A note on uniqueness of parameter identification in a jump diffusion model

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Abstract

In this note, we consider an inverse problem in a jump diffusion model. Using characteristic functions we prove the injectivity of the forward operator mapping the five parameters determining the model to the density function of the return distribution.

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1 Forward operator in a jump diffusion model

Let the random price process \((S_t; t \in [0, \infty))\) of a financial asset be described by a jump diffusion process. More precisely, we assume that the price process follows the stochastic differential equation

\[
dS_t = S_t((\mu - \lambda \nu)dt + \sigma dW_t) + S_t dN^c_t, \quad t \in (0, \infty), \quad S_0 = \xi,
\]

where \((W_t; t \in [0, \infty))\) is a standard Wiener process and \((N^c_t; t \in [0, \infty))\) an independent compound Poisson process with jump amplitudes \((Y_j - 1; j \in \mathbb{N})\). We assume that the random variables \(\ln Y_j, j \in \mathbb{N}\), are independent Gaussian variables with mean \(\mu_Y\) and variance \(\sigma^2_Y\). The parameter \(\lambda \geq 0\) expresses the intensity of the underlying Poisson process \((N_t; t \in [0, \infty))\). Furthermore we have \(\nu = e^{\mu_Y + \frac{1}{2} \sigma^2_Y} - 1\), \(\mu \in \mathbb{R}\), \(\sigma^2 > 0\) and an initial value \(\xi\). The model has been considered in detail in [1].

Then it holds for the logarithmic returns \(r_t = \ln \frac{S_t}{S_0}\) and fixed lag \(t > 0\)

\[
r_t = \tilde{\mu} t + \sigma W_t + \sum_{j=1}^{N_t} \ln Y_j
\]

with parameter \(\tilde{\mu} = \mu - \frac{1}{2} \sigma^2 - \lambda \nu\). The independence of the random processes \((W_t; t \in [0, \infty)), (N_t; t \in [0, \infty))\) and the random variables \((Y_j - 1; j \in \mathbb{N})\) as well as the distribution assumptions \(W_t \sim N(0, t), \ln Y_j \sim N(\mu_Y, \sigma^2_Y), N_t \sim \text{Poisson}(\lambda t)\) allow to calculate the distribution function

\[
F_{rt}(x, p) = \sum_{j=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^j j!}{\sqrt{\sigma^2 t + j \sigma^2_Y}} \Phi \left( \frac{x - (\tilde{\mu} t + j \mu_Y)}{\sqrt{\sigma^2 t + j \sigma^2_Y}} \right), \quad x \in \mathbb{R}, \tag{1.1}
\]

for the returns \(r_t\) and the associated probability density function

\[
f_{rt}(x, p) = \sum_{j=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^j j!}{\sqrt{\sigma^2 t + j \sigma^2_Y}} \phi \left( \frac{x - (\tilde{\mu} t + j \mu_Y)}{\sqrt{\sigma^2 t + j \sigma^2_Y}} \right), \quad x \in \mathbb{R}, \tag{1.2}
\]

as well as the characteristic function

\[
\varphi_{rt}(\theta, p) = \exp \left( i \tilde{\mu} t \theta - \frac{\sigma^2 t}{2} \theta^2 + \lambda t \left( \exp \left( i \mu \theta - \frac{\sigma^2_Y}{2} \theta^2 \right) - 1 \right) \right), \quad \theta \in \mathbb{R}. \tag{1.3}
\]

In the formulae (1.1) – (1.3) the corresponding functions explicitly depend on the parameter vector

\[
p = (\mu, \sigma, \lambda, \mu_Y, \sigma_Y)
\]

determining the model under consideration.
We are going to show that the inverse problem of identifying \( p \) from given density function \( f_r \) is, for fixed \( t > 0 \), uniquely solvable on some domain whenever a solution exists.

In this context, we consider the forward operator

\[
F : p \in D \mapsto f_r
\]

defined on the domain

\[
D = \{ p = (\mu, \sigma, \lambda, \mu_Y, \sigma_Y) : \mu \in \mathbb{R}, \sigma > 0, \lambda \geq 0, \mu_Y \in \mathbb{R}, \sigma_Y \geq 0 \}, \tag{1.4}
\]

where the nonlinear operator \( F \) is defined by formula (1.2).

Obviously, in the case \( \lambda = 0 \) no jumps occur, i.e., the values \( \mu_Y \in \mathbb{R} \) and \( \sigma_Y \geq 0 \) may be arbitrary and do not influence the value or the distribution of the returns. Similarly, for \( \mu_Y = \sigma_Y = 0 \), all jumps are of zero height. Then they also do not influence the value or the distribution of the returns, and the parameter \( \lambda \geq 0 \) may be arbitrary. In [1] it was conjectured that the following theorem is true.

**Theorem 1.1**

The operator \( F : p \in D \mapsto f_r \) defined by formula (1.2) is injective on the restricted domain

\[
\hat{D} = \{ p \in D : \lambda (\sigma_Y^2 + \mu_Y^2) \neq 0 \}. \tag{1.5}
\]

We prove theorem 1.1 in the subsequent paragraph setting for simplicity \( t = 1 \).

## 2 Proof of the theorem

Probability distributions of real valued random variables are uniquely determined by the corresponding characteristic functions (cf., e.g., [2]). Therefore we will show that for equal characteristic functions of returns either the no-jump-condition \( \lambda (\sigma_Y^2 + \mu_Y^2) = 0 \) (excluded in (1.5)) is valid or all parameters coincide.

Let us assume that for two parameter vectors

\[
{\bar{p}} = (\bar{\mu}, \bar{\sigma}, \bar{\lambda}, \bar{\mu}_Y, \bar{\sigma}_Y) \in D \quad \text{and} \quad \bar{p} = (\bar{\mu}, \bar{\sigma}, \bar{\lambda}, \bar{\mu}_Y, \bar{\sigma}_Y) \in D
\]

with \( D \) from (1.4) the distributions of the returns \( ^1r_t \) and \( ^2r_t \) with \( t = 1 \) and hence the corresponding characteristic functions

\[
\varphi_{^1r_t}(\theta, \bar{p}) = \exp \left(i \bar{\mu}_t \theta - \frac{1}{2} \sigma_Y^2 \theta^2 + \bar{\lambda} \left( \exp \left(i \bar{\mu}_Y \theta - \frac{1}{2} \sigma_Y^2 \theta^2 \right) - 1 \right) \right), \quad \theta \in \mathbb{R},
\]

\[
\varphi_{^2r_t}(\theta, \bar{p}) = \exp \left(i \bar{\mu}_t \theta - \frac{1}{2} \sigma_Y^2 \theta^2 + \bar{\lambda} \left( \exp \left(i \bar{\mu}_Y \theta - \frac{1}{2} \sigma_Y^2 \theta^2 \right) - 1 \right) \right), \quad \theta \in \mathbb{R},
\]
coincide. Then also the logarithms of the characteristic functions coincide, i.e.,
\[
i^\lambda \mu \theta - \frac{\sigma^2}{2} \theta^2 + \lambda \left( \exp \left( i \mu_Y \theta - \frac{\sigma^2}{2} \theta^2 \right) - 1 \right)
\]
\[
= i^\lambda \mu \theta - \frac{\sigma^2}{2} \theta^2 + 2 \lambda \left( \exp \left( i \mu_Y \theta - \frac{\sigma^2}{2} \theta^2 \right) - 1 \right), \quad \forall \theta \in \mathbb{R}.
\]
So we have for all real numbers \( \theta \in \mathbb{R} \)
\[
2\lambda - 1 + i (\mu - 2 \mu) \theta - \frac{\sigma^2 - \mu^2}{2} \theta^2 = 2 \lambda \exp \left( i \mu_Y \theta - \frac{\sigma^2}{2} \theta^2 \right) - \lambda \exp \left( i \mu_Y \theta - \frac{\sigma^2}{2} \theta^2 \right).
\]
The left hand side of this identity is a second order polynomial in \( \theta \), hence the third derivative with respect to \( \theta \) vanishes. The third order derivative of the right hand side is
\[
2A(\theta) \exp \left( 2B(\theta) \right) - A(\theta) \exp \left( A(\theta) \right)
\]
with polynomials
\[
A(\theta) = \lambda \left( -3 \sigma_Y^2 + (i \mu_Y - \sigma_Y^2) \right),
\]
\[
A(\theta) = 2 \lambda \left( -3 \sigma_Y^2 + (i \mu_Y - \sigma_Y^2) \right),
\]
\[
B(\theta) = i \mu_Y \theta - \frac{\sigma_Y^2}{2} \theta^2,
\]
\[
B(\theta) = i^2 \mu_Y \theta - \frac{\sigma_Y^2}{2} \theta^2.
\]
So we get the relation
\[
2A(\theta) \exp \left( 2B(\theta) \right) - A(\theta) \exp \left( A(\theta) \right) = 0, \quad \forall \theta \in \mathbb{R}.
\]
If one of the polynomials \( A(\theta) \) vanishes identically, say \( A(\theta) \), which is possible if and only if \( \lambda = 0 \) or \( \mu_Y = \sigma_Y^2 = 0 \), we get \( \lambda \left( \sigma_Y^2 + \mu_Y^2 \right) = 0 \) and consequently
\[
2A(\theta) \exp \left( 2B(\theta) \right) = 0, \quad \forall \theta \in \mathbb{R}.
\]
Then \( A(\theta) \) also vanishes identically and the no-jump-condition
\[
2\lambda \left( \sigma_Y^2 + \mu_Y^2 \right) = 0 \quad \text{equivalent to} \quad 2\lambda = 0 \quad \text{or} \quad 2\mu_Y = 2\sigma_Y^2 = 0
\]
must be fulfilled.

If \( \sigma_Y = 0 \) and \( \mu_Y \neq 0 \), then we have a nonzero constant \( A(\theta) = -i^4 \lambda \mu_Y \sigma_Y \), and \( A(\theta) \) must be the same constant yielding \( \sigma_Y = 0 \). This implies \( B(\theta) = -i \mu_Y \theta \equiv 2B(\theta) = i^2 \mu_Y \theta \) as well as \( \mu_Y = 2 \mu_Y \) and \( \lambda = \frac{\sigma_Y}{2} \).

If otherwise both polynomials
\[
A(\theta) = -i^4 \lambda \sigma_Y^6 \left( \theta - \frac{i \mu_Y}{\sigma_Y^2} - \frac{\sqrt{3}}{i \sigma_Y} \right) \left( \theta - \frac{i \mu_Y}{\sigma_Y^2} + \frac{\sqrt{3}}{i \sigma_Y} \right) \left( \theta - \frac{i \mu_Y}{i \sigma_Y^2} \right)
\]
and
\[ 2A(\theta) = -2\lambda \frac{\sigma_Y^6}{2} \left( \theta - \frac{i^2 \mu_Y}{\sigma_Y^2} - \sqrt{3} \right) \left( \theta - \frac{i^2 \mu_Y}{\sigma_Y^2} + \sqrt{3} \right) \left( \theta - \frac{i^2 \mu_Y}{\sigma_Y^2} \right) \]
do not vanish identically, their zeros must coincide due to fact that
\[ \left| \exp \left( iB(\theta) \right) \right| > 0, \quad \left| \exp \left( 2B(\theta) \right) \right| > 0, \quad \forall \theta \in \mathbb{R}. \]

So we get
\[ 2A(\theta) = c^1 A(\theta) \]
with a nonzero constant \( c \).

If \( \mu_Y = 0 \) and \( \lambda \sigma_Y \neq 0 \), then we have \( 2A(0) = 1A(0) = 0 \) and hence \( \mu_Y = 0 \). This implies \( \sigma_Y = \sigma_Y \) because the zeros coincide. Thus we have \( 2B(\theta) = 1B(\theta) \) and with (2.2) the equation \( \lambda = \lambda \).

In all other cases we set \( \theta = 0 \) in formula (2.2) and conclude \( c = 1 \), hence \( 2A(\theta) = 1A(\theta) \) for all \( \theta \in \mathbb{R} \). This leads to the relations
\[ i^1 \frac{\mu_Y}{\sigma_Y^2} = \frac{i^2 \mu_Y}{\sigma_Y^2}, \]
\[ i^1 \frac{\mu_Y}{\sigma_Y^2} - \sqrt{3} \frac{\mu_Y}{\sigma_Y^2} = \frac{i^2 \mu_Y}{\sigma_Y^2} - \sqrt{3} \frac{\mu_Y}{\sigma_Y^2}, \]
\[ -\lambda \frac{\sigma_Y^6}{2} = -\frac{\sigma_Y^6}{2}, \]
from which the identities
\[ \sigma_Y = \sigma_Y, \quad \mu_Y = \mu_Y, \quad \text{and} \quad \lambda = \lambda \]
follow.

This yields for equation (2.1)
\[ i \left( \tilde{\mu} - \frac{\sigma_Y}{2} \right) \theta - \frac{\sigma_Y^2 - \sigma_Y^2}{2} \theta^2 = 0 \quad \forall \theta \in \mathbb{R}, \]
consequently
\[ \tilde{\mu} = \frac{\sigma_Y}{2} \quad \text{and} \quad \sigma = \sigma. \]

Then the equations
\[ \tilde{\mu} = \mu - \frac{\sigma_Y^2}{2} - \lambda \left( e^{\mu_Y + \frac{\sigma_Y^2}{2}} - 1 \right), \quad \tilde{\mu} = \mu - \frac{\sigma_Y^2}{2} - 2\lambda \left( e^{\mu_Y + \frac{\sigma_Y^2}{2}} - 1 \right) \]
lead to
\[ \tilde{\mu} = \mu. \]

This proves theorem 1.1.

Note that for a forward operator \( F : p \mapsto F_p \) (see formula (1.1)) the injectivity can also be shown for an extended domain including the case \( \sigma = 0 \).
References
