A parabolic stochastic differential inclusion

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Abstract

Stochastic differential inclusions can be considered as a generalisation of stochastic differential equations. In particular a multivalued mapping describes the set of equations, in which a solution has to be found.

This paper presents an existence result for a special parabolic stochastic inclusion. The proof is based on the method of upper and lower solutions. In the deterministic case this method was effectively introduced by S. Carl.

Keywords: stochastic evolution inclusion, upper and lower solution, set-valued mapping, maximal monotone operator, Itô formula

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1 Introduction

As a part of the set-valued analysis we understand stochastic differential inclusions as an enlargement of stochastic differential equations. Here various stochastic equations that are characterised by a set-valued mapping have to be solved. Several applications are possible like modelling uncertainties, solving stochastic differential equations with discontinuous right hand side or describing events, whose dynamics are due to a large number of alternatives.

We are interested in the existence and uniqueness of solutions or the determination of the set of solutions by analysing stochastic differential inclusions.

One technique of proving the existence besides a semigroup approach or Galerkin approximations for example is the method of upper and lower solutions. This method has appeared to be very useful in the theory of nonlinear partial differential equations to encapsulate the solution. In the paper of S. Carl [3] the proof is done for a deterministic parabolic differential inclusion. It provides the basis for a promising adaption.

This paper is organised as follows. In section 2 some mathematical preliminaries are given. Section 3 contains the formulation of the problem and the main result with its proof.

2 Mathematical preliminaries

In our considerations we will use evolution triples and inclusions defined on these triples. At the beginning we repeat some definitions and facts concerning evolution triples.

An evolution triple \((V, H, V^*)\) consists of a real, separable, reflexive Banach space \(V\), its dual \(V^*\) and a real, separable Hilbert space \(H\) so that \(V \subseteq H \subseteq V^*\) holds, with all the embeddings being dense and continuous. An embedding \(V \subseteq H\) is called continuous, if \(u_n \to u\) in \(V\) implies \(u_n \to u\) in \(H\) as \(n \to \infty\). In the case of Sobolev spaces we have compactly embedded spaces. This means that the embedding is continuous and each bounded sequence \(u_n\) in \(V\) contains a subsequence that converges in \(H\).

For additional results and detailed proofs see Zeidler [6].

As mentioned before subject of the set-valued analysis is the investigation of set-valued mappings.

**Definition 2.1** Let \(X\) and \(Y\) be two sets. A set-valued or multivalued map \(F : X \to 2^Y\) from \(X\) into subsets of \(Y\) assigns to each \(x \in X\) a subset \(F(x) \subseteq Y\).

For example

\[
F(x) = \begin{cases} 
0 & \text{for } x < 0, \\
[0, 1] & \text{for } x = 0, \\
1 & \text{for } x > 0 
\end{cases}
\]

is a set-valued map.
An important element of characterisation is the selection.

**Definition 2.2** A single-valued function $f : X \to Y$ is called selection of the set-valued map $F : X \to 2^Y$, if

$$f(x) \in F(x) \quad \text{for all } x \in X.$$ 

In the example above

$$f(x) = \begin{cases} 
0 & \text{for } x \leq 0, \\
1 & \text{for } x > 0 
\end{cases}$$

is a selection of $F$.

Structures such as set-valued integrals, random variables or random processes are expressed with the help of selections. So the set-valued integral in the sense of Aumann is defined by the set of integrals of all integrable selections. Consequently the properties of the set-valued terms depend on the properties of the corresponding selections. A broad survey of set-valued structures can be found in the books of Hu and Papageorgiou [4], Aubin and Cellina [1] or in the one of Aubin and Frankowska [2].

In the theory of stochastic differential inclusions knowledge about the existence of selections and their properties is essential as we will see later in the definition of solutions. A simple but valuable selection theorem is the following.

**Theorem 2.3**

Let $(\Omega, \Sigma)$ be a measurable space, $Y$ a complete separable metric space and $F : \Omega \to 2^Y$ measurable with closed, non-empty values. Then $F$ admits a measurable selection. Further there exists a sequence $(f_n)_{n \geq 1}$ of measurable selections of $F$, such that for every $\omega \in \Omega$

$$F(\omega) = \{f_n(\omega)\}_{n \geq 1}.$$ 

We refer to chapter 2.2 in [4] for the proof.

Monotone and maximal monotone operators form a special class of set-valued mappings.

**Definition 2.4** Let $(H, (\cdot, \cdot))$ be a Hilbert space. An operator $A : H \to 2^H$ is said to be monotone, if for any $[x_i, y_i] \in GrA$, $i = 1, 2$

$$(y_1 - y_2, x_1 - x_2) \geq 0,$$

where $GrA = \{[x, y] \in H \times H : y \in A(x)\}$ is the graph of $A$.

The operator $A$ is called maximal monotone, if $[x_1, y_1] \in H \times H$ with $(y_1 - y_2, x_1 - x_2) \geq 0$ for all $[x_2, y_2] \in GrA$ implies $[x_1, y_1] \in GrA$. 

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3 Main result

Before we offer the main result with its proof in the second subsection, we formulate the problem and specify assumptions in part one.

3.1 Formulation of the problem

We consider the solvability of the parabolic stochastic differential inclusion

\[ dU(t, x) + AU(t, x)dt + g_1(t, x, U(t, x))dt + g_2(t, x, U(t, x))dW_1(t) \]
\[ \in \beta_1(t, x, U(t, x))dt + \beta_2(t, x, U(t, x))dW_2(t) \]

with the initial and boundary condition

\[ U(0, \cdot) = \psi(\cdot), \quad U(t, x)|_\Gamma = 0. \]  \hspace{1cm} (3.1)

Referring to this problem we assume

(a) \( O \subset \mathbb{R}^N \) is a bounded domain with Lipschitz boundary \( \partial O \), \( Q = [0, T] \times O \) and \( \Gamma = [0, T] \times \partial O \);

(b) \( (H_0^1(O), L^2(O), H^{-1}(O)) \) is an evolution triple denoted by \( (V, H, V^*) \);

(c) \( (\Omega, \mathcal{F}, P) \) is a complete probability space, \( \mathcal{F}_t \) is a filtration and \( (W_1(t))_{t \in [0, T]}, (W_2(t))_{t \in [0, T]} \) are real-valued Wiener processes with respect to \( \mathcal{F}_t \);

(d) \( A : V \rightarrow V^* \) is a nonlinear elliptic differential operator defined by

\[ Au = -\sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left( k(t, x, u) \frac{\partial u}{\partial x_j} \right), \]

where \( k : Q \times \mathbb{R} \rightarrow \mathbb{R} \) is a Carathéodory function such that

(i) for all \( s \in \mathbb{R}, \ (t, x) \mapsto k(t, x, s) \) is measurable,

(ii) for almost all \( (t, x) \in Q, \ s \mapsto k(t, x, s) \) is continuous;

(e) \( g_i : Q \times \mathbb{R} \rightarrow \mathbb{R}, \ i = 1, 2 \) are Carathéodory functions such that

(i) for all \( s \in \mathbb{R}, \ (t, x) \mapsto g_i(t, x, s) \) is measurable,

(ii) for almost all \( (t, x) \in Q, \ s \mapsto g_i(t, x, s) \) is continuous;
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\( \beta_i : \Omega \times [0, T] \times O \times \mathbb{R} \rightarrow 2^\mathbb{R}\setminus\{\emptyset\}, \ i = 1, 2 \) are set-valued, maximal monotone mappings satisfying

\[ \beta_i(\omega, t, x, s) = [f_i(\omega, t, x, s), \overline{f}_i(\omega, t, x, s)], \]

where \( s \mapsto f_i(\omega, t, x, s), \ i = 1, 2 \) are increasing random functions with

\[ f_i(\omega, t, x, s) = \lim_{\varepsilon \downarrow 0} f_i(\omega, t, x, s - \varepsilon), \]

\[ \overline{f}_i(\omega, t, x, s) = \lim_{\varepsilon \downarrow 0} f_i(\omega, t, x, s + \varepsilon). \]

for all \( t \) and \( x \) with probability 1;

\( \psi \) is an \( \mathcal{F}_0 \)-measurable random element with values in \( H \).

We introduce some basic definitions.

**Definition 3.1**

An \( \mathcal{F}_t \)-measurable, \( V \)-valued stochastic process \( U(t) \) with

\[ E \int_0^T ||U(t)||_V^2 \, dt < \infty, \quad E||U(t)||_H^2 < \infty \text{ for all } t \in [0, T] \]

is said to be a solution to (3.1), if:

(i) \( U(0, \cdot) = \psi(\cdot) \) a.s. in \( H \).

(ii) \( U(t) \) admits time-continuous trajectories \( P \)-a.s. in \( H \).

(iii) There exist \( H \)-valued, \( \mathcal{F}_t \)-measurable processes \( h_i(t, \cdot), i = 1, 2, \) \( h_i : \Omega \times [0, T] \times O \rightarrow \mathbb{R} \) such that for almost all \( t \) and \( x \), \( P \)-a.s.

\[ h_i(t, x) \in \beta_i(t, x, U(t, x)) \]

with

\[ \int_0^T ||h_1(t)||_H^2 \, dt < \infty, \quad \int_0^T ||h_2(t)||_H^2 \, dt < \infty. \]

(iv) It holds for all \( \phi \in C_0^\infty(O) \)

\[ \int_O U(t, x)\phi(x)dx + \int_0^t \int_O \sum_{j=1}^N k(s, x, U(s, x)) \cdot \frac{\partial U(s, x)}{\partial x_j} \cdot \frac{\partial \phi(x)}{\partial x_j} dx \, ds \]

\[ + \int_0^t \int_O g_1(s, x, U(s, x))\phi(x)dx \, ds + \int_0^t \int_O g_2(s, x, U(s, x)) \, dW_1(s)\phi(x) \, dx \]

\[ = \int_O \psi(x)\phi(x)dx + \int_0^t \int_O h_1(s, x)\phi(x)dx \, ds + \int_0^t \int_O h_2(s, x)dW_2(s)\phi(x)dx, \]

for all \( t \in [0, T], P \)-a.s.
Using the method of upper and lower solutions, we have to explain their meaning in the setting of differential inclusions.

**Definition 3.2**

1. An $\mathcal{F}_t$-measurable, $V$-valued stochastic process $\overline{U}(t)$ is said to be an upper solution to (3.1), if for all $\phi \in V \cap L^2_+(O)$

\[
\int_O \overline{U}(t,x)\phi(x)dx \\
+ \int_0^t \int_O \sum_{j=1}^N k(s,x,\overline{U}(s,x)) \cdot \frac{\partial \overline{U}(s,x)}{\partial x_j} \cdot \frac{\partial \phi(x)}{\partial x_j} dx ds \\
+ \int_0^t \int_O g_1(s,x,\overline{U}(s,x))\phi(x)dx ds + \int_O \int_0^t g_2(s,x,\overline{U}(s,x)) dW_1(s)\phi(x) dx \\
\geq \int_O \psi(x)\phi(x)dx + \int_0^t \int_O \overline{J}_1(s,x,\overline{U}(s,x))\phi(x)dx ds \\
+ \int_O \int_0^t \overline{J}_2(s,x,\overline{U}(s,x))dW_2(s)\phi(x)dx,
\]

for all $t \in [0,T]$, $P - a.s.$, $\overline{U}(t,x)|_\Gamma \geq 0$ and $\overline{U}(0,x) \geq \psi(x)$ for all $x \in O$.

2. An $\mathcal{F}_t$-measurable, $V$-valued stochastic process $\underline{U}(t)$ is said to be a lower solution to (3.1), if for all $\phi \in V \cap L^2_+(O)$

\[
\int_O \underline{U}(t,x)\phi(x)dx \\
+ \int_0^t \int_O \sum_{j=1}^N k(s,x,\underline{U}(s,x)) \cdot \frac{\partial \underline{U}(s,x)}{\partial x_j} \cdot \frac{\partial \phi(x)}{\partial x_j} dx ds \\
+ \int_0^t \int_O g_1(s,x,\underline{U}(s,x))\phi(x)dx ds + \int_O \int_0^t g_2(s,x,\underline{U}(s,x)) dW_1(s)\phi(x) dx \\
\leq \int_O \psi(x)\phi(x)dx + \int_0^t \int_O f_1(s,x,\underline{U}(s,x))\phi(x)dx ds \\
+ \int_O \int_0^t f_2(s,x,\underline{U}(s,x))dW_2(s)\phi(x)dx,
\]

for all $t \in [0,T]$, $P - a.s.$, $\underline{U}(t,x)|_\Gamma \leq 0$ and $\underline{U}(0,x) \leq \psi(x)$ for all $x \in O$.

**Remark 3.3**

In the previous definitions we have used the weak derivative with respect to $t$. Hence these definitions have to be interpreted in the weak sense.

Upper and lower solutions are defined by inserting special selections like the left- and right-hand limits of the functions $f_i$, $i = 1, 2$. 
Further we state the assumptions:

(H0) Initial condition \( \psi(\omega, \cdot) \in H, \mathcal{F}_0\)-measurable and \( E||\psi||^2_H < \infty \)

(H1) \( k : Q \times \mathbb{R} \rightarrow \mathbb{R} \) is a Carathéodory function such that for almost all \((t, x) \in Q\), for all \( s \in [U(t, x), \bar{U}(t, x)] \) there exists a \( q \in L^\infty(Q) \) and a constant \( k_0 > 0 \) such that

\[
|k(t, x, s)| \leq q(t, x) \quad \text{and} \quad k(t, x, s) \geq k_0 > 0.
\]

(H2) \( f_i : \Omega \times [0, T] \times O \times \mathbb{R} \mapsto \mathbb{R}, \ i = 1, 2 \) are \( \mathcal{F}_t\)-measurable functions such that

(i) for almost all \((t, x) \in Q, s \mapsto f_i(\omega, t, x, s)\) is increasing \( P\)-a.s.

(ii) with a constant \( \alpha > 0 \) and for almost all \((t, x) \in Q, s \in [\bar{U}(t, x) - \alpha, \bar{U}(t, x) + \alpha]\) there exist \( p_i \in L^2(Q), i = 1, 2 \) satisfying

\[
|f_i(t, x, s)| \leq p_i(t, x).
\]

(H3) \( g_i : Q \times \mathbb{R} \rightarrow \mathbb{R}, \ i = 1, 2 \) are Carathéodory functions such that for almost all \((t, x) \in Q \) and \( s \in [\bar{U}(t, x), \bar{U}(t, x)] \) there exist \( \tilde{p}_i \in L^2(Q), i = 1, 2 \) for which

\[
|g_i(t, x, s)| \leq \tilde{p}_i(t, x).
\]

### 3.2 Main theorem and proof

In general it is difficult to find solutions to a stochastic differential inclusion. The main result of this section, given below, provides information about the localisation of a solution, on condition that we know upper and lower solutions.

**Theorem 3.4**

Let \( \underline{U}, \bar{U} \) be lower and upper solution with \( \underline{U} \leq \bar{U}, P \) - a.s., for almost all \( t, x \).

If the assumptions (H0) - (H3) hold, then the problem (3.1) admits at least one solution \( U \in [\underline{U}, \bar{U}] \).
Proof.
Let us start with a brief sketch of the proof. The proof consists of the following four steps, based on the specifications in [3].

1. At first we introduce a regularised and truncated problem. The truncation operator $T$ is defined by:

$$
\tilde{U}(t,x) = TU(t,x) = \begin{cases} 
U(t,x) & \text{for } U(t,x) < U(t,x), \\
U(t,x) & \text{for } U(t,x) \leq U(t,x) \leq \overline{U}(t,x), \\
\overline{U}(t,x) & \text{for } U(t,x) > \overline{U}(t,x).
\end{cases}
$$

For $\varepsilon > 0$ we have the regularisation $f_i^\varepsilon$ of $f_i$, $i = 1,2$

$$
f_i^\varepsilon(t,x,s) = \frac{1}{\varepsilon} \int_{-\infty}^{\infty} f_i(t,x,s-\xi) J\left(\frac{\xi}{\varepsilon}\right) d\xi,
$$

where $J : \mathbb{R} \to \mathbb{R}$ is a mollifier with $\int_{\mathbb{R}} J(z) dz = 1$.

So the task reduces to show the existence of solutions to

$$
\int_O U_\varepsilon(t,x)\phi(x)dx \\
+ \int_0^t \int_O \sum_{j=1}^N k(s,x,TU_\varepsilon(s,x)) \cdot \frac{\partial U_\varepsilon(s,x)}{\partial x_j} \cdot \frac{\partial \phi(x)}{\partial x_j} dx ds \\
+ \int_0^t \int_O g_1(s,x,TU_\varepsilon(s,x))\phi(x)dx ds + \int_0^t \int_O g_2(s,x,TU_\varepsilon(s,x))dW_1(s)\phi(x) dx \\
= \int_O \psi(x)\phi(x)dx + \int_0^t \int_O f_1^\varepsilon(s,x,TU_\varepsilon(s,x))\phi(x)dx ds \\
+ \int_0^t \int_O f_2^\varepsilon(s,x,TU_\varepsilon(s,x))dW_2(s)\phi(x) dx,
$$

P-a.s., for all $\phi \in C_0^\infty(O)$ and for all $t \in [0,T]$.

2. Then we replace $TU_\varepsilon$ in the non-linearities of (3.3) by an arbitrary but fixed element $y$ and receive a linear problem. A theorem of Krylov-Rozovskij ensures the existence of a unique solution.

3. The third step concentrates on the application of the fixed-point theorem of Schauder.

4. After all it remains to pass to the limit of the simplified problem (3.3) and to show that the limiting process is a solution to the original problem.
We construct a simplified problem using truncation, regularisation and demonstrate the existence of solutions.

**Lemma 3.5**

The regularised and truncated problem (3.3) admits at least one solution for any \( \varepsilon, 0 < \varepsilon < \alpha \).

**Proof.**

The proof deals with the steps one to three from above. After transforming the problem (3.2) into (3.3), we treat the linear problem that arises, if we choose an arbitrary, but fixed \( y \in L^2(\Omega \times Q), \mathcal{F}_t \)-measurable and replace \( TU_\varepsilon \) with \( y \) in the non-linearities \( k, f_1^\varepsilon, f_2^\varepsilon, g_1, g_2 \) of equation (3.3). Thus we derive the following linear problem with respect to \( z_\varepsilon \):

\[
\int_\Omega z_\varepsilon(t, x)\phi(x)dx \\
+ \int_0^t \int_\Omega \left[ \sum_{j=1}^N k(s, x, y(s, x)) \cdot \frac{\partial z_\varepsilon(s, x)}{\partial x_j} \cdot \frac{\partial \phi(x)}{\partial x_j} \right] dx\, ds \\
+ \int_0^t \int_\Omega g_1(s, x, y(s, x))\phi(x)dx\, ds + \int_0^t \int_\Omega g_2(s, x, y(s, x))dW_1(s)\phi(x)\, dx \\
= \int_\Omega \psi(x)\phi(x)dx + \int_0^t \int_\Omega f_1^\varepsilon(s, x, y(s, x))\phi(x)dx\, ds \\
+ \int_0^t \int_\Omega f_2^\varepsilon(s, x, y(s, x))dW_2(s)\phi(x)\, dx,
\]

for all \( t \in [0, T] \), \( P \)-a.s.

According to theorem 2.1 in chapter 2 of Krylov-Rozovskij [5] there exists a unique solution \( z_\varepsilon(t, x) \) with \( P \)-a.s. time-continuous trajectories in \( H \) and

\[
E \int_\Omega (z_\varepsilon(t, x))^2dx < \infty \text{ for all } t \in [0, T], \quad E \int_0^T \|z_\varepsilon(t)\|^2_V dt < \infty.
\]

Having proved the existence of a solution to the linear equation (3.4), we go on solving the regularised and truncated problem (3.3). The key relation is the Schauder fixed-point theorem.

We transform equation (3.4) into an operator equation and check the assumptions, namely that the compact operator maps bounded, closed, convex and non-empty sets of a Banach space into itself [6].
The solution process $z_\varepsilon$ is expressed with the operator $S_\varepsilon$. This operator associates each $F_t$-measurable $y \in L^2(\Omega \times Q)$ with a solution of equation (3.4). Suppose $S_\varepsilon : L^2(\Omega \times Q) \mapsto L^2(\Omega \times Q)$, we obtain:

$$z_\varepsilon(t, x) = S_\varepsilon y(t, x) = y(0, x) + \int_0^t \sum_{j=1}^N \frac{\partial}{\partial x_j} \left( k(s, x, y(s, x)) \cdot \frac{\partial z_\varepsilon(s, x)}{\partial x_j} \right) \, ds$$

$$+ \int_0^t f_1^\varepsilon(s, x, y(s, x)) \, ds + \int_0^t f_2^\varepsilon(s, x, y(s, x)) \, dW_2(s)$$

$$- \int_0^t g_1(s, x, y(s, x)) \, ds - \int_0^t g_2(s, x, y(s, x)) \, dW_1(s),$$

$$z_\varepsilon(0, x) = y(0, x) = \psi(x).$$

Aside a unique solution $z_\varepsilon$ is assigned to each initial condition $y(0, x)$.

First of all we show that operator $S_\varepsilon$ is bounded. From equation (3.4), Itô's formula for the square of the norm (see appendix) and properties of the expected value we have:

$$E \int_O z_\varepsilon(t, x)^2 dx = E \int_O \psi(x)^2 dx$$

$$+ 2E \int_0^t \int_O (f_1^\varepsilon(s, x, y(s, x)) - g_1(s, x, y(s, x))) \cdot z_\varepsilon(s, x) \, dx \, ds$$

$$+ E \int_0^t \int_O f_2^\varepsilon(s, x, y(s, x))^2 \, dx \, ds$$

$$+ E \int_0^t \int_O g_2(s, x, y(s, x))^2 \, dx \, ds$$

$$- 2E \int_0^t \int_O \sum_{j=1}^N k(s, x, y(s, x)) \cdot \frac{\partial z_\varepsilon(s, x)}{\partial x_j} \cdot \frac{\partial z_\varepsilon(s, x)}{\partial x_j} \, dx \, ds.$$

The inequalities $2a \cdot b \leq \eta \cdot a^2 + \frac{1}{\eta} \cdot b^2$ for arbitrary $\eta > 0$ and $(a + b)^2 \leq 2a^2 + 2b^2$ as well as the application of Fubini’s theorem yield:

$$E \int_O z_\varepsilon(t, x)^2 dx + 2E \int_0^t \int_O \sum_{j=1}^N k(s, x, y(s, x)) \cdot \left( \frac{\partial z_\varepsilon(s, x)}{\partial x_j} \right)^2 \, dx \, ds$$

$$\leq E \int_O \psi(x)^2 dx$$

$$+ \eta \cdot E \int_0^t \int_O (f_1^\varepsilon(s, x, y(s, x)) - g_1(s, x, y(s, x)))^2 \, dx \, ds$$

$$+ \frac{1}{\eta} \cdot E \int_0^t \int_O z_\varepsilon(s, x)^2 \, dx \, ds$$

$$+ E \int_0^t \int_O f_2^\varepsilon(s, x, y(s, x))^2 \, dx \, ds + E \int_0^t \int_O g_2(s, x, y(s, x))^2 \, dx \, ds$$
\begin{align*}
\leq & \ E \int_O \psi(x)^2 \, dx + \frac{1}{\eta} \cdot E \int_0^t E \int_O z_\varepsilon(s, x)^2 \, dx \, ds \\
& \quad + \ 2\eta \cdot E \int_0^t \int_O f_1^\varepsilon(s, x, y(s, x))^2 \, dx \, ds + 2\eta \cdot E \int_0^t \int_O g_1(s, x, y(s, x))^2 \, dx \, ds \\
& \quad + \ E \int_0^t \int_O f_2^\varepsilon(s, x, y(s, x))^2 \, ds \, dx + E \int_0^t \int_O g_2(s, x, y(s, x))^2 \, ds \, dx \\
\leq & \ C + \frac{1}{\eta} \cdot \int_0^t E \int_O z_\varepsilon(s, x)^2 \, dx \, ds.
\end{align*}

Here \( C \) is a positive constant, independent from \( \varepsilon \), which follows from (H0), (H3) and the boundedness of the functions \( f_1^\varepsilon(t, x, y(t, x)), f_2^\varepsilon(t, x, y(t, x)) \).

From the last inequality and assumption (H1) we have

\[ E \int_O z_\varepsilon(t, x)^2 \, dx \leq C + \frac{1}{\eta} \cdot \int_0^t E \int_O z_\varepsilon(s, x)^2 \, dx \, ds. \]

Using Gronwall’s lemma, we determine the boundedness

\[ E \int_O z_\varepsilon(t, x)^2 \, dx \leq C + \int_0^t C \cdot \frac{1}{\eta} \cdot \exp \left\{ \int_\tau^t \frac{1}{\eta} \, ds \right\} \, d\tau \leq D, \]

where \( D \) is a constant independent of \( \varepsilon \) since the regularised functions are uniformly \( L^2 \)-bounded.

In addition the estimation of the partial derivative of \( z_\varepsilon \) with respect to \( x \) is deduced from (H1) and the last inequality

\begin{align*}
2k_0 \ E \int_0^t \|\nabla_x z_\varepsilon(s)\|^2_H \, ds \\
& \quad \leq 2E \int_0^t \int_O \sum_{j=1}^N k(s, x, y(s, x)) \cdot \frac{\partial z_\varepsilon(s, x)}{\partial x_j} \cdot \frac{\partial z_\varepsilon(s, x)}{\partial x_j} \, dx \, ds \\
& \quad \leq C + \frac{1}{\eta} \cdot \int_0^t E \int_O z_\varepsilon(s, x)^2 \, dx \, ds - E \int_O z_\varepsilon(t, x)^2 \, dx \\
& \quad \leq G.
\end{align*}

Altogether we have for all \( t \in [0, T] \)

\[ E\|z_\varepsilon(t)\|^2_H \leq D \quad \Rightarrow \quad E \int_0^t \|z_\varepsilon(s)\|^2_H \, ds \leq D \cdot t \]

and

\[ E \int_0^t \|z_\varepsilon(s)\|^2_{H^1(O)} \, ds = E \int_0^t \|z_\varepsilon(s)\|^2_V \, ds \leq \tilde{G}. \]
So $z_\varepsilon \in L^2(\Omega \times [0,T]; H)$ and $z_\varepsilon \in L^2(\Omega \times [0,T]; V)$. In particular $S_\varepsilon$ maps bounded sets into bounded sets.

We are now in a position to prove the compactness of $S_\varepsilon$. An operator is defined to be compact, if it is continuous and maps bounded sets into relatively compact sets. Let $B_R \subset L^2(\Omega \times Q)$ be the ball with radius $R > 0$, sufficiently large. Because $V$ is compactly embedded into $H$, each sequence of the bounded, convex, closed and non-empty set $B_R$ contains a subsequence, which is convergent in $L^2(\Omega \times [0,T]; H)$.

Thus it remains to verify the continuity of $S_\varepsilon$. Let $z_{0\varepsilon} = S_\varepsilon \cdot y_0$ and $z_{1\varepsilon} = S_\varepsilon \cdot y_1$. Then for any $\phi \in C^\infty_0(\Omega)$ the equation is true:

\[
\int_{\Omega} (z_{1\varepsilon}(t,x) - z_{0\varepsilon}(t,x)) \cdot \phi(x) dx \\
+ \int_0^t \int_{\Omega} \sum_{j=1}^N k(s,x,y_1(s,x)) \frac{\partial (z_{1\varepsilon}(s,x) - z_{0\varepsilon}(s,x))}{\partial x_j} \cdot \frac{\partial \phi(x)}{\partial x_j} dx \, ds \\
= \int_0^t \int_{\Omega} \sum_{j=1}^N k(s,x,y_0(s,x)) \frac{\partial z_{0\varepsilon}(s,x)}{\partial x_j} \cdot \frac{\partial \phi(x)}{\partial x_j} dx \, ds \\
- \int_0^t \int_{\Omega} \sum_{j=1}^N k(s,x,y_1(s,x)) \frac{\partial z_{0\varepsilon}(s,x)}{\partial x_j} \cdot \frac{\partial \phi(x)}{\partial x_j} dx \, ds \\
+ \int_0^t \int_{\Omega} (f_1^s(s,x,y_1(s,x)) - f_1^s(s,x,y_0(s,x))) \cdot \phi(x) dx \, ds \\
+ \int_0^t \int_{\Omega} \sum_{j=1}^N k(s,x,y_1(s,x)) \frac{\partial z_{0\varepsilon}(s,x)}{\partial x_j} \cdot \frac{\partial \phi(x)}{\partial x_j} dx \, ds \\
+ \int_0^t \int_{\Omega} (g_1(s,x,y_0(s,x)) - g_1(s,x,y_1(s,x))) \cdot \phi(x) dx \, ds \\
+ \int_0^t \int_{\Omega} (g_2(s,x,y_0(s,x)) - g_2(s,x,y_1(s,x))) dW_1(s) \cdot \phi(x) dx.
\]

It holds that $(z_{1\varepsilon}(0,x) - z_{0\varepsilon}(0,x)) = 0$ for all $x \in \Omega$.

We derive from Itô’s formula of the square of the norm, with similar considerations such as in proving the boundedness of $S_\varepsilon$:

\[
E \int_{\Omega} (z_{1\varepsilon}(t,x) - z_{0\varepsilon}(t,x))^2 dx \\
+ 2E \int_0^t \int_{\Omega} \sum_{j=1}^N k(s,x,y_1(s,x)) \left( \frac{\partial (z_{1\varepsilon}(s,x) - z_{0\varepsilon}(s,x))}{\partial x_j} \right)^2 dx \, ds.
\]
we omit this positive term. On the right hand side we improve the functions further the inequality is estimated with (H1) 

\[ \eta \]

By choosing regularised functions \( E \leq 1 \leq E + 2 + 1 \leq E + \eta E \int_{0}^{t} \left( z_{1c}(s, x) - z_{0c}(s, x) \right)^{2} dx \, ds \)

Further the inequality is estimated with (H1) and Lipschitz constants \( C_{1}, C_{2} \) for the regularised functions \( f_{1}^{\varepsilon} \) and \( f_{2}^{\varepsilon} \)

\[ E \int_{0}^{t} \left( z_{1c}(t, x) - z_{0c}(t, x) \right)^{2} dx + (2k_{0} - \eta) \cdot E \int_{0}^{t} \sum_{j=1}^{N} \left( \frac{\partial (z_{1c}(s, x) - z_{0c}(s, x))}{\partial x_{j}} \right)^{2} dx \, ds \]

\[ \leq \frac{1}{\eta} E \int_{0}^{t} \sum_{j=1}^{N} \left( k(s, x, y_{0}(s, x)) - k(s, x, y_{1}(s, x)) \right) \frac{\partial z_{0c}(s, x)}{\partial x_{j}} \right)^{2} dx \, ds \]

\[ + \left( \frac{1}{\eta} \cdot C_{1}^{2} + C_{2}^{2} \right) \cdot E \int_{0}^{t} \int_{0}^{t} (y_{1}(s, x) - y_{0}(s, x))^{2} dx \, ds \]

\[ + 2\eta E \int_{0}^{t} \sum_{j=1}^{N} \left( \frac{\partial (z_{1c}(s, x) - z_{0c}(s, x))}{\partial x_{j}} \right)^{2} dx \, ds \]

\[ + \frac{1}{\eta} E \int_{0}^{t} \int_{0}^{t} (g_{1}(s, x, y_{0}(s, x)) - g_{1}(s, x, y_{1}(s, x)))^{2} dx \, ds \]

\[ + E \int_{0}^{t} \int_{0}^{t} (g_{2}(s, x, y_{0}(s, x)) - g_{2}(s, x, y_{1}(s, x)))^{2} dx \, ds \, dx. \]

By choosing \( \eta \) such that \( 2k_{0} - \eta \geq 0 \), the left hand side of the inequality will decrease, if we omit this positive term. On the right hand side we improve the functions \( k, g_{1}, g_{2} \) being Carathéodory functions, that is continuous with respect to the third variable. Therewith the terms tend to 0 for \( ||y_{1} - y_{0}||_{L^{2}(\Omega \times Q)} \to 0. \)
The application of Gronwall’s lemma yields
\[
E \int_{Q} (z_{1\varepsilon}(t, x) - z_{0\varepsilon}(t, x))^2 dx \to 0 ,
\]
for \(|y_1 - y_0|_{L^2(\Omega \times Q)} \to 0\), that is \(E||z_{1\varepsilon}(t) - z_{0\varepsilon}(t)||^2_H \to 0\).

Analogously to the case of the boundedness we conclude
\[
E \left\| \frac{\partial (z_{1\varepsilon}(t) - z_{0\varepsilon}(t))}{\partial x} \right\|^2_H \to 0 \implies E \int_{0}^{t} ||(z_{1\varepsilon}(s) - z_{0\varepsilon}(s))||^2_V ds \to 0,
\]
for \(|y_1 - y_0|_{L^2(\Omega \times Q)} \to 0\). Finally the convergence in \(L^2(\Omega \times Q)\) is due to the theorem of Lebesgue. So we have determined the continuity of the mapping \(S_{\varepsilon}\) from \(L^2(\Omega \times Q)\) into itself.

With the fixed-point theorem of Schauder the equation \(y(t, x) = S_{\varepsilon}y(t, x)\) is fulfilled, i.e. there exists a fixed-point \(y \in L^2(\Omega \times Q)\). For this reason the regularised and truncated problem (3.3) admits a solution.

In the next step we are interested in the convergence of the solutions of equation (3.3).

Lemma 3.6
There exist a sequence \((\varepsilon_n)\), \(0 < \varepsilon_n < \alpha\) with \(\varepsilon_n \to 0\), \(n \to \infty\) and processes \(U(t, x), h_1(t, x)\) and \(h_2(t, x)\), such that:

(i) \(U_{\varepsilon_n} \to U\) in \(L^2(\Omega \times [0, T]; V)\),

(ii) \(U_{\varepsilon_n} \to U\) in \(L^2(\Omega \times Q)\),

(iii) \(f^{i,n}_{\varepsilon_n}(t, x, TU_{\varepsilon_n}(t, x)) \to h_1(t, x)\),

\[ f^{i,n}_{\varepsilon_n}(t, x, TU_{\varepsilon_n}(t, x)) \to h_2(t, x) \text{ in } L^2(\Omega \times Q). \]

Proof.
In reflexive Banach spaces each bounded sequence has a weak convergent subsequence. During the preceding paragraph it was shown that \(U_{\varepsilon} = S_{\varepsilon}U_{\varepsilon}\) and \(E \int_{0}^{T} ||U_{\varepsilon}(t)||^2_V dt \leq \tilde{G}\). So a weak convergent subsequence \(U_{\varepsilon_n}\) exists in the reflexive space \(L^2(\Omega \times [0, T]; V)\).

Furthermore the space \(V\) is compactly embedded in \(H\), i.e. each bounded sequence contains a subsequence which converges strongly in \(H\). The statement (ii) results from \(E||U_{\varepsilon}(t)||^2_H \leq D\) and the theorem of Lebesgue.

The regularisations \(f^{i,n}_{\varepsilon_n}(t, x, TU_{\varepsilon_n})\), \(i = 1, 2\) are uniformly bounded in \(L^2(\Omega \times Q)\). Consequently there are subsequences having weak limits in \(L^2(\Omega \times Q)\).
This is the point to establish the passage to the limit. Problem (3.3) converges for $\varepsilon_n \downarrow 0$ to the simply truncated problem

\[
\int_{O} U(t,x)\phi(x)dx \\
+ \int_{0}^{t} \int_{O} \sum_{j=1}^{N} k(s,x, TU(s,x)) \cdot \frac{\partial U(s,x)}{\partial x_j} \cdot \frac{\partial \phi(x)}{\partial x_j} dx \, ds \\
+ \int_{0}^{t} \int_{O} g_1(s,x, TU(s,x))\phi(x)dx \, ds + \int_{0}^{t} \int_{O} g_2(s,x, TU(s,x))dW_1(s)\phi(x) \, dx \\
= \int_{O} \psi(x)\phi(x)dx + \int_{0}^{t} \int_{O} h_1(s,x)\phi(x)dx \, ds \\
+ \int_{0}^{t} \int_{O} h_2(s,x)dW_2(s)\phi(x) \, dx,
\]

for all $\phi \in C_{0}^{\infty}(O)$, taking lemma 3.6 into account as well as the continuity of the truncation operator $T$.

Our task is now to show, that the weak limits $h_i$, $i = 1, 2$ are selections of $\beta_i$ and that each solution of (3.5) is a solution of (3.2). The next two lemmas carry out this facts.

First of all we consider the selection property.

**Lemma 3.7**

Let $\tilde{U}_n(t,x) \in [U(t,x), \overline{U}(t,x)]$ be a sequence of $\mathcal{F}_t$-measurable processes and $\tilde{U}(t,x)$ the strong limit in $L^2(\Omega \times Q)$ with

\[
E \int_{0}^{t} \int_{O} (\tilde{U}_n(s,x) - \tilde{U}(s,x))^2 dx \, ds \to 0 \text{ for } n \to \infty
\]

and $\varepsilon_n > 0$ a sequence, so that $\varepsilon_n \to 0$ for $n \to \infty$.

Then the weak limits $h_1, h_2$ in $L^2(\Omega \times Q)$ are selections of the multivalued mappings $\beta_1$ and $\beta_2$, more precisely

\[
h_n^1(t,x) = f^\varepsilon_1(t,x, \tilde{U}_n(t,x)) \rightrightarrows h_1(t,x) \quad \text{and} \quad h_n^2(t,x) = f^\varepsilon_2(t,x, \tilde{U}_n(t,x)) \rightrightarrows h_2(t,x)
\]

imply

\[
h_1(t,x) \in \beta_1(t,x, \tilde{U}(t,x)) \quad \text{and} \quad h_2(t,x) \in \beta_2(t,x, \tilde{U}(t,x)).
\]

**Proof.**

An important device is the monotony of the functions $f_i$, $i = 1, 2$ in the special form

\[
f_i(t,x,\tilde{U}_n - \varepsilon_n) \leq f^\varepsilon_i(t,x, \tilde{U}_n) \leq f_i(t,x, \tilde{U}_n + \varepsilon_n).
\]
To simplify matters the inequality is derived for \( f \), having the same characteristics as \( f_i \). Let \( \varepsilon_m \) be a sequence such that \( \varepsilon_m \to 0 \) for \( m \to \infty \). Then the definition of the regularisation leads to:

\[
\lim_{m \to \infty} \frac{1}{\varepsilon_m} \int_{-\infty}^{\infty} f(t, x, \tilde{U}_n - \xi) J \left( \frac{\xi}{\varepsilon_m} \right) d\xi = \lim_{m \to \infty} f^{\varepsilon_m}(t, x, \tilde{U}_n) = f(t, x, \tilde{U}_n).
\]

Using the definition of the limit it holds that for all \( \eta > 0 \) there exists \( m_0(\eta) \), so that for all \( m \geq m_0(\eta) \)

\[
|f^{\varepsilon_m}(t, x, \tilde{U}_n) - f(t, x, \tilde{U}_n)| < \eta
\]

and respectively

\[
-\eta < f^{\varepsilon_m}(t, x, \tilde{U}_n) - f(t, x, \tilde{U}_n) < \eta
\]

or equivalently

\[
-\eta + f(t, x, \tilde{U}_n) < f^{\varepsilon_m}(t, x, \tilde{U}_n) < \eta + f(t, x, \tilde{U}_n). \tag{3.7}
\]

Besides \( f \) is monotone with respect to the third variable, so

\[
f(t, x, \tilde{U}_n) - \eta \geq f(t, x, \tilde{U}_n - \varepsilon_m) - \eta,
\]

\[
f(t, x, \tilde{U}_n) + \eta \leq f(t, x, \tilde{U}_n + \varepsilon_m) + \eta.
\]

Inequality (3.7) effects

\[
f(t, x, \tilde{U}_n - \varepsilon_m) - \eta \leq f^{\varepsilon_m}(t, x, \tilde{U}_n) \leq f(t, x, \tilde{U}_n + \varepsilon_m) + \eta.
\]

For sufficient large \( n, \ n \geq m_0(\eta) \) it is

\[
f(t, x, \tilde{U}_n - \varepsilon_n) - \eta \leq f^{\varepsilon_n}(t, x, \tilde{U}_n) \leq f(t, x, \tilde{U}_n + \varepsilon_n) + \eta.
\]

Because \( \eta > 0 \) is supposed to be arbitrary

\[
f(t, x, \tilde{U}_n - \varepsilon_n) \leq f^{\varepsilon_n}(t, x, \tilde{U}_n) \leq f(t, x, \tilde{U}_n + \varepsilon_n).
\]

In detail we receive

\[
f_i(t, x, \tilde{U}_n(t, x) - \varepsilon_n) \leq f^{\varepsilon_1}_i(t, x, \tilde{U}_n(t, x)) \leq f_i(t, x, \tilde{U}_n(t, x) + \varepsilon_n), \ i = 1, 2
\]

for any \( \varepsilon_n \in (0, \alpha) \). Thus inequality (3.6) is just verified.
According to the assumption \( \hat{U}_n \) converges strongly in \( L^2(\Omega \times Q) \). So there is a subsequence of \( \hat{U}_n \) that converges almost everywhere to \( \hat{U} \), i.e. \( \hat{U}_{n_j}(t, x) \to \hat{U}(t, x) \) for \( P \)-almost all \( \omega \in \Omega \) in \( L^2(Q) \).

Yet another subsequence \( n_{j_r}(\omega, t, x) \) exists for almost all \( \omega \in \Omega \) and for Lebesgue-almost all \( t, x \), for which \( \hat{U}_{n_{j_r}}(\omega, t, x) \) converges Lebesgue-almost everywhere to \( \hat{U}(\omega, t, x) \) while \( \omega \) is fixed.

The theorem of Egorov states the existence of a measurable set \( Q' \subset Q \) with Lebesgue measure \( m(Q') < \delta \), such that \( \hat{U}_{n_{j_r}}(\omega, t, x) \to \hat{U}(\omega, t, x) \) uniformly in \( Q \setminus Q' \) for \( \varepsilon_{n_{j_r}} \to 0 \).

Let \( \rho \in (0, \alpha) \). Then there is an \( \varepsilon_{n_{j_0}} \in (0, \alpha) \) such that

\[
|\hat{U}_n(\omega, t, x) - \hat{U}(\omega, t, x)| < \frac{\rho}{2} \quad \text{in } Q/Q', \quad \text{for all } \varepsilon_{n_{j_r}} \text{ and } 0 < \varepsilon_{n_{j_r}} < \min \left\{ \varepsilon_{n_{j_0}}, \frac{\rho}{2} \right\}.
\]

From inequality (3.6) and the monotony of \( f_i \), \( i = 1, 2 \) in the fourth place, it follows

\[
f_i^{\varepsilon_{n_{j_r}}}(\omega, t, x, \hat{U}_{n_{j_r}}(\omega, t, x)) \leq f_i\left(\omega, t, x, \hat{U}(\omega, t, x) + \frac{\rho}{2} + \varepsilon_n\right) \leq f_i(\omega, t, x, \hat{U}(\omega, t, x) + \rho)
\]

and

\[
f_i(\omega, t, x, \hat{U}(\omega, t, x) - \rho) \leq f_i^{\varepsilon_{n_{j_r}}}(\omega, t, x, \hat{U}_{n_{j_r}}(\omega, t, x)).
\]

Concerning the two previous inequalities and \( h_{i_{n_{j_r}}}^{n_{j_r}}(\omega, t, x) = f_i^{\varepsilon_{n_{j_r}}}(\omega, t, x, \hat{U}_{n_{j_r}}(\omega, t, x)) \) the limit \( h_{i_{n_{j_r}}}^{n_{j_r}}(t, x) \to h_i(t, x) \) results for \( \varepsilon_{n_{j_r}} \to 0 \). Hence

\[
\int_{Q/Q'} f_i(t, x, \hat{U}(t, x) - \rho) \cdot \phi(x) dx \, dt \leq \int_{Q/Q'} h_i(t, x) \phi(x) dx \, dt \\
\leq \int_{Q/Q'} f_i(t, x, \hat{U}(t, x) + \rho) \cdot \phi(x) dx \, dt, \quad P \text{-a.s.}
\]

for all \( \phi \geq 0, \phi \in L^2_+(Q \setminus Q') \).

Now we apply the theorem of Lebesgue and consider \( \rho \to 0 \):

\[
\int_{Q/Q'} f_i(t, x, \hat{U}(t, x)) \cdot \phi(x) dx \, dt \leq \int_{Q/Q'} h_i(t, x) \phi(x) dx \, dt \leq \int_{Q/Q'} \bar{T}_i(t, x, \hat{U}(t, x)) \cdot \phi(x) dx \, dt.
\]

Since the chain of inequalities holds for any \( \phi \in L^2_+(Q \setminus Q') \), we can draw the conclusion \( h_i(t, x) \in [f_i(t, x, \hat{U}(t, x)), \bar{T}_i(t, x, \hat{U}(t, x))] = \beta_i(t, x, \hat{U}(t, x)), \quad P \text{-a.s. in } Q \setminus Q' \).

In particular \( \delta > 0 \) is chosen arbitrarily small, so that for almost all \( \omega, t, x \)

\[
h_i(t, x) \in \beta_i(t, x, \hat{U}(t, x)), \quad i = 1, 2.
\]

In other words \( h_i \) is a selection of \( \beta_i, \quad i = 1, 2. \)
Finally we have to prove that any solution to (3.5) is an element of the interval \([U, U]\). In this case \(TU = U\) and so the truncated problem transfers to the primary problem (3.2).

**Lemma 3.8**

Each solution to (3.5) is a solution to (3.2).

**Proof.**

The essential observation is that a solution \(U\) of (3.5) is in-between \(\underbar{U}\) and \(\overline{U}\).

Obviously for all \(t \in [0, T]\) and \(\phi \in C^\infty_0(O)\) (3.5) holds

\[
\begin{align*}
\int_O U(t, x)\phi(x)\,dx & + \int_0^t \int_O \sum_{j=1}^N k(s, x, TU(s, x)) \frac{\partial U(s, x)}{\partial x_j} \frac{\partial \phi}{\partial x_j}\,dx\,ds \\
& + \int_0^t \int_O g_1(s, x, TU(s, x))\phi(x)\,dx\,ds + \int_O \int_0^t g_2(s, x, TU(s, x))dW_1(s)\phi(x)\,dx \\
& = \int_0^t \psi(x)\phi(x)\,dx + \int_0^t \int_O h_1(s, x)\phi(x)\,dx\,ds \\
& + \int_0^t \int_O h_2(s, x)dW_2(s)\phi(x)\,dx,
\end{align*}
\]

with \(h_i(t, x) \in \beta_i(t, x, TU(t, x)), \ i = 1, 2\) using lemma 3.7 and \(\tilde{U} = TU\).

Remember the definition of an upper solution \(\overline{U}\)

\[
\begin{align*}
\int_O \overline{U}(t, x)\phi(x)\,dx & + \int_0^t \int_O \sum_{j=1}^N k(s, x, \overline{U}(s, x)) \frac{\partial \overline{U}(s, x)}{\partial x_j} \frac{\partial \phi}{\partial x_j}\,dx\,ds \\
& + \int_0^t \int_O g_1(s, x, \overline{U}(s, x))\phi(x)\,dx\,ds + \int_O \int_0^t g_2(s, x, \overline{U}(s, x))dW_1(s)\phi(x)\,dx \\
& \geq \int_0^t \psi(x)\phi(x)\,dx + \int_0^t \int_O \bar{f}_1(s, x, \overline{U}(s, x))\phi(x)\,dx\,ds \\
& + \int_0^t \int_O \bar{f}_2(s, x, \overline{U}(s, x))dW_2(s)\phi(x)\,dx
\end{align*}
\]

for all \(t \in [0, T]\) and \(\phi \in V \cap L^2(O)\).

In the difference of (3.8) and (3.5) we attach a special \(\phi(x) = (U(t, x) - \overline{U}(t, x))^+\) while \(t\) is fixed. That is \(\phi(x) = \max\{U(t, x) - \overline{U}(t, x); 0\}\) and in detail

\[
\begin{align*}
& \int_O (U(t, x) - \overline{U}(t, x)) \cdot (U(t, x) - \overline{U}(t, x))^+ \,dx \\
& + \int_0^t \int_O \sum_{j=1}^N k(s, x, TU(s, x)) \frac{\partial U(s, x)}{\partial x_j} \frac{\partial (U(s, x) - \overline{U}(s, x))^+}{\partial x_j} \,dx\,ds \\
& - \int_0^t \int_O \sum_{j=1}^N k(s, x, \overline{U}(s, x)) \frac{\partial \overline{U}(s, x)}{\partial x_j} \frac{\partial (U(s, x) - \overline{U}(s, x))^+}{\partial x_j} \,dx\,ds
\end{align*}
\]
+ \int_0^t \int_O [g_1(s,x,\sigma_1 U(s,x)) - g_1(s,x,\bar{U}(s,x))] (U(s,x) - \bar{U}(s,x))^+ dx \, ds
+ \int_0^t \int_O [g_2(s,x,\sigma_1 U(s,x)) - g_2(s,x,\bar{U}(s,x))] (U(s,x) - \bar{U}(s,x))^+ dW_1(s) \, dx
\leq \int_0^t \int_O [h_1(s,x) - \bar{f}_1(s,x,U(s,x))] (U(s,x) - \bar{U}(s,x))^+ dx \, ds
+ \int_0^t \int_O [h_2(s,x) - \bar{f}_2(s,x,U(s,x))] (U(s,x) - \bar{U}(s,x))^+ dW_2(s) \, dx.

Let \( O' = \{ x : U(t,x) > \bar{U}(t,x) \} \). From the properties of the expectation, the truncation and \( \phi(x) = 0 \) for all \( x \in O \setminus O' = \{ x : U(t,x) \leq \bar{U}(t,x) \} \) we obtain

\[
E \int_{O'} (U(t,x) - \bar{U}(t,x))^2 dx
+ E \int_0^t \int_{O'} \sum_{j=1}^N k(s,x,\bar{U}(s,x)) \left( \frac{\partial(U(s,x) - \bar{U}(s,x))}{\partial x_j} \right)^2 dx \, ds
\leq E \int_0^t \int_{O'} [h_1(s,x) - \bar{f}_1(s,x,\bar{U}(s,x))] (U(s,x) - \bar{U}(s,x)) dx \, ds.
\]

The first term on the left is positive in \( O' \). From (H1) we have

\[
E \int_0^t \int_{O'} \sum_{j=1}^N k(s,x,\bar{U}(s,x)) \left( \frac{\partial(U(s,x) - \bar{U}(s,x))}{\partial x_j} \right)^2 dx \, ds
\geq k_0 \cdot E \int_0^t ||\nabla_x(U(s) - \bar{U}(s))||_{2,\mathcal{H}}^2 ds \geq 0.
\]

Since \( f_1 \) is increasing with respect to the third variable it holds that

\[
\bar{f}_1(s,x,\bar{U}) = \lim_{\varepsilon \to 0} f_1(s,x,\bar{U} + \varepsilon) \geq h_1(s,x).
\]

So the product on the right hand side of the inequality is negative. Altogether it follows from above:

\[
0 \leq k_0 \cdot E \int_0^t ||\nabla_x(U(s) - \bar{U}(s))||_{2,\mathcal{H}}^2 ds \leq 0 \quad \text{for all } t \in [0,T]
\]

and

\[
E||\nabla_x(U - \bar{U})||_{2,\mathcal{H}(\mathcal{Q})}^2 = 0
\]

respectively. Thus \( (U - \bar{U})^+ = 0 \), i.e. \( U \leq \bar{U} \) P-a.s.

A similar study of the lower solution leads to the final conclusion

\[
\bar{U} \leq U \leq \bar{U}, \quad \text{P-a.s.}
\]

\[\square\]
In the end of the proof of theorem 3.4 each solution to the truncated problem (3.5) solves the original problem, because the equation shrinks with regard to the truncation operator to problem (3.2).

\[ \square \]

**Remark 3.9** Most of the stochastic differential inclusions are solved by a set of solutions. With the method of upper and lower solutions (theorem 3.4) we are able to locate a solution in the ordered interval \([\underline{U}, \overline{U}]\). But the knowledge of upper and lower solutions is the basic condition for application. Unfortunately a method for detecting upper and lower solutions in the context of stochastic differential inclusions is not known so far.

## 4 Appendix

**Itô’s formula of the square of the norm**

Let \((V, H, V^*)\) be an evolution triple, \(\| \cdot \|_H\) denote the norm, \((\cdot, \cdot)\) the inner product in \(H\) and \(\langle \cdot, \cdot \rangle\) the duality brackets of the pair \((V, V^*)\).

**Theorem 4.1**

Let \(v(t) \in V\), \(v^*(t) \in V^*\) and \(g(t) \in H\) be \(\mathcal{F}_t\)-measurable processes, such that for all \(\phi \in M\), \(M \subset V\) dense

\[
(v(t), \phi) = (v(0), \phi) + \int_0^t \langle v^*(s), \phi \rangle ds + \int_0^t \langle g(s), \phi \rangle dW(s),
\]

where \((W(t))_{t \geq 0}\) is a real-valued Wiener process.

Then

\[
\|v(t)\|_H^2 = \|v(0)\|_H^2 + 2 \int_0^t \langle v^*(s), v(s) \rangle ds + 2 \int_0^t \langle g(s), v(s) \rangle dW(s) + \int_0^t \|g(s)\|_H^2 ds.
\]

**References**


