Some remarks to large deformation elasto–plasticity (continuum formulation)

Abstract

The continuum theory of large deformation elasto–plasticity is summarized as far as it is necessary for the numerical treatment with the Finite–Element–Method. Using the calculus of modern differential geometry and functional analysis, the fundamental equations are derived and the proof of most of them is shortly outlined. It was not our aim to give a contribution to the development of the theory, rather to show the theoretical background and the assumptions to be made in state of the art elasto–plasticity.
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1 Introduction

From a theoretical point of view, a physical body can be treated as a set $\mathcal{B}$ of material points representing the atoms or molecules of the body’s material. Considering this body as a continuum, the theory shortly outlined in the following applies. For more details see [Tri81].

First we introduce a system $\mathcal{U}$ of open subsets of $\mathcal{B}$ with the following properties:

a) $\emptyset \in \mathcal{U}$ and $\mathcal{B} \in \mathcal{U}$.

b) Finite intersections of subsets from $\mathcal{U}$ are again subsets of $\mathcal{U}$.

c) Any unions of subsets from $\mathcal{U}$ are again subsets of $\mathcal{U}$.

$\mathcal{U}$ is called a topology of $\mathcal{B}$ and $\mathcal{A}_x \in \mathcal{U}$ is called neighbourhood of a point $x \in \mathcal{B}$ if $x \in \mathcal{A}_x$. $\mathcal{B}$ is called a Hausdorff space, if for each two points $x \in \mathcal{B}$ and $y \in \mathcal{B}$ there exist neighbourhoods $\mathcal{A}_x$ and $\mathcal{A}_y$ with $\mathcal{A}_x \cap \mathcal{A}_y = \emptyset$, i.e. the topology of $\mathcal{B}$ is rich enough to separate the points of $\mathcal{B}$.

The Hausdorff space $\mathcal{B}$ is said to have the dimension $n$, if for each $x \in \mathcal{B}$ there exists a neighbourhood $\mathcal{A}_x$ that can be mapped onto an open subset of $\mathbb{R}^n$ by a bijective mapping, called chart or local coordinate system, with components $\{x^k\}_{k=1..n}$.

If this mapping can be chosen to be $C^\infty$ and orientation preserving, the $n$-dimensional Hausdorff space $\mathcal{B}$ equipped with the chart $\{x^k\}$ is called a $n$-dimensional $C^\infty$-manifold $^1$.

The mathematical assumptions made above correspond to the common physical understanding of bodies and physical spaces. So, from a general point of view, including also shells, rods and ”exotic” materials like liquid crystals, a physical body $\mathcal{B}$ and the physical space $\mathcal{S}$ containing $\mathcal{B}$ can be considered to be special cases of manifolds.

To have a unified approach, as well as for conceptual clarity, it is useful to think geometrically and to represent bodies in terms of manifolds [MH83].

2 Some Differential Geometry

The tangent space

Let $\mathcal{M}$ be a $n$-dimensional $C^\infty$-manifold and let $x \in \mathcal{M}$. Then the tangent space $T_x \mathcal{M}$ to $\mathcal{M}$ at $x$ is the vector space $\mathbb{R}^n$ of vectors regarded as emanating from $x$. So vectors like the velocity and the acceleration (see ch. 3) can be understood as elements of $T_x \mathcal{S}$ to $\mathcal{S}$ at $x \in \mathcal{S}$, where $\mathcal{S}$ denotes the physical space described as $C^\infty$-manifold.

The dual space $T^*_x \mathcal{M}$ to $T_x \mathcal{M}$ is called the cotangent space. It’s elements $\alpha \in T^*_x \mathcal{M}$ will be called covectors. A covector $\alpha$ is a functional $\alpha(x): T_x \mathcal{M} \mapsto \mathbb{R}$. In this sense $T^*_x \mathcal{M}$ is dual to $T_x \mathcal{M}$ $^2$. A covector is said to be ”covariant”, and a vector to be ”contravariant”.

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$^1$ This is not the most general definition of a $C^\infty$ manifold, but it should be suitable for the understanding of the present subject (for more details see [Tri81]).

$^2$ After the introduction of cotangent spaces $T^*_x \mathcal{M}$, a vector $v$ also can be defined as a functional $v(x): T^*_x \mathcal{M} \mapsto \mathbb{R}$.
Tensors

A tensor $t$ of the type $(\stackrel{p}{q})$ at $x \in M$ is a multilinear mapping

$$t : T_x^* \mathcal{M} \times \ldots \times T_x^* \mathcal{M} \times T_x \mathcal{M} \times \ldots \times T_x \mathcal{M} \rightarrow \mathbb{R}.$$ 

It is said that $t$ is **contravariant** of rank $p$ and **covariant** of rank $q$.

Let $\{e_j\} \subset T_x \mathcal{M}$ and $\{e^i\} \subset T_x^* \mathcal{M}$ be the base vectors in $T_x \mathcal{M}$ and the dual base vectors in $T_x^* \mathcal{M}$ respectively. Then the components of $t$ are defined by

$$t^{i_1 \ldots i_p} := t(e^{i_1}, \ldots, e^{i_p}, e_{j_1}, \ldots, e_{j_q}),$$

and

$$t(w^1, \ldots, w^p, v_1, \ldots, v_q) = t^{i_1 \ldots i_p} w^{i_1} \ldots w^{i_p} v_{j_1} \ldots v_{j_q}$$

holds for all $w^i = w^i_a e^a \in T_x^* \mathcal{M}$ and $v_j = v^b_j e_b \in T_x \mathcal{M}$.

The Riemannian metric

For $x \in \mathcal{M}$ let $g$ be a **covariant tensor** of rank 2 (i.e., a tensor of type $(\frac{2}{0})$) with the properties

$$g(u, v) = g(v, u); \quad u, v \in T_x \mathcal{M} \quad (2.1)$$

$$g(u, u) > 0; \quad 0 \neq u \in T_x \mathcal{M}.$$ 

With this symmetric and positive definite (and therefore invertible) **metric tensor** $g$ the functional $\langle u, v \rangle := g(u, v)$ is an inner product on $T_x \mathcal{M}$. Written in components it reads

$$\langle u, v \rangle = u^a v^b g_{ab} \quad (2.2).$$

Introducing such a tensor $g$ for each $x \in \mathcal{M}$ we get a Riemannian metric on $\mathcal{M}$.

Tensor and vector fields

The associated tensor and vector fields

For $\alpha := \alpha_a e^b \in T_x^* \mathcal{M}$ the **associated vector field** $\alpha^i := \alpha^a e_a \in T_x \mathcal{M}$ is defined by it’s components $\alpha^a := g^{ab} \alpha_b$ with $g^{ab}$ denoting the components of $g^{-1}$. The **associated covector field** $u^i := u_a e^a \in T_x^* \mathcal{M}$ to $u := u^b e_b \in T_x \mathcal{M}$ is defined by $u_a := g^{ab} u^b$. Similarly $t^i$ means the **tensor associated** to $t$ with all indices lowered and $t_i$ with all indices raised, especially $g^{-1} \equiv g^i$ and $g \equiv g^i$.

Scalar products

Using this notation and equation (2.2), we define the dual paring $\langle \cdot, \cdot \rangle$ between elements of $T_x^* \mathcal{M}$ and $T_x \mathcal{M}$ and the scalar product $\langle \cdot, \cdot \rangle$ in $T_x \mathcal{M}$ by $\langle u, v \rangle := \langle u^i, v_i \rangle$ for $u \in T_x^* \mathcal{M}$, $v \in T_x \mathcal{M}$ and $\langle u, v \rangle := \langle u^i, v^i \rangle$ for $u, v \in T_x^* \mathcal{M}$. These definitions yield to the identity $\langle u^1, v^1 \rangle = \langle u, v \rangle$ for any $u, v \in T_x^* \mathcal{M}$ as well as $\langle u, v \rangle = \langle u^1, v^1 \rangle$ for any $u, v \in T_x \mathcal{M}$.
The dual base

In two or three dimensions, the dual base is defined by the vector product $e^a := \mathbf{e}_a \times \mathbf{e}_b / |\mathbf{e}_a \cdot \mathbf{e}_b|$. Due to this definition the equation $(e^a, e_b) = \delta^a_b$ holds, and once the base $\{e_i\}$ is given, the base vectors $e^a$ are uniquely defined. Since we deal here with spaces of arbitrary dimension, we have to go another way:

Let the base $\{e_i\}$ in $T_x\mathcal{M}$ be given. On page 2 we just introduced and used a base $\{e^a\}$ in $T^*_x\mathcal{M}$. From now on we postulate, that the base vectors $\{e^a\}$ fulfill $(e^a, e_b) = \delta^a_b$. This uniquely determines the $e^a$, since in case of $n$ dimensions $n^2$ conditions have to be fulfilled by choosing $n$ components of $n$ (co-)vectors. Note that this doesn’t mean to have any orthonormalized system\(^3\) in $T_x\mathcal{M}$ or in $T^*_x\mathcal{M}$.

The covariant derivative

Let $x \in \mathcal{M}$. The covariant derivative of a vector $v \in T_x\mathcal{M}$ along a vector $w \in T_x\mathcal{M}$ is a bilinear mapping defined by $gradfv : T_x\mathcal{M} \times T_x\mathcal{M} \rightarrow T_x\mathcal{M}$, fulfilling

\[
grad_{fw}v = f grad_wv
\]

\[
grad_{fw}v = f grad_wv + (\nabla f, w)v
\]

(2.3)

for all scalar functions $f$, where $\nabla f = \partial f / \partial x^a e^a \in T^*_x\mathcal{M}$ and $(\nabla f, w) = \partial f / \partial x^a w^a$\(^4\). Defining this for all $x$, $v$ and $w$ have to be regarded as vector fields and we get a connection on the manifold $\mathcal{M}$. The Christoffel symbols $\gamma^c_{ab}$ of this connection on $\mathcal{M}$ with the coordinate system $\{x^c\}$ are defined by $grad_{e_t} e_a = \gamma^c_{ab} e_c$. Using the properties (2.3) for $v = v^a e_a$ and $w = w^b e_b$, the identity $grad_wv = v^b [v^a grad_{e_t} e_a + (\nabla v^a, e_t) e_a]$ follows. Introducing the Christoffel symbols we get

\[
grad_wv = \left( \partial v^c / \partial x^b + \gamma^c_{ab} v^a \right) w^b e_c = v^b w^a e_c
\]

(2.4)

with $v^b := \partial v^c / \partial x^b + \gamma^c_{ab} v^a$.

Defining for each $v \in T_x\mathcal{M}$ another tensor $grad$ of type $(1, 1)$ with components

\[
(gradv)^a_b := \partial v^a / \partial x^b + \gamma^a_{bc} v^c = v^a_b,
\]

(2.5)

the covariant derivative can be expressed by $gradfwv \equiv (gradv, w)$\(^5\).

We define the covariant derivative of a covector $u \in T^*_x\mathcal{M}$ along a vector $w \in T_x\mathcal{M}$ via the

\[\text{As an example, in two dimensions, } e_1 = (1, 0)^T, e_2 = (1, 1)^T, e^1 = (1, 1)^T \text{ and } e^2 = (0, 1)^T \text{ fulfills these conditions.} \]

\[\text{Here and in the following we use the symbol } (\cdot, \cdot), \text{ originally defined for the dual pairing, also in the sense of } (gradv, w)^a := (gradv)^a_b w^b, \text{ since the components of the resulting vector can be regarded as dual pairings of the "columns" of } grad \text{ with } w. \]
associated vector fields:

\[ \nabla_w u := \left( \nabla_w u^i \right)^j, \quad u = u_a e^a, \quad w = w^b e_b. \] (2.6)

Using the rules from page 2, the identity \( g_{ab} \varepsilon^{ac} = \delta^c_b \), the properties (2.3) and the representation (2.4) some calculus shows that the components \( r_f \) of \( r := \nabla_w u = r_f e^f \) will be

\[ r_f = w^b \left[ \frac{\partial u_f}{\partial x^b} + u_d \left( g_{ef} \varepsilon^{ed} + g_{ef} \frac{\partial q_{de}}{\partial x^b} \right) \right]. \]

Applying (2.10) to \( g_{ef} \varepsilon^{ed} \) and taking for the rightmost term into account that \( 0 = \frac{\partial \delta^c_b}{\partial x^b} = \frac{\partial (g_{ef} \varepsilon^{ed})}{\partial x^b} = g_{ef} \frac{\partial q_{de}}{\partial x^b} + g_{ef} \frac{\partial q_{de}}{\partial x^b} \), we find

\[ r_f = w^b \left[ \frac{\partial u_f}{\partial x^b} - u_d g^{de} \frac{\partial q_{de}}{\partial x^b} \right]. \]

Thus \( r_f \) is a local tensor \( r_f \) as a multivector of \( p \) vectors and \( q \) covectors\(^6\), it is natural to define the components of the covariant derivative of a tensor by a generalization of (2.4) and (2.7):

\[ \nabla_w u = \left( \frac{\partial u_f}{\partial x^b} - \gamma^c_{cb} u_a \right) w^b e_c = u_{d} w^{b} e^{c} \] (2.7)

with \( u_{d} := \frac{\partial u_f}{\partial x^b} - \gamma^c_{cb} u_a \).

Regarding a tensor \( t \) of type \( (r, s) \) as a multivector of \( p \) vectors and \( q \) covectors\(^6\), it is natural to define the components of the covariant derivative of a tensor by a generalization of (2.4) and (2.7):

\[ \left( \nabla_w t \right)^a_{d, \ldots, b} := t^a_{d, \ldots, b} \] (2.8)

\[ \left[ \frac{\partial t^a_{d, \ldots, b}}{\partial x^f} + \left( \frac{\partial t^a_{d, \ldots, b}}{\partial x^f} \gamma^{c}_{ef} + \ldots + t^a_{d, \ldots, b} \gamma^{c}_{ef} \right) - \left( t^a_{d, \ldots, b} \gamma^{c}_{ef} + \ldots + t^a_{d, \ldots, b} \gamma^{c}_{ef} \right) \right] w^c. \]

The Riemannian space

If a connection is defined on a manifold \( M \), \( M \) is called an affine space. Introducing a Riemannian metric \( g \) on an affine space \( M \), \( M \) becomes torsion free, i.e. the Christoffel symbols are symmetric \( \gamma^c_{ab} = \gamma^c_{ba} \). A torsion free affine space is called a Riemannian space.

As can be shown [Tri81], in a Riemannian space \( M \) for each \( x \in M \) a local Cartesian coordinate system with the base vectors \( i_j \in T_x M \) can be found with \( \gamma^c_{ijk} = 0 \) and \( g(i_i, i_j) = \delta_{ij} \). Therefore the metric tensor \( g \) is uniquely determined. Its components may be computed by \( g_{ab} := g(e_a, e_b) = e_a^i e_b^j g(i_i, i_j) = \delta^c_{ab} = e_a^c e_b^c \). With components \( e^i_a \) of \( e_a \) with respect to the base \( \{ i_i \} \ e_a = e^i_a i_i \) and the components \( x^a \) and \( z^i \) for any \( x = x^a e_a = z^i i_i \), we get \( z^i = x^a e_a^i \). This gives \( e^i_a = \frac{\partial z^i}{\partial x^a} \), and the first part of (2.9) is proved. The Christoffel symbols \( \gamma^a_{bc} \) are defined by \( \nabla_b e_a = \gamma^a_{bc} e_c \). To compute \( \nabla_b e_a \), we formulate (2.4) in the Cartesian coordinate system and use it for \( w := e_a = e^i_a i_i \) and \( v := e_b = e^i_b i_i \) so

\[ \text{roughly spoken, write the vectors and covectors one besides the other} \]

\[ ^6 \text{roughly spoken, write the vectors and covectors one besides the other} \]
\[ \nabla_{e_k} e^i = \left( \frac{\partial e^i}{\partial z^j} + \gamma_{kj}^i \right) e^j e^k \]  

The resulting equation
\[ \frac{\partial e^i}{\partial z^j} e^j e^i = \gamma_{ab}^c e^i e^k \]
is equivalent to
\[ \gamma_{ab}^c \frac{\partial z^i}{\partial x^b} \delta_{ij} \]
and therefore as immediately can be seen, the components of a vector transform as
\[ v^i = \frac{\partial z^i}{\partial x^b} v^b \]
and consequently we have
\[ \frac{\partial}{\partial z^i} v^j = \frac{\partial x^a}{\partial z^j} \frac{\partial}{\partial x^a} \left( \frac{\partial z^i}{\partial x^b} v^b \right) = \frac{\partial x^a}{\partial x^j} \left( \frac{\partial^2 z^i}{\partial x^a \partial x^b} v^b + \frac{\partial z^i}{\partial x^a} \frac{\partial v^b}{\partial x^b} \right). \]

Using the notation from (2.9), (2.4) and (2.5) we get the divergence
\[ \nabla v = \gamma_{ab}^c v^a + \frac{\partial v^a}{\partial x^a} = v^a = \text{trace}(\nabla v). \]

**The divergence of a vector in noncartesian coordinates**

For any vector \( \textbf{v} \in T_x \mathcal{M} \) we have
\[ \textbf{v} = v^i e_i = v^b e_b = v^b \frac{\partial z^i}{\partial x^b} e^i \]
and therefore as immediately can be seen, the components of a vector transform as
\[ v^i = \frac{\partial z^i}{\partial x^b} v^b \]
and consequently we have
\[ \frac{\partial}{\partial z^i} v^j = \frac{\partial x^a}{\partial z^j} \frac{\partial}{\partial x^a} \left( \frac{\partial z^i}{\partial x^b} v^b \right) = \frac{\partial x^a}{\partial x^j} \left( \frac{\partial^2 z^i}{\partial x^a \partial x^b} v^b + \frac{\partial z^i}{\partial x^a} \frac{\partial v^b}{\partial x^b} \right). \]

Using the notation from (2.9), (2.4) and (2.5) we get the divergence
\[ \nabla v = \gamma_{ab}^c v^a + \frac{\partial v^a}{\partial x^a} = v^a = \text{trace}(\nabla v). \]

**Push forward and pull back**

Let \( \mathcal{B} \) and \( \mathcal{S} \) be two (not necessarily different) manifolds, \( \textbf{X} \in \mathcal{B}, \textbf{x} \in \mathcal{S} \) and \( \varphi : \mathcal{B} \rightarrow \mathcal{S} \) \((x = \varphi(X))\) a regular mapping in the sense, that \( \varphi \) has a \( C^1 \) inverse. For \( \textbf{U} \in T_X \mathcal{B} \) the vector field \( \varphi_* \textbf{U} \in T_{\varphi(X)} \mathcal{S} \) with components (2.12) is called the push forward of \( \textbf{U} \) by \( \varphi \):

\[ (\varphi_* \textbf{U})^a : = \left( \frac{\partial (\varphi)^a}{\partial X^A} U^A \right) |_{\varphi^{-1}(x)}. \]

The components of the pull back \( \varphi^* \textbf{u} \in T_X \mathcal{S} \) of some vector field \( \textbf{u} \in T_{\varphi(X)} \mathcal{S} \) by the mapping \( \varphi \) are defined in (2.13):

\[ (\varphi^* \textbf{u})^A : = \left( \frac{\partial (\varphi^{-1})^A}{\partial x^a} u^a \right) |_{\varphi(X)}. \]
In the same way for covectors \( \mathbf{V} \in T^*_x \mathcal{B} \) and \( \mathbf{v} \in T_{\varphi(x)} \mathcal{S} \) the push forward \( \varphi_\ast \mathbf{V} \in T^*_x \mathcal{B} \) and the pull back \( \varphi^\ast \mathbf{v} \in T_{\varphi(x)} \mathcal{S} \) are defined by their components

\[
(\varphi_\ast \mathbf{V})(x) := \left( \frac{\partial (\varphi^{-1})^A}{\partial x^a} \right)(x) \mathbf{V}^A \bigg|_{\varphi^{-1}(x)}, \quad (\varphi^\ast \mathbf{v})(x) := \left( \frac{\partial (\varphi)\partial x^a}{\partial (\varphi^{-1})^A} \right) \mathbf{v}^a \bigg|_{\varphi(x)} .
\] (2.14)

If \( T \) is a tensor of type \( (p, q) \) acting on \( \mathcal{B} \), its push forward \( \varphi_\ast T \) is a tensor of the same type on \( \varphi(\mathcal{B}) \) defined by:

\[
(\varphi_\ast T)(x)[v^1, \ldots, v^p, u^1, \ldots, u^q] := T_{\varphi^{-1}(x)}(\varphi^\ast v^1, \ldots, \varphi^\ast v^p, \varphi^\ast u^1, \ldots, \varphi^\ast u^q) \bigg|_{\varphi^{-1}(x)} .
\] (2.15)

with \( v^i \in T^*_x \mathcal{S} \) and \( u^i \in T_x \mathcal{S} \), and the pull back of a tensor \( t \) defined on \( \varphi(\mathcal{B}) \) is:

\[
(\varphi^\ast t)(x)[v^1, \ldots, v^p, u^1, \ldots, u^q] := t_{\varphi(x)}(\varphi_\ast v^1, \ldots, \varphi_\ast v^p, \varphi_\ast u^1, \ldots, \varphi_\ast u^q) \bigg|_{\varphi(x)} .
\] (2.16)

The pull back and the push forward of scalar functions \( f(x) \) and \( F(X) \) are defined by

\[
\varphi^\ast f(x) := f(\varphi(X)) , \quad \varphi_\ast F(X) := F(\varphi^{-1}(x)) .
\] (2.17)

The material time derivative

Let \( c(t) \) be a integral curve in \( \mathcal{M} \), i.e. the tangent to \( c(t) \) can be found as \( \mathbf{v} = \frac{dc}{dt} \in T_{c(t)} \mathcal{M} \). Then the material time derivative of \( \mathbf{a} = a^i e_i \in T_{c(t)} \mathcal{M} \) and \( \mathbf{b} = b^i e_i \in T_{c(t)} \mathcal{M} \) with \( a^i, b^i, e^i, e_i \) depending on \( \mathbf{x}(t) := c(t) \) is given by

\[
\frac{d}{dt} a^i = v^b \frac{\partial a_b^i}{\partial x^b} e_c + v^a \frac{\partial a^i_a}{\partial x^a} e_c = v^b \left( \frac{\partial a_b^i}{\partial x^b} + a^i_a \gamma^a_b \right) e_c = a^i_b v^b e_c = \text{grad}_x \mathbf{a}
\]

\[
\frac{d}{dt} b^i = v^b \frac{\partial b_b^i}{\partial x^b} e_c + v^a \frac{\partial b^i_a}{\partial x^a} e_c = v^b \left( \frac{\partial b_b^i}{\partial x^b} - b^i_a \gamma^a_b \right) e_c = b_{ab} v^b e^c = \text{grad}_x \mathbf{b}
\] (2.18)

with the covariant derivative "grad" from (2.4) and (2.7). To verify (2.18) we only have to remember\(^8\), that \( \frac{\partial}{\partial z^a} e_b^c = \frac{\partial^2 z^j}{\partial x^a \partial z^b} \frac{\partial z^c}{\partial z^j} e_c = \gamma^c_{ab} e_c \) and \( \frac{\partial}{\partial z^a} e^b = \frac{\partial}{\partial x^a} e^b = \frac{\partial}{\partial x^a} \frac{\partial x^c}{\partial z^j} \frac{\partial z^j}{\partial x^c} e^a = -\gamma^c_b \delta^a = \text{grad}_x \mathbf{v} \). In a similar way, the material time derivative of a tensor \( t \) can be proved to be

\[
\frac{d}{dt} t = \text{grad}_v t .
\] (2.19)

If \( t, a, b \) explicitly depend on \( t \), their material derivatives are given by

\[
\frac{d}{dt} t = \frac{\partial}{\partial t} t + \text{grad}_v t , \quad \frac{d}{dt} a = \frac{\partial}{\partial t} a + \text{grad}_v a , \quad \frac{d}{dt} b = \frac{\partial}{\partial t} b + \text{grad}_v b .
\] (2.20)

\(^8\)cf. pages 4, 5

\(^9\)taking into account \( \frac{\partial x^a}{\partial z^j} \frac{\partial x^b}{\partial x^c} = \delta^a_e \) and consequently

\[
0 = \frac{\partial}{\partial x^c} \delta^a_e = \frac{\partial}{\partial x^c} \left( \frac{\partial x^a}{\partial z^j} \frac{\partial x^b}{\partial x^c} \right) = \frac{\partial x^a}{\partial z^j} \frac{\partial x^b}{\partial x^c} + \frac{\partial x^a}{\partial x^c} \frac{\partial x^b}{\partial z^j} \frac{\partial x^c}{\partial z^j}
\]
The transport of vectors and tensors along curves

Let \( \psi_{t,s} : \mathcal{M} \to \mathcal{M} \) for real \( s \) and \( t \) be a collection of maps, such that for an integral curve \( x = c(s) \) of \( v \) (cf. page 6) \( c(t) := \psi_{t,s}(c(s)) \) is an integral curve of \( v \) again. Assume in addition, that \( \psi_{s,s}(x) = x \) and \( \psi_{t,s} \circ \psi_{s,r} = \psi_{t,r} \) holds¹⁰.

Based on this construction, we introduce a linear mapping \( \Psi_{t,s} : T_{c(s)} \mathcal{M} \to T_{c(t)} \mathcal{M} \) with \( \Psi_{t,s} \circ \Psi_{s,r} = \Psi_{t,r} \) and \( \Psi_{s,s} \) being the identical mapping. Then \( \Psi_{t,s} \) transports a vector \( a_s \in T_s \mathcal{M} \) emanating from \( x := c(s) \) to \( x' := c(t) \), i.e. \( a_t := \Psi_{t,s} a_s \in T_v \mathcal{M} \) with \( a_t^i := (\Psi_{t,s})_a^i a_s^i \).

Assuming the transport to be done in a parallel manner, i.e. \( \frac{d}{ds} (\Psi_{s,r} a_r) = \frac{d}{ds} a_s = 0 \), \( \Psi_{t,s} \) is called shifter and denoted by \( S_{t,s} \). For the case of parallel transport, we get from (2.18) \( 0 = \left( \frac{d}{ds} a_s \right)^a = \left[ \frac{d}{ds} (S_{s,r})_c^a \gamma_c^b v^c (S_{s,r})_r^b \right] a_s^b \), or \( \lim_{s \to t} \frac{d}{ds} (S_{s,t})_b^a = - \gamma_c^b v^c \). Since \( S_{t,s} = S_{s,t}^{-1} \), and therefore \( (S_{t,s})_a^b (S_{s,t})_b^c = \delta^a_c \), the relation \( \left. \frac{d}{ds} (S_{s,t})_c^a \right|_{s=t} (S_{s,t})_b^c = - (S_{s,t})_c^a \left. \frac{d}{ds} (S_{s,t})_b^d \right|_{s=t} \) can be found, tending to \( \lim_{s \to t} \frac{d}{ds} (S_{s,t})_c^a \delta^a_b = \delta^a_c \gamma^b v^c \) if \( s \) tends to \( t \). Applying this to the equation (2.18), (2.20), we get

\[
\lim_{s \to t} \frac{d}{ds} (S_{s,t} a_s)^a = \frac{d}{dt} a_t^a + \gamma_c^b v^c a_t^b = \left( \frac{d}{dt} a_t \right)^a . \tag{2.21}
\]

Applying the same calculus to a tensor \( t \) yields to

\[
\lim_{s \to t} \frac{d}{ds} (S_{s,t} t_s)^a = \frac{d}{dt} t_t . \tag{2.22}
\]

The Lie derivative

Using in (2.22) the push forward induced by \( \psi_{t,s} \) instead of a shifter, we get the Lie derivative:

Let the mapping \( \varphi \) used in (2.14) be \( \psi_{t,s} \). Then, the transport \( \Psi_{t,s} := \psi_{s,t} \) is well defined and

\[
L_v t := \lim_{s \to t} \frac{d}{ds} (\psi_{s,t} t_s) = \frac{d}{dt} t_t \tag{2.23}
\]

called the Lie derivative \( L_v t \) of the tensor \( t \). Owing to \( \psi_{s,t} t_s = \psi_{t,s}^* t_s \), the Lie derivative defined in (2.23) is equivalent to \( L_v t = \lim_{s \to t} \frac{d}{ds} (\psi_{s,t}^* t_s) \).

Holding \( s \) fixed in \( t_s \) at \( s = t \), i.e. \( \tilde{t}_t := t(t, c(s)) \), we get the autonomous Lie derivative

\[
\mathcal{L}_v t := \lim_{s \to t} \frac{d}{ds} (\psi_{s,t}^* \tilde{t}_t) . \tag{2.24}
\]

¹⁰Then \( \psi_{t,s} \) is called the flow or evolution operator of \( v \)
¹¹Note, that for \( a_t := S_{t,s} a_s \) we have

\[
\left( \frac{d}{ds} a_s \right)^a = \left( \frac{d}{ds} (S_{s,t})^a_b a_t^b (t) \right)^a = \left( \frac{d}{ds} (S_{s,t})^a_b \right)_b^a v^c (a_t^b (t)) = \frac{d}{ds} \left( (S_{s,t})^a_b a_t^b \right) = \frac{da_t^a}{ds} .
\]
In the general case, the autonomous Lie derivative is related to $L_v t$ by

$$L_v t = \frac{\partial}{\partial t} t + L_v t .$$  \hspace{1cm} (2.25)

The autonomous Lie derivative of a tensor

The components of the autonomous Lie derivative of a tensor $t$ are

$$(L_v t)^{a \ldots b} = t^{a \ldots b} + \left( t^{j \ldots h} v_f^a + \ldots + t^{a \ldots f} v^h_f \right) + \left( t^{a \ldots h} v^j_f + \ldots + t^{j \ldots f} v^h_f \right)$$  \hspace{1cm} (2.26)

(for notation see (2.18, 2.8)). In the special case of the metric tensor $g$ the autonomous Lie derivative can be simplified to

$$(L_v g)_{ab} = \frac{\partial g_{ab}}{\partial x^c} v^c + g_{ci} \frac{\partial v^c}{\partial x^a} + g_{ac} \frac{\partial v^c}{\partial x^b},$$  \hspace{1cm} (2.27)

taking into account (2.26, 2.8, 2.10).

The Lie derivative of a function

Let $f_s := f(s, c(s))$ be a function on $S$. Then, the push forward of $f_s$ induced by $\psi_{t,s}$ reads $\psi_{t,s} f_s = f(\psi_{t,s}^{-1} c(t), c(t)) = f(s, c(s))$. Due to $v_s := v (s, c(s)) = \frac{dc}{ds}$ we have

$$L_v f := \lim_{s \rightarrow t} \left( \frac{d}{ds} \psi_{t,s}^{-1} f_s \right) = \lim_{s \rightarrow t} \left( \frac{\partial f_s}{\partial s} \frac{\partial \psi_{t,s}}{\partial x^b} + \frac{\partial f_s}{\partial x^a} \frac{\partial (\psi_{t,s}^{-1})^a}{\partial s} \right) = \left( \frac{\partial f_s}{\partial t} + \frac{\partial f_t}{\partial x^i} v^i_t \right) = \frac{df}{dt}. \hspace{1cm} (2.28)$$

The Lie derivative of a vector field

For a vector field $w$ on $S$ we get $(\psi_{t,s} w)_s^a = \left( \frac{\partial (\psi_{t,s})^a}{\partial x^b} w_s^b \right)_{\psi_{t,s}^{-1}}$ for the push forward’s components. Some simple calculus gives $\frac{d}{ds} \left( \psi_{t,s}^{-1} w_s^b \right) = \frac{\partial (\psi_{t,s})^a}{\partial x^b} \frac{d}{ds} w_s^b + \frac{\partial w_s^b}{\partial x^b} \frac{d}{ds} \left( \psi_{t,s}^{-1} \right)^a$, with $\frac{\partial (\psi_{t,s})^a}{\partial x^b} \frac{d}{ds} w_s^b = \frac{\partial (\psi_{t,s})^a}{\partial x^b} \left( \frac{\partial w_s^b}{\partial s} + \frac{\partial w_s^c}{\partial x^c} \frac{d}{ds} \left( \psi_{t,s}^{-1} \right)^b \right) = \frac{\partial (\psi_{t,s})^a}{\partial x^b} \frac{d}{ds} \left( \psi_{t,s}^{-1} \right)^b$, and using

$$\frac{\partial (\psi_{t,s}^{-1})^b}{\partial x^c} \frac{d}{ds} \left( \psi_{t,s}^{-1} \right)^a = -\frac{\partial (\psi_{t,s}^{-1})^a}{\partial x^b} \frac{d}{ds} \left( \psi_{t,s}^{-1} \right)^b = -\frac{\partial (\psi_{t,s})^b}{\partial t} \frac{d}{ds} \left( \psi_{t,s}^{-1} \right)^a + \frac{\partial (\psi_{t,s})^b}{\partial x^c} \frac{d}{ds} \left( \psi_{t,s}^{-1} \right)^c = \frac{df}{dt} - \frac{\partial (\psi_{t,s})^a}{\partial x^b} \frac{d}{ds} \left( \psi_{t,s}^{-1} \right)^b$$  \hspace{1cm} (2.29)

The Lie derivative of a covector field

Let $u$ be a covector field on $S$. Then the components of its push forward can be found as $(\psi_{t,s}^{-1} u)_s^a = \left( \frac{\partial (\psi_{t,s}^{-1})^b}{\partial x^a} \right) c(t)|_{\psi_{t,s}^{-1}}$, an analogous computation as above leads to

\footnote{c.f. page 6}
\[
\frac{d}{ds} \left( \psi_{t,s}^* u_s \right)_a = \frac{d}{ds} \left( \frac{\partial (\psi_{t,s}^{-1})^b}{\partial x^a} \right) u_{sb} + \frac{\partial (\psi_{t,s}^{-1})^b}{\partial x^a} \frac{d u_{sb}}{ds} = \frac{\partial v^b}{\partial x^a} u_{sb} + \frac{\partial (\psi_{t,s}^{-1})^b}{\partial x^a} \left( u_c \frac{\partial u_{cb}}{\partial x^a} + \frac{\partial u_{cb}}{\partial s} \right)
\]

and consequently

\[
(I_v u)_a = \frac{\partial u_{ta}}{\partial t} + \frac{\partial u_{ta}}{\partial x^b} v^b_t + \frac{\partial u_{tb}}{\partial x^a}.
\]  

(2.30)

**Linearization of tensor fields**

The Taylor’s Theorem

\[
t_t = t_s + \left( \frac{d}{ds} t_s \right) (t - s) + \ldots + \frac{1}{n!} \left( \frac{d^n}{ds^n} t_s \right) (t - s)^n + \ldots
\]  

(2.31)

(used for \(t - s\) sufficiently small) remains valid also for sufficiently smooth tensor fields \(t\) over curves \(c(r)\) on manifolds, i.e. for \(t := t_s := t(c(r))\) ([MH83]). So, according to (2.19) and (2.3), the linearization \(\hat{t}\) of such a tensor field \(t\) can be written as

\[
\hat{t} = t + \text{grad}_t u_t,
\]  

(2.32)

with the tangent vector \(u := u_s := (t - s)v_s\) to the curve \(c(s)\) on which \(t_s\) is defined. With (2.22) an alternative formulation

\[
\tilde{t}_t = t_s + \lim_{t - s \to 0} \left( \frac{d}{dt} (S_s, t_r) \right) (t - s),
\]  

(2.33)

usefull for farther computations, can be gained from (2.31).

### 3 Kinematics of Finite Deformations

In the following the positions of the material points of a body shall be described by its reference configuration \(B \subset S\). \(B\) has to be an open set in the Riemannian space \(S\) with a piecewise smooth boundary. Material points in \(B\) are denoted by \(X = (X^1, \ldots, X^N)\), while spatial points in \(S\) are denoted by \(x = (x^1, \ldots, x^n)\). The dimensions of \(B\) and \(S\) are assumed to be the same \((n = N)\). Any motion of a body \(B\) may be regarded as a time-dependent family of configurations, defined as sufficiently smooth, orientation preserving and invertible mappings \(\Phi_t : B \to S\) (i.e. \(x := x_t = \Phi(X, t) := \Phi_t(X)\)) \(13\). According to this definition, the identification of the body \(B\) with the reference configuration \(\Phi_0(B)\) makes sense \((X \equiv \Phi_0(X))\). Let additionally \(\{X^A\}\) and \(\{x^a\}\) denote coordinate systems on \(B\) and \(S\), respectively. Component wise representations will be assumed always with respect to these coordinate systems in the following chapters.

---

\(13\)We denote the function \(\Phi(X, t)\) with \(t\) fixed by \(\Phi_t(X)\) and with \(X\) fixed by \(\Phi_X(t)\)
Velocity and acceleration

The material velocity $V_X(t)$ and the material acceleration $A_X(t)$ at some point $X$ are defined via its motion $x = \Phi_X(t)$ in $S$:

$$V_X(t) := \frac{d}{dt} \Phi_X(t) \quad , \quad A_X(t) := \frac{d}{dt} V_X(t). \quad (3.1)$$

They will be regarded as vectors based at the point $x = \Phi_X(t)$ with components

$$V^a = \frac{d}{dt} \Phi^a_X(t) \quad , \quad A^a = \frac{d}{dt} V^a + \gamma^a_{bc} V^b V^c. \quad (3.2)$$

The spatial velocity and spatial acceleration are defined as

$$v_X(t) := V_{\Phi_X^{-1}(t)}(t) \quad , \quad a_X(t) := A_{\Phi_X^{-1}(t)}(t). \quad (3.3)$$

Some calculus shows, that $a_X(t)$ is the material time derivative $\frac{d}{dt} v$ of $v$:

$$a_X(t) = \frac{d}{dt} v_X(t) = \frac{\partial v_X}{\partial t} + \text{grad}_v v$$

with the covariant derivative $\text{grad}_v v$ from (2.4) in the current configuration. The components of $v$ and $a$ are

$$v^a = V^a \quad \text{and} \quad a^a = \frac{\partial v^a}{\partial t} + v^b_a v^b \quad \text{with} \quad v^a_b := \frac{\partial v^a}{\partial x^b} + \gamma^a_{bc} v^c. \quad (3.5)$$

As in [Wri86] and using $\psi_{t,s} : t_s = \Phi_{t,s} \Phi_{t}^{-1} t_s$, the Lie derivative $L_v t$ from (2.23) of a tensor $t$ on $S$ with respect to the velocity $v$ can be found as

$$L_v t = \Phi_{t,s} \left( \frac{d}{dt} (\Phi_{t,s}) \right). \quad (3.6)$$

The displacements

Let $S := S(X) : T_X S \rightarrow T_{\Phi(X)} S$ be a shifter,\(^1\) transporting a vector emanating from $X$ to a vector emanating from $x = \Phi_t(X)$. Using the existence of local Cartesian coordinate systems $\{z^i\}$ and $\{Z^I\}$ corresponding to $\{x^a\}$ and $\{X^A\}$ the components of $S(X)$ are

$$S_{a}^{\alpha} = \frac{\partial x^a}{\partial z^i} \frac{\partial Z^{\alpha}}{\partial X^A} \delta^i_{\alpha}. \quad (3.7)$$

\(^1\)Note, that $A^a$ from (3.2) coincides with the material time derivative from (2.18) and (2.20), with the only difference, that the term $\frac{\partial V^a}{\partial X^A} \frac{d}{dt} X^A$, arising from (2.18) for $a := V$, disappears in (3.2) since the reference configuration does not change in time.

\(^1\)In the notation of page 7 it reads $S_{t,s}$ with $\psi_{t,s} := \Phi_t, t$.\hfill\(10\)
with $\delta^i_j$ denoting Kronecker's symbol. Note, that $\mathbf{S}$ is *orthogonal* ($\mathbf{S}^T = \mathbf{S}^{-1}$). Now, on the *reference configuration*, the displacements $\mathbf{U}$ can be defined as

$$\mathbf{U} := \mathbf{S}^T \mathbf{x}_t - \mathbf{X}$$

with components $U^A = S^A_a x^a_t - X^A$. (3.8)

In the *current configuration* the displacements $\mathbf{u}$ is

$$\mathbf{u} := \mathbf{x}_t - \mathbf{S} \mathbf{X}$$

with components $u^a = x^a_t - S^a_A X^A$. (3.9)

In (3.8) and (3.9) no difference is made in descriptors for $\mathbf{x}_t \in \mathcal{B}$ and $\mathbf{X} \in T_\mathcal{B} \mathcal{B}$ and also not for $\mathbf{x}_t \in \mathcal{S}$ and $\mathbf{x}_t \in T_{\Phi(x)} \mathcal{S}$. This is possible because of the supposed underlying local Euclidean structure of $\mathcal{B}$, implicating an isomorphism between $\mathcal{B}$ and $T_\mathcal{B} \mathcal{B}$.

The computation of the velocity and the acceleration using the displacements is possible, but seems to make not so much sense, as can be seen in the following. On page 7 was shown, that for a *shifter* $\mathbf{S}^T_{t,0}$ used here $\frac{d}{dt} S^A_a = S^A_a \gamma^a_b V^b$ is valid. The time derivative of $\mathbf{S}_{t,0}$ can be found, taking into account $\frac{d}{dt} S^a_A = - S^b_A S^a_B \frac{d}{dt} \delta^B_b$, as $\frac{d}{dt} S^a_A = - S^b_A \gamma^a_b v^c$. So the time derivative of *displacements* reads in components

$$\left( \frac{d}{dt} \mathbf{U} \right)^a = \frac{d}{dt} U^a + \gamma^a_b v^b u^c = v^a + S^a_A \gamma^a_b v^b X^A + \gamma^a_b v^b \left( x^c - S^c_A X^A \right) = v^a + \gamma^a_b v^b x^c.$$ 

The deformation gradient

Another kind of *mapping* between $T_\mathcal{B} \mathcal{B}$ and $T_{\Phi(x)} \mathcal{S}$ is the deformation gradient $\mathbf{F}$, $\mathbf{F} := \mathbf{F}(\mathbf{X}, t) : T_\mathcal{B} \mathcal{B} \rightarrow T_{\Phi(x)} \mathcal{S}$ with components

$$F^a_A = \frac{\partial \Phi^a}{\partial X^A}$$

(3.10)

In terms of displacements the *deformation gradient* $\mathbf{F}$, its inverse $\mathbf{F}^{-1}$ and their components can be expressed by

$$\mathbf{F} = \mathbf{S}(\mathbf{I} + \text{GRAD} \mathbf{U})$$
$$F^a_A = S^a_B (\delta^A_B + U^B_A)$$
$$\left( F^{-1} \right)_a^A = S^A_a \left( \delta^B_b - U^B_a \right)$$

(3.11)

with $\text{GRAD} \mathbf{U}$ and $\text{grad} \mathbf{u}$ according to (2.5), $U^B_A$ and $u^b_a$ from (3.5) and $\mathbf{I}$, $\mathbf{i}$ denoting the identity operator.

The *transpose*, or *adjoint* of $\mathbf{F}$ is the linear transformation $\mathbf{F}^T : T_{\Phi(x)} \mathcal{S} \rightarrow T_\mathcal{B} \mathcal{B}$ such that $\langle \mathbf{F} \mathbf{W}, \mathbf{v} \rangle = \langle \mathbf{W}, \mathbf{F}^T \mathbf{v} \rangle$ for all $\mathbf{W} \in T_\mathcal{B} \mathcal{B}$ and $\mathbf{v} \in T_{\Phi(x)} \mathcal{S}$. Consequently $\mathbf{F}^T(x, t)$ is given in components by

$$\left( F^T \right)_a^A = g_{ab} F^b_B G^{AB}.$$ 

(3.12)

The deformation gradient and its adjoint play a fundamental role in the subsequent theory.

---

16This can also be seen by the following calculation, using (3.7, 3.2, 2.9):

$$\frac{d}{dt} \delta^a_A = \frac{d}{dt} \left( \frac{\partial^2 V}{\partial x^A \partial x^A} \delta^a_A \right) = \frac{\partial^2 V}{\partial x^A \partial x^A} \frac{\partial x^A}{\partial x^A} \delta^a_A = \gamma^a_b V^b \frac{\partial x^A}{\partial x^A} \frac{\partial x^A}{\partial x^A} \delta^a_A = S^a_A \delta^b_c V^b.$$ 

17and any other linear transformation $A : T_\mathcal{B} \mathcal{B} \rightarrow T_{\Phi(x)} \mathcal{S}$
The deformation tensors

On the reference configuration we define the right Cauchy-Green tensor \( \mathbf{C}(\mathbf{X}, t) \), also called Green deformation tensor, to be

\[
\mathbf{C} := \mathbf{F}^T \mathbf{F}, \quad C^A_B = g_{ab} G^{AC} F^b_C F^a_B.
\]  

(3.13)

If \( \mathbf{C} \) is invertible, \( \mathbf{B} := \mathbf{C}^{-1} \) is called the Piola deformation tensor. On the current configuration the left Cauchy-Green tensor, also called Finger deformation tensor, \( \mathbf{b}(\mathbf{x}, t) \) is defined as

\[
\mathbf{b} := \mathbf{F} \mathbf{F}^T, \quad b^a_b = g_{bc} G^{AB} F^b_A F^a_B
\]

with the inverse \( \mathbf{c} := \mathbf{b}^{-1} \). The material or Lagrangian strain tensor \( \mathbf{E} \) is defined by

\[
\mathbf{E} := \frac{1}{2} (\mathbf{C} - \mathbf{I}), \quad E^A_B = \frac{1}{2} (C^A_B - \delta^A_B)
\]

(3.15)

and the spatial or Eulerian strain tensor \( \mathbf{e} \) by

\[
\mathbf{e} := \frac{1}{2} (\mathbf{i} - \mathbf{c}), \quad e^a_b = \frac{1}{2} (\delta^a_b - c^a_b).
\]

(3.16)

In terms of pull backs and push forwards the various deformation tensors (3.13)-(3.16) can be redefined by:

\[
\begin{align*}
\mathbf{C}^i & := \Phi^* \mathbf{g} & \mathbf{B}^i & := \Phi^* \mathbf{g}^i & \mathbf{E}^i & := \Phi^* \mathbf{e}^i \\
\mathbf{c}^i & := \Phi^* \mathbf{G} & \mathbf{b}^i & := \Phi^* \mathbf{G}^i & \mathbf{e}^i & := \Phi^* \mathbf{e}^i.
\end{align*}
\]

(3.17)

The material (or Lagrangian) rate of deformation tensor \( \mathbf{D} \) is defined by

\[
\mathbf{D} := \frac{1}{2} \frac{d}{dt} \mathbf{C}.
\]

(3.18)

Using the formulation of the Lie derivative from (3.6), the associated material rate of deformation tensor \( \mathbf{D}^i \) can be found as

\[
\mathbf{D}^i = \frac{1}{2} \Phi^* \mathbf{L}_\mathbf{v} \mathbf{g}.
\]

(3.19)

At last, the spatial (or Eulerian) rate of deformation tensor \( \mathbf{d} \) can be defined using

\[
\mathbf{d}^i := \frac{1}{2} \mathbf{L}_\mathbf{v} \mathbf{g}.
\]

(3.20)

Remark:

From an empirical point of view, the changes in the length of a line element during a motion of a body \( \mathcal{B} \) are a measure of deformation. Some computation gives

\[
dS^2 - ds^2 = G_{AB} \, dX^A \, dX^B - g_{ab} \, dx^a \, dx^b.
\]
From this we get
\[
dS^2 - ds^2 = [G_{AB} - g_{ab} F_A^c F_B^b] \, dX^A dX^B = G_{AC} F_B^C \, dX^A dX^B
\]
as well as
\[
dS^2 - ds^2 = [G_{AB} (F^{-1})_a^A (F^{-1})_b^B - g_{ab}] \, dx^a dx^b = g_{ac} \varepsilon_c^b \, dx^a dx^b,
\]
i.e., the deformation can completely be described in terms only related to the reference configuration or to the current configuration by using \( E \) or \( e \) from above and the corresponding metric tensors.

4 The stress tensor and balance of momentum

In the following we will assume that for a given sufficiently smooth motion \( \Phi(X, t) \) of a body \( B \subset S = \mathbb{R}^n \) there exist

- a mass density function \( \rho(x, t) \),
- a continuous vector field \( r(x, t, n) \), called the Cauchy traction vector (representing the force per unit area exerted on a surface element of \( \partial \Phi_i(A) \), oriented with unit outward normal \( n \)) and
- an external force field \( l(x, t) \).

Then, the balance of momentum is satisfied, if for every sufficiently smooth open set \( A \subset B \) the equation (4.1) is true:

\[
\frac{d}{dt} \int_{\Phi_i(A)} \rho \, v \, dv = \int_{\Phi_i(A)} \rho \, l \, dv + \int_{\partial \Phi_i(A)} r \, da .
\]

If (4.1) and conservation of mass \( \frac{d}{dt} \rho + \rho \text{div} v = 0 \) holds, there exists a unique\(^18\) symmetric Cauchy stress tensor \( \sigma = \sigma(x, t) \) satisfying

\[
r = \langle \sigma, n \rangle \quad \text{and} \quad \rho \frac{d}{dt} v = \rho \, l + \text{div} \, \sigma ,
\]
or, written in components\(^20\),

\[
r^a = \sigma^{ac} g_{bc} n^b \quad , \quad \rho \left( \frac{\partial v^a}{\partial t} + v^b \frac{\partial v^a}{\partial x^b} \right) = \rho \, l^a + (\text{div} \, \sigma)^a , \quad (\text{div} \, \sigma)^a = \sigma_{lb}^{ba} ,
\]

\[
\sigma_{lb}^{ba} = \frac{\partial \sigma_{lb}^{ba}}{\partial x^a} + \sigma_{ac}^{cb} \gamma_{lb}^{ac} + \sigma_{bc}^{ac} \gamma_{lb}^{bc} .
\]

\(^{18}\)For a proof see page 31 in the appendix.

\(^{19}\)Here and in the following we use the symbol \( \langle \cdot, \cdot \rangle \), originally defined for the inner product, also in the sense of the leftmost part of (4.3), since the components of the resulting vector may be regarded as inner products of the ”columns” of \( \sigma \) with \( n \).

\(^{20}\)See page 5 for a hint on how to derive the components of \( \text{div} \, \sigma \).
With the Jacobian (9.8) we define the *Piola transform* $P(X,t)$ of $\sigma$ with components

$$P^A := J \left( F^{-1} \right)_A^B \sigma^{ab} \quad (4.4)$$

which is called the *first Piola–Kirchhoff stress tensor*. This tensor is related to the Cauchy stress tensor $\sigma$ by means of the *Piola Identity*

$$\text{DIV} P = J \text{div} \sigma, \quad (4.5)$$

what can be proved by some calculus. Using the theorem of Gauss and Ostrogradski (9.7), taking into account the underlying Euclidean *structure* of $\Phi_t(A)$, the transformation behaviour of *domain integrals* (9.5), (9.6) and making use of (4.5) it can be shown, that the *balance of momentum* (4.1) is equivalent to

$$\frac{d}{dt} \int_A \rho_{\text{ref}} V dV = \int_A \rho_{\text{ref}} L dV + \int_{\partial A} R dA \quad (4.6)$$

with the *density* $\rho_{\text{ref}} := \rho J$ in the *reference configuration*, $N$ denoting the *unit outward normal* to $\partial A$, $L(X,t) = l(\Phi_t(X),t)$ and $R = (P,N) = P^A N_A$. The same analysis used to deduce (4.2) from (4.1) gives (4.7) from (4.6):

$$\rho_{\text{ref}} \frac{d}{dt} V = \rho_{\text{ref}} L + \text{DIV} P$$

in coordinates:

$$\rho_{\text{ref}} \left( \frac{dV^a}{dt} + \gamma^{ac}_{bc} V^b V^c \right) = \rho_{\text{ref}} L^a + P_{1A}^a \quad (4.7)$$

with

$$P_{1A}^a := \frac{\partial P^a}{\partial X^A} + F_A^a \gamma^{bc} p^{bA} + \Gamma_A^B p^{aC}. \quad (4.8)$$

The *second Piola–Kirchhoff stress tensor* $T(X,t)$ is defined by

$$T^{AB} := (F^{-1})_a^A P^{aB}. \quad (4.9)$$

The symmetry of $T$ follows from the symmetry of $\sigma$. The *first Piola–Kirchhoff stress tensor* is symmetric in the sense of

$$P^a F_A^b = P^b F_A^a. \quad (4.10)$$

On the *current configuration* it is also useful to introduce a fourth stress tensor $\tau$ called *Kirchhoff stress tensor*, defined by

$$\tau := J \sigma. \quad (4.11)$$
5 Balance of energy and principle of virtual work

Balance of momentum (4.1) explicitly uses the linear structure of $\mathbb{R}^n$, because vector functions are integrated. It is correct to interpret this equation component-by-component in Cartesian coordinates $\{z^i\}$ but not in a general coordinate system, because the assumption of total forces like $\mathbf{l}$ and $\mathbf{r}$ in (4.1) acting on a body doesn’t directly make sense, when the containing space $\mathcal{S}$ is curved. However, energy balance is sensefull on manifolds and can be used as a covariant basis for elasticity. Covariance may be explained in general terms in the following:

Suppose we have a theory described by a number of tensor fields $a, b, \ldots$ on some space $\mathcal{S}$, and the equations of our theory (partial differential equations, integral equations, ...) take the form $\Lambda(a, b, \ldots) = 0$. The equations are called covariant or form invariant, if for any diffeomorphism $\varphi : \mathcal{S} \mapsto \mathcal{S}$ the equation $\varphi^\ast \Lambda(a, b, \ldots) := \Lambda(\varphi^\ast a, \varphi^\ast b, \ldots) = 0$ holds with the pull back $\varphi^\ast a$ of some tensor $a$ by the mapping $\varphi$ as defined on page 5.

The balance of energy principle

We take into account only mechanical effects with functions $\rho(x, t), l(x, t)$ and $r(x, t, n)$, given for $x \in \Phi_t(\mathcal{B})$ and $n \in T_x\mathcal{S}$, as they were described at the beginning of chapter 4. Let $\epsilon := \epsilon(x, t)$ be the density of internal energy. Then, the balance of energy principle is satisfied if, for each sufficiently smooth $\mathcal{A} \subset \mathcal{B}$, the equation (5.1) holds:

$$ \frac{d}{dt} \int_{\Phi_t(\mathcal{A})} \rho \left[ \epsilon + \frac{1}{2} \langle \mathbf{v}, \mathbf{v} \rangle \right] \, dv = \int_{\Phi_t(\mathcal{A})} \rho(l, \mathbf{v}) \, dv + \int_{\partial \Phi_t(\mathcal{A})} \langle \mathbf{r}, \mathbf{v} \rangle \, da. \quad (5.1) $$

Superposed motions

Let the motion $\Phi_t, x := \Phi(x, t)$ of our body $\mathcal{S}$ be superposed by another motion or a change of observer $\tilde{\varphi}_t : \mathcal{S} \mapsto \mathcal{S}, \tilde{x}_t := \tilde{\varphi}_t(\varphi(x, t), t) = \Phi(\Phi^{-1}(t), t) = \tilde{\Phi}_t(t)$ with $\varphi(x, t_0) = \tilde{x}_t = \Phi^{-1}(t_0) = \Phi(t) = x_t$. Under this superposed motion the metric tensor $g$ changes to

$$ \tilde{g} = \varphi \cdot g \quad \text{with} \quad \tilde{g}_{ab} = \frac{\partial (\varphi^{-1})^i}{\partial \tilde{x}^a} \cdot \frac{\partial (\varphi^{-1})^j}{\partial \tilde{x}^b} g_{ij}. \quad (5.2) $$

To proof this, we start with (2.9) and get $\tilde{g}_{ab} := \frac{\partial \tilde{x}^i}{\partial x^a} \frac{\partial \tilde{x}^j}{\partial x^b} = \frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} \frac{\partial x^d}{\partial \tilde{x}^a} \frac{\partial x^c}{\partial \tilde{x}^b} \tilde{g}_{cd}$ with $x = \varphi^{-1}(\tilde{x})$. According to (3.1), the velocity $\tilde{\mathbf{V}}$ of $\tilde{\Phi}$ has the components

$$ \tilde{\mathbf{V}}^a(t) = \left( \frac{d}{dt} \Phi^{-1}(t) \right)^a = \frac{\partial \varphi^a}{\partial t} |_{\Phi(t)} + \frac{\partial \varphi^a}{\partial x^b} |_{\Phi(t)} \Phi^b(t). \quad (5.3) $$

$^{21}$a sufficiently smooth bijective mapping
Using (3.3) and (2.12) we get the spatial velocity \( \tilde{\mathbf{v}}(t) \) as
\[
\tilde{\mathbf{v}}(t) = \varphi \cdot \mathbf{v}(t) + \mathbf{\xi}(t) \quad \text{with} \quad \tilde{\mathbf{v}}^a = \frac{\partial \varphi^a}{\partial x^b} \mathbf{v}^b + \mathbf{\xi}^a
\]
where \( \mathbf{\xi} := \frac{d\varphi}{dt} \) is the velocity of \( \tilde{\mathbf{x}} \) relative to \( \mathbf{x} \).

As proved on page 32, the spatial acceleration \( \tilde{\mathbf{a}} \) reads
\[
\tilde{\mathbf{a}} = \varphi \cdot \mathbf{a} + \frac{\partial \mathbf{\xi}}{\partial t} + \text{grad} \, \mathbf{\xi} + 2 \text{grad} \,(\varphi \cdot \mathbf{\nu}) \mathbf{\xi},
\]
with \( \frac{\partial \mathbf{\xi}}{\partial t} + \text{grad} \, \mathbf{\xi} \) denoting the acceleration of \( \tilde{\mathbf{x}} \) relative to \( \mathbf{x} \). Due to (3.2)–(3.5) and (2.18), the components of \( \tilde{\mathbf{a}} \) are
\[
\tilde{a}^a = \frac{d}{dt} \tilde{v}^a + \tilde{g}_{cd} \tilde{v}^c \tilde{v}^d.
\]

We assume, that the forces and the Cauchy stress vector transform as in (5.7):
\[
\tilde{\mathbf{I}} - \tilde{\mathbf{a}} = \varphi \cdot (1 - \mathbf{a}) \quad \text{and} \quad \tilde{\mathbf{r}} = \varphi \cdot \mathbf{r}.
\]

At time \( t = t_0 \) equations (5.4–5.7) read
\[
\tilde{\mathbf{v}} = \mathbf{v} + \mathbf{\xi}, \quad \tilde{\mathbf{a}} = \mathbf{a} + \frac{\partial \mathbf{\xi}}{\partial t} + \text{grad} \, \mathbf{\xi} + 2 \text{grad} \,(\varphi \cdot \mathbf{\nu}) \mathbf{\xi}
\]
\[
\tilde{\mathbf{I}} - \tilde{\mathbf{a}} = 1 - \mathbf{a}, \quad \tilde{\mathbf{r}} = \mathbf{r}
\]

If the transformation \( \varphi \) is not a rigid body motion, \( \varphi \) changes the metric (cf. (5.2)) and influences the acceleration (cf. (5.5)). Therefore, the internal energy \( \varepsilon \) must depend parametrically on the metric \( \mathbf{g} \), and it is natural to suppose the transformation
\[
\tilde{\varepsilon} := \varepsilon(\varphi^{-1}(\tilde{\mathbf{x}}), t, \varphi^* \mathbf{g}).
\]

Then, as proved in the appendix page 33, the time derivative of \( \tilde{\varepsilon} \) at \( t = t_0 \) where \( \varphi = \text{identity} \) can be found as
\[
\left( \frac{d\tilde{\varepsilon}}{dt} \right)_{t_0} = \frac{d}{dt} \varepsilon + \frac{\partial \varepsilon}{\partial \mathbf{g}} (\mathcal{L}_{\xi} \mathbf{g})_{ab} = \frac{d}{dt} \varepsilon + \frac{\partial}{\partial \mathbf{g}} : \mathcal{L}_{\xi} \mathbf{g}
\]
with the autonomous Lie derivative \( \mathcal{L}_{\xi} \mathbf{g} \) from (2.27). Comparing the balance of energy principle in the original and in the transformed state, on page 33 the identity
\[
\int_{\Phi_i(A)} \left[ \left( \frac{d}{dt} \rho + \rho \text{div} \mathbf{v} \right) \left( \frac{1}{2} [\mathbf{\xi}, \mathbf{\xi}] + [\mathbf{v}, \mathbf{\xi}] \right) + \rho \left( \frac{\partial}{\partial \mathbf{g}} : \mathcal{L}_{\xi} \mathbf{g} + (\mathbf{a} - 1, \mathbf{\xi}) \right) \right] dv = \int_{\Phi_i(A)} [\mathbf{r}, \mathbf{\xi}] da
\]
is proved. Introducing the Cauchy stress tensor \( \mathbf{r} = \langle \mathbf{\sigma}, \mathbf{n} \rangle \) from (4.2) and applying the divergency theorem\(^{22}\)
\[
div(\mathbf{\sigma}, \mathbf{\xi}) = \langle \text{div} \, \mathbf{\sigma}, \mathbf{\xi} \rangle + \mathbf{\sigma} : \mathbf{\omega}_{\xi} + \frac{1}{2} \mathbf{\sigma} : \mathcal{L}_{\xi} \mathbf{g}
\]
with the spin \( \mathbf{\omega}_{\xi}, \mathbf{\omega}_{\xi ab} = \frac{1}{2} (g_{ac} \mathbf{\xi}_d^c - g_{bc} \mathbf{\xi}_d^c) = \frac{1}{2} \left( g_{ac} \mathbf{\xi}_b^c - g_{bc} \mathbf{\xi}_a^c \right),
\]
\(^{22}\)For a proof see page 31 in the appendix.
and \( \text{div } \boldsymbol{\sigma} \) from (4.3) to the right hand side of (5.11) it reads
\[
\int_{\Phi_i(A)} \left[ \left( \frac{d}{dt} \rho + \rho \text{div } \mathbf{v} \right) \left( \frac{1}{2} \mathbf{\xi} + \mathbf{v} \right) + \left( \rho \frac{\partial e}{\partial g} - \frac{1}{2} \mathbf{\sigma} \right) : \mathbf{\omega}_g^1 + \langle \rho \mathbf{a} - \mathbf{p} - \text{div } \mathbf{\sigma} , \mathbf{\xi} \rangle \right] dv = 0.
\]
\tag{5.13}
\]
Since \( \mathcal{A} \) is arbitrary, (5.13) results in a differential equation in \( \mathbf{\xi} \) at any point. This violates the assumption of the arbitrariness of \( \mathbf{\xi} \), unless the whole term to be integrated vanishes in each point. So (5.13) is valid only if we have
\[
\frac{d}{dt} \rho + \rho \text{div } \mathbf{v} = 0 \quad \Rightarrow \quad \text{conservation of mass}
\]
\[
\rho \mathbf{a} - \mathbf{p} - \text{div } \mathbf{\sigma} = 0 \quad \Rightarrow \quad \text{conservation of momentum}
\]
\[
\mathbf{\sigma} \text{ is symmetric} \quad \Rightarrow \quad \text{conservation of moment of momentum}
\]
\[
\mathbf{\sigma} = 2 \rho \frac{\partial e}{\partial g} \quad \Rightarrow \quad \text{Doyle–Ericksen–Formula}.
\]
So we see, that the conservation of mass, the conservation of momentum and the conservation of moment of momentum, as assumed in the previous chapter, can be shown to follow from balance of energy and the principle of covariance.

The principal of virtual work

Inserting the Doyle–Ericksen–Formula and the conservation of momentum from (5.14) into (5.11) with \( \mathcal{A} = \mathcal{B} \) we get the principal of virtual work
\[
\int_{\Phi_i(A)} \left[ \mathbf{\sigma} : \mathbf{d}_g^1 + \rho (\mathbf{a} - \mathbf{1}, \mathbf{\xi}) \right] dv - \int_{\partial \Phi_i(A)} \langle r, \mathbf{\xi} \rangle da = 0
\]
\[
\tag{5.15}
\]
with \( \mathbf{d}_g^1 := \frac{1}{2} \mathbf{\omega}_g^1 \) according to (3.20). Pulling back (5.1) to the reference configuration, it yields to
\[
\frac{d}{dt} \int_{\mathcal{A}} \rho_{\text{rel}} \left[ E + \frac{1}{2} \langle \mathbf{V}, \mathbf{V} \rangle \right] dV = \int_{\mathcal{A}} \rho_{\text{rel}} \langle \mathbf{L}, \mathbf{V} \rangle dV + \int_{\partial \mathcal{A}} \langle \langle \mathbf{P}, \mathbf{N} \rangle, \mathbf{V} \rangle dA,
\]
\[
\tag{5.16}
\]
the analogon to (5.1), with \( E := \Phi^* e = e(\Phi_i(\mathbf{X}), t, \Phi, \mathbf{C}^i) = E(\mathbf{X}, t, \mathbf{C}^i) \), as sketched on page 34. From this we get the equation (5.17):
\[
\int_{\mathcal{A}} \left[ \left( 2 \rho_{\text{rel}} \frac{\partial E}{\partial \mathbf{C}} - \mathbf{T} \right) : \mathbf{D}_g^1 - \mathbf{T} : \mathbf{\Omega}_g^1 + \langle \rho_{\text{rel}} \mathbf{A} - \rho_{\text{rel}} \mathbf{L} - \text{DIV} \mathbf{P}, \mathbf{\Xi} \rangle \right] dV = 0.
\]
\[
\tag{5.17}
\]
Following the argumentation used to derive (5.14) from (5.13) we see, that
\[ \rho_{\text{ref}} \mathbf{A} - \rho_{\text{ref}} \mathbf{L} - \text{DIV} \mathbf{P} = 0, \]

\[ \text{T is symmetric \quad \quad \text{and} \quad \quad } \]
\[ \text{T} = 2 \rho_{\text{ref}} \frac{\partial E}{\partial \mathbf{C}}. \]

Inserting the last line of (5.18) into (9.25) we finally get the principle of virtual work on the reference configuration:
\[ \int_{\tilde{\Omega}} \left( \mathbf{T} : \mathbf{D}_{\Xi} + \rho_{\text{ref}} \langle \mathbf{A} - \mathbf{L}, \Xi \rangle \right) dV - \int_{\partial \tilde{\Omega}} \langle \mathbf{R}, \Xi \rangle dA = 0 \quad (5.19) \]
with \( \mathbf{R} := (\mathbf{P}, \mathbf{N}) \).

6 The second law of thermodynamics

In thermodynamics of irreversible processes, one of the important objectives is to relate the change of specific entropy \( \eta \) to the various irreversible phenomena which may occur inside the system. The second law of thermodynamics is introduced by the ad-hoc dissipation inequality
\[ \frac{d}{dt} \int_{\Phi_i(A)} \rho \eta \, dv \geq \int_{\Phi_i(A)} \rho s \, dv + \int_{\partial \Phi_i(A)} \frac{h}{\vartheta} \, da, \quad (6.1) \]
for each sufficiently smooth \( A \subset \mathcal{B} \), with the heat supply per unit mass \( s(x, t) \), the heat flux (across a surface with normal \( n \)) \( h(x, t, n) \) and the absolute temperature \( \vartheta(x, t) \). The first law of thermodynamics, as given in (5.1), doesn’t reflect the influence of thermal effects as introduced now. So it has to be rewritten as
\[ \frac{d}{dt} \rho \left[ e + \frac{1}{2} (\mathbf{V}, \mathbf{V}) \right] = \int_{\Phi_i(A)} \rho [s + \mathbf{V} \cdot \mathbf{v}] \, dv + \int_{\partial \Phi_i(A)} (h + \mathbf{V} \cdot \mathbf{n}) \, da. \quad (6.2) \]
Assume, that there exists a heat flux vector \( \mathbf{q}(x, t) \) with \( h(x, t, n) = -\langle \mathbf{q}(x, t), n \rangle \) and that conservation of mass holds. Then
\[ \rho \frac{d}{dt} \eta \geq \frac{\rho s}{\vartheta} - \text{DIV} \left( \frac{\mathbf{q}}{\vartheta} \right) = \frac{\rho s}{\vartheta} - \frac{1}{\vartheta} \left[ \text{DIV} \mathbf{q} - \frac{1}{\vartheta} \langle \mathbf{q}, \nabla \vartheta \rangle \right], \quad (6.3) \]
and
\[ \rho \frac{d}{dt} e - \mathbf{\sigma} : \mathbf{d}^i - \rho s + \text{DIV} \mathbf{q} = 0 \quad (6.4) \]
can be shown \(^{23}\) to follow from (6.1) and (6.2). Combining (6.3) and (6.4) we get
\[ \rho \left[ \frac{d}{dt} - \vartheta \frac{d}{dt} \eta \right] - \mathbf{\sigma} : \mathbf{d}^i + \frac{1}{\vartheta} \langle \mathbf{q}, \nabla \vartheta \rangle \leq 0, \]

\(^{23}\)cf. page 37
and with the specific free energy $\zeta := e - \vartheta \eta$ the reduced dissipation inequality

$$\rho \left[ \frac{d}{dt} \zeta + \eta \frac{d}{dt} \vartheta \right] - \sigma : D + \frac{1}{\vartheta} \langle q, \nabla \vartheta \rangle \leq 0$$

(6.5)

follows.

Pulling back the inequalities (6.1) and (6.2), the second and the first law of thermodynamics on the reference configuration are obtained as

$$\frac{d}{dt} \int_A \rho_{\text{ref}} \mathcal{E} \, dV \geq \int_A \rho_{\text{ref}} \mathcal{T} \, dV + \int_{\partial A} H \, dA,$$

(6.6)

and

$$\frac{d}{dt} \int_A \rho_{\text{ref}} \left[ E + \frac{1}{2} \langle \mathbf{V}, \mathbf{V} \rangle \right] \, dV = \int_A \rho_{\text{ref}} \left[ \langle L, \mathbf{V} \rangle + S \right] \, dV + \int_{\partial A} \left[ \langle R, \mathbf{V} \rangle + H \right] \, dA,$$

(6.7)

with $\mathcal{E}(X, t, C^i, T) := \eta(\Phi_i(X), t, \Phi_i, C^i) = T(X, t) := \vartheta(\Phi_i(X), t) \cdot S(X, t) := s(\Phi_i(X), t)$, $H(X, t, \mathbf{N}) := -\langle Q(X, t), \mathbf{N} \rangle$, $Q := JF^{-1} q$, $R(X, t, \mathbf{N}) := \langle P(X, t), \mathbf{N} \rangle$, $P = JF^{-1} \sigma$. Following the ideas sketched on page 37, the localized forms of (6.6) and (6.7) can be found as

$$\rho_{\text{ref}} \frac{d}{dt} \mathcal{E} \geq \rho_{\text{ref}} \mathcal{S} - \text{DIV} \left( \frac{\mathcal{Q}}{T} \right),$$

(6.8)

$$\rho_{\text{ref}} \frac{d}{dt} E - \mathbf{T} : D^i - \rho_{\text{ref}} S + \text{DIV} \mathcal{Q} = 0,$$

(6.9)

and the reduced dissipation inequality with $\mathcal{Z} := E - T \mathcal{E}$ reads

$$\rho_{\text{ref}} \left[ \frac{d}{dt} \mathcal{Z} + \mathcal{E} \frac{d}{dt} T \right] - \mathbf{T} : D^i + \frac{1}{T} \langle \mathcal{Q}, \nabla T \rangle \leq 0.$$  

(6.10)

The inequalities (6.5) and (6.10) are also called spatial and material Clausius–Duhem inequality, respectively.

### 7 Linearization of nonlinear elasticity

Applying (2.33) and (2.21) to the deformation gradient $F$ defined in (3.10) we get

$$\tilde{F}^a = F^a_A + \lim_{r \to 1} \left[ \frac{d}{dr} \left( (S_r s)^a_{x} \frac{\partial \Phi^b_r}{\partial x^b} \right) \right] (t - s) = F^a_A + \frac{\partial \Psi^a}{\partial x^b} + \gamma^s \frac{\partial \Psi^b}{\partial x^c} = F^a_A + \Psi^a_{,s}.$$  

(7.1)

with $\Psi^a_s := \frac{d}{ds} \Phi^a_s (t - s)$ and assuming $s \approx t$ (cf. page 7), or, in a more compact notation,

$$\tilde{F} = F + \text{GRAD} \Psi.$$  

(7.2)

---

24. Specially using (9.5) and (9.6), for details see page 34 and the definitions made there.

25. Demanding the principle of covariance to apply to the second law of thermodynamics, the specific entropy $\eta$ is not permitted to depend on the metric [MH83], so we must write $\mathcal{E}(X, t, \mathcal{T}) := \eta(\Phi_i(X), t, \vartheta)$. But this doesn’t infer the following theory.
We saw that, assuming an infinitesimal deformation $\Psi$, the deformation gradient changes to (7.2).

The combination of (5.18) and (4.8) gives $P = 2\rho_{ref} F \frac{\partial E}{\partial C}$, with $E = E(X, t, C)$, showing us that, in the general case, the stresses $P = P(F(t), E)$ depend on the deformation $F$ as well as on space and time. Assuming the investigated material to be homogeneous in space and time means, that $P$ is assumed to be a tensorial function of $F$ only, and using the linear terms of (2.31) the first Piola–Kirchhoff stress tensor $P$ can be found as

$$\tilde{P} = P + \frac{\partial P}{\partial F} : \text{GRAD} \Psi.$$  \hfill (7.3)

Taking into account the linearization of $V$

$$\tilde{V}^a = V^a + \lim_{t \to s} \left[ \frac{d}{dt} \left( (S_{\alpha \nu})^a_{\nu} \frac{d}{dt} \Phi^a_r \right) \right] (t - s) = V^a + \frac{d}{dt} \Psi^a + \gamma^a_{bc} \Psi^b \Psi^c = V^a + \left( \frac{d}{dt} \Psi \right)^a$$ \hfill (7.4)

the linearization of the equation of motion (4.7) will be

$$\rho_{ref} \left[ \frac{d}{dt} \left( V + \frac{d}{dt} \Psi \right) - L \right] = \text{DIV} \left( P + \frac{\partial P}{\partial F} : \text{GRAD} \Psi \right).$$ \hfill (7.5)

As the next step we have to compute $\frac{\partial P}{\partial F}$:

With $C_{AB} = g_{ab} F_A^a F_B^b$ from (3.13) and (3.17), we get

$$\frac{\partial C_{AB}}{\partial F_C^c} = g_{ab} \left( F_A^a \delta_C^c + F_B^b \delta_C^a \right) = 2g_{ab} F_A^a \delta_C^a = 2g_{bc} F_A^a \delta_C^a \hfill (7.6)$$

due to the symmetry of $g$ and $C$. Since $E$ depends on $F$ only by $C$ we have

$$\frac{\partial E}{\partial F_A^a} = \frac{\partial E}{\partial C_{BC}} \frac{\partial C_{BC}}{\partial F_A^a} = 2 \frac{\partial E}{\partial C_{BC}} g_{ab} F_B^b \delta_C^a = 2 \frac{\partial E}{\partial C_{AB}} g_{ab} F_B^b \hfill (7.7)$$

Combining (5.18) and (4.8) we get

$$P^{eA} = 2\rho_{ref} F_C^e \frac{\partial E}{\partial C_{AC}} \hfill (7.8)$$

From (7.7) and (7.8) we get $g^{ee} P^{eE} = \frac{\partial E}{\partial F_E^e}$. Multiplying this by $g^{de}$ supplies for the components of $P$, $\rho_{ref} g^{ee} \frac{\partial E}{\partial F_E^e} = \delta^{de} P^{eE} = P^{dE}$. So we find the first expression for $\frac{\partial P}{\partial F}$:

$$\frac{\partial P^{eE}}{\partial F_B^b} = \rho_{ref} g^{ee} \frac{\partial E}{\partial F_E^e} \frac{\partial^2 E}{\partial F_B^b \partial F_E^e}. \hfill (7.9)$$
Using (7.7) and (7.6) helps us to compute
\[ \frac{\partial^2 E}{\partial F_B^a \partial F_A^e} = \frac{\partial E}{\partial F_A^e} \frac{\partial E}{\partial F_B^a} = 2 g_{bd} \frac{\partial E}{\partial C_{BC}} F_d^e F_d^a. \]

Inserting this into (7.9) we get the second expression for \( \frac{\partial P}{\partial F} \):
\[ \frac{\partial P_{\alpha A}}{\partial F_B^a} = \rho_{\alpha f} g^{ac} \frac{\partial^2 E}{\partial F_B^a \partial F_A^c} = \rho_{\alpha f} g^{ac} \left( 2 \frac{\partial E}{\partial C_{AB}} g_{cb} + 4 \frac{\partial^2 E}{\partial C_{AC} \partial C_{DB}} g_{cd} g_{cb} F_D^e F_D^c \right). \] (7.10)

A third representation for \( \frac{\partial P}{\partial F} \), introducing the components of the elasticity tensor \( C \), is gained from (4.8) and (7.6):
\[ \frac{\partial P_{\alpha A}}{\partial F_B^a} = \frac{\partial F_\alpha^i T_{i A}}{\partial F_B^a} = T_{AB} \delta_b^a + F_c^i \frac{\partial T^{AC}}{\partial C_{DE}} \frac{\partial C_{DE}}{\partial F_B^a} = T_{AB} \delta_b^a + 2 \frac{\partial T^{AC}}{\partial C_{BD}} g_{bc} F_D^e F_D^c \]
we have
\[ C_{b A} := \frac{\partial P_{\alpha A}}{\partial F_B^a} = 2 \frac{\partial T^{AC}}{\partial C_{BD}} F_D^e g_{bc} + T_{AB} \delta_b^a. \] (7.11)

Using the widely in common use notation \( \frac{\partial T^{AC}}{\partial C_{DB}} F_D^e g_{bc} \) and \( T_{AB} \delta_b^a := T \otimes 1 \) with \( (1)^a_b := \delta_b^a \), equation (7.5) becomes
\[ \rho_{\alpha f} \left[ \frac{d}{dt} \left( \mathbf{V} + \frac{d}{dt} \Psi \right) - L \right] = DIV \left[ P + 2 \frac{\partial T}{\partial C^i} \cdot F \cdot F \cdot g + T \otimes 1 \right] : GRAD \Psi. \] (7.12)

Next we linearize (4.2): From (3.3), (7.4) and \( \psi = \Psi(\Phi^{-1}(x)) \) we get
\[ \tilde{\mathbf{v}} = \mathbf{v} + \frac{d}{dt} \psi. \] (7.13)

Due to (4.4) we have \( \sigma = \frac{1}{J} F P \) and therefore
\[ \tilde{\sigma} = \frac{1}{J} F \tilde{P}. \] (7.14)

with \( \tilde{P} \) from (7.3). From (4.2) we see, that \( \rho \left( \frac{d}{dt} \mathbf{v} - 1 \right) = div \tilde{\sigma} \), and the insertion (7.13), (7.14) and (7.3) supplies
\[ \rho \left[ \frac{d}{dt} \left( \mathbf{v} + \frac{d}{dt} \psi \right) - 1 \right] = div \left[ \sigma + \frac{1}{J} F \frac{\partial P}{\partial F} : GRAD \Psi \right]. \] (7.15)

For the infinitesimal deformation \( \Psi(X) = \Psi(\Phi(X)) \) we have
\[ \Psi^a_{\mid B} = \frac{\partial \Psi^a}{\partial X^a} + \frac{\partial \Psi^a}{\partial \xi^b} F_B^b + \frac{\partial \Psi^a}{\partial \xi^b} \frac{\partial \Phi^b}{\partial X^a} + \left( \frac{\partial \psi^a}{\partial x^b} + \frac{\partial \psi^a}{\partial \xi^b} \right) F_B^b = F_B^b \psi^a_{\mid B}. \] (7.12)
Using the above equation and (7.10) we get the components of the inverse Piola transform of \( \frac{\partial \mathbf{P}}{\partial \mathbf{F}} : \text{GRAD} \Psi \) as

\[
\frac{1}{J} F_A^e \frac{\partial P^a_A}{\partial F_B^e} \Psi_{IB}^h = \frac{1}{J} F_A^e \frac{\partial P^a_A}{\partial F_B^e} F_B^d \psi_{bd}^h = \frac{\rho_{\text{rel}} F_A^e \left( \frac{\partial E}{\partial C_{AB}} \delta^a_b + \frac{\partial^2 E}{\partial C_{AC} \partial C_{DB}} g_{be} F_C^a F_D^e \right) F_B^d \psi_{bd}^h.}
\]

(7.16)

Supposing the internal energy to satisfy the principle of covariance, as done in the previous chapters, we have \( \Phi^* e(x, t, \mathbf{g}) = E(X, t, \mathbf{C}) \) and consequently

\[
\frac{\partial E}{\partial C} = \Phi^* \frac{\partial e}{\partial \mathbf{g}} \quad \text{and} \quad \frac{\partial^2 E}{\partial C^2} = \Phi^* \frac{\partial^2 e}{\partial \mathbf{g}^2}.
\]

(7.17)

Inserting (7.17) with (9.2) into (7.16) we can write

\[
\frac{1}{J} F_A^e \frac{\partial P^a_A}{\partial F_B^e} \Psi_{IB}^h = \rho \left( \frac{\partial e}{\partial g_{cd}} \delta^a_b + \frac{\partial^2 e}{\partial g_{ca} \partial g_{cd}} g_{be} \right) \psi_{bd}^h.
\]

Finally, from (5.14) and (4.10) we get \( 2 \rho \frac{\partial e}{\partial g_{cd}} = \sigma_{cd} \) as well as \( 2 \frac{\partial^2 e}{\partial g_{ca} \partial g_{cd}} = \frac{1}{\rho_{\text{rel}}} \frac{\partial \tau_{ca}}{\partial g_{cd}} \),

ending up with

\[
\frac{1}{J} F_A^e \frac{\partial P^a_A}{\partial F_B^e} \Psi_{IB}^h = \left( \sigma_{cd} \delta^a_b + \frac{2 \partial \tau_{ca}}{J g_{cd}} g_{be} \right) \psi_{bd}^h.
\]

(7.18)

Inserting (7.18) into (7.15), we get the desired linearization of (4.2) with notation explained in (7.12) and (7.18):

\[
\rho \left[ \frac{d}{dt} \left( \mathbf{v} + \frac{d}{dt} \psi \right) - \mathbf{1} \right] = \text{div} \left[ \sigma + \frac{2 \partial \tau}{J g} - \mathbf{g} + \mathbf{g} \otimes \mathbf{1} \right] : \text{GRAD} \psi.
\]

(7.19)

8 Multiplicative Elastoplasticity at Finite Strains

The following theory is founded on the basic assumption of the multiplicative split (cf. [Sim93]) of the deformation gradient \( \mathbf{F} \) in an elastic part \( \mathbf{F}^e \) and a plastic part \( \mathbf{F}^p \)

\[
\mathbf{F} = \mathbf{F}^e \cdot \mathbf{F}^p, \quad F_A^a = F_A^e F_A^p
\]

(8.1)

where the plastic part \( \mathbf{F}^p \) is obtained by elastic unloading all infinitesimal neighbourhoods of the body. This has the effect of introducing a new configuration with coordinates \( \{ \tilde{x}^a \} \) and the metric \( \tilde{g} \) into the formulation, commonly termed the intermediate configuration.

Obviously, the inverse tensor to (8.1) has the components

\[
(F^{-1})_A^a = (F^{-1})_a^e (F^{-1})_a^p.
\]

(8.2)
Under the above assumption and heeding (9.1), the right Cauchy–Green tensor from (3.17) can be found as

\[ C_{AB} = (\Phi^\sigma g)_{AB} = F_A^\alpha F_B^\beta g_{ab} = \frac{p}{F_A} \frac{p}{F_B} g_{ab} F_\alpha^e F_\beta^e = \frac{p}{F_A} \frac{p}{F_B} C_{\alpha\beta} \quad (8.3) \]

or

\[ C^i = \Phi^\sigma C^i, \quad C_{\alpha\beta} := g_{ab} F_\alpha^e F_\beta^e, \quad (8.4) \]

and the left Cauchy–Green tensor from (3.17) reads

\[ b^{ab} = (\Phi, G^i)^{ab} = F_A^\alpha F_B^\beta C^{AB} = F_A^\alpha e^\beta C^{AB} F_B^\alpha p_{\alpha} p_{\beta} C_{\alpha\beta} = F_A^\alpha e^\beta b^{\alpha\beta} \quad (8.5) \]

or

\[ b^i = \Phi^\sigma b^i, \quad b^{\alpha\beta} := C^{AB} F_B^\alpha p_{\beta}. \quad (8.6) \]

According to (3.18), the associated material rate of deformation tensor is given by

\[ D_{AB} = \frac{1}{2} \left[ \frac{p}{F_A} \frac{p}{F_B} d_{\alpha\beta} + C_{\alpha\beta} \left( \frac{p}{F_A} \frac{d}{dt} F_B^\alpha + \frac{p}{F_B} \frac{d}{dt} F_A^\alpha \right) \right], \quad (8.7) \]

and consequently, with (see e.g. [Hac92])

\[ \frac{e}{D} := \frac{1}{2} \frac{d}{dt} C \quad (8.8) \]

we may write

\[ D^i = \Phi^\sigma D^i + \Phi^i, \quad \text{with } D_{AB} := \frac{1}{2} C_{\alpha\beta} \left( \frac{p}{F_A} \frac{d}{dt} F_B^\alpha + \frac{p}{F_B} \frac{d}{dt} F_A^\alpha \right). \quad (8.9) \]

Combining (3.19) and (3.20) we see that \( d^i = \Phi^\sigma D^i \), and together with (8.9) we get the spatial rate of deformation tensor

\[ d^i = d^i + d^i \quad \text{with } d^i := \Phi^\sigma D^i = \Phi^\sigma D^i \quad \text{and } d^i := \Phi^\sigma D^i. \quad (8.10) \]

Using the equations from above, their components can be easily found as

\[ d_{ab} = \frac{1}{2} \left( g_{ac} (F^{-1})_b^c + g_{cb} (F^{-1})_a^c \right) F_A^\alpha F_B^\beta \quad \text{and } \quad d_{ab} = \frac{1}{2} \left( (F^{-1})_b^c (F^{-1})_a^c \right) d_{\alpha\beta} \quad (8.11) \]

Analogically to (3.6) we define the “elastic” Lie derivative

\[ \mathcal{L}_g := \frac{e}{dt} \left( \Phi^\sigma g \right) \quad (8.12) \]

and see from (9.1) that

\[ d^i = \frac{e}{2} \mathcal{L}_g \quad (8.13) \]

holds.
The yield criterion

Let $\mathbf{i}^{1}$ be a set of $k$ contravariant tensors of any rank, describing the hardening, and let the Stress space of $\left\{ \mathbf{T}, \mathbf{i}^{1} \right\}$ be defined to be the space $R^{m}$, where $m$ complies to the sum of the number of components in $\mathbf{T}$ and in all the $\mathbf{i}^{1}$’s, counting symmetrical components only once. For the sake of shortness and without lack of generality, we will restrict to $k = 1$ and drop the index $i$ in the sequel. Let’s assume, that the stress level of the second Piola–Kirchhoff stress tensor $\mathbf{T}$, at which plastic deformation begins, is determined by a convex hyperplane

$$\mathcal{Y}(\mathbf{T}, \mathbf{i}^{1}) = 0$$

in the stress space. For stress levels with $\mathcal{Y}(\mathbf{T}, \mathbf{i}^{1}) < 0$ the material is regarded to behave hyperelastic, that is the last line of (5.18) is assumed to hold, and $\mathcal{Y}(\mathbf{T}, \mathbf{i}^{1}) > 0$ will be forbidden. From (4.4) and (4.8) we get $\mathbf{T} = J\Phi^{*} \mathbf{\sigma}$ and therefore $\mathbf{\sigma} = \frac{1}{J} \Phi^{*} \mathbf{T}$ holds. According to this we define the internal variables describing hardening in the spatial formulation by

$$\mathbf{\pi}^{1} := \frac{1}{J} \Phi^{*} \mathbf{i}^{1}.$$  

With $\nu(\mathbf{\sigma}, \mathbf{\pi}^{1}) := \mathcal{Y}(J\Phi^{*} \mathbf{\sigma}, J\Phi^{*} \mathbf{\pi}^{1})$ the correlating yield criterion in the spatial formulation reads

$$\nu(\mathbf{\sigma}, \mathbf{\pi}^{1}) = 0.$$  

The principle of maximal dissipation

Since plastic deformation is an irreversible process, the internal energy as discussed on page 15 does not fully describe the appearing phenomena. Energy will be dissipated, the entropy of the system increases, although thermal effects further on are assumed to be neglectable. This can be taken into consideration by the free energy as follows from thermodynamics (cf. ch. 6).

As already stated on pages 17–19, the internal energy $E$ and the entropy $\mathcal{E}$ in the general case depend on $\mathbf{X}, t$ and $\mathbf{C}$. Restricting to homogeneous and stationary isothermal problems, we have for the free energy $\mathcal{Z} = E - T \mathcal{E}$, $\mathcal{Z} = \mathcal{Z}(\mathbf{C}^{i})$. Here we introduce some additional internal variables $\mathbf{\Theta}^{i}$ explained below, the specific free energy in the material formulation may depend on:

$$\mathcal{Z} = \mathcal{Z}(\mathbf{C}^{i}, \mathbf{\Theta}^{i}).$$  

Since the intermediate configuration is an appropriate configuration [Hac92] for describing the material behaviour, it is suggestive to formulate the free energy function

$$\mathcal{Z} := \Phi^{*} \tilde{\mathcal{Z}}(\mathbf{C}^{i}, \mathbf{\Theta}^{i})^{27}$$

\[\text{cf. (8.20).}\]

\[\text{cf. (8.24).}\]

\[\text{cf. (8.25).}\]
with $\mathbf{C}^i = \mathbf{P}^i, \mathbf{C}^i$ from (8.4) and $(\bar{\Theta}^i)_{a\ldots b} := (\mathbf{P}, \Theta^i)_{a\ldots b} = (F^{-1})_a^A \cdots (F^{-1})_B^B \Theta_{A\ldots B}$.

In the spatial configuration we have

$$
\zeta := \Phi, Z = \mathcal{Z}(\Phi^* g, \Phi^* \theta^i) = \zeta(g, \theta^i)
$$

(8.19)

with $(\theta^i)_{a\ldots b} := (\Phi, \Theta^i)_{a\ldots b} = (F^{-1})_a^A \cdots (F^{-1})_B^B \Theta_{A\ldots B}$.

Now we require the covariant tensors $\Pi^i$ and $\pi^i$ from the previous section to be conjugate to $\Theta^i$ and $\theta^i$, respectively:

$$
\Pi^i := -\rho_{rel} \frac{\partial Z}{\partial \Theta}, \quad \pi^i := -\rho \frac{\partial \zeta}{\partial \theta^i}.
$$

(8.20)

In components, this reads $\Pi^{A\ldots B} := -\rho_{rel} \frac{\partial \mathcal{Z}}{\partial \Theta_{A\ldots B}}$ and $\pi^{a\ldots b} := -\rho \frac{\partial \zeta}{\partial \theta_{a\ldots b}}$.

Combining (8.15) with (8.19), the transformations

$$
\Pi^i = -\rho \frac{\partial}{\partial \theta^i} = -\rho_{rel} \frac{\partial \mathcal{Z}}{\partial \Theta}
$$

can be found. So the assumptions (8.20) are in correspondence to each other.

The Drucker postulate or the principle of maximal dissipation implies, that the local dissipation function

$$
\mathcal{D}_M := T : D^i - \rho_{rel} \frac{d}{dt} \mathcal{Z} \geq 0, \quad \mathcal{D}_S := \sigma : d^i - \rho \frac{d}{dt} \mathcal{Z} \geq 0
$$

(8.21)

will become maximal during plastic deformation. Note, that (8.21) is the restriction of the reduced dissipation inequalities (6.10), (6.5) to isothermal processes.

Since $\frac{\partial \mathcal{Z}}{\partial \Theta_{A\ldots B}} = \frac{\partial \tilde{Z}}{\partial \Theta_{A\ldots B}} \frac{\partial C_{a\beta}}{\partial \Theta_{A\beta}} = \frac{\partial \tilde{Z}}{\partial \Theta_{A\beta}} (F^{-1})_a^A \frac{\partial F^{-1}}{\partial \Theta_{A\beta}}$ holds, we have $\frac{\partial \tilde{Z}}{\partial \Theta_{A\beta}} = \left[ \frac{\partial C_{a\beta}}{\partial \Theta_{A\beta}} \right]_{F^{-1}}$.

Using this, we get

$$
\frac{d \mathcal{Z}}{dt} \equiv \frac{d \tilde{Z}}{dt} = \frac{\partial \tilde{Z}}{\partial \Theta_{A\beta}} \frac{d C_{a\beta}}{dt} + \frac{\partial \tilde{Z}}{\partial \Theta_{a\ldots b}} \frac{d \tilde{\Theta}_{a\ldots b}}{dt} =
$$

$$
\left[ \frac{\partial C_{a\beta}}{\partial \Theta_{A\beta}} \right]_{F^{-1}} \frac{d \tilde{Z}}{dt} + \left[ \frac{\partial C_{a\beta}}{\partial \Theta_{a\ldots b}} \right]_{F^{-1}} \frac{d \tilde{\Theta}_{a\ldots b}}{dt} =
$$

$$
\left[ \frac{\partial C_{a\beta}}{\partial \Theta_{A\beta}} \right]_{F^{-1}} \frac{d \tilde{Z}}{dt} + \left[ \frac{\partial C_{a\beta}}{\partial \Theta_{a\ldots b}} \right]_{F^{-1}} \frac{d \tilde{\Theta}_{a\ldots b}}{dt} =
$$

$$
\left[ \frac{\partial C_{a\beta}}{\partial \Theta_{A\beta}} \right]_{F^{-1}} \frac{d \tilde{Z}}{dt} + \left[ \frac{\partial C_{a\beta}}{\partial \Theta_{a\ldots b}} \right]_{F^{-1}} \frac{d \tilde{\Theta}_{a\ldots b}}{dt} =
$$

This and an analogous calculation gives

$$
\frac{d \zeta}{dt} = 2 \left( \frac{\partial \zeta}{\partial \Theta^i} \right) \frac{d \zeta^i}{dt} + \frac{\partial \zeta}{\partial \Theta^i} \left[ \frac{\partial \zeta^i}{dt} \right]
$$

(8.22)

and

$$
\frac{d \zeta}{dt} = 2 \left( \frac{\partial \zeta}{\partial \Theta^i} \right) \frac{d \zeta^i}{dt} + \frac{\partial \zeta}{\partial \Theta^i} \left[ \frac{\partial \zeta^i}{dt} \right]
$$

(8.23)

with $\tilde{\zeta}^i$ from (8.8) and $\tilde{e}^i$ from (8.13).

Although it doesn’t coincide with the Lie derivative from (3.6) or (8.12), we define

$$
\mathbf{L}_\mathbf{v}, \Theta^i := \mathbf{P}^i \left[ \frac{d}{dt} \left( \mathbf{P}, \Theta^i \right) \right].
$$

(8.24)
Using this, (8.20), (8.9) and the evident identity $\mathbf{T}: (\Phi^* \mathbf{D}^i) = (\Phi, \mathbf{T}): \mathbf{D}^i$, the left part of (8.21) reads

$$D_M = \left[ \Phi, \left( \mathbf{T} - 2\rho_{rel} \frac{\partial \mathbf{Z}}{\partial \mathbf{C}} \right) \right]: \mathbf{D}^i + \mathbf{T}: \mathbf{D}^i + \Pi^i \mathbf{L}_\nu \Theta^i. \quad (8.25)$$

The same way, including (8.10), (8.20) and $\mathbf{L}_\nu \theta^i := \Phi, \frac{d}{dt} \left( \Phi^* \theta^i \right)$ the right part of (8.21) reads

$$D_S = \left( \sigma - 2\rho \frac{\partial \zeta}{\partial \mathbf{g}} \right): \mathbf{d}^i + \sigma: \mathbf{d}^i + \pi^i \mathbf{L}_\nu \theta^i. \quad (8.26)$$

Taking into account that, if no plastic deformation occurs, also no dissipation should take place, e.g. $D_S = D_M = 0$. Then

$$\mathbf{T} = 2\rho_{rel} \frac{\partial \mathbf{Z}}{\partial \mathbf{C}} \quad \text{and} \quad \sigma = 2\rho \frac{\partial \zeta}{\partial \mathbf{g}} \quad (8.27)$$

hold and (8.27) replaces the Doyle–Ericksen–Formula in (5.14) and (5.18) for plasticity problems. Now suppose, that (8.27) holds. Then, (8.25) and (8.26) reads

$$D_M = \mathbf{T}: \mathbf{D}^i + \Pi^i \mathbf{L}_\nu \Theta^i \quad \text{and} \quad D_S = \sigma: \mathbf{d}^i + \pi^i \mathbf{L}_\nu \theta^i. \quad (8.28)$$

Now suppose, that the yield criterions (8.14) and (8.16) are fulfilled, and that the stresses and internal variables $\mathbf{T}_{max}, \sigma_{max}, \Pi_{max}^i$ and $\pi_{max}^i$ adopt values, maximizing the dissipation (8.28). Then, as necessary conditions,

$$\frac{\partial}{\partial \mathbf{T}} (\Lambda \mathbf{Y} - D_M) = 0 \quad , \quad \frac{\partial}{\partial \Pi^i} (\Lambda \mathbf{Y} - D_M) = 0, \quad (8.29)$$

$$\frac{\partial}{\partial \sigma} (\lambda \mathbf{v} - D_S) = 0 \quad \text{and} \quad \frac{\partial}{\partial \pi^i} (\lambda \mathbf{v} - D_S) = 0 \quad (8.30)$$

must be fulfilled. Holding $\mathbf{D}^i, \mathbf{L}_\nu \Theta^i, \mathbf{d}^i, \mathbf{L}_\nu \theta^i$ tight, the extrem of (8.27) is described by the equations

$$\mathbf{D}^i = \lambda \frac{\partial \mathbf{Y}}{\partial \mathbf{T}}, \quad \mathbf{L}_\nu \Theta^i = \lambda \frac{\partial \mathbf{Y}}{\partial \Pi^i}, \quad (8.31)$$

$$\mathbf{d}^i = \lambda \frac{\partial \mathbf{v}}{\partial \sigma} \quad \text{and} \quad \mathbf{L}_\nu \theta^i = \lambda \frac{\partial \mathbf{v}}{\partial \pi^i}, \quad (8.32)$$

and the dissipation will be maximal if and only if the yield surface is convex, as assumed on page 24.

\textsuperscript{Note, that this complies to (3.6) and (8.12).}
The evolution of stresses

Applying (8.22) for $\frac{\partial Z}{\partial C_i}$ instead of $Z$, the material time derivative of the second Piola–Kirchhoff stress tensor (8.27) can be found as

$$\frac{dT}{dt} = 2\rho_{rel} \left[ \frac{\partial^2 Z}{\partial C_i \partial C_j} \Phi \frac{d}{dt} \left( \Phi, C_i \right) + \frac{\partial^2 Z}{\partial C_i \partial \Theta} \Phi_c \frac{d}{dt} \left( \Phi_c, \Theta \right) \right],$$

(8.33)

and therefore the Lie derivative of the Kirchhoff stress tensor reads

$$\mathcal{L}_\pi \tau = \Phi_c \frac{d}{dt} \left( \Phi_c \tau \right) = \Phi_c \frac{d}{dt} \left[ \frac{\partial^2 \zeta}{\partial \Theta^2} \Phi_c \frac{d}{dt} \left( \Phi_c, \tau \right) \right] + \frac{\partial^2 \zeta}{\partial \Theta \partial \tau} \Phi_c \frac{d}{dt} \left( \Phi_c, \tau \right).$$

(8.34)

Consequently using (8.9), (8.11), (8.20),

$$\frac{dT}{dt} = 2 \frac{\partial T}{\partial C_i} \left( D - D^i \right) + \frac{\partial T}{\partial \Theta} \mathcal{L}_\pi \Theta, \quad \text{or} \quad \frac{dT}{dt} = 2 \frac{\partial T}{\partial C_i} \left( D - D^i \right) - 2 \frac{\partial \Pi^i}{\partial C_i} \mathcal{L}_\pi \Theta, \quad \mathcal{L}_\pi \tau = 2 \frac{\partial \tau}{\partial g} \left( d^i - d^i \right) + \frac{\partial \tau}{\partial \Theta} \mathcal{L}_\pi \Theta, \quad \mathcal{L}_\pi \tau = 2 \frac{\partial \tau}{\partial g} \left( d^i - d^i \right) - 2 \frac{\partial \kappa}{\partial g} \mathcal{L}_\pi \Theta,$$

(8.35)

(8.36)

can be shown, with $\kappa := J \pi^i = -\rho_{rel} \frac{\partial \zeta}{\partial \Theta}$. 29

The plastic spin

As easy can be seen, the set of equations describing the plastic material behaviour in the material configuration (8.31) (8.35) and in the spatial configuration (8.32) (8.36) is not entirely complete, additional assumptions are necessary with respect to the plastic spin $\Omega^i$, $\omega^i$ to construct the plastic and elastic Lie derivative $\mathcal{L}_\pi \Theta^i$ and $\mathcal{L}_\pi \Theta^i$. Following the strategy splitting the rate of deformation tensor (cf. page 23) in an elastic and a plastic part, the skew symmetric spin $\Omega^i := skew(D^i)$ can be split by:

$$\Omega_{AB} := \frac{1}{2} g_{ab} \left( F_A^a \frac{d}{dt} F_B^b - F_B^a \frac{d}{dt} F_A^b \right)$$

$$= \frac{1}{2} g_{ab} \left[ F_A^a \frac{d}{dt} F_B^b \left( \frac{c^a}{dt} \frac{c^b}{dt} - \frac{c^b}{dt} \frac{c^a}{dt} \right) + F_A^a \frac{d}{dt} F_B^b \left( \frac{c^b}{dt} \frac{d}{dt} F_A^a - \frac{c^a}{dt} \frac{d}{dt} F_A^b \right) \right]$$

$$= \frac{p_a}{F_B^a} \frac{d}{dt} \left( \Omega^i \right) + \Omega_{AB}.$$

(8.37)

Or in a more compact notation

$$\Omega^i = \Phi_c \Omega^i + \Omega^i,$$

(8.37)

29like $\pi := J \sigma = 2\rho_{rel} \frac{\partial \zeta}{\partial g}$
with \( \Omega_{AB} := C_{a\beta} \left( F^a_A \frac{d}{dt} F^\beta_B - F^\beta_B \frac{d}{dt} F^a_A \right) \), \( \bar{\Omega}_{a\beta} := \frac{1}{2} g_{ab} \left( \frac{c}{d} \frac{d}{dt} F^b_{\beta} - \frac{c}{d} \frac{d}{dt} F^a_{\alpha} \right) \).

And using \( \omega^i := \Phi^i \Omega^j \) one finds

\[
\omega_{ab} := \frac{1}{2} \left[ \left( g_{ac} F^{-1} A_b - g_{cb} F^{-1} A_a \right) F^c_d \frac{d}{dt} F^a_B \right] + \left( g_{ac} F^{-1} A_b - g_{cb} F^{-1} A_a \right) F^c_d \frac{d}{dt} F^a_B , \quad \omega_{ab} := \frac{1}{2} \left( g_{ac} F^{-1} A_b - g_{cb} F^{-1} A_a \right) F^c_d \frac{d}{dt} F^a_B .
\]

The simplest assumptions is to let the plastic spin \( \bar{\Omega}^i, \omega^i \) to be zero (see [Hac92]) until more information is available.

A more detailed discussion of this subject, beside other models where the plastic spin is not explicitly included (e.g. [Sim93]), can be found in [MG98].

9 Appendix

Pull back, push forward and the deformation gradient

On several places of this article we use a representation of the pull back and the push forward of some tensors in terms of the deformation gradient (3.10).

Let \( \mathbf{t} \) and \( \mathbf{T} \) two tensors of type \( \left( \frac{1}{0} \right) \) and \( \mathbf{r} \) and \( \mathbf{R} \) of type \( \left( 0 \frac{1}{1} \right) \) without any special physical meaning in this chapter but connected by \( \mathbf{T} = \Phi^* \mathbf{t} \), \( \mathbf{t} = \Phi \mathbf{T} \), \( \mathbf{R} = \Phi^* \mathbf{r} \) and \( \mathbf{r} = \Phi \mathbf{R} \). Then, from the formulas given on page 5 we derive

\[
T_{AB} = (F^{-1})^A_B (F^{-1})^B_A t^{ab} , \quad t_{ab} = F^a_A F^b_B T_{AB} \]
\[
R_{AB} = F^a_A F^b_B r_{ab} , \quad r_{ab} = (F^{-1})^A_B (F^{-1})^B_A R_{AB} .
\]

In the same way for \( \mathbf{t} \) and \( \mathbf{T} \) of type \( \left( \frac{1}{0} \right) \) with \( \mathbf{T} = \Phi^* \mathbf{t} \) and \( \mathbf{t} = \Phi \mathbf{T} \) we have

\[
T_{ABCD} = (F^{-1})^A_a (F^{-1})^B_b (F^{-1})^C_c (F^{-1})^D_d t^{abcd} \quad \text{and} \quad t^{abcd} = F^a_A F^b_B F^c_C F^d_D T_{ABCD} .
\]

For "two–point–tensors" \( \mathbf{W} \) of type \( \left( \frac{1}{0} \frac{1}{1} \right) \) with components \( \mathbf{W}^a_B \), acting from \( T_X \mathcal{B} \times T_x \mathcal{M} \) to \( \mathcal{A} \), the push forward’s components are

\[
(\Phi \cdot \mathbf{W})^a_B = (F^{-1})^B_a \mathbf{W}^a_B
\]

and it’s pull back is

\[
(\Phi^* \mathbf{W})^A_B = (F^{-1})^A_B \mathbf{W}^a_B .
\]
Domain integrals

As well known [MK70], volume integrals transform under change of coordinate systems \( \{z^i\} \rightarrow \{x^a\} \) as

\[
\int_{A_z} f(z) \, dz^1 \cdots dz^n = \int_{A_X} f(x) \det \left( \frac{\partial z^i}{\partial x^a} \right) \, dx^1 \cdots dx^n = \int_{A_X} f(x) \sqrt{\det(g_{ab})} \, dx^1 \cdots dx^n,
\]

because, assuming the \( \{z^i\} \) to be Cartesian coordinates we have

\[
\sqrt{\det(g_{ab})} = \sqrt{\det \left( \frac{\partial z^1}{\partial x^a} \frac{\partial z^i}{\partial x^b} \right)} = \det \left( \frac{\partial z^i}{\partial x^a} \right).
\]

Therefore, in general coordinates, the volume element can be written as

\[
dv := \sqrt{\det(g_{ab})} \, dx^1 \cdots dx^n.
\]

The same way, considering the transformation behaviour of a volume integral under a motion \( \Phi_t \), we may write

\[
\int_{\Phi_t(A)} h(x) \, dx^1 \cdots dx^n = \int_{A} h(\Phi_t(X)) \det \left( \frac{\partial \Phi^a}{\partial X^A} \right) \, dX^1 \cdots dX^n,
\]

or, for \( h(x) := f(x)\sqrt{\det(g_{ab})} \),

\[
\int_{\Phi_t(A)} f(x) \, dv = \int_{A} f(\Phi_t(X)) \sqrt{\det(g_{ab})} \, \det \left( \frac{\partial \Phi^a}{\partial X^A} \right) \, dV. \tag{9.5}
\]

The surface integral of 2. kind

\[
\int_{\partial A_z} \langle b(z), n_z \rangle \, da_z := \int_{\partial A_z} \det \begin{pmatrix} \frac{b^1}{\partial p^1} & \cdots & \frac{b^n}{\partial p^n} \\ \frac{\partial z^1}{\partial p^1} & \cdots & \frac{\partial z^n}{\partial p^n} \\ \vdots & \cdots & \vdots \\ \frac{\partial z^1}{\partial p^n} & \cdots & \frac{\partial z^n}{\partial p^n} \end{pmatrix} \, dp^1 \cdots dp^{n-1}
\]

over some vector \( b \) with components \( b^i \) related to \( \{z^i\} \) transforms to

\[
\int_{\partial A_X} \langle b(z(x)), n_x \rangle \, da_x := \int_{\partial A_X} \det \begin{pmatrix} \frac{\beta^1}{\partial q^1} & \cdots & \frac{\beta^n}{\partial q^n} \\ \frac{\partial z^1}{\partial q^1} & \cdots & \frac{\partial z^n}{\partial q^n} \\ \vdots & \cdots & \vdots \\ \frac{\partial z^1}{\partial q^n} & \cdots & \frac{\partial z^n}{\partial q^n} \end{pmatrix} \, dq^1 \cdots dq^{n-1},
\]
with $\beta^a := \frac{\partial x^a}{\partial z^i}$ related to $\{x^a\}$. This follows from the transformation of volume integrals as stated above, from the matrix product
\[
\begin{pmatrix}
\frac{\partial x^1}{\partial p^1} & \cdots & \frac{\partial x^n}{\partial p^n} \\
\vdots & \ddots & \vdots \\
\frac{\partial x^n}{\partial p^n} & \cdots & \frac{\partial x^n}{\partial p^n}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial z^i}{\partial x^a} \\
\frac{\partial z^i}{\partial x^b} \\
\vdots \\
\frac{\partial z^i}{\partial x^n}
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial z^i}{\partial x^a} \\
\frac{\partial z^i}{\partial x^b} \\
\vdots \\
\frac{\partial z^i}{\partial x^n}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial p^1}{\partial z^1} & \cdots & \frac{\partial p^n}{\partial z^n} \\
\vdots & \ddots & \vdots \\
\frac{\partial p^n}{\partial z^n} & \cdots & \frac{\partial p^n}{\partial z^n}
\end{pmatrix}
\]

and the relation
\[
\det \begin{pmatrix}
\frac{\partial x^1}{\partial p^1} & \cdots & \frac{\partial x^n}{\partial p^n} \\
\vdots & \ddots & \vdots \\
\frac{\partial x^n}{\partial p^n} & \cdots & \frac{\partial x^n}{\partial p^n}
\end{pmatrix} = \det \begin{pmatrix}
\frac{\partial x^1}{\partial q^1} & \cdots & \frac{\partial x^n}{\partial q^n} \\
\vdots & \ddots & \vdots \\
\frac{\partial x^n}{\partial q^n} & \cdots & \frac{\partial x^n}{\partial q^n}
\end{pmatrix} \det \left( \frac{\partial q^a}{\partial p^i} \right).
\]

So, in general coordinates, the surface element reads
\[
da := \sqrt{\det(g_{ab})} \; dq^1 \cdots dq^{n-1},
\]
and with $\mathbf{B} := \Phi^* \mathbf{b}$, the transformation behaviour of a surface integral under a motion $\Phi_t$ can be written as
\[
\int_{\Phi_t(A)} \langle \mathbf{b}, \mathbf{n} \rangle \; da = \int_{\Phi_t(A)} \langle \mathbf{B}, \mathbf{n} \rangle \; \sqrt{\det(G_{AB})} \; \det \left( \frac{\partial \Phi^a}{\partial X^A} \right) \; dA.
\] (9.6)

Using the same calculus as above, the theorem of Gauss and Ostrogradski can be written in the form
\[
\int_{\Phi_t(A)} \text{div} \; \mathbf{b} \; dv = \int_{\Phi_t(A)} \langle \mathbf{b}, \mathbf{n} \rangle \; da.
\] (9.7)

The time derivative of the Jacobian $J$
\[
J := \sqrt{\det(g_{ab})} \; \det \left( \frac{\partial \Phi^a}{\partial X^A} \right) = \sqrt{\det(G_{AB})} \; \det(F^a_A)
\] (9.8)
can be found using the identities
\[
\frac{\partial}{\partial F^a_A} \det(F^a_A) = \det(F^a_A) (F^{-1})^A_a \quad \text{and} \quad \frac{\partial}{\partial g_{ab}} \det(g_{ab}) = \det(g_{ab}) g^{ab}
\] (9.9)

Since $G_{AB}$ doesn't depend on $t$, we have
\[
\frac{dJ}{dt} = \frac{1}{\sqrt{\det(G_{AB})}} \left[ \frac{1}{2} \frac{\partial}{\partial g_{ab}} \det(g_{ab}) \frac{dt}{dt} + \sqrt{\det(g_{ab})} \frac{d}{dt} \frac{\partial \Phi^a}{\partial X^A} \right] 
\]
\[
= J \left[ \frac{1}{2} g^{ab} \frac{dt}{dt} g_{ab} + \frac{\partial X^A}{\partial x^a} \frac{dt}{dt} \frac{\partial \Phi^a}{\partial X^A} \right] = J \left[ \frac{1}{2} g^{ab} \frac{\partial g_{ab}}{\partial x^c} + \frac{\partial \Phi^a}{\partial x^a} \right]
\]
due to \( \frac{d}{dt} g_{ab} = \frac{\partial g_{ab}}{\partial x^\alpha} v^\alpha \) and \( \frac{\partial X^A}{\partial x^\alpha} \frac{d}{dt} \frac{\partial \Phi^a}{\partial X^A} = \frac{\partial X^A}{\partial x^\alpha} \frac{\partial \Phi^a}{\partial X^A} \frac{d}{dt} \frac{\partial \Phi^a}{\partial X^A} = \frac{\partial v^a}{\partial x^\alpha} \). Inserting (2.10) and (2.11), we get
\[
\frac{\partial}{\partial t} J = J \text{div} \mathbf{v} .
\] (9.10)

Proof of equation (4.2)

Due to the theorem of Gauss and Ostrogradski (9.7), taking into account the underlying Euclidean structure of \( \Phi_i(A) \) the equation \( \int \text{div} \mathbf{\sigma} \, dv = \int \langle \mathbf{\sigma}, \mathbf{n} \rangle \, da \) holds\(^3\) component-by-component for every sufficiently smooth tensor \( \mathbf{\sigma} \), and \( \mathbf{\sigma} \) can be chosen to be symmetric and to fulfill \( \langle \mathbf{\sigma}, \mathbf{n} \rangle = \mathbf{r} \). Applying the transport theorem (9.11) to the left hand side of (4.1) and using the conservation of mass \( \left( \frac{d}{dt} \rho + \rho \text{div} \mathbf{v} = 0 \right) \) we get
\[
\frac{d}{dt} \int_{\Phi_i(A)} \rho \mathbf{v} \, dv = \int_{\Phi_i(A)} \left[ \rho \frac{d}{dt} \mathbf{v} + \mathbf{v} \left( \frac{d}{dt} \rho + \rho \text{div} \mathbf{v} \right) \right] \, dv = \int_{\Phi_i(A)} \rho \frac{d}{dt} \mathbf{v} \, dv.
\]
So (4.1) can be stated as \( \int_{\Phi_i(A)} \rho \frac{d}{dt} \mathbf{v} \, dv = \int_{\Phi_i(A)} [\rho \mathbf{l} + \text{div} \mathbf{\sigma}] \, dv \). Since \( \mathbf{\sigma} \) has to satisfy \( \text{div} \mathbf{\sigma} = \rho \frac{d}{dt} \mathbf{v} - 1 \) as well as \( \langle \mathbf{\sigma}, \mathbf{n} \rangle = \mathbf{r} \) we have 6 conditions to determine the components of \( \mathbf{\sigma} \). Due to the required symmetry of \( \mathbf{\sigma} \), the number of those components is reduced to 6. So \( \mathbf{\sigma} \) is unique.

Proof of equation (5.12)

Due to (2.11) and (4.3) we have \( \text{div} \langle \mathbf{\sigma}, \mathbf{\xi} \rangle = (\text{grad} \langle \mathbf{\sigma}, \mathbf{\xi} \rangle)_a^b = \frac{\partial \langle \mathbf{\sigma}, \mathbf{\xi} \rangle_a}{\partial x^b} + \gamma_{b}^{a} \langle \mathbf{\sigma}, \mathbf{\xi} \rangle = \frac{\partial \sigma_{ac} g_{bc} \xi^b}{\partial x^a} + \gamma_{ab}^{a} \sigma_{bc} \xi^c = \nu_c \left( \frac{\partial \sigma_{ac}}{\partial x^a} + \gamma_{ac}^{a} \sigma_{bc} \right) + \sigma_{ac} \frac{\partial \nu_c}{\partial x^a} + \sigma_{bc} \gamma_{ac}^{a} + \sigma_{ab} \gamma_{ac}^{a} = \sigma_{ac} \frac{\partial \nu_c}{\partial x^a} \right. \]
\]

The last addend is equal to \( \sigma_{ab} \nu_{bc} = \sigma_{ab} \nu_{bc} = \frac{1}{2} \sigma_{ab} \left( \nu_{ac} - \nu_{bc} + \nu_{bc} - \nu_{ac} \right) = \mathbf{\sigma} \cdot \mathbf{\omega} + \frac{1}{2} \sigma_{ab} \left( \nu_{ab} - \nu_{bc} + \nu_{bc} - \nu_{ab} \right) = \mathbf{\sigma} \cdot \mathbf{\omega} + \frac{1}{2} \sigma_{ab} \left( g_{bc} \frac{\partial \xi^c}{\partial x^a} + g_{ac} \frac{\partial \xi^c}{\partial x^b} + \xi^c \left( \frac{\partial g_{bc}}{\partial x^a} + \frac{\partial g_{ac}}{\partial x^b} - 2 \gamma_{ac}^{a} g_{bd} \right) \right) \].

Including (2.27), for the first line of (5.12), it remains to show, that
\[
\frac{\partial g_{bc}}{\partial x^a} + \frac{\partial g_{ac}}{\partial x^b} = \frac{\partial \Phi^a}{\partial x^b} \text{ and this is just (2.10). For the second line of (5.12), some simple computation gives} \]
\[
g_{ac} \xi_b - g_{ab} \xi_c = g_{ac} \frac{\partial \xi^c}{\partial x^b} - g_{ac} \frac{\partial \xi^c}{\partial x^b} + \xi^d \left( g_{ac} \xi^b - g_{ab} \xi^c \right) = \text{and (} g_{ac} \xi^b \right) = \text{and (} g_{ac} \xi^b \right),}
\]
\(^3\)see the footnote on page 13
\( g_{ac} \frac{\partial \xi^c}{\partial x^a} - g_{bc} \frac{\partial \xi^c}{\partial x^b} + \xi^d \left( \frac{\partial g_{ad}}{\partial x^d} - \frac{\partial g_{bd}}{\partial x^d} \right) \). Extracting the terms common to both expressions we see, that the proof is complete when \( \frac{\partial g_{ad}}{\partial x^d} - \frac{\partial g_{bd}}{\partial x^d} = g_{ac} \dot{\gamma}_c^d - g_{bc} \gamma_a^d \) has been shown, and this again is a consequence of (2.10).

The transport theorem

For any function \( f_t(x) \) we have \( \frac{d}{dt} \int_{\Phi_t(A)} f d\nu = \frac{d}{dt} \int_{\Phi_t(A)} f_t(\Phi_t(x))J_t(x) dV = \int_{\Phi_t(A)} \left( \frac{d}{dt} f_t + f_t \frac{\partial J}{\partial t} \right) dV = \int_{\Phi_t(A)} \left( \frac{d f_t}{d t} J + f_t \frac{d J}{d t} \right) dV \) with \( \frac{d J}{d t} = \frac{d J}{d t} = J \text{div} \nu \) from (9.10). This gives the transport theorem

\[
\frac{d}{d t} \int_{\Phi_t(A)} f_t d\nu = \int_{\Phi_t(A)} \left( \frac{d f_t}{d t} + f_t \text{div} \nu \right) d\nu \tag{9.11}
\]

with \( \nu \) denoting the velocity of the motion \( \Phi_t \).

For superposed motions \( \nu_t \) we have

\[
\int_{\nu_t(\Phi_t(A))} f_t(\nu_t(x)) d\nu = \int_{\Phi_t(A)} f_t(\Phi_t(x))J_t(x) dV \text{ with } J_t(x) \equiv 1 \text{ } \forall x \text{ and } \forall t \tag{9.12}
\]

because we get \( \text{det}(\dot{g}_{ab}) = \text{det}(g_{ab}) \text{det} \left( \frac{\partial (\nu_t^{-1})_a}{\partial x^b} \right) \text{det} \left( \frac{\partial (\nu_t^{-1})^a}{\partial x^b} \right) \) from (5.2)\(^{31} \), leading to

\[
\dot{J}_t = \sqrt{\frac{\text{det}(\dot{g}_{ab})}{\text{det}(g_{ab})}} \text{det} \left( \frac{\partial \nu_t^a}{\partial x^b} \right) = \text{det} \left( \frac{\partial (\nu_t^{-1})^a}{\partial x^b} \right) \text{det} \left( \frac{\partial (\nu_t^{-1})_a}{\partial x^b} \right) = \text{det} \left( \frac{\partial \nu_t^a}{\partial x^b} \right) \text{det} \left( \frac{\partial \nu_t^b}{\partial x^a} \right) = \text{det} \left( \delta^a_b \right) = 1.
\]

Using (9.12) and then (9.11) we find the transport theorem for superposed motions

\[
\frac{d}{d t} \int_{\nu_t(\Phi_t(A))} f_t(\nu_t(x)) d\nu = \int_{\Phi_t(A)} \left( \frac{d f_t(\nu_t(x))}{d t} + f_t(\nu_t(x)) \text{div} \nu \right) d\nu \tag{9.13}
\]

Proof of equation (5.5)

Using (5.4), (3.1) and (3.5), we get for the material acceleration \( \tilde{\mathbf{A}} \) the equation

\[
\tilde{\mathbf{A}} \mathbf{x}(t) = \frac{d}{d t} \tilde{\mathbf{V}}(t) = \frac{d}{d t} \left[ \left( \frac{\partial \xi^a}{\partial t} + \frac{\partial \xi^a}{\partial x^b} V^b \right) \tilde{\mathbf{e}}_a \right] \text{ with } \frac{d}{d t} \tilde{\mathbf{e}}_a = \gamma_a^{bc} \left( \frac{\partial \xi^b}{\partial x^c} + \frac{\partial \xi^b}{\partial x^c} V^d \right) \tilde{\mathbf{e}}_c \text{ (cf. (2.18))}
\]

and therefore we get

\[
\left( \frac{\partial \xi^a}{\partial t} + \frac{\partial \xi^a}{\partial x^b} V^b \right) \frac{d}{d t} \tilde{\mathbf{e}}_a = \gamma_a^{bc} \left( \xi^c + \frac{\partial \xi^c}{\partial x^a} V^d \right) \left( \xi^b + \frac{\partial \xi^b}{\partial x^c} V^e \right) \tilde{\mathbf{e}}_c.
\]

Since

\[
\frac{d}{d t} \left( \frac{\partial \xi^a}{\partial t} + \frac{\partial \xi^a}{\partial x^b} V^b \right) = \frac{d \xi^a}{d t} + \frac{\partial \xi^a}{\partial x^b} V^b \frac{d}{d t} \tilde{\mathbf{e}}_a + \frac{\partial \xi^a}{\partial x^b} V^b \frac{d \xi^b}{d t} + \frac{\partial \xi^a}{\partial x^b} \frac{d \xi^b}{d t} + \frac{\partial \xi^a}{\partial x^b} \left( \xi^b + \frac{\partial \xi^b}{\partial x^c} V^e \right) \tilde{\mathbf{e}}_c.
\]

\(^{31}\)Note, that this doesn’t apply to motions \( \Phi \) because these are mappings between different spaces.
and \( \frac{\partial \xi^a}{\partial x^b} V^b = \frac{\partial \xi^a}{\partial x^b} \frac{\partial \xi^c}{\partial x^d} V^d \) the material acceleration \( \overline{A} \) in the transformed state will be

\[
\overline{A} = \left[ \frac{\partial \xi^a}{\partial x^b} \frac{dV^b}{dt} + \left( \tilde{g}_{bc} \frac{\partial \xi^c}{\partial x^d} \frac{\partial \xi^d}{\partial x^e} + \frac{\partial^2 \xi^a}{\partial x^b \partial x^e} \right) V^b V^e \right] \tilde{e}_a +
\]

\[
\left[ \frac{\partial \xi^a}{\partial t} + \xi^b \left( \tilde{g}_{bc} \xi^c + \frac{\partial \xi^a}{\partial x^b} \right) + 2 \left( \tilde{g}_{bc} \xi^c + \frac{\partial \xi^a}{\partial x^b} \right) \frac{\partial \xi^d}{\partial x^e} \right] \tilde{e}_a.
\]

Due to the transformation behaviour of the Christoffel symbols \(^{32}\) we substitute

\[
\tilde{g}_{bc} \frac{\partial \xi^c}{\partial x^d} \frac{\partial \xi^d}{\partial x^e} + \frac{\partial^2 \xi^a}{\partial x^b \partial x^e} = \frac{\partial \xi^a}{\partial x^b} \frac{\partial}{\partial x^e} \gamma^d_{ef}.
\]

Note, that \( \frac{\partial \xi^a}{\partial x^b} \left( \frac{dV^d}{dt} + \gamma^d_{ef} V^e V^f \right) = \frac{\partial \xi^a}{\partial x^b} A^d = (\varphi, A)^a \) due to (2.18) and (2.12). According to (2.4) we introduce \( (\varphi, V)^c \left( \frac{\partial \xi^a}{\partial x^b} + \tilde{g}_{bc} \xi^c \right) = \left( \text{grad} (\varphi, V) \xi \right)^a \).

Finally we get

\[
\overline{A} = \varphi \cdot A + \frac{\partial \xi}{\partial t} + \text{grad} \xi + 2 \text{grad} (\varphi, V) \xi.
\]

**Proof of equation (5.10)**

We have \( \frac{d}{dt} \xi = \frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x^a} u^a + \frac{\partial \xi}{\partial g_{ab}} u^b v^c \), since the metric doesn’t depend explicitly on time. Starting from (5.9) we find for the material time derivative of the internal energy in the transformed state

\[
\left( \frac{d}{dt} \xi \right)_0 = \left( \frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x^a} u^a + \frac{\partial \xi}{\partial g_{ab}} \frac{d}{dt} \left( \frac{\partial \xi}{\partial x^b} \frac{\partial \xi}{\partial x^c} \tilde{g}_{cd} \right) \right)_0 = \frac{d}{dt} \xi + \frac{\partial \xi}{\partial g_{ab}} \frac{\partial g_{cd}}{\partial x^e} v^e +
\]

\[
\frac{\partial \xi}{\partial g_{ab}} \left( \frac{\partial \xi}{\partial x^a} \frac{\partial \xi}{\partial x^b} \frac{\partial \xi}{\partial x^c} \tilde{g}_{cd} v^e + \tilde{g}_{cd} \left[ \frac{\partial \xi}{\partial x^a} \left( \frac{\partial \xi}{\partial x^b} + \frac{\partial^2 \xi^a}{\partial x^d \partial x^e} v^e \right) \right] \frac{\partial g_{cd}}{\partial x^e} v^e + \frac{\partial^2 \xi^a}{\partial x^b \partial x^e} \right) +
\]

\[
\frac{d}{dt} \xi + \frac{\partial \xi}{\partial g_{ab}} \left( \delta_{a b} \xi^c v^c + \delta_{a b} \frac{\partial \xi}{\partial x^a} v^c + \delta_{a b} \frac{\partial \xi}{\partial x^a} v^c \right) = \frac{d}{dt} \xi + \frac{\partial \xi}{\partial g_{ab}} (\xi g)_{ab},
\]

cf. (2.27).

**Proof of equation (5.11)**

First we show identity (9.15):

\[
\frac{1}{2} \frac{d}{dt} \langle v, v \rangle = \langle a, v \rangle
\]

Using (3.5) we get \( \langle a, v \rangle = g_{ab} a^a v^b = g_{ab} v^b \left( \frac{\partial v^a}{\partial t} + \frac{\partial v^a}{\partial x^c} v^c + \gamma^a_{cd} v^d v^c \right). \)

Since \( \frac{1}{2} \frac{d}{dt} \langle v, v \rangle = \left( \frac{\partial v^a}{\partial t} + \frac{\partial v^a}{\partial x^c} v^c \right) v^b g_{ab} + \frac{1}{2} v^a v^b \frac{\partial g_{ab}}{\partial x^c} \) holds, it remains to check that

\[
g_{ab} \gamma^a_{cd} v^b v^c v^d = \frac{1}{2} \frac{\partial g_{ab}}{\partial x^c} v^b v^c v^d,
\]

and this can be done by means of (2.10). Next, we compare

\[
\tilde{A}_{bc} = \frac{\partial X^A}{\partial x^b} \left[ \Gamma^D_{EF} \frac{\partial X^E}{\partial x^F} \frac{\partial X^F}{\partial x^C} + \frac{\partial^2 X^D}{\partial x^E \partial x^C} \right]
\]

\(^{32}\)
the balance of energy on $\Phi_i(A)$ and on $\phi_i(\Phi_i(A))$ at $t = t_0$. On $\Phi_i(A)$ equation (5.1) combined with (9.11) and (9.15) leads to

$$\int_{\Phi_i(A)} \left[ \left( e + \frac{1}{2} \langle v, v \rangle \right) \left( \frac{d}{dt} \rho + \rho \text{div} v \right) + \rho \left( \frac{d}{dt} \varepsilon + \langle a - l, v \rangle \right) \right] dv = \int_{\partial \Phi_i(A)} \langle r, v \rangle da . \quad (9.16)$$

The analogon to (9.16) for the superposed motion is

$$\int_{\Phi_i(A)} \left[ \left( \tilde{e} + \frac{1}{2} \langle \tilde{v}, \tilde{v} \rangle \right) \left( \frac{d}{dt} \tilde{\rho} + \tilde{\rho} \text{div} \tilde{v} \right) + \tilde{\rho} \left( \frac{d}{dt} \tilde{\varepsilon} + \langle \tilde{a} - \tilde{l}, \tilde{v} \rangle \right) \right] dv = \int_{\partial \Phi_i(A)} \langle \tilde{r}, \tilde{v} \rangle da . \quad (9.17)$$

To prove (9.17), we formulate (5.1) on $\phi_i(\Phi_i(A))$:

$$\frac{d}{dt} \int_{\phi_i(\Phi_i(A))} \tilde{\rho} \left( \tilde{e} + \frac{1}{2} \langle \tilde{v}, \tilde{v} \rangle \right) d\tilde{v} = \int_{\phi_i(\Phi_i(A))} \tilde{\rho} \langle \tilde{l}, \tilde{v} \rangle d\tilde{v} + \int_{\partial \phi_i(\Phi_i(A))} \langle \tilde{r}, \tilde{v} \rangle d\tilde{a} . \quad (9.18)$$

Due to (9.12), the first term on the right of (9.18) is equal to $\int \tilde{\rho} \langle \tilde{l}, \tilde{v} \rangle d\tilde{v}$. To transform the second term on the right we apply Gaussian formula (cf. page 31) getting an integral over $\phi_i(\Phi_i(A))$, use (9.12) and apply Gaussian formula again on $\Phi_i(A)$ to get $\int \langle \tilde{r}, \tilde{v} \rangle da$.

After applying the transport theorem (9.13) to the integral on the left of (9.18) and using (9.15), taken in the transformed state, the equation (9.17) is proved by recombining the arising expressions. Due to (5.7)–(5.9) the balance of energy (9.17) for the superposed motion at $t = t_0$ reads

$$\int_{\Phi_i(A)} \left[ \left( e + \frac{1}{2} \langle v, v + \xi \rangle \right) \left( \frac{d}{dt} \rho + \rho \text{div} v \right) + \rho \left( \frac{d}{dt} \varepsilon + \langle a - l, v + \xi \rangle \right) \right] dv = \int_{\partial \Phi_i(A)} \langle r, v + \xi \rangle da . \quad (9.19)$$

Finally we subtract (9.16) from (9.19), regard (5.10) and get

$$\int_{\Phi_i(A)} \left[ \left( \frac{d}{dt} \rho + \rho \text{div} v \right) \left( \frac{1}{2} \langle \xi, \xi \rangle + \langle v, \xi \rangle \right) + \rho \left( \frac{\partial e}{\partial \xi} : \mathcal{L}_\xi \mathbf{g} + \langle a - l, \xi \rangle \right) \right] dv = \int_{\partial \Phi_i(A)} \langle r, \xi \rangle da . \quad (9.20)$$

**Proof of equations (5.16) and (5.17)**

Equation (5.1) with $r = \langle \sigma, n \rangle$ reads

$$\frac{d}{dt} \int_{\Phi_i(A)} \rho \left( e + \frac{1}{2} \langle v, v \rangle \right) dv = \int_{\Phi_i(A)} \rho \langle l, v \rangle dv + \int_{\partial \Phi_i(A)} \langle \langle \sigma, v \rangle, n \rangle da ,$$
where we used, that \( \langle \mathbf{\sigma}, \mathbf{n} \rangle \mathbf{v} = \langle \mathbf{\sigma}, \mathbf{v} \rangle \mathbf{n} \), what can be understood from simple computation. Applying (9.5) and (9.6) to this equation we get

\[
\frac{d}{dt} \int_{\mathcal{A}} \rho_{\text{ref}} \left[ \mathcal{E} + \frac{1}{2} \langle \mathbf{v}, \mathbf{v} \rangle \right] |_{\mathbf{x} = \mathbf{\Phi}_t(\mathbf{X})} dV = \int_{\mathcal{A}} \rho_{\text{ref}} \langle [\mathbf{I}, \mathbf{v}] \rangle |_{\mathbf{x} = \mathbf{\Phi}_t(\mathbf{X})} dV + \int_{\partial A} \langle \mathbf{\Phi}^* \mathbf{\sigma}, \mathbf{v} \rangle \mathbf{N} J dA
\]

with \( \rho(\mathbf{\Phi}_t(\mathbf{X})) J = \rho_{\text{ref}} \) from (4.6), \( J \) from (9.8) and

\[
E := \mathbf{\Phi}^* \mathcal{E}(\mathbf{x}, t, \mathbf{g}) = \mathcal{E}(\mathbf{\Phi}_t(\mathbf{X}), t, \mathbf{\Phi}_t(\mathbf{\Phi}^* \mathbf{g})) = \mathcal{E}(\mathbf{\Phi}_t(\mathbf{X}), t, \mathbf{\Phi}_t \mathbf{C}^t) = E(\mathbf{X}, t, \mathbf{C}^t)
\]

due to (5.8), (5.9) and (3.17). Since \( \mathbf{v} |_{\mathbf{x} = \mathbf{\Phi}_t(\mathbf{X})} = \mathbf{\Phi}_t(\mathbf{V}) \) (section 3) and \( \mathbf{l}(\mathbf{\Phi}_t(\mathbf{X}), t) = \mathbf{L}(\mathbf{X}, t) \) as in (4.6), this equation is equivalent to

\[
\frac{d}{dt} \int_{\mathcal{A}} \rho_{\text{ref}} \left[ E + \frac{1}{2} \langle \mathbf{L}, \mathbf{V} \rangle \right] dV = \int_{\mathcal{A}} \rho_{\text{ref}} \langle \mathbf{L}, \mathbf{V} \rangle dV + \int_{\partial A} \langle \mathbf{J} \mathbf{\Phi}^* \mathbf{\sigma}, \mathbf{v} \rangle \mathbf{N} dA.
\]

Equation (2.13) combined with (4.3) and (4.4), gives \( \mathbf{J} (\mathbf{\Phi}^* \mathbf{\sigma}) = \mathbf{J} \frac{d}{dt} (\mathbf{\Phi}^{-1})^A \mathbf{\sigma} = \mathbf{J} \mathbf{F}^{-1} \mathbf{a}^A \mathbf{g}^b \mathbf{v}^b = \mathbf{P}^A \mathbf{g}^b \mathbf{v}^b = (\mathbf{V}, \mathbf{V})^A \). So, the equation

\[
\frac{d}{dt} \int_{\mathcal{A}} \rho_{\text{ref}} \left[ E + \frac{1}{2} \langle \mathbf{L}, \mathbf{V} \rangle \right] dV = \int_{\mathcal{A}} \rho_{\text{ref}} \langle \mathbf{L}, \mathbf{V} \rangle dV + \int_{\partial A} \langle \mathbf{P}, \mathbf{V} \rangle \mathbf{N} dA \tag{9.21}
\]

with \( \langle \mathbf{P}, \mathbf{V} \rangle \mathbf{N} = \langle \mathbf{P}, \mathbf{N} \rangle, \mathbf{V} \) is obtained, and this is exactly (5.16). To prove (5.17), we permute differentiation and integration\(^{33}\) include (9.15) and are led to

\[
\int_{\mathcal{A}} \rho_{\text{ref}} \left( \frac{d}{dt} E + \langle \mathbf{A} - \mathbf{L}, \mathbf{V} \rangle \right) dV = \int_{\partial A} \langle \mathbf{P}, \mathbf{V} \rangle \mathbf{N} dA. \tag{9.22}
\]

As on page 34, we formulate the analogon to (9.22) for an arbitrary superposed motion with the material velocity \( \mathbf{E} := \frac{\partial \mathbf{\Sigma}}{\partial t} = \mathbf{\xi} \):

From (5.3) and (5.4) we use \( \hat{\mathbf{V}} = \varphi_\ast \mathbf{V} + \mathbf{\Xi} \). Like above, we have \( \hat{\mathbf{E}}(\mathbf{X}, t) = \hat{\mathbf{\Phi}^*} \hat{\mathbf{e}}(\hat{\mathbf{X}}, t, \hat{\mathbf{g}}) = \hat{\mathbf{e}}(\hat{\mathbf{\Phi}_t(\mathbf{X}), t, \hat{\Phi}_t(\hat{\mathbf{\Phi}^*} \hat{\mathbf{g}})}) = \hat{\mathbf{e}}(\hat{\mathbf{\Phi}_t(\mathbf{X}), t, \hat{\Phi}_t \mathbf{C}^t) = \hat{\mathbf{E}}(\mathbf{X}, t, \mathbf{C}^t) \) with \( \mathbf{C}^t := \hat{\mathbf{\Phi}^*} \hat{\mathbf{g}} \), cf. (3.17) and \( \hat{\mathbf{L}}(\mathbf{X}, t) = \hat{\mathbf{L}}(\hat{\mathbf{\Phi}_t(\mathbf{X}), t}) \). The definition (3.3) gives \( \hat{\mathbf{A}}(\mathbf{X}, t) = \mathbf{\hat{a}}(\hat{\mathbf{\Phi}_t(\mathbf{X}), t}) \), and from (5.7) we deduce \( \hat{\mathbf{L}} - \hat{\mathbf{A}} = \hat{\mathbf{L}} - \mathbf{\hat{a}} = \varphi_\ast (1 - \mathbf{a}) = \varphi_\ast (\mathbf{L} - \mathbf{A}) \). With (4.4) we have \( \mathbf{P} = \mathbf{J} \mathbf{\Phi}^* \mathbf{\sigma} \) and \( \hat{\mathbf{P}} = \hat{\mathbf{J}} \hat{\mathbf{\Phi}^*} \hat{\mathbf{\sigma}} \) with \( \hat{\mathbf{J}} = \mathbf{J} \) as shown near (9.12). With those preliminaries we note (9.22) for the superposed motion as

\[
\int_{\mathcal{A}} \rho_{\text{ref}} \left( \frac{d}{dt} \hat{\mathbf{E}} + \langle \hat{\mathbf{A}} - \hat{\mathbf{L}}, \hat{\mathbf{V}} \rangle \right) dV = \int_{\partial A} \langle \hat{\mathbf{P}}, \hat{\mathbf{V}} \rangle \mathbf{N} dA \tag{9.23}
\]

\(^{33}\)In contrast to (5.1), here is no need to heed (9.11), since \( \mathcal{A} \) does not depend on the time.
Subtracting \(9.22\) from \(9.23\) and selecting the time \(t_0\) for which \(\dot{\Phi}_{t_0} = \Phi_{\xi}^0\) holds, we get
\[
\int_A \rho_{\kappa_\ell} \left( \frac{d}{dt}(\dot{E} - E) \right) \bigg|_{t_0} + \langle A - L, \Xi \rangle \bigg|_{t_0} \right) dV = \int_{\partial A} \langle \langle P, N \rangle, \Xi \rangle dA ,
\]
where \((\dot{A} - \dot{L})|_{t_0} = A - L, \dot{\mathbf{V}}|_{t_0} = \mathbf{V} + \Xi\), and from \((4.2), (5.8)\) we get \(\dot{\sigma}|_{t_0} = \sigma\), delivering \(\dot{\mathbf{P}}|_{t_0} = \mathbf{P}\). For the term \(\frac{d}{dt}(\dot{E} - E) = \frac{\partial \dot{E}}{\partial t} - \frac{\partial E}{\partial t} + \frac{\partial \dot{E}}{\partial C_{AB}} \frac{d}{dt} C_{AB} - \frac{\partial E}{\partial C_{AB}} \frac{d}{dt} C_{AB}\) we see, that
\[
\frac{\partial \dot{E}}{\partial t} = \frac{\partial E}{\partial t} \quad \text{and} \quad \frac{\partial \dot{E}}{\partial C_{AB}} = \frac{\partial E}{\partial C_{AB}} \quad \text{for} \quad t = t_0 \quad \text{holds}.
\]
So we get \(\frac{d}{dt}(\dot{E} - E)|_{t_0} = \frac{\partial E}{\partial C_{AB}} \frac{d}{dt} C_{AB} - C_{AB} = 2 \frac{\partial E}{\partial C_{AB}} (\dot{D}_{AB}|_{t_0} - D_{AB})\). Due to \((3.19)\)
we have \(\dot{D}^i|_{t_0} - D^i = \frac{1}{2} \Phi^* \left( (\mathcal{L}_g \mathbf{g})|_{t_0} - \mathcal{L}_g \mathbf{g} \right) = \frac{1}{2} \Phi^* \mathcal{L}_g \mathbf{g} =: \mathbf{D}_g^\perp\), cf. \((2.29)\). So the equation
\((9.25)\) is proven:
\[
\int_A \rho_{\kappa_\ell} \left( \frac{\partial E}{\partial C} : \mathbf{D}_g^\perp + \langle A - L, \Xi \rangle \right) dV = \int_{\partial A} \langle \langle P, N \rangle, \Xi \rangle dA .
\]
To continue, we need an analogon to the divergency theorem \((5.12)\) formulated in the
reference configuration. This reads
\[
\text{DIV} \langle P, \Xi \rangle = \langle \text{DIV} \mathbf{P}, \Xi \rangle + (\mathbf{P} \cdot \mathbf{F}^{-1}) : \mathbf{D}_g^\perp
\]
with the spin \(\Omega^\perp_{\Xi}\) defined by
\[
\Omega^\perp_{\Xi} := \frac{1}{2} \left( (g_{bc} \Xi^c|_A F^b_B - (g_{ac} \Xi^c|_B F^a_B) = \frac{1}{2} \left( g_{bc} \Xi^c|_A F^b_B - g_{ac} \Xi^c|_B F^a_B \right) \right) ,
\]
the \textit{rate of deformation} \(\mathbf{D}_g^\perp\) with components
\[
D^\perp_{AB} = \left( \frac{1}{2} (g_{bc} \Xi^c|_A F^b_B + g_{ac} \Xi^c|_B F^a_B \right) ,
\]
and \((\mathbf{PF}^{-1})^A_B = P^a_A (F^{-1})^b_a = T^{AB}\) with \(T^{AB}\) from \((4.8)\). This shall be proved in the sequel:
Following the proof on page 31, including \((4.7)\) and introducing the temporary substitution \(\mathcal{N}_e := g_{ab} \Xi^e\) we get
\[
\text{DIV} \langle \mathbf{P}, \Xi \rangle = \mathcal{N}_e \left( \frac{\partial P^a_A}{\partial X^A} + \Gamma^A_{BC} P^a_B \right) + \mathcal{N}_e \left( \frac{\partial P^e_A}{\partial X^A} + \Gamma^A_{BC} \frac{\partial g^e_B}{\partial X^A} + \frac{\partial g^b_A}{\partial X^A} F^a_B \right)
\]
and \((\mathbf{PF}^{-1})^A_B = P^a_A (F^{-1})^b_a = T^{AB}\) with \(T^{AB}\) from \((4.8)\). This shall be proved in the sequel:
Following the proof on page 31, including \((4.7)\) and introducing the temporary substitution \(\mathcal{N}_e := g_{ab} \Xi^e\) we get
\[
\text{DIV} \langle \mathbf{P}, \Xi \rangle = \mathcal{N}_e \left( \frac{\partial P^a_A}{\partial X^A} + \Gamma^A_{BC} P^a_B \right) + \mathcal{N}_e \left( \frac{\partial P^e_A}{\partial X^A} + \Gamma^A_{BC} \frac{\partial g^e_B}{\partial X^A} + \frac{\partial g^b_A}{\partial X^A} F^a_B \right)
\]
and \((\mathbf{PF}^{-1})^A_B = P^a_A (F^{-1})^b_a = T^{AB}\) with \(T^{AB}\) from \((4.8)\). This shall be proved in the sequel:
Following the proof on page 31, including \((4.7)\) and introducing the temporary substitution \(\mathcal{N}_e := g_{ab} \Xi^e\) we get
\[
\text{DIV} \langle \mathbf{P}, \Xi \rangle = \mathcal{N}_e \left( \frac{\partial P^a_A}{\partial X^A} + \Gamma^A_{BC} P^a_B \right) + \mathcal{N}_e \left( \frac{\partial P^e_A}{\partial X^A} + \Gamma^A_{BC} \frac{\partial g^e_B}{\partial X^A} + \frac{\partial g^b_A}{\partial X^A} F^a_B \right)
\]
is true.
To start with the latter, we state, that \( \Xi^B_\alpha = F^B_\alpha \Xi^B_\alpha \) follows from simple calculations. So, rearranging the terms, we get \( g_{bc} \Xi^B_\alpha F^B_B - g_{ac} \Xi^B_\alpha F^B_A = g_{ab} \Xi^B_\alpha (F^B_A F^B_B - F^B_B F^B_A) \). In a similar manner the following transformations

\[
N^B_\alpha F^B_B - N^B_\alpha B F^B_A = F^B_\alpha N^B_\beta F^B_B - F^B_\beta N^B_\alpha F^B_A = (F^B_\alpha F^B_B - F^B_B F^B_A)(\Xi^B \frac{\partial g_{ab}}{\partial x^\nu} + g_{ab} \frac{\partial \Xi^B}{\partial x^\nu} - \gamma^d_{ac} g_{de} \Xi^e) = (F^A_\alpha F^B_B - F^B_B F^A_\alpha)(\Xi^B \frac{\partial g_{ab}}{\partial x^\nu} + g_{ab} \frac{\partial \Xi^B}{\partial x^\nu} - \gamma^d_{ac} g_{de} \Xi^e) \]

are obtained and it remains to verify, that \( \Xi^B (\frac{\partial g_{ab}}{\partial x^\nu} - g_{ad} \gamma^d_{bc} - g_{db} \gamma^d_{ac}) = 0 \), what is a straight consequence of (2.10).

To complete the proof to (9.26) we compute

\[
(\Omega^A_\alpha + D^A_\alpha)_{\alpha} = g_{bc} \Xi^B_\alpha F^B_B \quad \text{and} \quad (P \cdot F^{-1})_A (\Omega^A_\alpha + D^A_\alpha) = P^{\alpha A} g_{bc} \Xi^B_\alpha F^B_B (F^{-1})^B_\alpha = P^{\alpha A} g_{bc} \Xi^B_\alpha \delta^B_\alpha = P^{\alpha A} g_{bc} \Xi^A_\alpha = P^{\alpha A} g_{ac} \Xi^A_\alpha = P^{\alpha A} F^A_\alpha \Xi^A_\alpha \] and compare it to

\[
P^{\beta A} F^A_\alpha \Xi^A_\alpha (\frac{\partial N^A_\alpha}{\partial x^\nu} - \gamma^d_{bc} g_{ad}) = P^{\beta A} F^A_\alpha (\frac{\partial g_{ab}}{\partial x^\nu} + g_{ab} \frac{\partial \Xi^B}{\partial x^\nu} - \gamma^d_{ac} g_{de} \Xi^e) \]

in a similar way, using (4.2), the divergence theorem (5.12), the conservation of mass (5.14), (9.7), (9.11) and (9.15), from (6.2) we get

\[
\int_{\Phi_i(A)} \left\{ \rho \frac{d}{dt} \eta - \frac{\rho \sigma}{\gamma} + div (\frac{q}{\gamma}) + \eta \left[ \frac{d}{dt} \rho + \rho div \mathbf{v} \right] \right\} dv \geq 0.
\]

with \( \frac{d}{dt} \rho + \rho div \mathbf{v} \) vanishing due to the supposed conservation of mass. Taking into account the arbitrariness of \( \mathcal{A} \), the first part of (6.3) is found. The second part of (6.3) follows from simple calculus.

In a similar way, using (4.2), the divergence theorem (5.12), the conservation of mass (5.14), (9.7), (9.11) and (9.15), from (6.2) we get

\[
\int_{\Phi_i(A)} \left\{ \rho \frac{d}{dt} + \rho (a-1) - div \mathbf{v} \right\} dv = 0.
\]

According to the symmetry of \( \sigma \) we have \( \sigma : \omega^1 = 0 \), and the conservation of momentum from (5.14) gives \( \rho(a-1) - div \mathbf{v} = 0 \). Inserting the definition of the spatial rate of
deformation tensor $d^i$ from (3.20), we find

$$\int_{\Phi_i(A)} \left[ \rho \frac{d}{dt} \varepsilon - \sigma : d^i - \rho_s + \text{div} \, q \right] dv = 0,$$

and the arbitraryness of $A$ supplies (6.4).

References


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