Price models with weakly correlated processes

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Abstract

Empirical autocorrelation functions of returns of stochastic price processes show phenomena of correlation on small intervals of time, which decay to zero after a short time. The paper deals with the concept of weakly correlated random processes to describe a mathematical model which takes into account this behaviour of statistical data. Weakly correlated functions have been applied to model numerous problems of physics and engineering. The main idea is, that the values of the functions at two points are uncorrelated if the distance between the points exceeds a certain quantity $\varepsilon > 0$. In contrast to the white noise model, for distances smaller than $\varepsilon$ a correlation between the values is permitted.

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1 Introduction

From statistical studies of the returns of price processes in liquid markets it is well-known, that the autocorrelation functions of these returns rapidly decay to zero (cf. [1], [2], [3]). Nevertheless, empirical autocorrelation functions of logarithmic returns show phenomena of correlation on small intervals of time. This short-range correlation effects have been the subject of theoretical as well as empirical research activities over the past two decades, see for instance [3], [4]. The well-known Black-Scholes model is an extreme case for these considerations, it results in a degenerate autocorrelation function, i.e. the single-period returns are Gaussian and uncorrelated. In contrast to the traditional models also the presence of long-range dependencies has been studied (see for instance [5]), which is not a subject of this paper.

There is a great variety of price models to explain correlation effects like these. However, one must not argue, that because the empirically observed logarithmic returns do not have the independent normal distributions assumed for the Black-Scholes formula it is therefore necessary to consider extensions in order to price options accurately. It is well known (cf. [6]), that as long as it is only possible to observe the returns in a discrete scheme, it is (under smooth regularity conditions) always possible to find a price model based on a geometric Brownian motion with a stochastic drift term and an arbitrary constant volatility coefficient having identically distributions as the observed returns, see also [7].

The aim of the present paper is to make an attempt to introduce the concept of weakly correlated random processes into finance research by describing a mathematical model which takes into account the non-zero behaviour of the autocorrelation function in small time intervals. Weakly correlated random functions have been applied to model numerous problems of physics and engineering, it is referred for instance to the textbooks [8], [9] and [10]. The concept of weakly correlated functions is based on the idea that the values of the functions at two points are uncorrelated if the distance between the points exceeds a certain quantity $\varepsilon > 0$. In contrast to the white noise model, for distances smaller than $\varepsilon$ a correlation between the values is permitted. The above idea is closely related to the theory of differential equations with random parameters, where a probabilistic analysis of the pathwise solutions is considered. It is well-known, that there are relations but also differences of these methods to the utilization of stochastic integrals. There is also a close relation to the classical Black-Scholes model, which makes the described model desirable to be in the interest of further research.

We consider a frictionless market with continuous trading. In order to take into account the mentioned correlation phenomena we model the price process similar to the classical Black-Scholes model but replace the white noise process by a weakly correlated process with continuous paths. A consequence of the described model consists in very strong assumptions with regard to the smoothness of the trajectories of the price processes. Resulting from this smoothness, an investor, who can trade continuously and without transaction costs would be able to spot the direction in which prices are moving and exploit this in trading. From the mathematical point of view, the used stochastic processes
possess absolutely continuous sample paths, that means, that the rate of price change is well defined for almost each time point. In case of price processes having continuous sample paths of bounded variation the existence of arbitrage (which is a riskless plan to make profit without any investment) is well known, cf. [11], [12].

The paper is organized as follows. In Section 2 the basic definitions and applications concerning the theory of weakly correlated processes are explained. The concepts of $\varepsilon$-dependent, $\varepsilon$-correlated and weakly correlated functions are introduced. Basic relations between these definitions as well as the convergence of integral functionals of weakly correlated processes to Gaussian processes are discussed. In Section 3 some basic concepts of the description of financial markets are given. Because we are interested in a pathwise approach, the main ideas of [11] and [12] are recalled, which deal with the concept of stochastic price processes with bounded $p$-variation ($1 \leq p < 2$) and its consequences with regard to arbitrage opportunities. Section 4 contains the mentioned stochastic price model with $\varepsilon$-correlated returns. Hereby, the price process is modelled with the help of integral functionals of weakly correlated processes. On the one hand, this has the mentioned disadvantage of the existence of arbitrage, on the other hand exist advantages in the close relation to the classical model of a geometric Brownian motion as well as in the opportunity of the description of the correlation behaviour of the returns. Finally, Section 5 presents a model, which is a natural consequence of the problems described in the previous Section and overcomes the arbitrage problems. To state precise, a geometric Brownian motion model with a stochastic drift term (which incorporates the desired correlation behaviour) is considered.

2 Weakly correlated processes

Throughout the paper we consider a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ (where the $\sigma$-algebra $\mathcal{F}$ is $P$-complete, $(\mathcal{F}_t)_{t \geq 0}$ is an increasing family of $\sigma$-algebras with $\mathcal{F}_t \subseteq \mathcal{F}, t \geq 0$, $\mathcal{F}_0$ contains all sets $N \in \mathcal{F}$ with $P(N) = 0$ and $(\mathcal{F}_t)_{t \geq 0}$ is assumed to be right continuous). Furthermore, stochastic processes $X = (X_t)_{t \in D}, D \subseteq \mathbb{R}^+$ are considered, which are families of random variables on the sample space $(\Omega, \mathcal{F})$, and take values in the space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. These processes are assumed to be adapted to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, i.e. for each $t \in D$, the random variable $X_t$ is assumed to be $\mathcal{F}_t$-measurable. Throughout the paper the considered processes possess continuous trajectories $t \mapsto X_t(\omega), \omega \in \Omega$. Let $\varepsilon$ be a (small) positive real number.

**Definition 2.1**

Let $\mathcal{P} = (x_i \in \mathbb{R}; i \in I)$ be a family of points of $\mathbb{R}$ (i.e. a mapping of the index set $I$ into $\mathbb{R}$). For $\tilde{I} \subset I$ the set

$$\mathcal{J}(\tilde{I}) := \bigcup_{i \in \tilde{I}} \{j \in I : |x_i - x_j| \leq \varepsilon\}$$

consists of the indices of those points, which are in the $\varepsilon$-neighbourhood of the points belonging to $\tilde{I}$.
A family of a single point, \( P = (x_1) \), is also called \( \varepsilon \)-neighbouring.

**Definition 2.2**

Let \( P = (x_i \in \mathbb{R}; i \in I) \) be a family of points of \( \mathbb{R} \). A subfamily \( P^* \) of \( P \), \( P^* = (x_i \in \mathbb{R}; i \in I^* \subset I) \) is called maximally \( \varepsilon \)-neighbouring with respect to \( P \), if \( P^* \) is \( \varepsilon \)-neighbouring, but \( P^{**} = (x_i \in \mathbb{R}; i \in I^* \cup \{r\}) \) is not \( \varepsilon \)-neighbouring for all \( r \in I \setminus I^* \). If a family \( P \) is \( \varepsilon \)-neighbouring, it is called maximally \( \varepsilon \)-neighbouring with respect to itself.

In [8] it is proved, that every family \( P \) of real numbers has an (up to changing of indices) unique decomposition into maximally \( \varepsilon \)-neighbouring subfamilies.

As mentioned in Section 1, the main idea of the concept of weakly correlated functions is, that the values of the function in some points do not interact in a stochastic sense, if the points are “far away” from each other. The next definition gives a mathematical description of this fact.

**Definition 2.3**

Let \( D \subseteq \mathbb{R}^+ \) and let \( \varepsilon > 0 \) be a real number. The stochastic process \( \varepsilon X = (\varepsilon X_t)_{t \in D} \) is called \( \varepsilon \)-dependent, if for every finite family \( P \) of real numbers in \( D \), \( P = (t_i; i \in I) = (t_1, \ldots, t_k), k \in \mathbb{N}, I = \{1, 2, \ldots, k\} \), and the corresponding decomposition into maximally \( \varepsilon \)-neighbouring subfamilies \( (t_i; i \in I_j); j = 1, 2, \ldots, p; I = \bigcup_{j=1}^p I_j; I_{j_1} \cap I_{j_2} = \emptyset \) for \( j_1 \neq j_2 \), the random vectors

\[
(\varepsilon X_{t_i}; i \in I_j), \quad j = 1, \ldots, p
\]

are stochastically independent.

**Definition 2.4**

Let \( X = (X_t)_{t \in D}, D \subseteq \mathbb{R}^+ \) be a stochastic process. \( \varepsilon X = (\varepsilon X_t)_{t \in D} \) is called \( \varepsilon \)-correlated with correlation length \( \varepsilon > 0 \), if for all \( t_1, t_2 \in D \) with \( |t_1 - t_2| > \varepsilon \) it holds

\[
\text{cov}(\varepsilon X_{t_1}, \varepsilon X_{t_2}) = \mathbb{E}(\varepsilon X_{t_1} - \mathbb{E}\varepsilon X_{t_1}) (\varepsilon X_{t_2} - \mathbb{E}\varepsilon X_{t_2}) = 0.
\]

**Remark 2.5**

For a real-valued Gaussian process \( X = (X_t)_{t \in D} \) the terms \( \varepsilon \)-dependent and \( \varepsilon \)-correlated are equivalent.

In the textbooks [8] and [9] the following definition of a weakly correlated process is used, which shows minor differences to the definition of \( \varepsilon \)-dependent processes.
Definition 2.6
For \( D \subseteq \mathbb{R}^+ \) let \( \varepsilon X = (\varepsilon X_t)_{t \in D} \) be a stochastic process with
\[
E \varepsilon X_t = 0, \quad \forall t \in D.
\]
\( \varepsilon X = (\varepsilon X_t)_{t \in D} \) is called weakly correlated with correlation length \( \varepsilon > 0 \), if for every finite family \( \mathcal{P} \) of real numbers in \( D \), \( \mathcal{P} = (t_i; i \in \mathcal{I}) = (t_1, \ldots, t_k), \ k \in \mathbb{N}, \ \mathcal{I} = \{1, 2, \ldots, k\} \), and the corresponding decomposition into maximally \( \varepsilon \)-neighbouring subfamilies \( (t_i; i \in \mathcal{I}_j); j = 1, 2, \ldots, p; \mathcal{I} = \bigcup_{j=1}^p \mathcal{I}_j; \mathcal{I}_{j_1} \cap \mathcal{I}_{j_2} = \emptyset \) for \( j_1 \neq j_2 \), it holds
\[
E(\varepsilon X_{t_1} \cdot \varepsilon X_{t_2} \cdot \ldots \cdot \varepsilon X_{t_k}) = \prod_{j=1}^p E \left( \prod_{i \in \mathcal{I}_j} \varepsilon X_{t_i} \right),
\]
that means, every moment of order \( k, k \in \mathbb{N} \), can be decomposed into a product of the moments corresponding to the decomposition of \( (t_1, \ldots, t_k) \) into maximally \( \varepsilon \)-neighbouring subfamilies.

Remark 2.7 A Gaussian weakly correlated process with correlation length \( \varepsilon \) is \( \varepsilon \)-correlated and therefore \( \varepsilon \)-dependent. An \( \varepsilon \)-correlated Gaussian process \( \varepsilon X \) with \( E \varepsilon X_t = 0 \) for every \( t \in D \) is weakly correlated with correlation length \( \varepsilon \), see [9].

Also in the non-Gaussian case there are close relations between the terms introduced in the above definitions.

Proposition 2.8
Let \( \varepsilon X = (\varepsilon X_t)_{t \in D} \) be an \( \varepsilon \)-dependent stochastic process with \( E \varepsilon X_t = 0 \) for every \( t \in D \) and let all moment functions of \( \varepsilon X \) exist, \( E(|\varepsilon X_t|^m) < \infty, \forall t \in D, \forall m \in \mathbb{N} \). Then \( \varepsilon X \) is weakly correlated with correlation length \( \varepsilon \).

Let \( \varepsilon X = (\varepsilon X_t)_{t \in D} \) be a weakly correlated stochastic process with correlation length \( \varepsilon > 0 \) with \( \sum_{m=0}^{\infty} E(|\varepsilon X_t|^m)/m! < \infty, \forall t \in D \). Then \( \varepsilon X \) is \( \varepsilon \)-dependent.

Remark 2.9 In order to motivate the basic idea of the introduced processes, Definition 2.3 and Definition 2.4 were given. In the following, in this paper the term of weakly correlated processes is used.

The terms \( \varepsilon \)-dependent, \( \varepsilon \)-correlated and weakly correlated can be defined also in the case of vector processes \( \varepsilon X : \mathbb{R} \to \mathbb{R}^n \) as well as in the general case of random fields \( \varepsilon X : \mathbb{R}^m \to \mathbb{R}^n \), cf. [8], [9].

Definition 2.10
Let \( \varepsilon X = (\varepsilon X_t)_{t \in D}, \varepsilon > 0 \), \( D \subseteq \mathbb{R}^+ \), be a family of weakly correlated processes. Let \( \hat{D} \subseteq D \) denote the set of inner points of \( D \) and \( t_0 \in \hat{D} \). The intensity \( a(t_0) \) is defined by
\[
a(t_0) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} E(\varepsilon X_{t_0} \varepsilon X_{t_0+s}) \ ds,
\]
if the integral and the limit exist.
Remark 2.11 Weakly correlated wide-sense stationary processes \( (\varepsilon X_t)_{t \in \mathbb{R}} \) possess constant intensities, \( a(t) \equiv a, t \in \mathbb{R} \).

In applications of weakly correlated processes limit theorems for integral functionals of such functions play an important role, cf. [8], [9] and [10]. The fundamental result consists in a convergence of these functionals to Gaussian processes. We present a limit theorem given in [10].

Theorem 2.12
Let \( D_i, i = 1, 2, \ldots, n, \) be intervals in \( \mathbb{R}^+ \) and let \( (\varepsilon X) = ((\varepsilon X_t)_{t \in \mathbb{R}}, \varepsilon > 0) \) be a family of weakly correlated processes, where the intensity \( a(t) \) of \( \varepsilon X \) exists on \( D_i \cap D_j, i, j = 1, 2, \ldots, n \). Let \( E|\varepsilon X_t|^p \leq c_p < \infty \) for all \( t \in \mathbb{R}^+ \), for all \( \varepsilon \) and for all \( p \geq 1 \), where the sample functions of \( \varepsilon X_t \) are assumed to be continuous. We consider integral functionals

\[
\varepsilon r_i(\omega) = \int_{D_i} G_i(t) \varepsilon X_t(\omega) \, dt, \quad i = 1, 2, \ldots, n,
\]

where \( G_i(t) \) are deterministic functions on \( D_i \) fulfilling the conditions

\[
\int_{D_i} |G_i(t)| \, dt < \infty \quad \text{and} \quad \int_{D_i} (G_i(t))^2 \, dt < \infty, \quad i = 1, 2, \ldots, n.
\]

Then it follows the convergence in distribution

\[
\lim_{\varepsilon \to 0} \frac{1}{\sqrt{\varepsilon}} (\varepsilon r_1, \varepsilon r_2, \ldots, \varepsilon r_n) = (\xi_1, \xi_2, \ldots, \xi_n),
\]

where \( (\xi_1, \xi_2, \ldots, \xi_n) \) is a Gaussian vector with mean

\[
E\xi_i = 0, \quad i = 1, 2, \ldots, n
\]

and covariance relations

\[
E(\xi_i \xi_j) = \int_{D_i \cap D_j} G_i(t) G_j(t) a(t) \, dt, \quad i, j = 1, 2, \ldots, n.
\]

3 Financial models containing bounded variation processes

Consider a frictionless market. Besides further assumptions, the term frictionless means especially, that there are no transaction costs and borrowing as well as short-selling are allowed without restrictions. The aim is the consideration of stochastic processes that reflect economically meaningful ideas, for instance with respect to the description of the price of financial assets. To state more precisely, we consider a market in which \( K \) assets are traded during the time interval \([0, T]\). Usually, financial markets are modelled by the vector \((S^1, S^2, \ldots, S^K)\), where \( S^k = (S^k_t)_{t \in [0,T]} \). \( S^k_t \) stands for the price of one share of
the $k$-th asset ($k = 1, \ldots, K$) at time $t \in [0, T]$. For details of the used concepts it is referred for instance to the textbooks [3] and [13].

In the following we assume, that $S^k_t = (S^k_t(t)_{t \in [0, T]}$, $k = 1, \ldots, K$, are continuous, strictly positive functions. Moreover, we make the (uncommon) assumption, that the processes $S^k_t = (S^k_t(t)_{t \in [0, T]}$ are of bounded variation.

**Definition 3.1**

Let $p$ be a real number, $1 \leq p < 2$. A process $X = (X_t)_{t \in [0, T]}$ is said to be of bounded $p$-variation, if $P$-almost surely the paths of $X$ are of bounded $p$-variation on each compact interval $[a, b] \subseteq [0, T]$. For a trajectory, given a partition $\Pi = (t_1, t_2, \ldots, t_n)$ with $a \leq t_1 < t_2 \ldots < t_n \leq b$, the $p$-variation is defined by $v^p_X(\Pi) := \sum_{i=2}^n |X_{t_i} - X_{t_{i-1}}|^p$ and consequently $V^p_X([a, b]) := \sup\{v^p_X(\Pi) : \Pi$ is a finite partition of $[a, b]\} < \infty$

is assumed. In case of $p = 1$ the process $X$ is said to be of bounded variation.

It is clear, that for the description of price processes the above assumption is critical. It is well-known, that there exist arbitrage opportunities in markets where the sample paths have bounded variation (cf. [11]), or more general bounded $p$-variation for some $p \in [1, 2)$ (cf. [12]). Nevertheless, processes of bounded ($p$-)variation, $p \in [1, 2)$, are used for data-oriented models of stock price fluctuations. Here the model of a fractional Brownian motion should be mentioned, which has been used to describe long-range dependence in empirical data, see for instance [5]. In contrast to this we want to describe short-range correlation effects. The consequences of models with price processes of bounded variation shall be described more detailed.

**Definition 3.2**

Consider a vector process $(S^1, S^2, \ldots, S^K)$, whose components satisfy Definition 3.1. Consider the adapted stochastic processes $\Phi^k = (\Phi^k_t)_{t \in [0, T]}$, $k = 1, 2, \ldots, K$, which represent the number of shares of asset $k$ hold at time $t$. The vector process $\Phi = (\Phi^1, \Phi^2, \ldots, \Phi^K)$, is called trading strategy. Thereby it is assumed, that, for almost all fixed $\omega \in \Omega$, each $\Phi^k$ is Riemann-Stieltjes integrable with respect to $S^k$, $k = 1, 2, \ldots, K$.

The stochastic process $U^\Phi, S = (U^\Phi, S_t)_{t \in [0, T]}$ defined by

$$U^\Phi, S_t := \sum_{k=1}^K \Phi^k_t \cdot S^k_t$$

represents the total value of the considered portfolio at time $t$, $t \in [0, T]$.

The concepts of self-financing and admissible trading strategies play an important role.
Definition 3.3
A trading strategy $\Phi$ is said to be self-financing, if it holds

$$U_t^{\Phi,S} = U_0^{\Phi,S} + \sum_{k=1}^K \int_0^t \Phi_u^k dS_u^k.$$ 

A trading strategy $\Phi$ is said to be admissible, if

$$U_t^{\Phi,S} \geq 0, \text{ for every } t \in [0,T].$$

Arbitrage is defined as a riskless plan to make profits without investment.

Definition 3.4
An admissible, self-financing trading strategy $\Phi$ is an arbitrage opportunity, if it holds

$$U_0^{\Phi,S} = 0 \quad \text{and} \quad \mathbb{P}(U_T^{\Phi,S} > 0) > 0.$$ 

In [12] under the assumption of price processes with bounded $p$-variation the existence of arbitrage opportunities is proved by the explicit construction of such strategies.

Theorem 3.5
In case that the price processes $S^k$, $k = 1, 2, \ldots, K$ (units are chosen so that $S_0^k := 1$) are continuous, strictly positive functions of bounded $p$-variation ($1 \leq p < 2$), for each $\alpha > 0$ the trading strategy $\alpha \Phi = (\alpha \Phi^1, \alpha \Phi^2, \ldots, \alpha \Phi^K)$ defined by

$$\alpha \Phi_t^k := \alpha U_t \frac{(S_t^k)^{\alpha-1}}{K} \sum_{i=1}^K (S_t^i)^\alpha,$$

where

$$\alpha U_t := \left( \frac{1}{K} \sum_{k=1}^K (S_t^k)^\alpha \right)^{\frac{1}{\alpha}},$$

is a self-financing trading strategy.

The proof of Theorem 3.5 can be found in [12], in [11] the case $p = 1$ is considered. The corresponding arbitrage property can be illustrated as follows. Consider arbitrary $\alpha, \beta$ with $0 < \alpha < \beta$. To make arbitrage profits, buy $\frac{1}{K}$ shares of each asset at time $t = 0$ and manage the portfolio according to the $\beta$-strategy and sell $\frac{1}{K}$ shares of each asset and continue selling according to the $\alpha$-strategy. Due to the fact, that it holds $\alpha U_t \leq \beta U_t$ and equality holds only in case $S_t^1 = S_t^2 = \ldots = S_t^K$ (which follows immediately from the corresponding properties of power means), we get a riskless profit of $\beta U_T - \alpha U_T$ at terminal time $T$ with no investment.
**Remark 3.6** Consider a market with price processes of bounded variation (i.e. $p = 1$). Because of the fact, that there are no non-trivial continuous processes of bounded variation, which are additionally local martingales (see for instance [13]), by the non-existence of an equivalent local martingale measure it is easy to deduce, that the so-called property of “No Free Lunch with Vanishing Risk” is not fulfilled. This property, which is a slightly stronger version of the arbitrage property, is used for instance in [14] for the discussion of arbitrage opportunities. The ideas of equivalent local martingale measures are pursued in [2] in order to show, that there exist “Free Lunch” opportunities in markets, where the price processes have been modelled with the help of integral functionals of weakly correlated processes.

## 4 A price model using weakly correlated functions

For simplification, we restrict the following considerations to the description of the price process $S$ of one financial asset.

**Definition 4.1**

Let $S = (S_t)_{t \in [0,T]}$ be a stochastic (price-) process with $S_t > 0$ for every $t \in [0,T]$. Let $\Delta t > 0$. The **logarithmic return** $\Delta t Z_t$ is defined by

$$\Delta t Z_t = \ln \left( \frac{S_t + \Delta t}{S_t} \right).$$

We remark, that $\Delta t$ stands for a representative time interval depending on the availability of the financial data.

**Remark 4.2** Assume the well-known model of a geometric Brownian motion for the description of a price process,

$$S_t = S_0 \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma B_t \right),$$

where $B = (B_t)_{t \in [0,T]}$ denotes a standard Brownian motion and $\mu$ and $\sigma > 0$ are real constants. In that case, the logarithmic return $\Delta t Z_t$ is given by

$$\Delta t Z_t = \left( \mu - \frac{\sigma^2}{2} \right) \Delta t + \sigma (B_{t+\Delta t} - B_t),$$

especially $\Delta t Z_s$ and $\Delta t Z_t$ are uncorrelated if and only if it holds $|t - s| \geq \Delta t$.

The aim is now, to describe a model of a price process, which takes into account the non-zero behaviour of the autocorrelation function of the returns and allows correlation between $\Delta t Z_s$ and $\Delta t Z_t$ for (small) time intervals $[s,t]$ with $\Delta t < |t - s| < \varepsilon$, where $\varepsilon$ is a fixed number.
As mentioned, the paper pursues the idea to introduce the concept of weakly correlated functions to the description of stochastic price processes. The idea of $\varepsilon$-correlated functions is, that for some $\varepsilon > 0$ it holds $R(t) \equiv 0$ for $|t| \geq \varepsilon$. This behavior entails advantages with regard to the opportunity of the analytic determination of stochastic characteristics as well as to the consideration of limit theorems ($\varepsilon \to 0$), cf. Theorem 2.12. For details, it is referred to [8].

Consider now a family of wide-sense stationary, weakly correlated processes, $(\varepsilon X_t) = (\varepsilon X_{t_i})_{t \in [0,T]}, \varepsilon > 0$ fulfilling the assumptions of Theorem 2.12 and possessing intensity $a = 1$. The existence of such processes is shown in [8]. Furthermore, we define for $0 \leq a_i \leq b_i \leq T$ $(i = 1, 2, \ldots, n)$ the integral functionals

$$\varepsilon r_i(\omega) = \int_{a_i}^{b_i} \varepsilon X_t(\omega) \, dt, \quad i = 1, 2, \ldots, n.$$  

Then from Theorem 2.12 it follows the convergence in distribution of

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\sqrt{\varepsilon}} (\varepsilon r_1, \varepsilon r_2, \ldots, \varepsilon r_n) = (\xi_1, \xi_2, \ldots, \xi_n),$$

where $(\xi_1, \xi_2, \ldots, \xi_n)$ is a Gaussian vector with mean

$$E\xi_i = 0, \quad i = 1, 2, \ldots, n$$

and covariance relations

$$E(\xi_i \xi_j) = \lambda([a_i, b_i] \cap [a_j, b_j]), \quad i, j = 1, 2, \ldots, n.$$  

Thereby, $\lambda$ is the Lebesgue measure on $\mathbb{R}$.

**Proposition 4.3**

We define the process $\varepsilon \tilde{B} = (\varepsilon \tilde{B}_t)_{t \in [0,T]}$ by

$$\varepsilon \tilde{B}_t(\omega) := \frac{1}{\sqrt{\varepsilon}} \int_{0}^{t} \varepsilon X_s(\omega) \, ds$$

and consider the behaviour of $\varepsilon \tilde{B}$ for $\varepsilon \downarrow 0$.

It holds $P(\varepsilon \tilde{B}_0 = 0) = 1$ uniformly with respect to $\varepsilon$. By Theorem 2.12, it follows, that for $0 = t_0 < t_1 < \ldots < t_n \leq T$ the vector of increments $(\varepsilon \tilde{B}_{t_1} - \varepsilon \tilde{B}_{t_0}, \varepsilon \tilde{B}_{t_2} - \varepsilon \tilde{B}_{t_1}, \ldots, \varepsilon \tilde{B}_{t_n} - \varepsilon \tilde{B}_{t_{n-1}})$ converges to a Gaussian random vector $(\xi_1, \xi_2, \ldots, \xi_n)$, whose components are stochastically independent. Furthermore, for $0 \leq s < t$, the difference $\varepsilon \tilde{B}_t - \varepsilon \tilde{B}_s$ converges in distribution to a Gaussian random variable $\xi$ with $\xi \sim N(0, t - s)$.

A comparison of the properties from Proposition 4.3 with the definition of a Standard Brownian motion $B$ shows the close relation of $\varepsilon \tilde{B}$ to the Brownian motion and motivates the replacement of $B$ by $\varepsilon \tilde{B}$ in the model of a geometric Brownian motion (cf. Remark 4.2).
Remark 4.4  We are not only interested in the asymptotic behaviour of $\tilde{\varepsilon}B$ but also in the properties for fixed $\varepsilon > 0$. The following properties can be found for fixed $\varepsilon > 0$.

- $P(\tilde{\varepsilon}B_0 = 0) = 1$,
- $E\tilde{\varepsilon}B_t = 0$ for all $t \geq 0$,
- $\tilde{\varepsilon}B$ possesses continuous trajectories,
- the variance of the increments $\tilde{\varepsilon}B_t - \tilde{\varepsilon}B_s$ depends for $0 \leq s < t \leq T$ only on the difference $t - s$.

The latter statement follows from

$$E \left( \tilde{\varepsilon}B_t - \tilde{\varepsilon}B_s \right)^2 = \frac{1}{\varepsilon} E \left( \int_s^t \int_s^t \varepsilon X_u \varepsilon X_v \, du \, dv \right) = \frac{1}{\varepsilon} \int_s^t \int_s^t E (\varepsilon X_u \varepsilon X_v) \, du \, dv$$

where the property of $\varepsilon X$ to be wide-sense stationary has been used.

According to Remark 4.2, in [2] the following model for the price $\varepsilon S = (\varepsilon S_t)_{t \in [0, T]}$ of a financial asset is considered,

$$\varepsilon S_t = \varepsilon S_0 \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma \varepsilon B_t \right) = \varepsilon S_0 \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) t + \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t \varepsilon X_s \, ds \right).$$

We consider the properties of $\varepsilon S$. Let $\varepsilon > 0$ be fixed. Then it holds

$$\frac{d\varepsilon S_t}{dt} = \varepsilon S_0 \left( \left( \mu - \frac{\sigma^2}{2} \right) + \frac{\sigma}{\sqrt{\varepsilon}} \varepsilon X_t \right) \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) t + \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t \varepsilon X_s \, ds \right),$$

i.e. the trajectories $t \mapsto \varepsilon S_t(\omega)$ are not only of bounded variation, but even absolutely continuous. It is clear, that the smoothness of the trajectories of the modelled process is a very critical assumption, which leads to arbitrage opportunities. Corresponding arbitrage strategies can easily be constructed with the help of the ideas shown in Section 3.

One advantage of the model lies on the correlation behaviour of the returns $\Delta t Z_s$ and $\Delta t Z_t$ (cf. Remark 4.2) under the model of $\varepsilon S$. It holds

$$\Delta t Z_t = \ln \left( \frac{\varepsilon S_{t+\Delta t}}{\varepsilon S_t} \right) = \left( \mu - \frac{\sigma^2}{2} \right) \Delta t + \frac{\sigma}{\sqrt{\varepsilon}} \left( \int_0^t \varepsilon X_s \, ds \right),$$

and because of the assumptions to $\varepsilon X$ the returns $\Delta t Z_s$ and $\Delta t Z_t$ are uncorrelated in case of $|t - s| \geq \varepsilon + \Delta t$, i.e. $(\Delta t Z_t)_{t \geq 0}$ is (for fixed $\Delta t > 0$) an $\varepsilon$-correlated process with $\varepsilon = \varepsilon + \Delta t$. In case of $|t - s| < \varepsilon$ the correlation between $\Delta t Z_s$ and $\Delta t Z_t$ is described by the correlation function of $X$. The problem, whether it is possible to adapt the behaviour of this correlation function to statistical data is in the interest of further research.


5 An advanced model

In Section 4, the Brownian motion $B = \{B_t\}_{t \in [0,T]}$ in the classical model

$$S_t = S_0 \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma B_t \right),$$

was just replaced with the process $\tilde{\tilde{B}} = \{\tilde{\tilde{B}}_t\}_{t \in [0,T]}$. Naturally, the next step to overcome the disadvantage of the existence of arbitrage opportunities could be to split up the Brownian motion term in the following way,

$$S_t = S_0 \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma_1 B_t + \sigma_2 \tilde{\tilde{B}}_t \right),$$

where $B = \{B_t\}_{t \in [0,T]}$ and $\tilde{\tilde{B}} = \{\tilde{\tilde{B}}_t\}_{t \in [0,T]}$ are stochastically independent Brownian motions on the considered stochastic basis and $\sqrt{\sigma_1^2 + \sigma_2^2} = \sigma$; $\sigma_1, \sigma_2 > 0$. Subsequently, $\tilde{\tilde{B}}$ is replaced by $\tilde{\tilde{B}}$ (cf. Proposition 4.3), where the underlying weakly correlated process is chosen to be stochastically independent from $B$. This leads to the model

$$\hat{\hat{S}}_t = \hat{\hat{S}}_0 \exp \left( \left( \mu - \frac{\sigma_1^2 + \sigma_2^2}{2} \right) t + \sigma_1 B_t + \frac{\sigma_2}{\sqrt{\varepsilon}} \int_0^t \varepsilon X_s ds \right).$$

In terms of the corresponding stochastic differential equation this means

$$d\hat{\hat{S}}_t = \hat{\hat{S}}_t \left( \left( \mu - \frac{\sigma_1^2 + \sigma_2^2}{2} + \frac{\sigma_2}{\sqrt{\varepsilon}} X_t \right) dt + \sigma_1 dB_t \right).$$

We discuss the properties of this model. Of course, the consideration of two independent sources of randomness is a new aspect and impacts the model for instance with respect to questions of completeness. On the other hand, we just introduced in the classical model of geometric Brownian motion an independent drift term, which leads to the desired behaviour of the returns. This is, for instance, in the spirit of [15] of incorporating predictability into the Black-Scholes-Model with the help of drift terms. In [15] not only independent drift terms are considered, the main focus lies on a multivariate trending Ornstein-Uhlenbeck process as a general model for return processes.

The logarithmic return $\Delta t Z_t$ under the model of $\hat{\hat{S}}$ is given by

$$\Delta t Z_t = \ln \left( \frac{\hat{\hat{S}}_{t+\Delta t}}{\hat{\hat{S}}_t} \right) = \left( \mu - \frac{\sigma_1^2 + \sigma_2^2}{2} \right) \Delta t + \sigma_1 (B_{t+\Delta t} - B_t) + \frac{\sigma_2}{\sqrt{\varepsilon}} \int_0^{\Delta t} \varepsilon X_s ds,$$

consequently it follows

$$E\left( (\Delta t Z_s - E\Delta t Z_s) (\Delta t Z_t - E\Delta t Z_t) \right)$$

$$= \frac{\sigma_2^2}{\varepsilon} \int_0^{\Delta t} \int_0^{\Delta t} E(\varepsilon X_u \varepsilon X_{v+\Delta t-s}) \, du \, dv + \sigma_1^2 \cdot \left\{ \begin{array}{ll} \frac{\Delta t}{\Delta t} - |t-s| & |t-s| < \Delta t \\ 0 & |t-s| \geq \Delta t \end{array} \right..$$
Consequently, similar to the model of Section 4, the returns $\Delta t Z_s$ and $\Delta t Z_t$ are uncorrelated in case of $|t - s| \geq \varepsilon + \Delta t$.

The trajectories $t \mapsto \tilde{S}_t(\omega)$ accord now to the classical semimartingale-framework, where the concepts of trading strategies introduced in Section 3 can be defined in a well-known manner. A consequence of such models is, that (under smooth regularity conditions) option pricing formulas are shown to be functionally independent of the drift of the price processes. In [16] this is shown (in a different context) for the case of a continuous drift term which is stochastically independent of the included Brownian motion.

On the other hand it is clear, that for a fixed unconditional variance of the true returns (i.e. the observed data) by introducing correlation effects via the drift, the value of the diffusion coefficient must change as to keep the unconditional variance constant. Therefore, even though the option pricing formula is unaffected by changes in the drift term, option prices do change (cf. [15]). Also in [6] it is pointed out, that the study of corresponding models is meaningful, whereby the estimate of volatility which is obtained from data has to be reinterpreted in the light of the specific model which is assumed.

However, the introduced model with $\varepsilon$-correlated returns shows close relations to the classical model of a geometric Brownian motion. A further advantage (especially in comparison with the multivariate Ornstein-Uhlenbeck models considered in [15]) consists in the wide variety of autocorrelation functions which can be modeled and in the effect, that the correlation vanishes for time intervals which are sufficiently large.

While the theory of weakly correlated functions has been established for the consideration of numerous problems concerning differential equations of physics and engineering, a great deal of further work is still necessary to improve the properties of the model to make it practicable for the description of stochastic price processes as well as to adapt the model to financial time series. Nevertheless, the described relation of the model using weakly correlated processes to the classical price model as well as the opportunity of generalizations seem to make it desirable to study the application of weakly correlated functions in mathematical finance.

References


