the vertices by Lemma 13.12. So $k_{l+1} = \tilde{k}_{l+1}$ as claimed.
Finally assume \( d(P) > 0 \) and let \( k \in P_0 \) be the unique vertex such that \( d(\mu_k(P)) < d(P) \). Let \( \hat{k} \) be the image of \( k \) under the isomorphism from \( P \) to \( \hat{P} \). Then of course \( \mu_k(P) \sim \mu_k(\hat{P}) \), and by induction on the (common) distance we get a unique \( \circ \)-reduced sequence \( S_l \) (\( l \geq 0 \)) not starting with \( k_0 \) or \( k'_0 \) such that \( P(l) \sim \mu_k(P) \sim \mu_k(\hat{P}) \). The considerations above show that then also the \( \circ \)-reduced sequences leading to \( P \) and \( \hat{P} \) are the same.

We summarise the results about polynomial semi-forks in the following

**Theorem 13.19.** Let \( P^{(0)} \) be a polynomial semi-fork with point of return \( k_0 \) and, if \( k_0 \) is not abundant, \( k'_0 \) the unique vertex with \( |p_{kk'_0}| = 1 \). We set \( R := \{k_0, k'_0\} \). Then \( P^{(0)} \) is \( R \)-mutable, i.e. each \( R \)-avoiding \( \circ \)-reduced sequence is admissible. For any polynomial quiver obtained by mutations along such a sequence there is a unique \( R \)-avoiding \( \circ \)-reduced sequence leading to it.

*Proof.* We use induction on the length \( l \) of the \( R \)-avoiding sequences. For \( l = 0 \) there is nothing to show. So suppose the claim holds for all \( R \)-avoiding sequences of length \( l \) and consider one of length \( l+1 \), say \( S_{l+1} \). By the induction hypothesis \( S_l \) is admissible, and there is a unique \( R \)-avoiding \( \circ \)-reduced sequence leading to \( P(l) \). If \( P(l) \sim P^{(0)} \), then \( S_{l+1} \) is admissible because it is \( R \)-avoiding, and \( P^{(l+1)} \) can be reached by a single mutation (and hence a \( \circ \)-reduced sequence) which is unique by Lemma 13.18. Otherwise \( d(P(l)) > 0 \), and Corollary 13.17 implies that \( S_{l+1} \) is admissible and \( P^{(l+1)} \) can be reached by a \( \circ \)-reduced sequence which is again unique by Lemma 13.18.

We state several consequences.

**Corollary 13.20.** Let \( P \) be a polynomial semi-fork with point of return \( r \) and, if \( r \) is not abundant, \( r' \) the unique vertex with \( |p_{rr'}| = 1 \). Let \( \Gamma \) be the polynomial exchange graph of \( P \). If \( \mu_r \) or \( \mu_{r'} \) are admissible, delete the corresponding edges, and denote by \( \Gamma' \) the connected component containing \( P \) in the resulting graph. Then \( \Gamma' \) has the following properties: All polynomial quivers in \( \Gamma' \) are polynomial semi-forks, each of them except \( P \) has \( |P_0| \) neighbours, and the edges corresponding to mutations in \( \circ \)-reduced sequences not starting with \( r \) or \( r' \) form a maximal tree in \( \Gamma' \). In particular the fundamental group of \( \Gamma' \) is freely generated by the pentagons induced by simple arrows.

*Proof.* Any vertex in \( \Gamma' \) can by definition be reached via an \( R \)-avoiding sequence (with \( R := \{r, r'\} \)) and hence also via an additionally \( \circ \)-reduced sequence by Theorem 13.19. Therefore it is a polynomial semi-fork by Proposition 13.14 and every such vertex except \( P \) has \( |P_0| \) neighbours by Corollary 13.17. The \( \circ \)-reduced sequences induce a tree because there is a unique sequence leading to a given vertex in \( \Gamma' \), and this tree is maximal.
because all vertices can be reached. Clearly each missing edge belongs to a pentagon, so the last claim follows with Theorem 6.8.

**Corollary 13.21.** A polynomial semi-fork \(P\) with abundant point of return \(r\) is almost completely mutable. Furthermore, if \(\mu_r\) is admissible, the edge between \(P\) and \(\mu_r(P)\) in the polynomial exchange graph is a bridge.

**Proof.** The first claim follows at once from Theorem 13.19; the second from Corollaries 4.3 and 13.20 because \(P\) has \(|P_0| - 1\) neighbours except \(\mu_r(P)\).

A particular application finishes the treatment of pre-forks as promised in Remark 6.10. Recall from Definition 3.7 that a pre-fork with point of return \(r\) is a quiver \(Q\) with two vertices \(k' \neq k\) such that \(Q - \{k'\}\) and \(Q - \{k\}\) are both forks with common point of return \(r\) and, moreover, the set \(k'Qk\) of stopovers is empty. We have seen in Lemma 6.1 that such a pre-fork with \(|q_{kk'}| \geq 2\) is actually a fork with point of return \(r\). The case \(|q_{kk'}| = 0\) yields a \(\diamond\)-root and is treated in Lemma 6.7. The remaining case is now a special case of a semi-fork, i.e. a constant polynomial semi-fork.

**Lemma 13.22.** A pre-fork \(Q\) as above with \(|q_{kk'}| \geq 1\) and point of return \(r\) is a semi-fork with abundant point of return \(r\).

**Proof.** Actually the proof is almost literally identical with that of Lemma 6.1. Just note that \(r\) is abundant by assumption, and so condition (S1) is equivalent to (F1). Moreover, \(Q\) is clearly almost abundant.

Another consequence further supports Conjecture 12.28. In contrast to Lemma 2.5, we have not allowed an acyclic polynomial quiver \(P^{(0)}\) in Proposition 13.14 although the first step with \(k_1\) being neither source nor sink would have been also covered by case 1 of the proof. However, if \(k_1\) is not abundant, we would be left with a mutation whose admissibility is not covered by Proposition 13.14. Luckily we can avoid analysing this situation by applying a simple trick and obtain

**Theorem 13.23.** Each almost abundant acyclic polynomial quiver \(P\) is completely mutable; notably Conjectures 12.28 and 8.16 hold in this case.

**Proof.** We embed \(P\) in a polynomial semi-fork \(\tilde{P}\) by adding two vertices \(r\) and \(r'\) and setting \(\tilde{p}_{rr} := 2\), \(\tilde{p}_{rr} := 2\) and \(\tilde{p}_{ri} := 3\) for all \(i \in P_0\). It is easily checked that \(\tilde{P}\) is indeed a polynomial semi-fork with point of return \(r\), hence it is \(\{r\}\)-mutable. In particular all mutation sequences with only vertices from \(P\) are admissible. As those mutations commute with deleting \(r\) and \(r'\) by (the polynomial version of) Lemma 1.4, \(P\) is completely mutable as required by Conjecture 12.28. Hence the last claim follows by Theorem 12.29.
Corollary 13.24. Let \( P^{(0)} \) be almost abundant acyclic and \( S_l \) an arbitrary \( \circ \)-reduced sequence. Then \( P^{(l)} \) is either almost abundant acyclic or a polynomial semi-fork with point of return \( k_l \).

Proof. For \( l = 0 \) there is nothing to show, so suppose \( l \geq 1 \). As in the proof of Theorem 13.23 we embed \( P^{(0)} \) in a polynomial semi-fork \( \tilde{P}^{(0)} \) by adding two vertices \( r, r' \). Then \( S_l \) becomes a \( \circ \)-reduced sequence not starting with \( r \) or \( r' \), so \( P^{(l)} \) is a polynomial semi-fork with point of return \( k_l \) by Proposition 13.14. Again we use (the polynomial version of) Lemma 1.4 to see that \( P^{(l)} \) is a full polynomial subquiver of \( \tilde{P}^{(l)} \), hence the claim follows by Lemma 13.13.

Theorem 13.25. Let \( P \) be an almost abundant acyclic quiver (!). Then the fundamental group of its exchange graph \( \Gamma(P) \) is generated by pentagons.

Proof. If \( |P_0| \leq 2 \), then \( P \) is either abundant and its exchange graph a tree by Theorem 9.4, or just a simple arrow; then the exchange graph is a pentagon by Lemma 7.4. So suppose \( |P_0| \geq 3 \), which implies that \( P \) is representation-infinite. Hence source- and sink-mutations yield an infinite chain of canonical transjectives by Lemma 9.1. Note that this is even unique as \( P \) has a unique admissible numbering. As in the proof of Theorem 10.3 we start to construct \( \Gamma(P) \) from this “spine” of almost abundant acyclic seeds.

We need to analyse what happens when we mutate in a vertex \( r \) of (say) \( P \) that is neither source nor sink. This generates 3-cycles, so \( \mu_r(P) \) is not acyclic and thus a semi-fork with point of return \( r \) by Corollary 13.24. Now there are basically two cases.

If \( r \) is abundant, then the edge between \( P \) and \( \mu_r(P) \) is a bridge by Corollary 13.21. Furthermore we know from Corollary 13.20 that the part of \( \Gamma(P) \) “beyond” that bridge consists of polynomial semi-forks and has a maximal tree induced by the \( \circ \)-reduced sequences starting in \( \mu_r(P) \) but not with \( r \). It follows that each vertex in this part has a unique shortest path to an acyclic seed (namely \( P \)); of course the same holds analogously for all other canonical transjectives.

If \( r \) is not abundant, there is a unique vertex \( r' \) with \( |p_{rr'}| = 1 \), and we set \( R := \{r, r'\} \). In this case the edge between \( P \) and \( \mu_r(P) \) is part of a unique pentagon, induced by the simple arrow between \( r \) and \( r' \), which we add to the spine of acyclic seeds. For simplicity we will again keep the notation of quivers rather than that of seeds; so the mutations corresponding to the edges of the pentagon are those in \( r \) and \( r' \) for any of the seeds on the pentagon though the corresponding cluster variables of \( P \) are exchanged. (This will cause at least less confusion than starting to introduce the exchanged variables.) There are different possibilities for the number of canonical transjectives on such a pentagon (one, two or three), but we can treat them all with one argument.
Namely, any seed on the pentagon that is not canonical transjective can be reached by at most two mutations in $r$ and $r'$ from a seed lying on the spine. We claim that it is therefore a polynomial semi-fork with point of return in $R$.

We can again assume without loss of generality that we start from $P$ and the first mutation is in $r$. Then it is easy to see that neither $\mu_r(P)$ nor $\mu_{r'}(\mu_r(P))$ can be acyclic. Indeed, as $|P_0| \geq 3$ and $r$ is neither source nor sink, $P$ has (up to duality) a full subquiver of the form $\begin{tikzcd} k \arrow[r,b] \& b \arrow[r,k] \& a \end{tikzcd}$ with $a, b > 0$. Then the considered mutations yield $\begin{tikzcd} r \arrow[r,a+b] \& k \arrow[r,b] \& a \end{tikzcd}$, so the corresponding seeds are not acyclic. Hence they are semi-forks with point of return in $R$ by Corollary 13.24 as claimed.

In the next step we analyse what happens if we leave the considered pentagon from such a semi-fork. By construction the mutations in $R$ correspond to mutations along the pentagon, so we can only leave the pentagon from one of the semi-forks with an $R$-avoiding sequence. In this situation we can once more invoke Corollary 13.20, which tells us that any seed we reach by such a sequence is again a semi-fork that can be reached by a unique ($R$-avoiding) $\circ$-reduced sequence of mutations. In particular, with such a sequence we can never return to the spine of transjectives, which in turn implies that a semi-fork on this pentagon cannot lie on another pentagon that shares a vertex with the spine. Hence, if we add all these pentagons, all the added semi-forks are pairwise distinct and thus belong to a unique added pentagon each.

Finally, from each added pentagon we choose precisely one edge that is not contained in the spine. If we delete these edges, each semi-fork in one of the added pentagons is now reachable by a unique shortest path from one of the canonical transjectives. As we have just seen, all other vertices of the exchange graph are also semi-forks with a unique shortest path – given by a $\circ$-reduced sequence – to one of the semi-forks on the pentagons. Altogether, taking the edges on the spine, the remaining edges on the added pentagons, the edges corresponding to mutations along a $\circ$-reduced sequence leaving one of the added pentagons from a semi-fork, and, not to be forgotten, the bridges of the first case together with the edges corresponding to the $\circ$-reduced sequences continuing beyond the bridge, we get a maximal tree in $\Gamma(P)$, since an arbitrary vertex can now be reached by a unique reduced path from a fixed canonical transjective. By construction all edges not contained in this maximal tree belong to a pentagon, so the claim of the theorem follows with Theorem 6.8.

So we finally get a considerable extension of Corollary 9.6 and the partial answer to Happel’s question 8.9 for almost abundant acyclic quivers.
Corollary 13.26. Let \( Q \equiv Q' \) be almost abundant acyclic \( n \)-point-quivers. Then \( \Delta(Q) \cong \Delta(Q') \), \( \Sigma(Q) \cong \Sigma(Q') \) and \( \mathcal{K}_{KQ} \cong \mathcal{K}_{KQ'} \). Notably there is a canonical bijection between the sets of isomorphism classes of exceptional modules over \( KQ \) and \( KQ' \).

Proof. This follows from Corollary 8.14 with Theorems 13.23 and 13.25.

We summarise the classes of completely mutable polynomial quiver in the following

Theorem 13.27. A polynomial quiver \( P \) is completely mutable if it is almost abundant acyclic, basic acyclic with 3 vertices, or a polynomial fork with point of return \( r \) such that \( \mu_r(P) \) is again a polynomial fork with point of return \( r \).

Proof. This is Theorem 13.23, Corollary 12.27 and Corollary 12.22.
Bibliography


Theses

This thesis deals with the question of Dieter Happel whether finite acyclic quivers that differ at most in the multiplicities of their multiple arrows have isomorphic associated structures on the level of representation theory. We call such quivers basically equal. We take a combinatorial approach to this question via the link between representation theory and cluster combinatorics; specifically we study the behaviour of certain quivers under quiver mutation and derive consequences for the associated structures, in particular the exchange graph and the cluster complex.

1. We define a certain class of quivers called forks and show that their behaviour under quiver mutation allows us to describe large parts of the exchange graph as trees that consist of forks only.

2. We show that forks occur in all exchange graphs of mutation-infinite quivers, furthermore an arbitrary vertex in such an exchange graph has at most distance $D(n)$ from a fork for some bound $D(n)$ that only depends on the number $n$ of vertices of the quiver. This implies that most of the vertices in the exchange graph are forks and one has a good picture of its global structure.

3. Transferred to the representation-theoretic side, the given construction for obtaining forks yields a confirmation of a conjecture by Unger in all but finitely many cases.

4. We state some (together) sufficient conditions for a partial positive answer to Happel’s question. One of them is that basically equal acyclic quivers remain basically equal under mutation.

5. We give examples of quivers for which the results about forks lead to a complete description of the exchange graph and show that this yields a classification that supports a positive answer to Happel’s question.
6. We extend the described classification of the exchange graphs to the cluster complexes by showing the sufficient conditions and derive partial positive answers to Happel’s question.

7. We observe a possible reason for the behaviour of basically equal acyclic quivers under mutation and suggest to express it in terms of “polynomial quivers” in which we have arrow polynomials instead of arrow numbers. We give precise definitions and examples of polynomial quivers.

8. We introduce a class of polynomial quivers that generalises the class of forks. By refining our previous arguments we obtain similar classification results and can give our partial positive answer to Happel’s question for this larger class.
Erklärungen gemäß §6 der Promotionsordnung

Hiermit erkläre ich an Eides Statt, dass ich die von mir eingereichte Arbeit „Exchange Graphs via Quiver Mutation“

• selbständig und nur unter Benutzung der in der Arbeit angegebenen Quellen und Hilfsmittel angefertigt habe.

• in dieser oder ähnlicher Form an keiner anderen Stelle zum Zwecke eines Promotionsverfahrens vorgelegt habe.

Hiermit erkläre ich an Eides Statt, dass ich keine weiteren Promotionsverfahren bei anderen Stellen beantragt hatte bzw. beantragt habe.

Chemnitz, den 8. Januar 2014