Moving-Average approximations of random \(\varepsilon\)-correlated processes

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Abstract

The paper considers approximations of time-continuous \(\varepsilon\)-correlated random processes by interpolation of time-discrete Moving-Average processes. These approximations are helpful for Monte-Carlo simulations of the response of systems containing random parameters described by \(\varepsilon\)-correlated processes. The paper focuses on the approximation of stationary \(\varepsilon\)-correlated processes with a prescribed correlation function. Numerical results are presented.

*Keywords:* \(\varepsilon\)-correlated process, Moving-Average process, Monte-Carlo simulation, integral functional

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1 Introduction

This paper is devoted to the approximation of time-continuous $\varepsilon$-correlated random processes by interpolation of time-discrete Moving-Average processes. The essential property of $\varepsilon$-correlated processes is that the correlation of the random function does not reach far. That means, the values of the process at two points are not correlated if the distance of the two points exceeds a certain quantity $\varepsilon > 0$. The quantity $\varepsilon$ is denoted as correlation length. In applications $\varepsilon$ is always assumed to be sufficiently small. Hence, $\varepsilon$-correlated processes can also be characterized as processes without "distant effect". In contrast to the white noise model which is frequently used in stochastic analysis, $\varepsilon$-correlated processes may possess trajectories of arbitrary smoothness.

In engineering applications $\varepsilon$-correlated processes can be used to describe the random behaviour of fluctuating external loads or heterogeneous material properties. Typically, the system dynamics is mathematically described by some differential equation containing random parameters and the problem is to determine the statistical properties of the system response to prescribed statistical properties of the random parameters modelled by $\varepsilon$-correlated processes (see e.g. [6, 8, 13, 14, 16, 17]).

Often only approximate solutions, e.g. by means of asymptotic expansions with respect to the correlation length $\varepsilon$, are available (see [15]). Another approach uses Monte-Carlo simulations of the system response. It is an alternative procedure to determine approximations of the desired statistical properties and can also be used to verify the above mentioned analytically derived approximations (see e.g. [3, 6, 9, 13, 14]).

For the application of Monte-Carlo simulations it is necessary to generate sample paths of the $\varepsilon$-correlated processes which are involved as parameters of the system. For this task the paper uses the interpolation of time-discrete processes as it has been suggested in [13]. In contrast to this monograph where a time-discrete white noise is interpolated here a Moving-Average process is used. This approach is motivated by the fact, that the correlation function of a Moving-Average process of order $q \in \mathbb{N}_0$ vanishes outside a $q$-neighbourhood of zero. This property corresponds to the defining property of an $\varepsilon$-correlated process whose correlation function vanishes outside an $\varepsilon$-neighbourhood of zero. The approach used here can be considered as an extension to the above mentioned approach in [13] so that approximations of $\varepsilon$-correlated processes with a prescribed form of the correlation function are possible. The classical method is contained as the special case of a Moving-Average process of order $q = 0$.

The paper is organized as follows. Section 2 introduces the concept of time-discrete Moving-Average processes. Especially the computation of the coefficients of these processes to a prescribed correlation function is considered. Based on these findings in Section 3 the properties of time-continuous processes which result from the interpolation of stationary time-discrete Moving-Average processes are studied. It is found that they are $\varepsilon$-correlated and periodically distributed. In order to obtain stationary $\varepsilon$-correlated processes two modifications of the approximation technique are considered. Section 4 deals with integral functionals of stationary $\varepsilon$-correlated processes, which arise e.g. in the representation of solutions to differential equations containing $\varepsilon$-correlated influences.
Second-order moments of these integral functionals are found by means of asymptotic expansions with respect to the correlation length \( \varepsilon \). The expansion terms contain so-called correlation moments which depend on the correlation function of the underlying \( \varepsilon \)-correlated process. These quantities are evaluated for the case of the interpolated approximation process. Finally Section 5 presents some numerical examples and illustrates the findings of the preceding sections by results of Monte-Carlo simulations concerning a random boundary value problem for an ordinary differential equation.

The following notation will be used throughout this paper. All random variables and processes are defined on a probability space \((\Omega, \mathcal{A}, P)\) where \(\mathcal{A}\) denotes a \(\sigma\)-algebra of subsets of \(\Omega\) on which a probability measure \(P\) is defined. The expectation with respect to \(P\) is denoted by \(E\{\cdot\}\).

Let \(f : \mathbb{R} \times \Omega \rightarrow \mathbb{R}\) be a continuous-time random process. Then

- \(m(x) = Ef(x)\) is called mean value function,
- \(\sigma^2(x) = \text{var} f(x)\) is called variance function,
- \(R_{ff}(x, y) = \text{cov}(f(x), f(y))\) is called correlation function of \(f\).

Thereby it is \(\text{cov}(X, Y) = E \{(X - EX)(Y - EY)\}\) and \(\text{var}X = \text{cov}(X, X)\) for random variables \(X\) and \(Y\).

A stochastic process \(f : \mathbb{R} \times \Omega \rightarrow \mathbb{R}\) is called strict-sense stationary if for every sequence \(x_1, \ldots, x_n \in \mathbb{R}\), \(n \in \mathbb{N}\), the joint distribution of the random variables \(f(x_1 + l), \ldots, f(x_n + l)\) is independent of \(l \in \mathbb{R}\), i.e. \(\forall B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R})\) it holds

\[
P(f(x_1) \in B_1, \ldots, f(x_n) \in B_n) = \mathbf{P}(f(x_1 + l) \in B_1, \ldots, f(x_n + l) \in B_n). \tag{1.1}
\]

The process is called wide-sense stationary if

- \(m(x) = m = \text{const}\),
- \(R_{ff}(x, y) = R_{ff}(y - x)\), i.e. the correlation function depends only on the difference of the arguments \(x\) and \(y\).

If property (1.1) holds for integer multiples of a fixed number \(\theta \in \mathbb{R}\{0\}\), i.e. \(l = k\theta, k \in \mathbb{Z}\), then \(f\) is called periodically distributed with the period \(\theta\) or \(\theta\)-periodic. A stationary process is periodic with arbitrary period \(\theta \in \mathbb{R}\).

A random process is called \(\varepsilon\)-correlated, if its correlation function \(R(x, y)\) vanishes if the distance of \(x\) and \(y\) exceeds the correlation length \(\varepsilon > 0\). That means for \(|x - y| \geq \varepsilon\), the values \(f(x)\) and \(f(y)\) are uncorrelated.

Contrary to the notation of continuous-time random processes discrete-time processes \(\eta : \mathbb{Z} \times \Omega \rightarrow \mathbb{R}\) are considered as sequences of random variables denoted by \((\eta_t)_{t \in \mathbb{Z}}\). For the mean value, variance and correlation function of \(\eta\) the corresponding definitions are used. The notation of the correlation function in the discrete-time case is specified to

\[
\gamma_\eta(t, s) = \text{cov}(\eta_s, \eta_t), \quad s, t \in \mathbb{Z}.
\]
The concepts of strict-sense, wide-sense stationarity and periodicity are used accordingly. In case of a wide-sense stationary process \( \eta \) then it holds for the correlation function 
\[
\gamma(s, t) = \gamma(t - s).
\]

2 Moving-Average processes

2.1 Definition and properties

Let \( (\xi_t)_{t \in \mathbb{Z}} \) be a sequence of independent and identically distributed random variables with \( \mathbb{E}\xi_t = 0 \) and \( \text{var}\xi_t = \sigma^2_\xi, \forall t \), then \((\xi_t)_{t \in \mathbb{Z}}\) is called discrete-time white noise.

Definition 2.1

A stochastic process \((\eta_t)_{t \in \mathbb{Z}}\) with
\[
\eta_t = \sum_{i=0}^{q} a_i \xi_{t-i}, \quad t \in \mathbb{Z}, q \in \mathbb{N}_0
\]

is called Moving-Average process of order \( q \) – MA\([q]\) process for short. Here \((\xi_t)_{t \in \mathbb{Z}}\) is white noise with the parameters \( \mathbb{E}\xi_t = 0 \) and \( \text{var}\xi_t = \sigma^2_\xi \) and \((a_0, \ldots, a_q)\) a sequence of real valued coefficients.

Remark 2.2 It is possible to define a MA process as a two-sided process \( \eta_t = \sum_{i=-q}^{q} a_i \xi_{t-i}, t \in \mathbb{Z} \). This process can be transformed into a one-sided MA process of order \( 2q \) by replacing the variable \( \xi_t \) by \( \hat{\xi}_t = \xi_{t+q} \) in equation (2.1). It yields
\[
\eta_t = \sum_{i=-q}^{q} a_i \xi_{t-i} = \sum_{j=0}^{2q} a_{j-q} \hat{\xi}_{t-j} = \sum_{j=0}^{2q} \hat{a}_j \hat{\xi}_{t-j}, \text{ where } \hat{a}_j = a_{j-q}.
\]

So without loss of generality, only one sided processes are considered.

Generally Moving Average processes with \( q = \infty \) can be defined, but in this paper finite MA processes are considered, only.

Remark 2.3 Without loss of generality it is set \( a_0 = 1 \). This case can be obtained by the transformation \( \tilde{a}_i = \frac{a_i}{a_0} \), \( i = 0, \ldots, q \) and \( \tilde{\xi}_t = a_0 \xi_t \).

Theorem 2.4

Let \((\eta_t)_{t \in \mathbb{Z}}\) be a MA\([q]\) process with a sequence of coefficients \((1, a_1, a_2, \ldots, a_q)\). Then it holds

1. the process is strict-sense stationary,
2. \( \mathbb{E}\eta_t = 0, t \in \mathbb{Z} \),
3. The correlation function has the form

\[
\gamma_\eta(\tau) = \begin{cases} 
\sigma_\xi^2 \sum_{i=0}^{q-|\tau|} a_i a_{i+|\tau|}, & \text{for } 0 \leq |\tau| \leq q, \\
0, & \text{for } |\tau| > q,
\end{cases}
\]

Especially \(\sigma_\eta^2 = \gamma_\eta(0) = \sigma_\xi^2 \sum_{i=0}^{q} a_i^2\).

Proof.

i) In order to show strict-sense stationarity relation (1.1) has to be proved. It holds

\[
P(\eta_1 \in B_1, \eta_2 \in B_2, \ldots, \eta_n \in B_n) = P\left(\sum_{i=0}^{q} a_i \xi_{1-i} \in B_1, \sum_{i=0}^{q} a_i \xi_{2-i} \in B_2, \ldots, \sum_{i=0}^{q} a_i \xi_{n-i} \in B_n\right).
\]

Using the property of independent and identical distributions of the random variables in the sequence \((\xi_t)_{t \in \mathbb{Z}}\) it follows for arbitrary \(l \in \mathbb{Z}\)

\[
P(\eta_1 \in B_1, \eta_2 \in B_2, \ldots, \eta_n \in B_n) = P\left(\sum_{i=0}^{q} a_i \xi_{1+i-l} \in B_1, \sum_{i=0}^{q} a_i \xi_{2+i-l} \in B_2, \ldots, \sum_{i=0}^{q} a_i \xi_{n+i-l} \in B_n\right) = P(\eta_{1+l} \in B_1, \eta_{2+l} \in B_2, \ldots, \eta_{n+l} \in B_n).
\]

ii) \(E\eta_t = E\left(\sum_{i=0}^{q} a_i \xi_{t-i}\right) = \sum_{i=0}^{q} a_i E\xi_{t-i} = 0\), because \(E\xi_t = 0, \forall t \in \mathbb{Z}\).

iii) Let \(t \geq s\) then it holds

\[
\gamma_\eta(s, t) = E\{\eta_s \eta_t\} = E\left(\sum_{i=0}^{q} a_i \xi_{s-i} \sum_{j=0}^{q} a_j \xi_{t-j}\right) = \sum_{i,j=0}^{q} a_i a_j E\{\xi_{s-i} \xi_{t-j}\}
\]

\[
= \sigma_\xi^2 \sum_{i,j=0}^{q} \delta_{s-i,t-j} a_i a_j = \sigma_\xi^2 \sum_{i=0}^{q-\tau} a_i a_{i+\tau}, \text{ with } \tau = t - s \geq 0.
\]

For \(t \leq s\) it yields

\[
\gamma_\eta(s, t) = \gamma_\eta(t, s) = \sigma_\xi^2 \sum_{i=0}^{q+\tau} a_i a_{i-\tau}, \text{ with } \tau = t - s < 0.
\]

This leads to equation (2.2). \(\square\)
Remark 2.5 For \( q = 0 \) the MA[\( q \)] process \((\eta_t)_{t \in \mathbb{Z}}\) coincides with the white noise process \((\xi_t)_{t \in \mathbb{Z}}\) because \( a_0 = 1 \).

Formula (2.2) shows that the correlation function \( \gamma_\eta(\tau) \) vanishes if the absolute value of the difference \( \tau \) between the arguments is greater than the order of the MA process. This property corresponds to the property of the correlation function of \( \varepsilon \)-correlated processes, which vanishes outside an \( \varepsilon \)-neighbourhood of zero. Afterwards MA[\( q \)] processes will be used for the approximation of time-continuous \( \varepsilon \)-correlated processes. The construction of the approximating MA[\( q \)] process requires to find appropriate coefficients \( a_0, a_1, \ldots, a_q \) and the parameter \( \sigma^2_\xi \) such that the correlation function of the MA[\( q \)] process coincides with a prescribed correlation function. For this purpose the characteristic polynomial and the covariance generating function are introduced.

Definition 2.6
For a MA[\( q \)] process with the coefficients \( (a_0, a_1, \ldots, a_q) \) the polynomial
\[
a(z) = \sum_{i=0}^{q} a_i z^i, \quad z \in \mathbb{C}
\]
is called characteristic polynomial.

Definition 2.7
Let \((\varsigma_t)_{t \in \mathbb{Z}}\) be a random process possessing the correlation function \((\gamma_\varsigma(\tau))_{\tau \in \mathbb{Z}}\) then the function
\[
\Gamma_\eta(z) = \sum_{\tau=-q}^{q} \gamma_\varsigma(\tau) z^\tau, \quad z \in \mathbb{C} \setminus \{0\}
\]
is called covariance generating function.

For the covariance generating function \( \Gamma_\eta(z) \) of a MA[\( q \)] process \((\eta_t)\) Eq. (2.2) implies
\[
\Gamma_\eta(z) = \sum_{\tau=-q}^{q} \gamma_\eta(\tau) z^\tau = \sigma^2_\xi \sum_{\tau=-q}^{q} \sum_{i=0}^{q-|\tau|} a_i a_{i+|\tau|} z^\tau = \sigma^2_\xi \left( \sum_{k=0}^{q} a_k z^k \right) \left( \sum_{k=0}^{q} a_k z^{-k} \right)
\]
\[
= \sigma^2_\xi a(z) a \left( z^{-1} \right). \tag{2.3}
\]

Next the spectral density of the MA[\( q \)] process, which is defined as the Fourier transformation of the correlation function, e. g. \( f_\eta(\lambda) = \sum_{\tau=-q}^{q} \gamma_\eta(\tau) e^{i2\pi\lambda \tau} \), is considered.

Theorem 2.8
A MA[\( q \)] process \((\eta_t)_{t \in \mathbb{Z}}\) possesses the spectral density
\[
f_\eta(\lambda) = \sigma^2_\xi \left| \sum_{k=0}^{q} a_k e^{i2\pi k \lambda} \right|^2. \tag{2.4}
\]
Proof.
Applying Eq. (2.3) it follows
\[ f_\eta(\lambda) = \sum_{\tau=-q}^{q} \gamma_\eta(\tau)e^{i2\pi\lambda\tau} = \Gamma_\eta(e^{i2\pi\lambda}) = \sigma_\xi^2 \sum_{\tau=-q}^{q} \eta(\tau)e^{i2\pi\lambda\tau} = \sigma_\xi^2 \sum_{k=0}^{q} a_k e^{i2\pi\lambda k}. \]

\[ = \sigma_\xi^2 \left| a(e^{i2\pi\lambda}) \right|^2 = \sigma_\xi^2 \left| \sum_{k=0}^{q} a_k e^{i2\pi\lambda k} \right|^2. \]

Theorem 2.9
Let \((\gamma(\tau))_{\tau \in \mathbb{Z}}\) be a correlation function with finite support and a strictly positive \(\gamma(0)\).
Then there exist a \(q \in \mathbb{N}_0\) and a MA\([q]\) process \(\eta_t = \sum_{i=0}^{q} a_i \xi_{t-i}\) with the property \(\gamma(\tau) = \gamma_\eta(\tau)\).

Proof.
Let \(q = \max \text{supp } \gamma\), where \(\text{supp}\) denotes the support set of a function. Then using the symmetry of \(\gamma\) it holds \(-q = \min \text{supp } \gamma\) and for the covariance generating function of \(\gamma\) it follows
\[ \Gamma(z) = \sum_{\tau \in \mathbb{Z}} \gamma(\tau)z^\tau = \sum_{\tau=-q}^{q} \gamma(\tau)z^\tau, \quad z \in \mathbb{C} \setminus \{0\}, \]
hence
\[ \Gamma(z) = z^{-q} \sum_{k=0}^{2q} \gamma(k-q)z^k. \]
Denoting the zeros of the polynomial \(\tilde{\Gamma}(z) = \sum_{k=0}^{2q} \gamma(k-q)z^k\) by \(z_1, \ldots, z_{2q}\) the function \(\Gamma(z)\) can be written as
\[ \Gamma(z) = z^{-q} \gamma(q) \prod_{j=1}^{2q} (z - z_j). \]
For the coefficient in front of \(z^{-q}\) it holds
\[ \gamma(-q) = \gamma(q) \prod_{j=1}^{2q} z_j. \]
Since \(\gamma(-q) = \gamma(q) \neq 0\) because of \(q = \max \text{supp } \gamma\) it follows \(\prod_{j=1}^{2q} z_j = 1\) which implies that \(z_j \neq 0\) for all \(j = 1, \ldots, 2q\). The symmetry of \(\gamma\) yields \(\Gamma(z) = \Gamma(z^{-1})\) for \(z \neq 0\) which means that if \(z_j, \quad j = 1, \ldots, 2q\) is a zero of \(\Gamma(z)\) then \(z_j^{-1}\) is a zero, too. It follows that the zeros always can be arranged as \(z_1, \ldots, z_q, z_1^{-1}, \ldots, z_q^{-1}\) and it holds
\[ \Gamma(z) = z^{-q} \gamma(q) \prod_{j=1}^{q} (z - z_j) \prod_{j=1}^{q} (z - z_j^{-1}). \]
The relation \( z - z_j^{-1} = -zz_j^{-1}(z^{-1} - z_j), \ z \neq 0 \), leads to

\[
\Gamma(z) = z^{-q}\gamma(q)(-1)^q z^q \left( \prod_{j=1}^{q} z_j^{-1} \right) \left( \prod_{j=1}^{q} (z - z_j) \right) \left( \prod_{j=1}^{q} (z^{-1} - z_j) \right), \ z \neq 0.
\]

Defining

\[
c = (-1)^q \gamma(q) \prod_{j=1}^{q} z_j^{-1} \quad \text{and} \quad \bar{a}(z) = \prod_{j=1}^{q} (z - z_j)
\]

\( \Gamma(z) \) can be written as

\[
\Gamma(z) = c \bar{a}(z)\bar{a}(z^{-1}), \ z \neq 0.
\]

Finally it is set

\[
a(z) = \frac{(-1)^q}{q} \bar{a}(z)
\]

which results in a polynomial \( a(z) = \sum_{i=0}^{q} a_i z^i \) with \( a_0 = 1 \) and the representation

\[
\Gamma(z) = c \left( \prod_{j=1}^{q} z_j \right)^2 a(z)a(z^{-1}) = \gamma(q)(-1)^q \left( \prod_{j=1}^{q} z_j \right) a(z)a(z^{-1}), \ z \neq 0.
\]

Setting \( \sigma^2_\xi = \gamma(q)(-1)^q \prod_{j=1}^{q} z_j \) it follows

\[
\Gamma(z) = \sigma^2_\xi a(z)a(z^{-1}).
\]

The term \( (-1)^q \gamma(q) \prod_{j=1}^{q} z_j \) which is set to be the variance \( \sigma^2_\xi \) is indeed a positive number. This can be deduced from a comparison of the coefficients in front of \( z^0 \) from

\[
\Gamma(z) = \sum_{\tau=-q}^{q} \gamma(\tau)z^\tau = \sigma^2_\xi \sum_{k=0}^{q} a_k z^k \sum_{k=0}^{q} a_k z^{-k}
\]

and therefore it yields \( \gamma(0) = \sigma^2_\xi \sum_{k=0}^{q} a_k^2 \). From the assumption of the Theorem it follows \( \gamma(0) > 0 \) and this implies \( \sigma^2_\xi > 0 \).

From Eq. (2.3) it can be deduced that the MA[q] process \( \eta_t = \sum_{i=0}^{q} a_i \xi_{t-i} \) possesses the given correlation function.

Because the arrangement of the zeros \( z_j \) for the definition of the characteristic polynomial is arbitrary, the parameters \( (\sigma^2_\xi, a_0, \ldots, a_q) \) of a MA[q] process can not be
Moving-Average approximations of random $\varepsilon$-correlated processes

uniquely estimated by a given correlation function. Box and Jenkins introduced in [1] the criterion of invertibility. This property means that the process $(\xi_t)_{t \in \mathbb{Z}}$ can be reconstructed from the MA$[q]$ process $(\eta_t)_{t \in \mathbb{Z}}$. A criterion for the invertibility is the property of the zeros of the characteristic polynomial $a(z) = \sum_{i=0}^{q} a_i z^i$ to lie outside the unit circle.

It turns out that in case of an invertible MA$[q]$ process there exists a unique mapping of the given correlation $\gamma$ function to the coefficients.

**Theorem 2.10**

Let $(\gamma(\tau))_{\tau \in \mathbb{Z}}$ be a correlation function with finite support such that for the covariance generating function it holds $\Gamma(z) \neq 0$ for all $z \in \mathbb{C}$ with $|z| = 1$. Then there exist $q \in \mathbb{N}_0$ and an unique invertible MA$[q]$ process $\eta_t = \sum_{i=0}^{q} a_i \xi_{t-i}$ with the property $\gamma(\tau) = \gamma_{\eta}(\tau)$.

**Proof.**

$\gamma(0) > 0$ can be assumed. If $\gamma(0) = 0$ then $\gamma(\tau) = 0$, $\forall \tau \in \mathbb{Z}$ and so $\Gamma(z) = 0$, $\forall z \in \mathbb{C}$, especially for $z$ with $|z| = 1$. In Theorem 2.9 the existence of a MA$[q]$ process is shown using the roots $(z_1, \ldots, z_q, z_1^{-1}, \ldots, z_q^{-1})$ of the polynomial $\tilde{\Gamma}(z)$. Because it is required that $\Gamma_{\eta}(z) \neq 0, \forall z$ with $|z| = 1$ the $2q$ roots of the covariance generating function can be arranged according to $(z_1, \ldots, z_q, z_1^{-1}, \ldots, z_q^{-1})$ where $|z_i| > 1$ for $i = 1, \ldots, q$.

The $q$ zeros with property $|z_i| > 1$, $i = 1, \ldots, q$ are used to define the characteristic polynomial $a(z) = a_q \prod_{i=1}^{q} (z - z_i)$. By this procedure the characteristic polynomial is uniquely determined and it follows that the zeros of $a(z)$ lie outside the unit circle, i.e. the corresponding MA$[q]$ process $\eta_t = \sum_{i=0}^{q} a_i \xi_{t-i}$ is invertible and uniquely defined. \hfill \Box

In the next section four different methods to estimate the coefficients of such an invertible MA$[q]$ process are introduced.

### 2.2 Computation of coefficients

#### 2.2.1 Factorization of $\Gamma(z)$

The intention of this method is to factorize the covariance generating function on the one hand using the zeros of the polynomial $\tilde{\Gamma}(z)$ and on the other hand using the corresponding correlation function. From the comparison of the coefficients the parameters $(a_0, \ldots, a_q)$ can be determined. It yields

$$
\Gamma_{\eta}(z) = \sum_{\tau = -q}^{q} \gamma(\tau) z^\tau = \sigma_\xi^2 a(z) a(z^{-1}) = \sigma_\xi^2 a_q \left( \prod_{i=1}^{q} (z - z_i) \right) \left( \prod_{i=1}^{q} (z^{-1} - z_i) \right) \\
= \sigma_\xi^2 \left( \sum_{i=0}^{q} a_i z_i^i \right) \left( \sum_{i=0}^{q} a_i z^{-i} \right),
$$

(2.6)
with \( a_q = \frac{(-1)^q}{\prod_{i=1}^q z_i} \) (see (2.5)) and \( \sigma^2_x = \gamma_q(-1)^q \prod_{i=1}^q z_i \). That means for the calculation of the coefficients \((a_0, \ldots, a_q)\) the 2q zeros of the covariance generating function \( \Gamma_q \) have to be calculated. In a second step the polynomial \( \tilde{a}(z) = \prod_{i=1}^q(z - z_i) = \sum_{i=0}^q \tilde{a}_i z^i \) is formed. Thereby the q zeros \((z_1, \ldots, z_q)\) which fulfil the condition \(|z_i| > 1\) are chosen so that an invertible process is obtained. Due to Eq. (2.5) the comparison of the coefficients 

\[
a_i = \frac{(-1)^q}{q} \prod_{j=1}^q z_j, \quad i = 0, \ldots, q
\]

leads to the required parameters \((1, a_1, \ldots, a_q)\).

### 2.2.2 Algorithm of Wilson

The algorithm of Wilson [18] describes another approach to determine the coefficients. This method is based on the iterative numerical calculation of the coefficients \(a_i, \; i = 0, \ldots, q\), from the given values of the correlation function \(\gamma(\tau), \; \tau = 0, \ldots, q\). The idea is to calculate correction terms \(\delta^t_i, \; i = 0, \ldots, q\) on the basis of approximate values \(\theta_i = \sigma_x a_i, \; i = 0, \ldots, q\), so that \(\theta^{t+1}_i = \theta^t_i + \delta^t_i, \; i = 0, \ldots, q\), reflects as exactly as possible the values of the given correlation function. So the intention is to find parameters which fulfil the system of equations

\[
g(\tau) = \sum_{j=0}^{q-\tau} \theta_j \theta_{j+\tau} - \gamma(\tau) = 0, \quad \tau = 0, \ldots, q
\]

using Eq. (2.3). This system of equations is linearized according to a modified Newton method, the so-called Newton Raphson method.

The algorithm of Wilson looks for a solution for which the zeros of the characteristic polynomial \(a(z) = \sum_{i=0}^q a_i z^i\) lie outside the unit circle. That means the determined coefficients \((a_0, a_1, \ldots, a_q)\) generate an invertible \(\text{MA}[q]\) process. Due to Theorem 2.10 the solution is unique.

In the following the method of estimation of the coefficients according to the algorithm of Wilson is described. The superscript \(t\) of the parameters denotes the step of iteration. It holds for \(\tau = 0, \ldots, q\) and \(j = 0, \ldots, q\)

\[
\frac{\partial g(\tau)}{\partial \theta_j} = \theta_j + \tau \mathbb{1}_{\{j+\tau \in \{0, \ldots, q\}\}} + \theta_{j-\tau} \mathbb{1}_{\{j-\tau \in \{0, \ldots, q\}\}}.
\]

Then the upper triangular matrices

\[
T_1 = (\theta_j + \tau \mathbb{1}_{\{j+\tau \in \{0, \ldots, q\}\}})_{\tau,j=0,\ldots,q}, \quad T_1^t = (\theta^t_j + \tau \mathbb{1}_{\{j+\tau \in \{0, \ldots, q\}\}})_{\tau,j=0,\ldots,q},
\]
and lower triangular matrices

\[
T_2 = (\theta_{j-r} \mathbf{1}_{\{j-r \in \{0, \ldots, q\}\}})_{r, j=0, \ldots, q}, \\
T_2^* = (\theta_{j-r}^* \mathbf{1}_{\{j-r \in \{0, \ldots, q\}\}})_{r, j=0, \ldots, q},
\]

are so defined that equation (2.8) can be written as \(\frac{\partial g_\tau}{\partial \theta_j}\big|_{\tau, j=0, \ldots, q} = T\) with \(T = T_1 + T_2\). According to the Newton Raphson procedure the iteration equation is given by

\[
\theta^{t+1} = \theta^t - (T^t)^{-1} g^t, \quad \text{with } \theta^t = \begin{pmatrix} \theta_0^t \\ \theta_1^t \\ \vdots \\ \theta_q^t \end{pmatrix} \quad \text{and } f^t = \begin{pmatrix} g(0)^t \\ g(1)^t \\ \vdots \\ g(q)^t \end{pmatrix}. \tag{2.9}
\]

The vector \((T^t)^{-1} g^t\) is equal to the correction term \(\delta^t\). Because there holds the relation \(T^t \theta^t = 2\gamma^t = \sum_{i=0}^{q-r} \theta_i^t \theta_{i+r}^t\) equation (2.9) can be written as

\[
T^t \theta^{t+1} = T^t \theta^t - g^t = T^t \theta^t - \begin{pmatrix} \gamma^t_0(0) \\ \gamma^t_0(1) \\ \vdots \\ \gamma^t_0(q) \end{pmatrix} + \begin{pmatrix} \gamma^t_1(0) \\ \gamma^t_1(1) \\ \vdots \\ \gamma^t_1(q) \end{pmatrix} = \gamma^t_0 + \gamma^t_1.
\]

Finally the following iteration equation results

\[
\theta^{t+1} = (T^t)^{-1} \left(\gamma^t_1 + \gamma^t_0\right). \tag{2.10}
\]

It can be shown that if the starting values \((\theta_0^0, \ldots, \theta_q^0)\) are chosen such that the roots of the characteristic polynomial \(\theta^0(z) = \sum_{i=0}^{q} \theta_i^0 z^i, \ z \in \mathbb{C}\), lie outside the unit circle, the method is self-correcting and always converges to the required solution [18]. The convergence is second order, which is a consequence of using the Newton Raphson method.

### 2.2.3 Decomposition of the spectral density

This approach tries to determine the coefficients of a stationary MA process from the given spectral density instead of its correlation function. The disadvantage of the method is that only infinite MA processes are obtained. Equation (2.2) shows that the correlation function of an infinite MA process in general vanishes for no lag \(\tau\). So these processes are not suitable to generate \(\varepsilon\)-correlated functions. Therefore only a short description of the idea is given below.
If the correlation function is given in a sufficiently large number of points \( \tau \) with \(-H \leq \tau \leq H\), \( H \in \mathbb{N} \), it’s Fourier transform, i.e. the spectral density, can be specified with a high accuracy [7] over the equation
\[
f(\lambda) = \sum_{\tau=-\infty}^{\infty} \gamma(\tau)e^{i2\pi\lambda\tau}.
\]
Expanding \( f(\lambda)^{1/2} \) into a Fourier series the Fourier coefficients are used to define the coefficients of an infinite MA process. The function \( f(\lambda)^{1/2} \) exists, because it holds \( f(\lambda) \geq 0 \).

### 2.2.4 Algorithm of Ehlgen

This fourth algorithm, a recursive method to find the Moving-Average coefficients from the values of the correlation function by using standard matrix operations, is only mentioned. The exact procedure can be found in [2].

### 3 Approximation of \( \varepsilon \)-correlated processes

The aim of this section is to find methods to approximate a continuous-time, \( \varepsilon \)-correlated random process \( f : \mathbb{R} \times \Omega \to \mathbb{R} \), \( \varepsilon > 0 \), satisfying the following conditions.

**Assumption 3.1**

(A1) \( \varepsilon f \) is wide-sense stationary.

(A2) \( \varepsilon f \) is centered, i.e. \( \mathbf{E}\varepsilon f = 0 \).

(A3) \( \varepsilon f \) is \( \varepsilon \)-correlated, i.e. \( R_{\varepsilon f}(\tau) = 0 \) for \( |\tau| \geq \varepsilon \).

(A4) The correlation function \( R_{\varepsilon f} \) is generated by a correlation function \( \varrho \) of a 1-correlated weakly stationary process: \( R_{\varepsilon f}(\tau) = \varrho(\tau^\varepsilon) \).

(A5) The sample paths of \( \varepsilon f \) are continuous.

(A6) The correlation function \( \varrho(\cdot) \) is continuous on \( \mathbb{R} \), i.e. the processes \( \varepsilon f \) are continuous in mean square on \( \mathbb{R} \).

In order to ensure that the approximation process \( \varepsilon f_q \) possesses these characteristics a suitable approximation method must be chosen.

**Remark 3.2** It is sufficient to restrict the consideration to the case of a 1-correlated process \( f \). By a simple scaling transformation \( \varepsilon f(x, \omega) = f(x^\varepsilon, \omega) \) for \( \varepsilon > 0 \) an \( \varepsilon \)-correlated process \( \varepsilon f \) with an arbitrary correlation length \( \varepsilon > 0 \) is obtained from a 1-correlated process \( f \). For the sake of shorter notation the index \( \varepsilon \) for \( \varepsilon = 1 \) is suppressed and \( f \) is written instead of \( f \).

The used approximation idea is based on the properties of Moving-Average processes. A continuous-time process \( f_q : \mathbb{R} \times \Omega \to \mathbb{R} \) can be obtained from the discrete-time MA[q]
process \((\eta_i)_{i \in \mathbb{Z}}, \ q \in \mathbb{N}_0\) by setting
\[ f_q(x_i) = \eta_i, \text{ for grid points } x_i = ih_q, i \in \mathbb{Z}, \]
and interpolating between the grid points as explained below. Thereby \(h_q > 0\) is an appropriate grid parameter denoting the grid distance. The coefficients \((a_0, \ldots, a_q)\) of the MA\([q]\) process, the grid parameter \(h_q\) and the interpolation function must be determined such that the process \(f_q\) fulfils the above mentioned assumptions as good as possible.

### 3.1 Interpolation of Moving-Average processes

Let for \(i \in \mathbb{Z}\) the interpolation functions \(p_i : \mathbb{R} \rightarrow \mathbb{R}\) be defined by
\[ p_i(x) = p \left( \frac{x - x_i}{h_q} \right), \ x_i = ih_q. \]

Thereby \(p : \mathbb{R} \rightarrow \mathbb{R}\) is assumed to fulfil the properties

- \(p(x) \geq 0\) for \(\forall x \in \mathbb{R}\),
- \(p(x) = 0\) for \(|x| \geq 1\),
- \(p(0) = 1\),
- \(p(-x) = 1 - p(1 - x), \ x \in [0, 1]\).

Under these assumptions the interpolation functions \(p_i\) have the following properties,

- \(p_i(x_j) = \delta_{ij}\),
- \(p_i(x) + p_{i+1}(x) = 1\) for \(x \in [x_i, x_{i+1})\),
- \(p_i(x) = 0\) for \(x \in \mathbb{R} \setminus (x_{i-1}, x_{i+1})\),
- \(\sum_i p_i(x) = 1, \forall x \in \mathbb{R}\).

**Definition 3.3**

Let \((\eta_i)_{i \in \mathbb{Z}}\) be a MA\([q]\) process and \(p_i(x), \ i \in \mathbb{Z}\), denote the above defined interpolation functions. Then the time-continuous process
\[ f_q(x) = \sum_{i \in \mathbb{Z}} p_i(x)\eta_i, \ x \in \mathbb{R}, \tag{3.1} \]
is called MA\([q]\) approximation process.
Due to the properties of the interpolation function, the approximation process $f_q(x)$ can be written for $x \in [x_i, x_{i+1})$ as

$$f_q(x) = p_i(x)\eta_i + p_{i+1}(x)\eta_{i+1} = p_i(x)\eta_i + (1 - p_i(x))\eta_{i+1}. \quad (3.2)$$

For $p(x) = \begin{cases} 1 & x \in [0, 1) \\ 0 & \text{otherwise} \end{cases}$ the MA[q] approximation process $f_q(x)$ is a step function.

The interpolation function $p(x) = (1 - |x|)^+$ yields the case of linear interpolation.

The above defined approximation process $f_q(x)$ possesses the following properties.

**Theorem 3.4**

Let $f_q(x)$ be an approximation process to the MA[q] process $(\eta_i)_{i \in \mathbb{Z}}$ with the correlation function $\gamma_q(\tau), \tau \in \mathbb{Z}$. Then it holds

1. $f_q$ is centered, i.e. $\mathbf{E} f_q(x) = 0$,

2. the correlation function has the form

$$R_{f_qf_q}(x, y) = \mathbf{E}\{f_q(x)f_q(y)\}$$

$$= (p_i(x)p_j(y) + p_{i+1}(x)p_{j+1}(y))\gamma_q(\tau) + p_i(x)p_{j+1}(y)\gamma_q(\tau + 1)$$

$$+ p_{i+1}(x)p_j(y)\gamma_q(\tau - 1), \quad (3.3)$$

where $i = \left[\frac{x}{h_q}\right], \ j = \left[\frac{y}{h_q}\right], \ \tau = j - i$ and the symbol $[\cdot]$ denotes the entier function,

3. $f_q(x)$ is $\varepsilon$-correlated with correlation length $\varepsilon = (q+2)h_q$, i.e. it holds $R_{f_qf_q}(x, y) = 0$ for $|x - y| \geq (q + 2)h_q$.

**Proof.**

1. From the representation $f_q(x) = \sum_m p_m(x)\eta_m$ it follows

$$\mathbf{E} f_q(x) = \sum_m p_m(x)\mathbf{E}\eta_m = 0 \quad \text{using} \quad \mathbf{E}\eta_m = 0 \quad \forall m \in \mathbb{Z}.$$

2. It holds

$$R_{f_qf_q}(x, y) = \mathbf{E}\{f_q(x)f_q(y)\} = \sum_{m,n} p_m(x)p_n(y)\mathbf{E}\{\eta_m\eta_n\}.$$

Since $x \in [ih_q, (i+1)h_q)$ and $y \in [jh_q, (j+1)h_q)$ it follows, that the interpolation functions $p_m(x)$ and $p_n(y)$ vanish for $m \neq i, i + 1$ and $n \neq j, j + 1$. Further it holds $\mathbf{E}\{\eta_m\eta_n\} = \gamma(n - m)$ and hence assertion 2 yields.
3. Let be \( y = x + z \) with \( |z| \geq (q + 2)h_q \) then

\[
\begin{align*}
k &= j - i \geq \left\lfloor \frac{x}{h_q} + q + 2 \right\rfloor - \left\lfloor \frac{x}{h_q} \right\rfloor = q + 2, \quad \text{for } z \geq (q + 2)h_q, \\
k &= j - i \leq \left\lfloor \frac{x}{h_q} - (q + 2) \right\rfloor - \left\lfloor \frac{x}{h_q} \right\rfloor = -(q + 2), \quad \text{for } z \leq -(q + 2)h_q. 
\end{align*}
\]

Applying \( \gamma(\tau) = 0 \) for \( |\tau| \geq q+1 \) (see (2.2)) and using (3.3) it follows \( R_{f_qf_q}(x, y) = 0 \). \( \square \)

According to Assumption 3.1 the process \( f \), approximated by \( f_q \), should be (wide-sense) stationary. Unfortunately, the MA\([q]\) approximation process \( f_q \) does not satisfy this condition. The consideration of the variance function proves this assertion. Relation (3.3) yields to

\[
\sigma^2_{f_q}(x) = R_{f_qf_q}(x, x) = (p^2_i(x) + p^2_{i+1}(x))\gamma_\eta(0) + 2p_i(x)p_{i+1}(x)\gamma_\eta(1).
\]

Using \( p_{i+1}(x) = 1 - p_i(x) \) and setting \( z = p_i(x) \) it holds

\[
\begin{align*}
\sigma^2_{f_q}(x) &= (z^2 + (1 - z)^2)\gamma_\eta(0) + 2z(1 - z)\gamma_\eta(1) \\
&= \gamma_\eta(0) - 2z(1 - z)(\gamma_\eta(0) - \gamma_\eta(1)).
\end{align*}
\]

From \( z \in [0, 1], \ 0 \leq z(1 - z) \leq \frac{1}{4} \) and \( \gamma_\eta(0) \geq \gamma_\eta(1) \) it follows

\[
\frac{1}{2}(\gamma_\eta(0) + \gamma_\eta(1)) \leq \sigma^2_{f_q}(x) \leq \gamma_\eta(0).
\]

The variance function is not constant, and therefore the process \( f_q \) is not wide-sense stationary. Afterwards in Subsection 3.2 two methods to generate stationary approximation processes are introduced.

![Figure 3.1: \( \sigma^2_{f_q}(x) \) for \( q = 0, 1, 5 \)](image-url)
Figure 3.2: Sampling of the correlation function $\gamma$ for $q = 1, 2, 5$

Figure 3.1 shows the variance function $\sigma^2_{f_q}(x)$ of a 1-correlated process $f$ with the correlation function $\varrho(\tau) = (1 - |\tau|)^2$ for $q = 0, 1, 5$. It can be seen that the deviations of $\sigma^2_{f_q}$ from the constant $1 = \gamma(0)$ decrease for increasing $q$. Figure 3.2 shows the prescribed correlation function $\gamma$ together with the sampling points $\left(\frac{k}{q + 2}, \varrho\left(\frac{k}{q + 2}\right)\right)$, $k = 0, \ldots, q$.

The higher the order $q$ of the MA process the more sampling points are used. In particular, the distance $\gamma(0) - \gamma(1) = \varrho(0) - \varrho\left(\frac{1}{q + 2}\right)$ which controls the deviation of the variance of $f_q$ from the prescribed variance $\varrho(0)$ becomes smaller.

Figure 3.3 and 3.4 show the correlation function $R_{f_q f_q}(x, x + z)$ of the process $f_q$ for the orders $q = 0$ and $q = 2$. Just like in the case of the variance the deviations from $R_{f_q f_q}(x, x + z)$ to $\gamma(z)$ decrease for increasing $q$.

Although the approximation process $f_q$ is not wide-sense stationary the next Theorem shows that it possesses the weaker property of $h_q$-periodicity.

**Theorem 3.5**

Let $f_q$ be an approximation process to the MA$q$ process $(\eta_i)_{i \in \mathbb{Z}}$. Then $f_q$ is periodically distributed with period $\theta = h_q$. 
Proof.
The MA\([q]\) approximation process \(f_q\) is defined using a strict-sense stationary MA\([q]\) process (see Theorem 2.4) and it yields for \(\forall n \in \mathbb{N}, x_1, \ldots, x_n \in \mathbb{R}, B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R}), k \in \mathbb{N}\)

\[
P(f_q(x_1) \in B_1, \ldots, f_q(x_n) \in B_n) = P \left( \sum_m p_m(x_1) \eta_m \in B_1, \ldots, \sum_m p_m(x_n) \eta_m \in B_n \right) = P \left( \sum_m p_{m+k}(x_1 + kh_q) \eta_{m+k} \in B_1, \ldots, \sum_m p_{m+k}(x_n + kh_q) \eta_{m+k} \in B_n \right) = P(f_q(x_1 + kh_q) \in B_1, \ldots, f_q(x_n + kh_q) \in B_n).
\]

This proves the periodicity of \(f_q\). \(\Box\)

In Theorem 3.4, iii) it has been proven that \(f_q\) is \(\varepsilon\)-correlated with the correlation length \(\varepsilon = (q + 2)h_q\), i.e. two values \(f_q(x)\) and \(f_q(y)\) are uncorrelated for \(|y - x| > \varepsilon\). This is a property of the two-dimensional distributions of the process \(f_q\) only. In the following the so-called \(\varepsilon\)-dependence of \(f_q\) is considered. It is a stronger property and refers to the complete distribution of the process \(f_q\).

A random process \(f : \mathbb{R} \times \Omega \rightarrow \mathbb{R}\) is called \(\varepsilon\)-dependent with dependence length \(\varepsilon > 0\), if for every family of nonempty subsets \((\mathcal{X}_j)_{j=1, \ldots, p}\), \(p \geq 2\) with \(\mathcal{X}_j \subset \mathbb{R}\), \(j = 1, \ldots, p\), and \(d(\mathcal{X}_i, \mathcal{X}_j) > \varepsilon\) for \(i \neq j\) the random processes \(f_j : \mathcal{X}_j \times \Omega \rightarrow \mathbb{R}\), with \(f_j(x) = f(x)\) for \(x \in \mathcal{X}_j\), \(j = 1, \ldots, p\) are independent. Thereby it is defined

\[
d(\mathcal{X}_i, \mathcal{X}_j) := \inf \{|x_i - x_j|, x_i \in \mathcal{X}_i, x_j \in \mathcal{X}_j\}.
\]

The \(\varepsilon\)-dependence of the process \(f\) implies that for \(|y - x| > \varepsilon\) the values \(f(x)\) and \(f(y)\) are independent and not only uncorrelated as in case of an \(\varepsilon\)-correlated process.

Theorem 3.6
Let \(f_q\) be an MA\([q]\) approximation process to the MA\([q]\) process \((\eta_t)_{t \in \mathbb{Z}}\). Then \(f_q\) is \(\varepsilon\)-dependent with dependence length \(\varepsilon = (q + 2)h_q\).

Proof.
Let \((\mathcal{X}_j)_{j=1, \ldots, p}\), \(p \geq 2\) be a family of nonempty subsets with \(\mathcal{X}_j \subset \mathbb{R}\), \(j = 1, \ldots, p\), and \(d(\mathcal{X}_i, \mathcal{X}_j) > \varepsilon = (q + 2)h_q\) for \(i \neq j\). Then due to the construction of the MA\([q]\) approximation process the values \(f_q(x)\) for arbitrary \(x \in \mathcal{X}_j\) can be represented as (linear) measurable functions of a subsequence \((\xi_k)_{k \in \mathcal{J}_j}\) of the white noise \((\xi_k)_{k \in \mathbb{Z}}\).

For the index sets \(\mathcal{J}_j \subset \mathbb{Z}\) the condition \(d(\mathcal{X}_i, \mathcal{X}_j) > (q + 2)h_q\) for \(i \neq j\) implies \(\mathcal{J}_i \cap \mathcal{J}_j = \emptyset\) for \(i \neq j\). Hence the random processes \(f_{q,j} : \mathcal{X}_j \times \Omega \rightarrow \mathbb{R}\) with \(f_{q,j}(x) = f_q(x)\) for \(x \in \mathcal{X}_j, j = 1, \ldots, p\), are independent which proves the \(\varepsilon\)-dependence. \(\Box\)

The next theorem considers the convergence of the correlation function of \(f_q\) to the correlation function of \(f\) for \(q \rightarrow \infty\). 

Theorem 3.7
Let $\varrho$ be a correlation function of a 1-correlated wide-sense stationary process. Further, let $(f_q)_{q \in \mathbb{N}_0}$ be a family of MA[q]-approximation processes such that for the correlation function $\gamma_\eta$ of the underlying MA[q] process $(\eta_i)_{i \in \mathbb{Z}}$ it holds

$$\gamma_\eta(\tau) = \varrho(\tau h_q), \quad |\tau| = 0, \ldots, q.$$ 

Let for the sequence of discretization parameters $(h_q)_{q \in \mathbb{N}_0}$ the condition $\lim_{q \to \infty} q h_q = 1$ be fulfilled. Then for the correlation functions $R_{f_q f_q}(x, y) = \mathbb{E}\{f_q(x)f_q(y)\}$ of $f_q$ it holds

$$\lim_{q \to \infty} R_{f_q f_q}(x, y) = \varrho(y - x), \quad \forall x, y \in \mathbb{R}.$$ 

Proof.
See [11].

Theorem 3.7 shows that the approximation of the given correlation function becomes exact for $q \to \infty$. For finite $q$ the approximation of the correlation functions is exact if $x$ and $y$ are chosen as suitable grid points as the next corollary shows.

Corollary 3.8 Let $(\eta_i)_{i \in \mathbb{Z}}$ be a MA[q] process with the correlation function $\gamma_\eta$ satisfying $\gamma_\eta(\tau) = \varrho(\tau h_q), \quad \tau = 0, \ldots, q, \quad h_q > 0$, where $\varrho$ denotes the correlation function of a 1-correlated process. Then it follows for the correlation function of $f_q(x)$

$$R_{f_q f_q}(x, y) = \varrho(y - x) \text{ for } x = ih_q, \quad y = jh_q, \quad i, j \in \mathbb{Z}, \quad |j - i| \leq q.$$ 

Proof.
Let $\tau = j - i$. Then for the grid points $x, y$ Eq. (3.3) leads to

$$R_{f_q f_q}(x, y) = \gamma_\eta(\tau) = \varrho(\tau h_q) = \varrho((j - i) h_q) = \varrho(y - x).$$ 

Remark 3.9 The problem of defining the grid point distance $h_q$ can be discussed as following.

- If it is set $h_q = \frac{1}{q+1}$ then the coefficients of the MA[q] process fulfil the system of equations $\gamma_\eta(\tau) = \varrho\left(\frac{k}{q+1}\right), \quad k = 0, \ldots, q$. But consequently for the correlation length of the approximation process $f_q$ it yields

$$\varepsilon = (q + 2)h_q = \frac{q + 2}{q + 1} = 1 + \frac{1}{q + 1}.$$ 

That means not a 1-correlated process but a $\left(1 + \frac{1}{q+1}\right)$-correlated process is obtained. However, according to Corollary 3.8 the approximation of the given correlation function $\varrho$ is exact.
• If it is set \( h_q = \frac{1}{q+2} \) then the coefficients of the MA[q] process fulfil the system of equations \( \gamma_h(\tau) = \varrho\left(\frac{k}{q+2}\right) \), \( k = 0, \ldots, q \) and it yields
\[
\varepsilon = (q + 2)h_q = 1.
\]

But in this case the approximation of the given correlation function \( \varrho \) is only exact for the first \( q \) grid points. It holds
\[
R_{f_qf_q}(ih_q, (i + k)h_q) = \varrho\left(\frac{k}{q+2}\right), \ k = 0, \ldots, q,
\]
while
\[
R_{f_qf_q}(ih_q, (i + (q + 1))h_q) = 0 \neq \varrho\left(\frac{q + 1}{q+2}\right) \quad \text{(see (3.3))}.
\]

Because of assumption (A3) which requires \( R_{f_qf_q}(x, x + z) = 0 \) for \( |z| \geq 1 \) in view of Theorem 3.4 (3) in the following the discretization parameter is chosen as \( h_q = \frac{1}{q+2} \).

### 3.2 Approximation of stationary processes

The MA[q] approximation process \( f_q \) introduced above is \( h_q \)-periodic but not (wide-sense) stationary, i.e. assumption (A1) is not fulfilled. This section deals with two modifications of the introduced approximation procedure which result in strict-sense respectively wide-sense stationary approximation processes.

The idea of the first method consists in adding a random variable \( \alpha \) to the argument \( x \). Thereby \( \alpha \) is uniformly distributed on \([0, h_q)\) and independent of the MA[q] process \((\eta_i)_{i \in \mathbb{Z}}\). The new approximation process is defined by
\[
\tilde{f}_q(x, \omega) = f_q(x + \alpha(\omega), \omega).
\]

This operation can also be interpreted as a scattering of the underlying grid. All grid points \( x_k, \ k \in \mathbb{Z} \), are shifted by a distance given by the random variable \( \alpha \). The grid remains equidistant.

The second method is based on the shifting of each interpolation function \( p_i \) by a random variable \( \alpha_k, k \in \mathbb{Z} \), separately. While the first method leads to a strict-sense stationary, \( \varepsilon \)-correlated but not \( \varepsilon \)-dependent process, the second method results in an \( \varepsilon \)-dependent (and consequently \( \varepsilon \)-correlated), wide-sense but not strict-sense stationary process.

#### 3.2.1 Shifting grid points using one noise variable

In this subsection the properties of the process
\[
\tilde{f}_q(x, \omega) = f_q(x + \alpha(\omega), \omega)
\]
are studied.
The MA\([q]\) approximation process \(f_q\) can be rewritten as follows

\[
f_q(x, \omega) = \sum_{i \in \mathbb{Z}} p_i(x) \sum_{j \in \mathbb{Z}} a_j \xi_{i-j}(\omega) = \sum_{k \in \mathbb{Z}} \left( \sum_{i \in -k \mathbb{Z}} a_i p_i(x) \right) \xi_k(\omega)
\]

\[
= \sum_{k \in \mathbb{Z}} g_k(x) \xi_k(\omega),
\]

where \(g_k(x) := \sum_{i \in \mathbb{Z}} a_{i-k} p_i(x)\). Using the above representation of \(f_q\) the random shift of the interpolation functions \(g_k\) results in the following process

\[
\tilde{f}_q(x, \omega) = \sum_{k \in \mathbb{N}} g_k(x + \alpha(\omega)) \xi_k(\omega).
\]

The next lemma contains an auxiliary result needed for the subsequent derivations.

**Lemma 3.10**

Let \(g : \mathbb{R} \to \mathbb{R}\) be a \(h_q\)-periodic function, i.e. \(g(x) = g(x + h_q), \forall x \in \mathbb{R}\) and let there exists \(\int_0^{h_q} g(s) \, ds\). Then it holds

\[
\int_0^{h_q} g(s) \, ds = \int_0^{h_q} g(s + a) \, ds, \quad \forall a \in \mathbb{R}.
\]

**Proof.**

See [4], p. 141.

As the main result the next theorem shows that a random shift of the argument of the \(h_q\)-periodic MA\([q]\) approximation process \(f_q\) results in a strict-sense stationary process \(\tilde{f}_q\).

**Theorem 3.11**

Let \(\alpha\) be a random variable which is uniformly distributed on \([0, h_q]\) and independent from the underlying white noise process \((\xi_t)_{t \in \mathbb{Z}}\) of the approximation process \(f_q(x, \omega)\). Then the process

\[
\tilde{f}_q(x, \omega) = f_q(x + \alpha(\omega), \omega)
\]

1. is strict-sense stationary,

2. its correlation function has the form

\[
R_{\tilde{f}_q \tilde{f}_q}(x, x + z) = \frac{1}{h_q} \int_0^{h_q} R_{f_q f_q}(u, u + z) \, du =: R_{\tilde{f}_q \tilde{f}_q}(z).
\]
Moving-Average approximations of random \( \varepsilon \)-correlated processes

3. is \( \varepsilon \)-correlated with the correlation length \( \varepsilon = (q + 2)h_q \).

**Proof.**

1. It is to prove that relation (1.1) holds for \( \forall n, \forall x_1, \ldots, x_n, \forall B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R}) \) and \( l \in \mathbb{R} \). Eq. (3.6) implies

\[
P(\tilde{f}_q(x_1 + l) \in B_1, \ldots, \tilde{f}_q(x_n + l) \in B_n) = P(f_q(x_1 + l + \alpha) \in B_1, \ldots, f_q(x_n + l + \alpha) \in B_n).
\]

For the purpose of simplification it is defined

\[ C = \{ f_q(x_1 + l + \alpha) \in B_1, \ldots, f_q(x_n + l + \alpha) \in B_n \}, \]

then it holds

\[
P(\tilde{f}_q(x_1 + l) \in B_1, \ldots, \tilde{f}_q(x_n + l) \in B_n) = P(C) = \mathbb{E}1_C = \mathbb{E}\{ \mathbb{E}\{1_C | \alpha\}\} = \mathbb{E}\{ \mathbb{P}\{C | \alpha\}\} = \frac{1}{h_q} \int_0^{h_q} \mathbb{P}(C | \alpha = s) \, ds.
\]

Using the property, that the random variable \( \alpha \) and the approximation process \( f_q(x) \) are independent it follows

\[
P(\tilde{f}_q(x_1 + l) \in B_1, \ldots, \tilde{f}_q(x_n + l) \in B_n) = \frac{1}{h_q} \int_0^{h_q} P(f_q(x_1 + l + s) \in B_1, \ldots, f_q(x_n + l + s) \in B_n) \, ds.
\]

Now it is set \( g(l + s) = P(f_q(x_1 + l + s) \in B_1, \ldots, f_q(x_n + l + s) \in B_n) \) and Theorem 3.5 and Lemma 3.10 imply

\[
P(\tilde{f}_q(x_1 + l) \in B_1, \ldots, \tilde{f}_q(x_n + l) \in B_n) = \frac{1}{h_q} \int_0^{h_q} g(l + s) \, ds = \frac{1}{h_q} \int_0^{h_q} g(s) \, ds
\]

\[
= \frac{1}{h_q} \int_0^{h_q} P(f_q(x_1 + s) \in B_1, \ldots, f_q(x_n + s) \in B_n) \, ds
\]

\[
= \mathbb{P}(\tilde{f}_q(x_1) \in B_1, \ldots, \tilde{f}_q(x_n) \in B_n).
\]

2. Again due to independence of the random number \( \alpha \) and the approximation process
It follows

\[ R_{\tilde{f}_q \tilde{f}_q}(z) = E\{\tilde{f}_q(x)\tilde{f}_q(x + z)\} = E\{f_q(x + \alpha)f_q(x + \alpha + z)\} \]

\[ = \frac{1}{h_q} \int_0^{h_q} E\{f_q(x + s)f_q(x + s + z)\} ds \]

\[ = \frac{1}{h_q} \int_0^{h_q} R_{f_qf_q}(x + s, x + s + z) ds. \]

It is set \( g(x + s) = R_{f_qf_q}(x + s, x + s + z) \), then Lemma 3.10 implies

\[ R_{\tilde{f}_q \tilde{f}_q}(z) = \frac{1}{h_q} \int_0^{h_q} g(x + s) ds = \frac{1}{h_q} \int_0^{h_q} g(s) ds = \frac{1}{h_q} \int_0^{h_q} R_{f_qf_q}(s, s + z) ds. \]

3. Corresponding to Theorem 3.4 (3) it yields

\[ R_{f_qf_q}(u, u + z) = 0 \quad \text{for} \quad |z| \geq (q + 2)h_q \]

and therefore

\[ \int_0^{h_q} R_{f_qf_q}(u, u + z) du = 0 \quad \text{for} \quad |z| \geq (q + 2)h_q. \]

\[ \square \]

**Remark 3.12** While the process \( \tilde{f}_q \) is \( \varepsilon \)-correlated it is not \( \varepsilon \)-dependent, since for all \( x_0 \in \mathbb{R} \) the random variables \( \tilde{f}_q(x_0, \omega) = f_q(x_0 + \alpha, \omega) \) depend on \( \alpha \).

**Corollary 3.13**

Let \( q \) be a correlation function of an 1-correlated wide-sense stationary process. Further, let \( (f_q)_{q \in \mathbb{N}_0} \) be a family of stationary MA[\( q \)]-approximation processes such that for the correlation function \( \gamma_\eta \) of the underlying MA[\( q \)] process \( (\eta_i)_{i \in \mathbb{Z}} \) it holds

\[ \gamma_\eta(\tau) = \varrho(\tau h_q), \quad |\tau| = 0, \ldots, q. \]

Let for the sequence of discretization parameters \( (h_q)_{q \in \mathbb{N}_0} \) the condition

\[ \lim_{q \to \infty} q h_q = 1 \]

be fulfilled. Then for the correlation functions \( R_{\tilde{f}_q \tilde{f}_q}(x, y) = E\{\tilde{f}_q(x)\tilde{f}_q(y)\} \) of \( \tilde{f}_q \) it holds

\[ \lim_{q \to \infty} R_{\tilde{f}_q \tilde{f}_q}(x, y) = \varrho(y - x), \quad \forall x, y \in \mathbb{R}. \]
Moving-Average approximations of random \( \varepsilon \)-correlated processes

**Proof.**
The assertion follows from Theorem 3.7 and Theorem 3.11 (Eq. (3.7)).

It holds

\[
R_{\tilde{f}_q\tilde{f}_q}(z) = \frac{1}{h_q} \int_{0}^{h_q} R_{f_qf_q}(u, u + z)du.
\]

As a result of the periodicity of the process \( f_q \) it yields

\[
\frac{1}{h_q} \int_{0}^{h_q} R_{f_qf_q}(u, u + z)du = \frac{1}{q + 2} \int_{0}^{1} R_{f_qf_q}(u, u + z)du = \int_{0}^{1} R_{f_qf_q}(u, u + z)du,
\]

using \( (q + 2)h_q = 1 \). Since \( R_{f_qf_q}(u, u + z) \leq \sigma^2 \) Lebesque’s Theorem on dominating convergence and Theorem 3.7 imply

\[
\lim_{q \to \infty} R_{\tilde{f}_q\tilde{f}_q}(z) = \lim_{q \to \infty} \int_{0}^{1} R_{f_qf_q}(u, u + z)du = \int_{0}^{1} \lim_{q \to \infty} R_{f_qf_q}(u, u + z)du = \varrho(z).
\]

Figure 3.5 illustrates the behaviour of \( R_{\tilde{f}_q\tilde{f}_q}(z) \) and compares the prescribed correlation function \( \varrho(z) = (1 - |z|)^2 \) with the correlation functions of \( \tilde{f}_q \) for \( q = 0, 1 \) and 5. It can be seen that the deviation of \( R_{\tilde{f}_q\tilde{f}_q}(z) \) to \( \varrho(z) \) gets smaller for increasing \( q \) which corresponds to the behaviour of \( R_{\tilde{f}_q\tilde{f}_q}(z) \) described in Corollary 3.13.

![Figure 3.5](image.png)

**3.2.2 Shifting interpolation functions using independent noise variables**

Contrary to the previous subsection now a sequence of independent random variables \( (\alpha_k)_{k \in \mathbb{Z}} \) is used to shift each interpolation function separately.
Using representation (3.4) of $f_q$ the random shift of the interpolation functions $g_k$ results in the following process

$$\tilde{f}_q(x, \omega) := \sum_{k \in \mathbb{Z}} g_k(x + \alpha_k(\omega)) \xi_k(\omega).$$

**Theorem 3.14**

Let $f_q$ be a $h_q$-periodic process and $(\alpha_i)_{i \in \mathbb{Z}}$ a sequence of independent and on $[0, h_q)$ uniformly distributed random variables which are independent from the underlying white noise process $(\xi_i)_{i \in \mathbb{Z}}$ of the approximation process. Then the process

$$\tilde{f}_q(x, \omega) = \sum_{k \in \mathbb{Z}} g_k(x + \alpha_k(\omega)) \xi_k(\omega)$$

(3.8) possesses the following properties.

1. The correlation function of $\tilde{f}_q$ has the form

$$R_{\tilde{f}_q \tilde{f}_q}(x, x + z) = \frac{1}{h_q} \int_0^{h_q} R_{f_q f_q}(u, u + z) \, du =: R_{\tilde{f}_q \tilde{f}_q}(z).$$

(3.9)

2. $\tilde{f}_q$ is wide-sense stationary.

3. $\tilde{f}_q$ is $\varepsilon$-correlated with correlation length $\varepsilon = (q + 2)h_q$.

4. $\tilde{f}_q$ is $\varepsilon$-dependent with dependence length $\varepsilon = (q + 2)h_q$.

**Proof.**

1. For the correlation function $R_{\tilde{f}_q \tilde{f}_q}$ it holds

$$R_{\tilde{f}_q \tilde{f}_q}(x, x + z) = \mathbb{E} \left\{ \sum_{k,l \in \mathbb{Z}} g_k(x + \alpha_k) \xi_k g_l(x + z + \alpha_l) \xi_l \right\} = \sum_{k \in \mathbb{Z}} \mathbb{E}\{g_k(x + \alpha_k) \xi_k g_k(x + z + \alpha_k) \xi_k\} + \sum_{k \neq l, k,l \in \mathbb{Z}} \mathbb{E}\{g_k(x + \alpha_k) \xi_k g_l(x + z + \alpha_l) \xi_l\}.$$  

The terms in the two sums can be written as

$$\mathbb{E}\{g_k(x + \alpha_k) \xi_k g_k(x + z + \alpha_k) \xi_k\}$$

$$= \mathbb{E}\{\mathbb{E}\{g_k(x + \alpha_k) \xi_k g_k(x + z + \alpha_k) \xi_k\} | \alpha_k\}$$

$$= \frac{1}{h_q} \int_0^{h_q} \mathbb{E}\{g_k(x + u) \xi_k g_k(x + z + u) \xi_k\} \, du$$

$$= \frac{\mathbb{E} \xi_k^2}{h_q} \int_0^{h_q} g_k(x + u) g_k(x + z + u) \, du$$
and using $k \neq l$

$$
\mathbb{E}\{g_k(x + \alpha_k)\xi_k g_l(x + z + \alpha_l)\xi_l\} \\
= \mathbb{E}\{\mathbb{E}\{g_k(x + \alpha_k)\xi_k g_l(x + z + \alpha_l)\xi_l\}\,|\,\alpha_k, \alpha_l\} \\
= \frac{1}{h_q^2} \int_0^{h_q} \int_0^{h_q} \mathbb{E}\{g_k(x + u_1)\xi_k g_l(x + z + u_2)\xi_l\} \, du_1 \, du_2 \\
= \mathbb{E}\{\xi_k \xi_l\} \frac{1}{h_q^2} \int_0^{h_q} \int_0^{h_q} g_k(x + u_1) g_l(x + z + u_2) \, du_1 \, du_2 = 0.
$$

From this it follows

$$
R_{f_q f_q}(x, x + z) = \sum_{k \in \mathbb{Z}} \frac{\mathbb{E} \xi_k^2}{h_q} \int_0^{h_q} g_k(x + u) g_k(x + z + u) \, du
$$

and applying Lemma 3.10 further on

$$
\frac{1}{h_q} \int_0^{h_q} R_{f_q f_q}(u, u + z) \, du = \frac{1}{h_q} \int_0^{h_q} R_{f_q f_q}(x + u, x + z + u) \, du \\
= \sum_{k, l \in \mathbb{Z}} \frac{\mathbb{E} \xi_k \xi_l}{h_q} \int_0^{h_q} g_k(x + u) g_l(x + z + u) \, du \\
= \sum_{k \in \mathbb{Z}} \frac{\mathbb{E} \xi_k^2}{h_q} \int_0^{h_q} g_k(x + u) g_k(x + z + u) \, du \\
+ \sum_{k \neq l, k, l \in \mathbb{Z}} \frac{\mathbb{E} \xi_k \xi_l}{h_q} \int_0^{h_q} g_k(x + u) g_l(x + z + u) \, du \\
= \sum_{k \in \mathbb{Z}} \frac{\mathbb{E} \xi_k^2}{h_q} \int_0^{h_q} g_k(x + u) g_k(x + z + u) \, du.
$$

Eq.(3.9) is proved.

2. The condition of wide-sense stationarity is fulfilled if the mean value function $\mathbb{E}\overline{f_q}(x)$ is constant and the correlation function $R_{\overline{f_q} \overline{f_q}}(x, x + z)$ depends only on difference of the arguments. The second property is proven above.
For the mean value function it holds
\[
\mathbb{E} \mathcal{J}_q(x) = \mathbb{E} \left\{ \sum_{k \in \mathbb{Z}} g_k(x + \alpha_k) \xi_k \right\} = \sum_{k \in \mathbb{Z}} \mathbb{E} \{ g_k(x + \alpha_k) \xi_k \} = \sum_{k \in \mathbb{Z}} \mathbb{E} \mathbb{E} \{ g_k(x + \alpha_k) \xi_k \mid \alpha_k \} = \frac{1}{h_q} \sum_{k \in \mathbb{Z}} \mathbb{E} \xi_k \int_0^{h_q} g_k(x + u) \, du = 0.
\]

3. The proof of this property is given with Theorem 3.11 (3).

4. The \( \varepsilon \)-dependence is proved similarly to the proof of Theorem 3.6. In this case the independent subsequences \((\xi_k)_{k \in J_i}\) and \((\alpha_k)_{k \in J_j}\) are considered.

Remark 3.15 For the processes \( \mathcal{J}_q \) a similar result to Corollary 3.13 can be proven.

The following example shows that processes of the form
\[
\varphi(t) = \sum_{k \in \mathbb{Z}} g_k(t + \alpha_k) \xi_k, \quad t \in \mathbb{R}
\]
in general are not strict-sense stationary.

Let \( g_k(t) \) be a hatlike function on \([k, k + 1]\), i.e.
\[
g_k(t) = \begin{cases} 2(t - k), & k \leq t \leq k + 0.5, \\ 2(k + 1 - t), & k + 0.5 \leq t \leq k + 1, \\ 0, & \text{otherwise.} \end{cases}
\]

\((\xi_k)_{k \in \mathbb{Z}}\) is a sequence of independent, identical distributed and centered random variables with a continuous distribution and a finite variance. Thus the centered random process \( \varphi(t) \) possesses 1-periodic, finite dimensional distributions. Due to Theorem 3.14 this process is wide-sense stationary under the assumption, that the random variables \( \alpha_k \) are independent, on \([0, 1)\) uniformly distributed and independent of the sequence \((\xi_k)_{k \in \mathbb{Z}}\). This random process can vanish identical with a positive probability on finite intervals. It yields for \(0 \leq s_1 \leq s_2 \leq 1\)
\[
P(\varphi(t) \equiv 0 \ \forall t \in [k + s_1, k + s_2]) = P(\{\alpha_k \geq 1 - s_1\} \cap \{\alpha_{k+1} \leq 1 - s_2\}) = s_1(1 - s_2).
\]

This probability depends not only on the distance \( s_2 - s_1 \) between \( s_1 \) and \( s_2 \) of the considered interval. Consequently the process \( \varphi \) can not be strict-sense stationary.
4 Integral functionals of stationary \( \varepsilon \)-correlated processes

In this section integral functionals of the form \( \tau(\omega) = \frac{1}{0} Q(x) f(x, \omega) dx \) are considered, where \( f \) is a stationary \( \varepsilon \)-correlated process and \( Q \) is some continuous, non-random function. Such integral functionals arise for example in representations of solutions to differential equations containing \( \varepsilon \)-correlated random parameters. In this context it is a typical problem to find the distribution or the moments of \( \tau \) to given \( Q \) and given distribution or moments of \( f \). Since exact solutions to this problem often can not be given or are hardly to find, here an approximate computation of moments of \( \tau \) for small \( \varepsilon > 0 \) is investigated. For this purpose families of \( \varepsilon \)-correlated processes \( (\varepsilon f)_{\varepsilon > 0} \) and the corresponding families of integral functionals \( (\tau)_{\varepsilon > 0} \) are considered and asymptotic expansions of second-order moments of \( \tau \) in powers of \( \varepsilon \) are studied. For the formulation of these expansions the following quantities are introduced.

**Definition 4.1**

Let \( (\varepsilon f)_{\varepsilon > 0} \) be a family of stationary \( \varepsilon \)-correlated processes \( \varepsilon f \) fulfilling Assumption 3.1 for all \( \varepsilon > 0 \). Moreover let be for \( j \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \)

\[
\int_{-1}^{1} |s|^j g(s) ds < \infty,
\]

where \( g \) denotes the correlation function of the 1-correlated process according to Assumption 3.1 (A4). Then

\[
\mu_j = \int_{-1}^{1} s^j g(s) ds \quad \text{and} \quad \nu_j = \int_{-1}^{1} |s|^j g(s) ds \quad (4.1)
\]

are called ordinary and absolute correlation moments of order \( j \) of the family \( (\varepsilon f)_{\varepsilon > 0} \), respectively.

From the symmetry of the correlation function \( g \), i.e. \( g(s) = g(-s) \) for \( s \in \mathbb{R} \) for the ordinary correlation moment it holds:

\[
\mu_j = \begin{cases} 
\nu_j &= 2 \int_{0}^{1} s^j g(s) ds \quad \text{for even } j, \\
0 &= \quad \text{for odd } j.
\end{cases} \quad (4.2)
\]

**Remark 4.2** The correlation moment of order \( j = 0 \), i.e.

\[
\mu_0 = \nu_0 = \int_{-1}^{1} g(s) ds
\]
is also called *intensity* of the family $(\varepsilon f)_{\varepsilon > 0}$. It can be interpreted as a measure of the decrease of the correlation $E \{\varepsilon f(x)\varepsilon f(y)\} = R_{\varepsilon f}(y - x) = \rho \left(\frac{y - x}{\varepsilon}\right)$ in an $\varepsilon$-neighborhood around $x$.

Since the process $\varepsilon f$ is centered it holds

$$E \varepsilon r = \int_0^1 Q(x) E \varepsilon f(x) dx = 0, \quad \forall \varepsilon > 0,$$

i.e. the integral functionals are centered, too.

For the variance function it holds

$$E \varepsilon r^2 = E \left\{ \left( \int_0^1 Q(x) \varepsilon f(x) dx \right)^2 \right\} = \int_0^1 \int_0^1 Q(x_1) Q(x_2) E \{\varepsilon f(x_1)\varepsilon f(x_2)\} dx_1 dx_2$$

$$= \int_0^1 \int_0^1 Q(x_1) Q(x_2) \rho \left(\frac{x_2 - x_1}{\varepsilon}\right) dx_1 dx_2.$$

The substitution $u = \frac{x_2 - x_1}{\varepsilon}$ and $v = x_2$ leads to

$$E \varepsilon r^2 = \varepsilon \left( \int_{\frac{1}{\varepsilon}}^0 g(u) \int_0^{1+\varepsilon u} Q(v - \varepsilon u)Q(v) dv du + \int_{\frac{1}{\varepsilon}}^1 g(u) \int_{\frac{1}{\varepsilon}}^{1-\varepsilon u} Q(v)Q(v + \varepsilon u) dv du \right)$$

$$= \varepsilon \left( \int_{\frac{1}{\varepsilon}}^0 g(u) \int_0^{1+\varepsilon u} Q(v - \varepsilon u)Q(v) dv du + \int_{\frac{1}{\varepsilon}}^1 g(u) \int_{\frac{1}{\varepsilon}}^{1-\varepsilon u} Q(v)Q(v + \varepsilon u) dv du \right)$$

$$= \varepsilon \int_{-1}^0 g(u) \int_0^{1-|u|} Q(v + \varepsilon |u|)Q(v) dv du.$$

A further analytical evaluation of this integral is possible, but connected with some difficulties. An approach to calculate the variance function is the following asymptotic expansion of $E \varepsilon r^2$ in powers of $\varepsilon$, which is given in [15]. The main idea is a Taylor expansion of the inner integral.

**Theorem 4.3**

Let $(\varepsilon f)_{\varepsilon > 0}$ be a family of stationary $\varepsilon$-correlated random processes $\varepsilon f$ on $[0, 1]$ fulfilling
Assumption 3.1 and \( Q \) a non-random function on \([0, 1]\) with \( Q \in \mathcal{C}^n([0, 1]) \) and \( Q^{(n)} \) is absolutely continuous. Then for \( \varepsilon \downarrow 0 \) it holds

\[
\mathbb{E} \varepsilon^2 = \sum_{i=0}^{n} \frac{\varepsilon^{i+1}}{i!} \kappa_i \nu_i + o(\varepsilon^{n+1}). \tag{4.3}
\]

Thereby \( \nu_i \) denotes the absolute correlation moment of \( j \)th order of the family \((\varepsilon f)_{\varepsilon > 0}\) and \( \kappa_i = \Phi^{(i)}(0) \) with \( \Phi(z) = \int_0^z Q(u)Q(u+z) \, du \).

In order to evaluate the expansion terms of the above power series for the case of \( \varepsilon \)-correlated processes \( \varepsilon f \) resulting from MA-approximations and the corresponding strict respectively wide-sense stationary versions \( \tilde{f}_q \) and \( \overline{f}_q \), it is necessary to determine the correlation moments for the families \((\tilde{f}_q)_{\varepsilon > 0}\) and \( (\overline{f}_q)_{\varepsilon > 0}\). The relations (3.7) and (3.9) show that the correlation functions of the processes \( f_q \) and \( \overline{f}_q \) coincide, therefore it is sufficient to determine the correlation moments of \((\tilde{f}_q)_{\varepsilon > 0}\) denoted by \( \tilde{\mu}_{q,j} \) and \( \tilde{\nu}_{q,j} \).

It is mentioned again that an \( \varepsilon \)-correlated function with arbitrary correlation length \( \varepsilon \) can be obtained by a 1-correlated function using the scaling transformation \( 1 f(x, \omega) = \varepsilon f(\frac{x}{\varepsilon}, \omega) \) for \( \forall \varepsilon > 0 \). Further according to Theorem 3.4 (3) it yields the relation \( \varepsilon = (q+2)h_q \) hence in the case of 1-correlated functions \( 1 = (q+2)h_q \).

The next corollary proves the convergence of the correlation moments of the process \( \tilde{f}_q = 1 f_q \) for \( q \to \infty \).

**Corollary 4.4**

For the ordinary and absolute correlation moments

\[
\tilde{\mu}_{q,j} = \int_{-1}^{1} \tilde{R}_{f_q}(z) z^j \, dz \quad \text{and} \quad \tilde{\nu}_{q,j} = \int_{-1}^{1} \tilde{R}_{f_q}(z) |z|^j \, dz
\]

of the 1-correlated, stationary approximation process \( \tilde{f}_q \) it holds

\[
\lim_{q \to \infty} \tilde{\mu}_{q,j} = \mu_j \quad \text{and} \quad \lim_{q \to \infty} \tilde{\nu}_{q,j} = \nu_j.
\]

**Proof.**

According to Corollary 3.13 and by using Lesbeque’s theorem on dominating convergence it yields

\[
\lim_{q \to \infty} \int_{-1}^{1} \tilde{R}_{f_q}(z) g(z) \, dz = \int_{-1}^{1} \lim_{q \to \infty} \tilde{R}_{f_q}(z) g(z) \, dz = \int_{-1}^{1} g(z) \, dz,
\]

and the corollary is proved with \( g(z) = z^j, |z|^j \).
In Theorem 4.5 for even \( j \) the correlation moments \( \tilde{\mu}_{q,j} \), which are equal to the absolute correlation moments \( \tilde{\nu}_{q,j} \) (see (4.2)) are given. Afterwards, for odd \( j \) the absolute correlation moments \( \tilde{\nu}_{q,j} \) are derived. In this case it holds \( \tilde{\mu}_{q,j} = 0 \).

**Theorem 4.5**

Let be \( \tilde{f}_q \) a stationary, 1-correlated process with the underlying MA\([q]\) approximation process \( f_q \). Then it yields for the ordinary correlation moments

\[
\tilde{\mu}_{q,j} = \frac{1}{h_q} \sum_{k=-q}^{q} \gamma(k)d_{kj}
\]

with

\[
d_{kj} = \int_{0}^{h_q} \left( [a_{kj}(u)(2p_0(u) - 1) + b_{kj}(u)(1 - p_0(u))] \\
+ p_0(u)(b_{k-1,j}(u) - a_{k-1,j}(u)) + a_{k+1,j}(u)(1 - p_0(u)) \right) du,
\]

\[
a_{kj}(u) = h_q \int_{0}^{1} p(z)((k + z)h_q - u)^j dz,
\]

\[
b_{kj}(u) = h_q \int_{0}^{1} ((k + z)h_q - u)^j dz.
\]

**Proof.**

Using relation (3.7) the correlation moments of the process \( \tilde{f}_q \) are given as

\[
\tilde{\mu}_{q,j} = \int_{-1}^{1} R_{\tilde{f}_q \tilde{f}_q}(z)z^j dz = \frac{1}{h_q} \int_{-1}^{1} \left[ \int_{-1}^{u} R_{f_q f_q}(u,u+z)z^j dz \right] du,
\]

\( R_{f_q f_q}(x,y) \) denotes the correlation function (3.3) of the approximation process \( f_q \).

For fixed \( u \in [0, h_q] \) the inner integral \( \int_{-1}^{1} R_{f_q f_q}(u,u+z)z^j dz \) can be written by substituting \( y = u + z \) as

\[
\int_{-1}^{1} R_{f_q f_q}(u,u+z)z^j dz = \int_{u-1}^{u+1} R_{f_q f_q}(u,y)(y-u)^j dy.
\]

It is noticed that the correlation function \( R_{f_q f_q}(u,y), u \in [0, h_q], \) vanishes for \( y \in [u - 1, -(q + 1)h_q] \) and \([1, u + 1]\). This results from the property of the MA\([q]\) process, whose correlation function \( \gamma(\tau) \), \( \tau \in \mathbb{Z} \), equals zero if the argument \( \tau \) is greater than \( q \) (see Theorem 2.4). According to Eq. (3.3) it holds for \( y \in [1, u + 1] \)

\[
R_{f_q f_q}(u,y) = \left[ (p_0(u)p_{q+2}(y) + p_1(u)p_{q+3}(y))\gamma(q + 2) \\
+ p_0(u)p_{q+3}(y)\gamma(q + 3) + p_1(u)p_{q+2}(y)\gamma(q + 1) \right] = 0
\]
and for \( y \in [u - 1, -(q + 1)h_q] \)

\[
R_{f_q f_q}(u, y) = \left[ (p_0(u)p_{-(q+2)}(y) + p_1(u)p_{-(q+1)}(y))\gamma(q + 2) + p_0(u)p_{-(q+1)}(y)\gamma(q + 1) + p_1(u)p_{-(q+2)}(y)\gamma(q + 3) \right] = 0.
\]

Therefore the domain of integration in the above integral can be restricted from \([u - 1, u + 1]\) to \([-(q + 1)h_q, 1]\) and it is

\[
\int_{-1}^{1} R_{f_q f_q}(u, y + z)z^j dz = \int_{-(q+1)h_q}^{1} R_{f_q f_q}(u, y)(y - u)^j dy
\]

\[
= \sum_{k=-(q+1)}^{q+1} \int_{kh_q}^{(k+1)h_q} R_{f_q f_q}(u, y)(y - u)^j dy = \sum_{k=-(q+1)}^{q+1} t_{kj}(u),
\]

where

\[
t_{kj}(u) = \int_{kh_q}^{(k+1)h_q} R_{f_q f_q}(u, y)(y - u)^j dy.
\]

The next step of the derivation is to evaluate the integral \(t_{kj}(u)\). By substituting representation (3.3) for the correlation function \(R_{f_q f_q}\) it yields

\[
t_{kj}(u) = \int_{kh_q}^{(k+1)h_q} \left[ (p_0(u)p_k(y) + p_1(u)p_{k+1}(y))\gamma(k) + p_0(u)p_{k+1}(y)\gamma(k + 1) + p_1(u)p_k(y)\gamma(k - 1) \right] (y - u)^j dy
\]

\[
= \gamma(k) \left[ p_0(u) \int_{kh_q}^{(k+1)h_q} p_k(y)(y - u)^j dy + p_1(u) \int_{kh_q}^{(k+1)h_q} p_{k+1}(y)(y - u)^j dy \right]
\]

\[
+ \gamma(k + 1)p_0(u) \int_{kh_q}^{(k+1)h_q} p_{k+1}(y)(y - u)^j dy
\]

\[
+ \gamma(k - 1)p_1(u) \int_{kh_q}^{(k+1)h_q} p_k(y)(y - u)^j dy.
\]

For the sake of shorter notation it is defined

\[
a_{kj}(u) = \int_{kh_q}^{(k+1)h_q} p_k(y)(y - u)^j dy = h_q \int_{0}^{1} p(z) ((k + z)h_q - u)^j dz.
\]
Using
\[ p_{k+1}(y) = 1 - p_k(y) \quad \text{for} \quad y \in [kh_q, (k + 1)h_q] \]
the integral \( \int_{kh_q}^{(k+1)h_q} p_{k+1}(y)(y - u)^j dy \) can be written in terms of \( a_{kj}(u) \) as
\[
\int_{kh_q}^{(k+1)h_q} p_{k+1}(y)(y - u)^j dy = \int_{kh_q}^{(k+1)h_q} (1 - p_k(y))(y - u)^j dy
\]

\[
= \int_{kh_q}^{(k+1)h_q} (y - u)^j dy - a_{kj}(u) = b_{kj}(u) - a_{kj}(u),
\]

with
\[
b_{kj}(u) = \int_{kh_q}^{(k+1)h_q} (y - u)^j dy = h_q \int_0^1 ((k + z)h_q - u)^j dz.
\]

Hence, the integral \( t_{kj}(u) \) can be written as
\[
t_{kj}(u) = \gamma(k)p_0(u)a_{kj}(u) + p_1(u)(b_{kj}(u) - a_{kj}(u))
\]+ \( \gamma(k + 1)p_0(u)(b_{kj}(u) - a_{kj}(u)) + \gamma(k - 1)p_1(u)a_{kj}(u) \)
\[
= \gamma(k)(a_{kj}(u)(2p_0(u) - 1) + b_{kj}(u)(1 - p_0(u)))
\]+ \( \gamma(k + 1)p_0(u)(b_{kj}(u) - a_{kj}(u)) + \gamma(k - 1)a_{kj}(u)(1 - p_0(u)). \)

Furthermore, Eqs. (4.4) and (4.6) lead to
\[
\bar{\mu}_{q,j} = \frac{1}{h_q} \int_0^{h_q} \left[ \int_{-1}^{1} R_{f_r f_q}(u, u + z)z^j dz \right] du
= \frac{1}{h_q} \int_0^{h_q} \sum_{k=-(q+1)}^{q+1} t_{kj}(u)du
\]
\[
= \frac{1}{h_q} \int_0^{h_q} \sum_{k=-(q+1)}^{q+1} \left[ \gamma(k)[a_{kj}(u)(2p_0(u) - 1) + b_{kj}(u)(1 - p_0(u))] \right.
\]
\[
+ \gamma(k + 1)p_0(u)(b_{kj}(u) - a_{kj}(u)) + \gamma(k - 1)a_{kj}(u)(1 - p_0(u)) \bigg] du
\]
\[
= \frac{1}{h_q} \int_0^{h_q} \left[ \sum_{k=-q}^{q} \gamma(k)[a_{kj}(u)(2p_0(u) - 1) + b_{kj}(u)(1 - p_0(u))] \right.
\]
\[
+ \sum_{k=-(q+1)}^{q-1} \gamma(k + 1)p_0(u)(b_{kj}(u) - a_{kj}(u))
\]+ \( \sum_{k=-q+1}^{q+1} \gamma(k - 1)a_{kj}(u)(1 - p_0(u)) \bigg] du
\]
Hence the representation of the intensity can be found by
\[ \mu_q, \] has been obtained. According to Remark 4.2 the intensity equals the correlation moment \( \tilde{\mu}_{q,j} \).

Thus, an explicit representation of the correlation moments \( \tilde{\mu}_{q,j} \) of the strict-sense stationary process \( f_q \) which coincides with those of the wide-sense stationary process \( f_q \) has been obtained. According to Remark 4.2 the intensity equals the correlation moment \( \tilde{\mu}_{q,0} \). For this case \((j = 0)\) there hold the relations
\[
a_{k0}(u) = h_q \int_0^u p(z)dz = a_{k0} \quad \forall k \quad \text{and} \quad b_{k0}(u) = h_q = b_{k0} \quad \forall k.
\]
Hence the representation of the intensity can be found by
\[
\tilde{\mu}_{q,0} = h_q \sum_{k=-q}^{q} \gamma(k) = \frac{1}{q + 2} \sum_{k=-q}^{q} \gamma(k).
\] (4.8)

**Theorem 4.6**
Under the assumptions of Theorem 4.5 it yields for the absolute correlation moments

\[
j \quad \text{odd:} \quad \tilde{\nu}_{q,j} = \frac{1}{h_q} \left[ \sum_{k=-q}^{q} \gamma(k) \int_0^{h_q} \left( a_{kj}(u) \int_0^{h_q} R_{f_q,f_q}(u,y)(y-u)^j dy du - \sum_{k=0}^{q} \gamma(k) \left( \left( 2p_0(u) - 1 \right) a_{kj}(u) + (1 - p_0(u))b_{kj}(u) \right) dy du - a_{-k+1,j}(u) + a_{k+1,j}(u) \right) du \right]
\]

Finally, it follows
\[
\tilde{\mu}_{q,j} = \frac{1}{h_q} \sum_{k=-q}^{q} \gamma(k) d_{kj} \quad (4.7)
\]

with
\[
d_{kj} = \int_0^{h_q} \left( a_{kj}(u)(2p_0(u) - 1) + b_{kj}(u)(1 - p_0(u)) \right) du
\]

Thus, an explicit representation of the correlation moments \( \tilde{\mu}_{q,j} \) of the strict-sense stationary process \( f_q \) which coincides with those of the wide-sense stationary process \( f_q \) has been obtained. According to Remark 4.2 the intensity equals the correlation moment \( \tilde{\mu}_{q,0} \). For this case \((j = 0)\) there hold the relations
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\]

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\]

with
\[
d_{kj} = \int_0^{h_q} \left( a_{kj}(u)(2p_0(u) - 1) + b_{kj}(u)(1 - p_0(u)) \right) du
\]

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a_{k0}(u) = h_q \int_0^u p(z)dz = a_{k0} \quad \forall k \quad \text{and} \quad b_{k0}(u) = h_q = b_{k0} \quad \forall k.
\]
Hence the representation of the intensity can be found by
\[
\tilde{\mu}_{q,0} = h_q \sum_{k=-q}^{q} \gamma(k) = \frac{1}{q + 2} \sum_{k=-q}^{q} \gamma(k).
\] (4.8)
\( \tilde{\nu}_{q,j} = \tilde{\mu}_{q,j} \),

with

\[
\begin{align*}
    a_{kj}(u) &= h_q \int_0^1 p(z)((k + z)h_q - u)^j dz, \\
    b_{kj}(u) &= h_q \int_0^1 ((k + z)h_q - u)^j dz, \\
    \alpha_{kj}(u) &= a_{kj}(u) - a_{-k,j}(u), \\
    \beta_{kj}(u) &= b_{kj}(u) - b_{-k,j}(u).
\end{align*}
\]

**Proof.**

Corresponding to Definition (4.1) the absolute correlation moments are given by

\[
\tilde{\nu}_{q,j} = \int_{-1}^{1} \left| z \right|^j R_{f_q f_q}(z) dz.
\]

For even order \( j = 2k \) it holds

\[
\tilde{\nu}_{q,j} = \tilde{\nu}_{q,2k} = \int_{-1}^{1} \left| z \right|^{2k} R_{f_q f_q}(z) dz = \tilde{\mu}_{q,2k}.
\]

So the subsequent derivation is restricted to odd orders \( j \). Using results from the investigations for \( \tilde{\mu}_{q,j} \) it follows

\[
\tilde{\nu}_{q,j} = \int_{-1}^{1} \left| z \right|^j R_{f_q f_q}(z) dz = \int_{-1}^{h_q} \left[ \int_{-1}^{1} R_{f_q f_q}(u, u + z) \left| z \right|^j dz \right] du
\]

\[
= \int_{-1}^{h_q} \left[ \int_{-q}^{-q+1} R_{f_q f_q}(u, y) |y - u|^j dy \right] du.
\]

Furthermore it holds in this case

\[
\int_{-q}^{-q+1} R_{f_q f_q}(u, y) |y - u|^j dy = \int_{-q}^{-q+1} R_{f_q f_q}(u, y) |y - u|^j dy - \int_{-q}^{-q+1} R_{f_q f_q}(u, y) |y - u|^j dy.
\]  \quad (4.9)

Analogously to the derivation of the correlation moments \( \tilde{\mu}_{q,j} \) the integrals

\[
\int_{-q}^{-q+1} R_{f_q f_q}(u, y) |y - u|^j dy \quad \text{and} \quad \int_{-q}^{-q+1} R_{f_q f_q}(u, y) |y - u|^j dy \quad \text{with} \quad u \in [0, h_q]
\]

can be decomposed using relation (4.5),

\[
\int_{-q}^{-q+1} R_{f_q f_q}(u, y) |y - u|^j dy = \int_{-q}^{-q+1} R_{f_q f_q}(u, y) |y - u|^j dy + \int_{h_q}^{h_q} R_{f_q f_q}(u, y) |y - u|^j dy
\]

\[
= \int_{h_q}^{h_q} R_{f_q f_q}(u, y) |y - u|^j dy + \sum_{k=1}^{q+1} t_{kj}(u)
\]
and
\[
\int_{-(q+1)h_q}^{u} R_{f_q f_q}(u,y)(y-u)^2 dy = \int_{0}^{u} R_{f_q f_q}(u,y)(y-u)^2 dy + \int_{-(q+1)h_q}^{0} R_{f_q f_q}(u,y)(y-u)^2 dy
\]
\[
= \int_{0}^{u} R_{f_q f_q}(u,y)(y-u)^2 dy + \sum_{k=-(q+1)}^{-(q+1)} t_{kj}(u).
\]

Then Eq. (4.9) can be written as
\[
\int_{-(q+1)h_q}^{1} R_{f_q f_q}(u,y)|y-u|^2 dy
\]
\[
= \int_{0}^{h_q} R_{f_q f_q}(u,y)(y-u)^2 dy - \int_{0}^{u} R_{f_q f_q}(u,y)(y-u)^2 dy + \sum_{k=1}^{q+1} (t_{kj}(u) - t_{-k,j}(u)).
\]

By inserting expression (4.6) in the above equation and using the symmetry of the correlation function \(\gamma\), \(\gamma(-k) = \gamma(k)\), and the property of \(\gamma(k)\) to vanish for \(k > q\), a straightforward calculation proves the result of the theorem.

The representation for the \(j\)th absolute correlation moment \(\bar{\nu}_{q,j}\) given in Theorem 4.6 can be inserted in the asymptotic expansion for the second order moment \(E \gamma^2\), which now can be calculated.

## 5 Monte-Carlo simulation

This section provides some numerical examples which illustrate the results of the preceding sections. Especially, results of Monte-Carlo simulations based on the MA\([q]\) approximation processes are presented.

Let \(\varepsilon f\) be an stationary \(\varepsilon\)-correlated process satisfying Assumption 3.1 and let its correlation function be given by

\[
R_{\varepsilon f, \varepsilon f}(\tau) = \varphi \left( \frac{\tau}{\varepsilon} \right) \quad \text{where} \quad \varphi(s) = \begin{cases} 
(1 - |s|)^2 & |s| < 1 \\
0 & |s| \geq 1 
\end{cases}
\]

For the construction of a MA\([q]\) approximation process to \(\varepsilon f\) it is necessary to determine the coefficients \(a_0, \ldots, a_q\) of the MA\([q]\) process \(\eta\). To this end, the correlation function \(\varphi\) is sampled at grid points \(ih_q, \ i = 0, \ldots, q\) with \(h_q = \frac{1}{q+2}\). Then the correlation function \(\gamma_\eta\) of the process \(\eta\) is set to be \(\gamma_\eta(i) = \varphi(ish_q), \ i = 0, \ldots, q\). The coefficients \(a_0, \ldots, a_q\) are found from the factorization of the covariance generating function \(\Gamma(z) = \sum_{i=-q}^{p} \gamma_\eta(i)z\).

This procedure has been described in Subsection 2.2.1.
Table 5.1 presents for \( q = 0, 1, 5 \) the coefficients \( a_0, \ldots, a_q \) and the relevant parameters needed for their computation.

<table>
<thead>
<tr>
<th>( q )</th>
<th>0</th>
<th>1</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>grid points ( ih_q )</td>
<td>( \left( \frac{1}{2} \right)_{i=0} )</td>
<td>( \left( \frac{1}{3} \right)_{i=0,1} )</td>
<td>( \left( \frac{1}{7} \right)_{i=0,\ldots,5} )</td>
</tr>
<tr>
<td>values of ( \gamma_\eta )</td>
<td>(1)</td>
<td>(1, 0.44)</td>
<td>(1, 0.73, 0.51, 0.33, 0.18, 0.08)</td>
</tr>
<tr>
<td>roots ( z_i )</td>
<td>( (-1.64, -0.61) )</td>
<td>( (0.32 \pm 0.62i, -0.35 \pm 0.61i, -0.72, -0.70 \pm 1.22i, 0.66 \pm 1, 26i, -1.38) )</td>
<td></td>
</tr>
<tr>
<td>coefficients ( a_i )</td>
<td>(1)</td>
<td>(1, 0.61)</td>
<td>(1, 0.78, 0.58, 0.41, 0.27, 0.18)</td>
</tr>
</tbody>
</table>

Table 5.1: Parameters for the approximation of \( \varepsilon f \)

For given \( \varepsilon \) the MA[\( q \)] approximation process \( \varepsilon f_q \) is obtained from \( f_q = 1 f_q \) by

\[
\varepsilon f_q(x, \omega) = f_q \left( \frac{x}{\varepsilon}, \omega \right) \quad \text{where} \quad f_q(z, \omega) = \sum_{i \in \mathbb{Z}} p_i(z) \eta(\omega).
\]

Figure 5.1 shows for \( \varepsilon = 0.1 \) a sample path of \( \varepsilon f_q \) for \( q = 0 \) and \( q = 10 \), respectively. The grid points have a distance of \( \varepsilon h_q = \frac{\varepsilon}{q+2} \), i.e. for \( q = 0 \) it is \( \varepsilon h_q = \frac{\varepsilon}{2} = 0.05 \) and for \( q = 10 \) it is \( \varepsilon h_q = \frac{\varepsilon}{12} = 0.0083 \). For \( q = 0 \) the MA[\( q \)] process \( \eta \) coincides with the underlying white noise process and the values of \( \varepsilon f_q \) in subsequent grid points (with distance \( \varepsilon h_0 = 0.05 \)) constitute independent random variables. In case \( q = 10 \) the grid is refined such that \( \varepsilon h_{10} = \frac{\varepsilon}{12} = \frac{1}{6} \varepsilon h_0 \) and additional values of the MA[10] process, which are now dependent random variables according to the prescribed correlation function, are incorporated into the interpolation.

![Figure 5.1](image.png)

Figure 5.1: Realizations of \( \varepsilon f_q \) for \( q = 0, 10 \) and \( \varepsilon = 0.1 \)
The MA[$q$] approximation process $\varepsilon f_q$ is periodically distributed but not stationary. For the generation of sample paths of stationary $\varepsilon$-correlated processes Subsection 3.2 suggests two procedures. Here the first method is considered. It generates from a sample path of $\varepsilon f_q(x, \omega)$ a sample path of a stationary process by

$$\varepsilon \tilde{f}_q(x, \omega) = \varepsilon f_q(x + \alpha(\omega), \omega),$$

i.e. the underlying grid $(ih_q)_{i \in \mathbb{N}_0}$ of the approximation process $\varepsilon f_q$ is shifted by the value of a random variable $\alpha$, which is uniformly distributed on the interval $[0, h_q)$ and independent of the MA[$q$] process $\eta$.

The correlation function of $\varepsilon \tilde{f}_q$ can be obtained from in Theorem 3.11 (Eq. (3.7)) by

$$R_{\varepsilon \tilde{f}_q} (z) = R_{\varepsilon f_q} \left( \frac{z}{\varepsilon} \right) = \frac{1}{h_q} \int_0^{h_q} R_{f_q f_q} \left( u, u + \frac{z}{\varepsilon} \right) du.$$  

Figure 5.2 compares statistical estimates of the correlation function $R_{\varepsilon \tilde{f}_q} (z)$ with the theoretical values. The estimates result from Monte-Carlo simulations of $N = 1000$ (left) and $N = 10,000$ (right) sample paths of $\varepsilon \tilde{f}_q$ and are denoted by $\hat{R}_{\varepsilon \tilde{f}_q, N}$. The figure shows a very good coincidence of the statistical estimates and the theoretical values in case of $N = 10,000$ sample paths.

Finally a random boundary value problem of the form

$$Lu = -u''(x) + bu(x) = \varepsilon f(x, \omega), \quad 0 \leq x \leq 1,$$

$$u(0) = u(1) = 0$$

is considered. A detailed description of the theory can be found in [8]. Here $L$ denotes the differential operator, $\varepsilon f$ an stationary $\varepsilon$-correlated function with the correlation function $R_{\varepsilon f f} (\tau) = \varrho \left( \varepsilon \frac{\tau}{\varepsilon} \right)$ and $b > 0$ some parameter. For the sake of simplicity this parameter
is set to be \( b = 1 \). The above problem arises e.g. in the mathematical modeling of the deflection of a string with a linear restoring force described by \( bu \), subjected to an external random load \( \varepsilon f \). The solution of this problem is given by the integral functional

\[
u(x, \omega) = \varepsilon u(x, \omega) = \int_0^1 G(x, z) \varepsilon f(z, \omega) dz,
\]

where the deterministic function \( G(x, z) \) is the Green function of the problem (5.1) given by

\[
G(x, z) = \begin{cases} 
\frac{\sinh x \sinh(1 - z)}{\sinh 1} & 0 \leq x \leq z \leq 1 \\
\frac{\sinh z \sinh(1 - x)}{\sinh 1} & 0 \leq z \leq x \leq 1 
\end{cases}
\]

It holds \( E \varepsilon u = 0 \) and for the variance function of \( \varepsilon u \) the asymptotic expansion

\[
\varepsilon \sigma^2(x) = E\{\varepsilon u^2(x)\} = \varepsilon \kappa_0(x) \nu_0 + o(\varepsilon) \quad \text{with} \quad \kappa_0(x) = \int_0^1 G^2(x, z) dz
\]

can be derived from Theorem 4.3. Here, \( \nu_0 \) denotes the absolute correlation moment of the family \( (f)_{\varepsilon > 0} \) which coincides with the ordinary correlation moment \( \mu_0 \),

\[
\nu_0 = \mu_0 = \int_{-1}^1 g(z) dz = \int_{-1}^1 (1 - |z|)^2 dz = \frac{2}{3}.
\]

Figure 5.3: Scaled variance functions \( \frac{1}{\varepsilon} \sigma^2(x) \) and approximation \( \kappa_0(x) \nu_0 \) for \( \varepsilon = 0.1, 0.2 \)

In Figure 5.3 the (exact) variance function scaled by \( \frac{1}{\varepsilon} \), i.e. \( \frac{1}{\varepsilon} \sigma^2(x) \) is compared with the first-order approximation \( \kappa_0(x) \nu_0 \) derived from the above expansion by truncating
after the leading term. It can be seen, that the deviations of the approximation to the
effect values become smaller for decreasing $\varepsilon$. Next, the (scaled) variance function will
be compared with estimates resulting from Monte-Carlo simulations. To this end, $\varepsilon f$
is replaced by its stationary MA[$q$] approximation process $\varepsilon \tilde{f}_q$ which leads to the following
approximation of the solution of the boundary value problem

$$
\varepsilon \tilde{u}_q(x) = \int_{0}^{1} G(x, z) \varepsilon \tilde{f}_q(z, \omega) \, dz.
$$

For the variance of $\varepsilon \tilde{u}_q(x)$ the above asymptotic expansion reads as

$$
\varepsilon \sigma^2_q(x) = \mathbb{E}\{\varepsilon \tilde{u}^2_q(x)\} = \varepsilon \kappa_0(x) \tilde{\nu}_{q,0} + o(\varepsilon)
$$

where $\tilde{\nu}_{q,0} = \tilde{\mu}_{q,0}$ denote the correlation moments of order zero of the family $(\varepsilon \tilde{f}_q)_{\varepsilon>0}$. From Eq. (4.8) it follows

$$
\tilde{\nu}_{q,0} = \tilde{\mu}_{q,0} = \frac{1}{q+2} \sum_{k=-q}^{q} \gamma_q(k) = \frac{1}{q+2} \sum_{k=-q}^{q} q \left( \frac{k}{q+2} \right)
= \frac{1}{q+2} \sum_{k=-q}^{q} \left( 1 - \frac{|k|}{q+2} \right)^2
= \frac{2}{3} + \frac{1}{3(q+2)^2} - \frac{2}{(q+2)^3}.
$$

Figure 5.4: Correlation moments $\tilde{\nu}_{q,0}$, $q = 0, \ldots, 30$ and $\nu_0$

It can be seen that $\tilde{\nu}_{q,0} \to \nu_0 = \frac{2}{3}$ for $q \to \infty$. Figure 5.4 shows the correlation moments $\tilde{\nu}_{q,0}$ for various $q$ and compares with the correlation moment $\nu_0$

Let the estimate of the variance of $\varepsilon \tilde{u}_q(x)$ resulting from Monte-Carlo simulation using
$N$ sample paths of $\varepsilon \tilde{f}_q$ be denoted by

$$
\varepsilon \sigma^2_{q,N}(x) := \frac{1}{N-1} \sum_{i=1}^{N} \left( \varepsilon \tilde{u}_q(x, \omega_i) - \frac{1}{N} \sum_{i=1}^{N} \varepsilon \tilde{u}_q(x, \omega_i) \right)^2.
$$
Figure 5.5 compares the scaled variance function $\frac{1}{\varepsilon} \sigma^2(x)$ with its first-order approximation $\kappa_0(x)\tilde{\nu}_{q,0}$ and its statistical estimate $\frac{1}{\varepsilon} \hat{\sigma}^2_{q,N}(x)$ for $q = 0$ and $q = 1$, $N = 10,000$ and $\varepsilon = 0.05$.

Finally Figure 5.6 compares the first-order approximation $\kappa_0(x)\tilde{\nu}_{q,0}$ for the scaled variance function $\frac{1}{\varepsilon} \sigma^2(x)$ with statistical estimates $\frac{1}{\varepsilon} \hat{\sigma}^2_{q,N}(x)$ for $\varepsilon = 0.1, 0.2$. The order of the MA[q] process is chosen to be $q = 5$, the number of sample paths is $N = 10,000$. The figure shows the decreasing deviation of the estimates to the approximation for decreasing $\varepsilon$ which indicates the convergence of the scaled variance to its first-order approximation $\kappa_0(x)\tilde{\nu}_{q,0}$ as $\varepsilon \downarrow 0$. 

Figure 5.5: Scaled variance function $\frac{1}{\varepsilon} \sigma^2(x)$, approximation $\kappa_0(x)\tilde{\nu}_{q,0}$ and statistical estimate $\frac{1}{\varepsilon} \hat{\sigma}^2_{q,N}(x)$ for $q = 0, 1$, $N = 10,000$, $\varepsilon = 0.05$

Figure 5.6: Scaled variance functions $\frac{1}{\varepsilon} \sigma^2(x)$, approximation $\kappa_0(x)\tilde{\nu}_{q,0}$ and statistical estimate $\frac{1}{\varepsilon} \hat{\sigma}^2_{q,N}(x)$ for $q = 5$, $N = 10,000$, $\varepsilon = 0.1, 0.2$
References


