Stationary solutions of linear ODEs with a randomly perturbed system matrix and additive noise

H.-J. Starkloff [a], R. Wunderlich [b]

[a]Martin-Luther-Universität Halle Wittenberg, Fachbereich für Mathematik und Informatik, 06099 Halle (Saale), Germany

[b]Westsächsische Hochschule Zwickau (FH), Fachgruppe Mathematik, PSF 201037, 08056 Zwickau, Germany

Abstract

The paper considers systems of linear first-order ODEs with a randomly perturbed system matrix and stationary additive noise. For the description of the long-term behavior of such systems it is necessary to study their stationary solutions. We deal with conditions for the existence of stationary solutions as well as with their representations and the computation of their moment functions.

Assuming small perturbations of the system matrix we apply perturbation techniques to find series representations of the stationary solutions and give asymptotic expansions for their first- and second-order moment functions. We illustrate the findings with a numerical example of a scalar ODE, for which the moment functions of the stationary solution still can be computed explicitly. This allows the assessment of the goodness of the approximations found from the derived asymptotic expansions.

Keywords: stationary solution, randomly perturbed system matrix, perturbation method, asymptotic expansions, correlation function

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1 Introduction

The present paper considers systems of first-order ODEs

$$\dot{\mathbf{z}}(t,\omega) = \mathbf{A}(\omega)\mathbf{z}(t,\omega) + \mathbf{f}(t,\omega) \tag{1.1}$$

for the random function $\mathbf{z} = \mathbf{z}(t,\omega)$ on $\mathbb{R} \times \Omega$ with values in $\mathbb{C}^n$, $n \in \mathbb{N}$, which is defined on a probability space $(\Omega, \mathcal{G}, \mathbf{P})$, where $\mathcal{G}$ denotes a suitable σ-algebra of subsets of Ω on which a probability measure $\mathbf{P}$ is defined. Further, $\mathbf{A}$ denotes an $n \times n$ matrix with random complex entries. The inhomogeneous term contains the random excitation function $\mathbf{f}(t,\omega)$ defined on $\mathbb{R} \times \Omega$ with values in $\mathbb{C}^n$. It is assumed that $\mathbf{f}$ is strict- and wide-sense stationary, pathwise and mean-square continuous and is independent of $\mathbf{A}$. In technical applications the random inhomogeneous term $\mathbf{f}$ is called additive noise.

The present paper deals with conditions for the existence of stationary solutions $\mathbf{z}$ to (1.1) as well as their representation and with the computation of their moment functions to given $\mathbf{A}$ and $\mathbf{f}$.

A random function $\boldsymbol{\xi} : \mathbb{R} \times \Omega \to \mathbb{C}^m$, $m \in \mathbb{N}$, is said to be stationary (in the strict sense) if for every sequence $t_1, \dots, t_N \in \mathbb{R}$, $N \in \mathbb{N}$, the joint distribution of the random vectors $\boldsymbol{\xi}(t_1 + \tau), \dots, \boldsymbol{\xi}(t_N + \tau)$ is independent of $\tau \in \mathbb{R}$. Further, a random function $\mathbf{z}(t,\omega)$ is called a stationary solution of (1.1) if $\mathbf{z}$ pathwise satisfies Eq. (1.1) and if $(\mathbf{z}, \mathbf{f})$ is a stationary random function, i.e., $\mathbf{z}$ and $\mathbf{f}$ are stationarily related.

The above problem arises e.g. in the investigation of the long-term behaviour of the response of discrete vibration systems with permanently acting random external excitations (see Soong, Grigoriu [16], Preumont [6] and [7, 13, 14]). Moreover equations of this type arise as result of the semi-discretization of some kinds of partial differential equations (PDEs) with respect to the spatial variables using finite difference or finite element methods. For PDEs describing random heat propagation in heterogeneous media (see e.g. [4]) the random matrix $\mathbf{A}$ represents a random spatially varying heat conductivity while the random process $\mathbf{f}$ represents random external heat fluxes on the boundary, heat sources or ambient temperatures. For PDEs describing random vibrations of continuous vibration systems we refer to [10, 11]. In this case the matrix $\mathbf{A}$ represents a random spatially varying bending stiffness and $\mathbf{f}$ describes external excitations.

Especially in the mathematical modelling of vibration phenomena the modal analysis and the use of so-called modal coordinates leads to equations of type (1.1) with complex state variables and parameters. Therefore we consider this general case throughout this paper and mention that the real-valued case is contained as a special case.

In the case of a non-random matrix $\mathbf{A}$ the stability of this matrix, i.e., all eigenvalues of $\mathbf{A}$ possess strictly negative real parts, guarantees that a unique stationary solution exists and solutions to initial value problems for Eq. (1.1) converge for $t \to \infty$ to this stationary solution (see Arnold, Wihstutz [2], Bunke [3], p. 45ff). For the computation of moment functions of the stationary solution there exist numerous methods, see e.g. Soong, Grigoriu [16], Preumont [6], [7, 13, 14]. In general, these methods can not be applied to the present case of a randomly perturbed matrix.

The paper is organized as follows. Section 2 gives an explicit analytic representation of the stationary solution of Equation (1.1) and proves its uniqueness. This representation is the starting point of the computation of first- and second-order moment functions for the stationary solution in Section 3. Decomposing the random parameters involved in (1.1) into their respective means and centered fluctuation terms the stationary solution can be decomposed in a similar way. Using this decomposition Subsection 3.1 gives general representations of the considered moment functions. It turns out that an explicit evaluation of the derived formulas is possible only in a very limited number of special cases. Therefore we present in Subsection 3.2 an approximate computation applying perturbation techniques and derive the leading as well as first (non-zero) correction terms of the corresponding expansions. These expansion terms are evaluated explicitly in Section 4. Thereby we restrict to a class of Equation (1.1) for which the correlation function of the stationary solution in the case of an unperturbed matrix $\mathbf{A}$ is exponentially decaying.

Finally Section 5 presents some numerical results for the case of a scalar Equation (1.1) with real random parameters. This is one of the rare cases where the moment functions can be computed explicitly and we can study the goodness of the approximations using the perturbation techniques.

Throughout the paper we use the following notation.

For a vector $\mathbf{x} \in \mathbb{C}^n$ and a matrix $\mathbf{M} \in \mathbb{C}^{n\times n}$ we denote by $\|\mathbf{x}\|$ the norm of $\mathbf{x}$ and by $\|\mathbf{M}\|$ some matrix norm of $\mathbf{M}$ which is compatible with the chosen vector norm, i.e., there holds the relation $\|\mathbf{Mx}\| \le \|\mathbf{M}\| \, \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{C}^n$. If not specified otherwise the vector norm $\|.\|$ can be chosen arbitrarily. With $\mathbf{x}^*$ and $\mathbf{M}^*$ we denote the adjoint vector and matrix.

The expectation w.r.t. the probability measure $\mathbf{P}$ is denoted by $\mathbf{E}\{.\}$, the conditional expectation by $\mathbf{E}\{.|.\}$ and the conditional probability by $\mathbf{P}\{.|.\}$. For random vectors $\boldsymbol{\xi}_1(\omega)$, $\boldsymbol{\xi}_2(\omega)$ with values in $\mathbb{C}^n$ the covariance matrix is denoted by

$$\mathbf{cov}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = \mathbf{E}\{(\boldsymbol{\xi}_1 - \mathbf{E}\{\boldsymbol{\xi}_1\})(\boldsymbol{\xi}_2 - \mathbf{E}\{\boldsymbol{\xi}_2\})^*\}$$

and for a stationary random vector function $\mathbf{y}(t,\omega)$, $t \in \mathbb{R}$, with values in $\mathbb{C}^n$ the correlation function is defined as

$$\mathbf{R_{yy}}(\tau) = \mathbf{cov}(\mathbf{y}(t), \mathbf{y}(t+\tau)), \quad \tau \in \mathbb{R}.$$

2 Stationary solution

In this section we give an explicit analytic representation of the stationary solution of Equation (1.1) and prove its uniqueness. This representation is the starting point of the computation of first- and second-order moment functions for the stationary solution in Section 3. First we provide the corresponding result for the case of a non-random matrix $\mathbf{A}$ which is known from the literature. (see e.g. Bunke [3], Theorem 3.5, p. 45ff; Arnold, Wihstutz [2])

Lemma 2.1
Assuming a stationary, a.s. pathwise and mean-square continuous excitation function $\mathbf{f}$ *and a non-random stable matrix* $\mathbf{A}$, *i.e., all eigenvalues of* $\mathbf{A}$ *possess negative real parts, then the linear first-order system*

$$\dot{\mathbf{z}}(t,\omega) = \mathbf{A}\mathbf{z}(t,\omega) + \mathbf{f}(t,\omega)$$

possesses the unique stationary solution

$$\mathbf{z}(t,\omega) = \int_{-\infty}^{t} e^{\mathbf{A}(t-u)}\mathbf{f}(u,\omega)du = \int_{0}^{\infty} e^{\mathbf{A}u}\mathbf{f}(t-u,\omega)du \tag{2.1}$$

in the a.s. pathwise as well as means-square sense.

If the matrix $\mathbf{A}$ is random and stochastically independent of the excitation function $\mathbf{f}$ a similar result can be derived (see also Khasminskij [5]).

Theorem 2.2
Assuming a stationary, a.s. pathwise and mean-square continuous excitation function $\mathbf{f}$ *and a random matrix* $\mathbf{A}$ *with* $\mathbf{E}\{\|\mathbf{A}^{-2}\|\} < \infty$, *which is a.s. stable and independent of* $\mathbf{f}$, *then the first-order system (1.1) possesses the unique stationary solution*

$$\mathbf{z}(t,\omega) = \int_{0}^{\infty} e^{\mathbf{A}(\omega)u}\mathbf{f}(t-u,\omega)du \tag{2.2}$$

in the a.s. pathwise as well as means-square sense.

Proof.
The assumptions on $\mathbf{A}$ and $\mathbf{f}$ ensure that the integral in (2.2) exists and is well-defined in the a.s. pathwise as well as mean-square sense. Obviously, $\mathbf{z}(t,\omega)$ given in (2.2) satisfies Eq. (1.1), i.e., $\mathbf{z}$ is a solution in the a.s. pathwise as well as means-square sense.

In order to prove that $\mathbf{z}(t,\omega)$ is a stationary solution of Eq. (1.1), one has to show that for arbitrary $N \in \mathbb{N}$, $t_1,\ldots,t_N \in \mathbb{R}$ and arbitrary Borel sets $B_1,\ldots,B_N$ of $\mathbb{C}^{2n}$ the probability of the events $C_s \in \mathcal{G}$ defined by

$$\begin{aligned} C_s &:= \bigcap_{i=1}^{N} \left\{ (\mathbf{z}(t_i+s,\omega), \mathbf{f}(t_i+s,\omega)) \in B_i \right\} \\ &= \bigcap_{i=1}^{N} \left\{ \Big(\int_0^\infty e^{\mathbf{A}(\omega)u}\mathbf{f}(t_i+s-u,\omega)du, \mathbf{f}(t_i+s,\omega) \Big) \in B_i \right\}, \quad s \in \mathbb{R}, \end{aligned}$$

does not depend on s, i.e., $\mathbf{P}(C_s) = \mathbf{P}(C_0)$.

Denote by $F_{\mathbf{A}}$ the ($n \times n$-dimensional) distribution function of the random matrix $\mathbf{A}$. Then for the probability of the events C_s it can be derived

$$\mathbf{P}(C_s) = \mathbf{E}\{\mathbf{P}\{C_s|\mathbf{A}\}\} = \int_{\mathbb{C}^{n\times n}} \mathbf{P}\{C_s|\mathbf{A}=\mathbf{M}\}dF_{\mathbf{A}}(\mathbf{M}).$$

Using that $\mathbf{A}$ is independent of $\mathbf{f}$ and that $\int_0^\infty e^{\mathbf{M}u}\,\mathbf{f}(t-u,\omega)du$ is a stationary solution of (1.1) for a fixed matrix $\mathbf{A}(\omega)=\mathbf{M}$ it follows

$$\begin{aligned}\mathbf{P}\{C_s|\mathbf{A}=\mathbf{M}\} &= \mathbf{P}\left(\bigcap_{i=1}^{N}\left\{\Big(\int_0^\infty e^{\mathbf{M}u}\mathbf{f}(t_i+s-u,\omega)du,\mathbf{f}(t_i+s,\omega)\Big)\in B_i\right\}\right)\\ &= \mathbf{P}\left(\bigcap_{i=1}^{N}\left\{\Big(\int_0^\infty e^{\mathbf{M}u}\mathbf{f}(t_i-u,\omega)du,\mathbf{f}(t_i,\omega)\Big)\in B_i\right\}\right)\\ &= \mathbf{P}\{C_0|\mathbf{A}=\mathbf{M}\},\end{aligned}$$

which implies

$$\mathbf{P}(C_s)=\int\limits_{\mathbb{C}^{n\times n}}\mathbf{P}\{C_0|\mathbf{A}=\mathbf{M}\}dF_{\mathbf{A}}(\mathbf{M})=\mathbf{E}\left\{\mathbf{P}\{C_0|\mathbf{A}\}\right\}=\mathbf{P}(C_0)$$

and proves the stationarity of $\mathbf{z}$. In the evaluation of the conditional expectation in the above derivation the random matrix $\mathbf{A}$ has been replaced by $\mathbf{M}$ which is possible due to the assumed independence of $\mathbf{A}$ and $\mathbf{f}$ (see e.g. Shiryaev [15], §7, p. 221).

It remains to prove the uniqueness of the solution. To this end two stationary solutions $\mathbf{z}_1$ and $\mathbf{z}_2$ of Eq. (1.1) are considered and it is proven that the difference $\mathbf{r}=\mathbf{z}_1-\mathbf{z}_2$ vanishes a.s. for all $t\in\mathbb{R}$.

Consider initial value problems for functions $\mathbf{x}$ on $[t_0,\infty)\times\Omega$, $t_0\in\mathbb{R}$, given by

$$\dot{\mathbf{x}}(t,\omega)=\mathbf{A}(\omega)\mathbf{x}(t,\omega)+\mathbf{f}(t,\omega),\qquad \mathbf{x}(t_0,\omega)=\mathbf{x}_0(\omega),$$

possessing for almost all $\omega\in\Omega$ the unique solution

$$\mathbf{x}(t,\omega)=e^{\mathbf{A}(\omega)(t-t_0)}\mathbf{x}_0(\omega)+\int\limits_{t_0}^{t}e^{\mathbf{A}(\omega)(t-u)}\,\mathbf{f}(u,\omega)\,du.$$

Since $\mathbf{z}_1$ and $\mathbf{z}_2$ satisfy Eq. (1.1) for all $t\in\mathbb{R}$ they coincide on $[t_0,\infty)$ with the solutions of the above initial value problem with initial values $\mathbf{x}_0(\omega)=\mathbf{z}_1(t_0,\omega)$ and $\mathbf{x}_0(\omega)=\mathbf{z}_2(t_0,\omega)$, respectively. For the difference $\mathbf{r}=\mathbf{z}_1-\mathbf{z}_2$ it holds for $t\in[t_0,\infty)$

$$\mathbf{r}(t,\omega)=\mathbf{z}_1(t,\omega)-\mathbf{z}_2(t,\omega)=e^{\mathbf{A}(\omega)(t-t_0)}(\mathbf{z}_1(t_0,\omega)-\mathbf{z}_2(t_0,\omega)).$$

Since t_0 can be chosen arbitrarily it holds

$$\mathbf{r}(t)=e^{\mathbf{A}(t-s)}\mathbf{r}(s),\quad \text{for all } s,t\in\mathbb{R} \text{ with } s\le t. \tag{2.3}$$

There hold the following assertions

1. $e^{\mathbf{A}(t-s)}\xrightarrow{a.s.}\mathbf{0}$ for $s\to-\infty$, since $\mathbf{A}$ is assumed to be a.s. stable.

2. $\mathbf{r}(s)$ is stochastically bounded, i.e., $\sup_{s\in\mathbb{R}} \mathbf{P}(\|\mathbf{r}(s)\| > c) \to 0$ for $c \to \infty$, which follows from the relation

$$\begin{aligned}
\mathbf{P}(\|\mathbf{r}(s)\| > c) &= \mathbf{P}(\|\mathbf{z}_1(s) - \mathbf{z}_2(s)\| > c) \leq \mathbf{P}(\|\mathbf{z}_1(s)\| + \|\mathbf{z}_2(s)\| > c) \\
&\leq \mathbf{P}\Big(\|\mathbf{z}_1(s)\| > \frac{c}{2}\Big) + \mathbf{P}\Big(\|\mathbf{z}_2(s)\| > \frac{c}{2}\Big) \\
&= \mathbf{P}\Big(\|\mathbf{z}_1(0)\| > \frac{c}{2}\Big) + \mathbf{P}\Big(\|\mathbf{z}_2(0)\| > \frac{c}{2}\Big) \underset{c\to\infty}{\longrightarrow} 0 \quad \forall s \in \mathbb{R},
\end{aligned}$$

where the stationarity of $\mathbf{z}_1$ and $\mathbf{z}_2$ has been used. Hence $\sup_{s\in\mathbb{R}} \mathbf{P}(\|\mathbf{r}(s)\| > c) \to 0$ for $c \to \infty$.

3.
$$e^{\mathbf{A}(t-s)}\, \mathbf{r}(s) \xrightarrow{\mathbf{P}} \mathbf{0} \quad \text{for } s \to -\infty,$$

since for any random function $\mathbf{M} : \mathbb{R}\times\Omega \to \mathbb{C}^{n\times n}$ with $\|\mathbf{M}(s)\| > 0$ a.s. for arbitrary $c > 0$ and $\varepsilon > 0$ it holds

$$\begin{aligned}
\mathbf{P}\big(\|\mathbf{M}(s)\, \mathbf{r}(s)\| > \varepsilon\big) &\leq \mathbf{P}\big(\|\mathbf{M}(s)\|\, \|\mathbf{r}(s)\| > \varepsilon\big) \\
&= \mathbf{P}\Big(\Big\{ \|\mathbf{r}(s)\| > \frac{\varepsilon}{\|\mathbf{M}(s)\|}\Big\} \cap \Big\{\frac{\varepsilon}{\|\mathbf{M}(s)\|} \leq c\Big\}\Big) + \\
&\quad\ \mathbf{P}\Big(\Big\{ \|\mathbf{r}(s)\| > \frac{\varepsilon}{\|\mathbf{M}(s)\|}\Big\} \cap \Big\{\frac{\varepsilon}{\|\mathbf{M}(s)\|} > c\Big\}\Big) \\
&\leq \mathbf{P}\Big(\frac{\varepsilon}{\|\mathbf{M}(s)\|} \leq c\Big) + \mathbf{P}(\|\mathbf{r}(s)\| > c).
\end{aligned}$$

Since $\mathbf{r}(s)$ is stochastically bounded it holds $\forall \delta > 0\ \exists c = c_\delta$ such that $\mathbf{P}(\|\mathbf{r}(s)\| > c_\delta) < \frac{\delta}{2}$.
Moreover with $\mathbf{M}(s) = e^{\mathbf{A}(t-s)}$ it follows $\forall \varepsilon > 0, \forall c_\delta > 0\ \exists s_0$ such that $\forall s \leq s_0$

$$\mathbf{P}\Big(\frac{\varepsilon}{\|\mathbf{M}(s)\|} \leq c_\delta\Big) = \mathbf{P}\Big(\|\mathbf{M}(s)\| \geq \frac{\varepsilon}{c_\delta}\Big) = \mathbf{P}\Big(\big\|e^{\mathbf{A}(t-s)}\big\| \geq \frac{\varepsilon}{c_\delta}\Big) \leq \frac{\delta}{2}$$

and it yields

$$\forall \varepsilon > 0, \forall \delta > 0\ \exists s_0 \text{ such that } \forall s \leq s_0 \text{ it holds } \mathbf{P}\big(\big\|e^{\mathbf{A}(t-s)}\, \mathbf{r}(s)\big\| > \varepsilon\big) < \delta$$

which proves the above assertion.

Relation (2.3) and assertion 3) imply $\mathbf{r}(t) \equiv \mathbf{0}$ a.s. $\forall t \in \mathbb{R}$, i.e., $\mathbf{z}_1$ and $\mathbf{z}_2$ coincide which proves the uniqueness of the stationary solution. □

3 Moment functions of the stationary solution

3.1 Decomposition of the stationary solution

For the subsequent investigation of stationary solutions of Eq. (1.1) we require that the assumptions of Theorem 2.2 are fulfilled and we consider the decomposition of the inhomogeneous term

$$\mathbf{f}(t,\omega) = \widehat{\mathbf{f}} + \widetilde{\mathbf{f}}(t,\omega)$$

into the non-random constant mean $\widehat{\mathbf{f}}$ and the random fluctuation term $\widetilde{\mathbf{f}}(t,\omega)$. This random process is stationary, pathwise and mean-square continuous, its correlation function coincides with the correlation function of $\mathbf{f}$.

Substituting the decomposition $\mathbf{f}(t,\omega) = \widehat{\mathbf{f}} + \widetilde{\mathbf{f}}(t,\omega)$ of the excitation function into the representation (2.2) of the stationary solution $\mathbf{z}$ of system (1.1) the following representation of $\mathbf{z}$ can be derived

$$\begin{aligned} \mathbf{z}(t,\omega) &= \widehat{\mathbf{z}}(\omega) + \widetilde{\mathbf{z}}(t,\omega) \\ \text{where} \quad \widehat{\mathbf{z}}(\omega) &= \int_0^\infty e^{\mathbf{A}(\omega)u}\ \widehat{\mathbf{f}}\,du = -\mathbf{A}^{-1}(\omega)\,\widehat{\mathbf{f}} \\ \text{and} \quad \widetilde{\mathbf{z}}(t,\omega) &= \int_0^\infty e^{\mathbf{A}(\omega)u}\ \widetilde{\mathbf{f}}(t-u,\omega)du. \end{aligned}$$

Here, the random vector $\widehat{\mathbf{z}}$ can be considered as the response of the system to the mean excitation $\widehat{\mathbf{f}}$ and the random process $\widetilde{\mathbf{z}}$ as the response to the random fluctuations of the excitation $\widetilde{\mathbf{f}}$. Theorem 2.2 implies that $\widetilde{\mathbf{z}}$ is the stationary solution of the system

$$\dot{\widetilde{\mathbf{z}}}(t,\omega) = \mathbf{A}(\omega)\widetilde{\mathbf{z}}(t,\omega) + \widetilde{\mathbf{f}}(t,\omega). \tag{3.1}$$

Lemma 3.1
Under the assumptions of Theorem 2.2 it holds $\mathbf{E}\{\widetilde{\mathbf{z}}(t)\} = \mathbf{0}$.

Proof.
It holds $\mathbf{E}\{\widetilde{\mathbf{z}}(t)\} = \int\limits_0^\infty \mathbf{E}\left\{e^{\mathbf{A}u}\,\widetilde{\mathbf{f}}(t-u)\right\} du$. For the integrand it can be derived

$$\mathbf{E}\left\{e^{\mathbf{A}u}\,\widetilde{\mathbf{f}}(t-u)\right\} = \mathbf{E}\left\{e^{\mathbf{A}u}\right\}\ \mathbf{E}\left\{\widetilde{\mathbf{f}}(t-u)\right\} = \mathbf{0}$$

since $\mathbf{A}$ and $\widetilde{\mathbf{f}}$ are independent and $\widetilde{\mathbf{f}}$ has zero mean. This implies $\mathbf{E}\{\widetilde{\mathbf{z}}(t)\} = \mathbf{0}$.

□

The next theorem states a decomposition of first- and second-order moment functions of the stationary solution $\mathbf{z}$ to system (1.1) which is helpful for the subsequent computation of moment functions of $\mathbf{z}$.

Theorem 3.2
Let $\mathbf{A}^{-1}$ possess finite second-order moments and let the assumptions of Theorem 2.2 be fulfilled. Then for the mean and the correlation function of the stationary solution $\mathbf{z}$ of system (1.1) it holds

$$\mathbf{E}\{\mathbf{z}(t)\} = \mathbf{E}\{\widehat{\mathbf{z}}\} = -\mathbf{E}\{\mathbf{A}^{-1}\}\widehat{\mathbf{f}} \tag{3.2}$$

$$\text{and} \quad \mathbf{R}_{\mathbf{zz}}(\tau) = \mathbf{cov}(\widehat{\mathbf{z}},\widehat{\mathbf{z}}) + \mathbf{R}_{\widetilde{\mathbf{z}}\widetilde{\mathbf{z}}}(\tau). \tag{3.3}$$

Proof.
Using the decomposition $\mathbf{z} = \widehat{\mathbf{z}} + \widetilde{\mathbf{z}}$ introduced above it follows $\mathbf{E}\{\mathbf{z}(t)\} = \mathbf{E}\{\widehat{\mathbf{z}}\} + \mathbf{E}\{\widetilde{\mathbf{z}}(t)\}$. Since $\mathbf{A}^{-1}$ possesses finite second-order moments $\mathbf{E}\{\mathbf{A}^{-1}\}$ exists and for the first term it holds $\mathbf{E}\{\widehat{\mathbf{z}}\} = -\mathbf{E}\{\mathbf{A}^{-1}\}\widehat{\mathbf{f}}$ while Lemma 3.1 implies that the second term vanishes. This proves Eq. (3.2).

To prove Eq. (3.3) again the decomposition $\mathbf{z} = \widehat{\mathbf{z}} + \widetilde{\mathbf{z}}$ is used. It holds

$$\begin{aligned}\mathbf{R}_{\mathbf{zz}}(\tau) &= \mathbf{cov}(\mathbf{z}(t),\mathbf{z}(t+\tau))\\ &= \mathbf{cov}(\widehat{\mathbf{z}},\widehat{\mathbf{z}}) + \mathbf{cov}(\widetilde{\mathbf{z}}(t),\widetilde{\mathbf{z}}(t+\tau)) + \mathbf{cov}(\widetilde{\mathbf{z}}(t),\widehat{\mathbf{z}}) + \mathbf{cov}(\widehat{\mathbf{z}},\widetilde{\mathbf{z}}(t+\tau))\\ &= \mathbf{cov}(\widehat{\mathbf{z}},\widehat{\mathbf{z}}) + \mathbf{R}_{\widetilde{\mathbf{z}}\widetilde{\mathbf{z}}}(\tau),\end{aligned}$$

since the two last covariances vanish because of

$$\begin{aligned}\mathbf{cov}(\widetilde{\mathbf{z}}(t),\widehat{\mathbf{z}}) &= \mathbf{E}\{\widetilde{\mathbf{z}}(t)\,\widehat{\mathbf{z}}^*\} - \mathbf{E}\{\widetilde{\mathbf{z}}(t)\}\,\mathbf{E}\{\widehat{\mathbf{z}}^*\}\\ &= \mathbf{E}\mathbf{E}\{\widetilde{\mathbf{z}}(t)\widehat{\mathbf{z}}^*|\mathbf{A}\} - \mathbf{0}\cdot\mathbf{E}\{\widehat{\mathbf{z}}^*\}\\ &= \mathbf{E}\{\mathbf{E}\{\widetilde{\mathbf{z}}(t)|\mathbf{A}\}\;\widehat{\mathbf{z}}^*\}\\ &= \mathbf{E}\left\{\int_0^\infty e^{\mathbf{A}u}\,\mathbf{E}\left\{\widetilde{\mathbf{f}}(t-u)\big|\mathbf{A}\right\}du\,\widehat{\mathbf{z}}^*\right\}\\ &= \mathbf{0}.\end{aligned}$$

Here it has been used that $\widehat{\mathbf{z}} = -\mathbf{A}^{-1}\widehat{\mathbf{f}}$ is $\mathbf{A}$-measurable and that $\widetilde{\mathbf{f}}$ is centered and independent of $\mathbf{A}$. Analogously $\mathbf{cov}(\widehat{\mathbf{z}},\widetilde{\mathbf{z}}(t+\tau)) = 0$ can be proven. □

The above theorem shows that the influence of the random parameters $\mathbf{A}$ and $\mathbf{f}$ on the moments of $\mathbf{z}$ can be separated. The mean $\mathbf{Ez}$ depends only on the distribution of $\mathbf{A}$ via $\mathbf{E}\{\mathbf{A}^{-1}\}$ and linearly on the mean excitation $\widehat{\mathbf{f}}$. For a centered excitation $\mathbf{f}$, i.e., for $\widehat{\mathbf{f}} = \mathbf{0}$, the stationary solution is centered, too.

The correlation function $\mathbf{R}_{\mathbf{zz}}(\tau)$ can be decomposed into the covariance of $\widehat{\mathbf{z}}$ and the correlation function of $\widetilde{\mathbf{z}}$. The first term $\mathbf{cov}(\widehat{\mathbf{z}},\widehat{\mathbf{z}})$ is invariant w.r.t. τ and depends only on the distribution of $\mathbf{A}$ and on $\widehat{\mathbf{f}}$. It vanishes in case of a centered excitation $\mathbf{f}$. The second term $\mathbf{R}_{\widetilde{\mathbf{z}}\widetilde{\mathbf{z}}}(\tau)$ depends on the distributions of $\mathbf{A}$ and the fluctuating part $\widetilde{\mathbf{f}}$ of the excitation but not on the mean excitation $\widehat{\mathbf{f}}$.

The explicit computation of the mean of $\mathbf{z}$ requires the computation of $\mathbf{E}\{\mathbf{A}^{-1}\}$, i.e., the mean of the inverse of $\mathbf{A}$. In the multi-dimensional case this is often very cumbersome or even impossible. The same holds for the computation of

$$\mathbf{cov}(\widehat{\mathbf{z}},\widehat{\mathbf{z}}) = \mathbf{E}\left\{\mathbf{A}^{-1}\widehat{\mathbf{f}}\,\widehat{\mathbf{f}}^{*}\mathbf{A}^{*^{-1}}\right\} - \mathbf{E}\widehat{\mathbf{z}}\,(\mathbf{E}\widehat{\mathbf{z}})^{*}.$$

For the correlation function of $\widetilde{\mathbf{z}}$ it holds

$$\begin{aligned}\mathbf{R}_{\widetilde{\mathbf{z}}\widetilde{\mathbf{z}}}(\tau) &= \mathbf{E}\left\{\widetilde{\mathbf{z}}(t)\,\widetilde{\mathbf{z}}^{*}(t+\tau)\right\} = \mathbf{E}\left\{\int_0^\infty\int_0^\infty e^{\mathbf{A}u}\widetilde{\mathbf{f}}(t-u)\,\widetilde{\mathbf{f}}^{*}(t+\tau-v)e^{\mathbf{A}^{*}v}\,du\,dv\right\}\\ &= \int_0^\infty\int_0^\infty \mathbf{E}\left\{e^{\mathbf{A}u}\mathbf{R}_{\widetilde{\mathbf{f}}\widetilde{\mathbf{f}}}(\tau+u-v)e^{\mathbf{A}^{*}v}\right\}\,du\,dv,\end{aligned}$$

where the independence of $\mathbf{A}$ and $\widetilde{\mathbf{f}}$ had been used. It can be seen that the evaluation $\mathbf{R}_{\widetilde{\mathbf{z}}\widetilde{\mathbf{z}}}(\tau)$ requires only second-order moments of $\widetilde{\mathbf{f}}$ but the complete distribution of $\mathbf{A}$, since it is involved in the matrix exponentials. In general, an explicit computation of expectations of these matrix exponentials fails.

Remark 3.3 Another way for the computation of the above correlation function makes use of conditional expectations w.r.t. the random matrix $\mathbf{A}$. This approach might be useful for the computation of approximations of the correlation function $\mathbf{R}_{\widetilde{\mathbf{z}}\widetilde{\mathbf{z}}}(\tau)$ based on Monte-Carlo simulations. It holds

$$\begin{aligned}\mathbf{R}_{\widetilde{\mathbf{z}}\widetilde{\mathbf{z}}}(\tau) &= \mathbf{E}\mathbf{E}\left\{\widetilde{\mathbf{z}}(t)\,\widetilde{\mathbf{z}}^{*}(t+\tau)\middle|\mathbf{A}\right\}\\ &= \int\limits_{\mathbb{C}^{n\times n}} \mathbf{E}\left\{\int_0^\infty\int_0^\infty e^{\mathbf{A}u}\widetilde{\mathbf{f}}(t-u)\,\widetilde{\mathbf{f}}^{*}(t+\tau-v)e^{\mathbf{A}^{*}v}\,du\,dv\middle|\mathbf{A}=\mathbf{M}\right\}\,dF_{\mathbf{A}}(\mathbf{M})\\ &= \int\limits_{\mathbb{C}^{n\times n}} \mathbf{E}\left\{\int_0^\infty\int_0^\infty e^{\mathbf{M}u}\widetilde{\mathbf{f}}(t-u)\,\widetilde{\mathbf{f}}^{*}(t+\tau-v)e^{\mathbf{M}^{*}v}\,du\,dv\right\}\,dF_{\mathbf{A}}(\mathbf{M})\\ &= \int\limits_{\mathbb{C}^{n\times n}} \mathbf{R}^{\mathbf{M}}_{\widetilde{\mathbf{z}}\widetilde{\mathbf{z}}}(\tau)\,dF_{\mathbf{A}}(\mathbf{M})\end{aligned}$$

where $F_{\mathbf{A}}(.)$ denotes the distribution function of the random matrix $\mathbf{A}$ and

$$\begin{aligned}\mathbf{R}^{\mathbf{M}}_{\widetilde{\mathbf{z}}\widetilde{\mathbf{z}}}(\tau) &= \mathbf{E}\left\{\int_0^\infty\int_0^\infty e^{\mathbf{M}u}\,\widetilde{\mathbf{f}}(t-u)\,\widetilde{\mathbf{f}}^{*}(t+\tau-v)e^{\mathbf{M}^{*}v}\,du\,dv\right\}\\ &= \int_0^\infty\int_0^\infty e^{\mathbf{M}u}\,\mathbf{R}_{\widetilde{\mathbf{f}}\widetilde{\mathbf{f}}}(\tau+u-v)e^{\mathbf{M}^{*}v}\,du\,dv\end{aligned}$$

denotes the correlation function of the stationary solution of $\dot{\widetilde{\mathbf{z}}} = \mathbf{M}\,\widetilde{\mathbf{z}} + \widetilde{\mathbf{f}}$, i.e., a system with a non-random matrix $\mathbf{M}$. In the evaluation of the conditional expectation in the above derivation the random matrix $\mathbf{A}$ has been replaced by $\mathbf{M}$ which is possible due to the assumed independence of $\mathbf{A}$ and $\mathbf{f}$ (see e.g. Shiryaev [15], §7, p. 221).

For special choices of the correlation function of the fluctuating part $\widetilde{\mathbf{f}}$ of the excitation (e.g. weakly or exponential correlated) it is possible to simplify the above double integral and to find explicit representations or power series expansions (see Soong, Grigoriu [16], [7, 8, 12]).

In many cases only partial information about the distribution of the random parameters (e.g. only first- and second-order moments) is available. Therefore, in the next section approximate representations of the moments of $\mathbf{z}$ in terms of first- and second-order moments of $\mathbf{A}$ and $\mathbf{f}$ are derived by applying perturbation methods.

3.2 Perturbation methods

This section deals with the approximative computation of the mean and the correlation function of the stationary solution $\mathbf{z}$ of Eq. (1.1). The main idea of the approximation is the decomposition of the random matrix $\mathbf{A}(\omega)$ into its constant mean $\widehat{\mathbf{A}}$ and a random fluctuating part which is scaled by a non-negative perturbation parameter η, i.e., it is set

$$\mathbf{A}(\omega) = \widehat{\mathbf{A}} + \eta \mathbf{C}(\omega), \quad \text{with } \eta \geq 0.$$

We consider the random matrix $\mathbf{A}$ as a perturbation of the constant matrix $\widehat{\mathbf{A}}$. The desired moments of $\mathbf{z}$ are expanded in powers of η and for sufficiently small values of η the appropriately truncated power series can be used as approximations of the exact moments functions.

If in Eq. (1.1) the random matrix $\mathbf{A}$ is replaced by its mean while the excitation term $\mathbf{f}$ remains unchanged the system

$$\dot{\mathbf{y}}(t,\omega) = \widehat{\mathbf{A}}\mathbf{y}(t,om) + \mathbf{f}(t,\omega) \tag{3.4}$$

arises. We call the above system the ”unperturbed system” since it contains the unperturbed matrix $\widehat{\mathbf{A}}$. From Lemma 2.1 it is known that

$$\mathbf{y}(t,\omega) = \int_0^\infty e^{\widehat{\mathbf{A}}u}\, \mathbf{f}(t-u,\omega)du$$

is the unique stationary solution of Eq. (3.4) since $\mathbf{A}$ and consequently $\widehat{\mathbf{A}} = \mathbf{E}\{\mathbf{A}\}$ is a stable matrix.

If in addition to the random matrix $\mathbf{A}(\omega)$ also the random inhomogeneous term $\mathbf{f}(t,\omega)$ is replaced by its constant mean, we get the so-called averaged system

$$\dot{\mathbf{x}}(t) = \widehat{\mathbf{A}}\mathbf{x}(t) + \widehat{\mathbf{f}}. \tag{3.5}$$

This is a non-random system possessing the trivial stationary solution $\mathbf{x}(t) \equiv -\widehat{\mathbf{A}}^{-1}\,\widehat{\mathbf{f}}$ which is non-random and does not depend on t. We note that a non-random function is stationary only if it is a constant function.

The mean of stationary solution $\mathbf{y}$ of the unperturbed system (3.4) coincides with the solution of the averaged problem (3.5), i.e., it holds

$$\mathbf{E}\{\mathbf{y}(t)\} = \int_0^\infty e^{\widehat{\mathbf{A}}u}\, \mathbf{E}\{\mathbf{f}(t-u)\}\, du = -\widehat{\mathbf{A}}^{-1}\,\widehat{\mathbf{f}} = \mathbf{x}.$$

The so-called "averaging problem" arises if one compares the solution $\mathbf{x}$ of the above averaged system with the average of the stationary solution of the original system (1.1). We will come back to this problem in Remark 3.5 below.

For the existence and uniqueness of the stationary solution $\mathbf{z}$ the matrix $\mathbf{A}(\omega) = \widehat{\mathbf{A}} + \eta\mathbf{C}(\omega)$ is supposed to be stable and for the existence of first- and second-order moments of $\mathbf{z}$ finite second-order moments of the inverse matrix $\mathbf{A}^{-1}(\omega)$ are required. It is noted, that the latter condition implies the existence of first-order moments of $\mathbf{A}^{-1}(\omega)$.

In order to check these conditions in terms of the perturbation parameter η it is assumed that $\widehat{\mathbf{A}} = \mathbf{E}\mathbf{A}$ is a stable matrix which implies the existence of $\widehat{\mathbf{A}}^{-1}$. Moreover it is assumed that the matrix $\mathbf{C}$ is bounded, i.e., there exists a positive real number c_0 such that $\|\mathbf{C}(\omega)\| \leq c_0$ a.s.. Then for sufficiently small $\eta > 0$ the matrix $\mathbf{A}(\omega) = \widehat{\mathbf{A}} + \eta\mathbf{C}(\omega)$ is stable and its inverse possesses finite second-order moments. Define

$$\begin{aligned} \eta_S &:= \sup\{\eta > 0 : \quad \widehat{\mathbf{A}} + \eta\mathbf{C}(\omega) \text{ is a.s. stable}\} \\ \eta_M &:= \sup\{\eta > 0 : \quad \mathbf{E}\left\{\left\|(\widehat{\mathbf{A}} + \eta\mathbf{C})^{-1}\right\|^2\right\} < \infty\} \end{aligned}$$

then for $\eta < \eta_S$ the matrix $\mathbf{A}$ is stable and for $\eta < \eta_M$ its inverse possesses finite second-order moments. It is noted that the a.s. boundedness of the matrix $\mathbf{C}$ implies the existence of first- and second-order moments of $\mathbf{C}$ as well as of $\mathbf{A} = \widehat{\mathbf{A}} + \eta\mathbf{C}$.

3.2.1 Moments of $\widehat{\mathbf{z}}$

For the evaluation of Formulas (3.2) and (3.3) the mean and the covariance of $\widehat{\mathbf{z}} = -\mathbf{A}^{-1}\widehat{\mathbf{f}}$ are required. Substituting the representation $\mathbf{A} = \widehat{\mathbf{A}} + \eta\mathbf{C}$ the inverse of $\mathbf{A}$ can be represented using a Neumann series

$$\begin{aligned} \mathbf{A}^{-1} &= (\widehat{\mathbf{A}} + \eta\mathbf{C})^{-1} = (\widehat{\mathbf{A}}(\mathbf{I} + \eta\widehat{\mathbf{A}}^{-1}\mathbf{C}))^{-1} \\ &= \sum_{p=0}^{\infty}(-\widehat{\mathbf{A}}^{-1}\mathbf{C})^p\,\eta^p\,\widehat{\mathbf{A}}^{-1}, \end{aligned} \tag{3.6}$$

which is convergent for $\eta < \left\|\widehat{\mathbf{A}}^{-1}\mathbf{C}\right\|^{-1}$ a.s. . Let

$$\eta_N := \sup\{\eta > 0 : \quad \eta < \left\|\widehat{\mathbf{A}}^{-1}\mathbf{C}(\omega)\right\|^{-1} \text{ a.s.}\},$$

then the inequality

$$\left\|\widehat{\mathbf{A}}^{-1}\mathbf{C}\right\| \leq \left\|\widehat{\mathbf{A}}^{-1}\right\| \|\mathbf{C}\| \leq c_0 \left\|\widehat{\mathbf{A}}^{-1}\right\|$$

leads to the lower bound $\eta_N \geq \frac{1}{c_0\left\|\widehat{\mathbf{A}}^{-1}\right\|}$ for the radius of convergence of the Neumann series.

The next theorem provides expansions of the first- and second-order moments of $\widehat{\mathbf{z}}$ including the leading terms and the first correction terms which are of order η^2.

Theorem 3.4
For $\eta < \min\{\eta_S, \eta_M\}$ there hold the following expansions for $\eta \downarrow 0$

$$\mathbf{E}\{\widehat{\mathbf{z}}\} = \left(\mathbf{I} + \widehat{\mathbf{A}}^{-1}\,\mathbf{E}\left\{\mathbf{C}\widehat{\mathbf{A}}^{-1}\mathbf{C}\right\}\eta^2\right)\mathbf{x} + o(\eta^2) \tag{3.7}$$

and

$$\mathbf{cov}(\widehat{\mathbf{z}}, \widehat{\mathbf{z}}) = \widehat{\mathbf{A}}^{-1}\mathbf{E}\{\mathbf{C}\,\mathbf{x}\,\mathbf{x}^*\mathbf{C}^*\}\,\widehat{\mathbf{A}}^{*^{-1}}\eta^2 + o(\eta^2) \tag{3.8}$$

where $\mathbf{x} = -\widehat{\mathbf{A}}^{-1}\,\widehat{\mathbf{f}}$.

Proof.
The Neumann series expansion (3.6) for $\mathbf{A}^{-1}$ yields for $\eta \downarrow 0$

$$\begin{aligned}\widehat{\mathbf{z}} &= -\mathbf{A}^{-1}\,\widehat{\mathbf{f}} = -\left(\mathbf{I} - \widehat{\mathbf{A}}^{-1}\mathbf{C}\eta + (\widehat{\mathbf{A}}^{-1}\mathbf{C})^2\eta^2 + o(\eta^2)\right)\widehat{\mathbf{A}}^{-1}\,\widehat{\mathbf{f}} \\ &= \left(\mathbf{I} - \widehat{\mathbf{A}}^{-1}\mathbf{C}\eta + (\widehat{\mathbf{A}}^{-1}\mathbf{C})^2\eta^2\right)\,\mathbf{x} + o(\eta^2) \end{aligned} \tag{3.9}$$

For $\eta < \min\{\eta_S, \eta_M\}$ the moments $\mathbf{E}\{\widehat{\mathbf{z}}\}$ and $\mathbf{cov}(\widehat{\mathbf{z}}, \widehat{\mathbf{z}})$ are well-defined and from Eq. (3.9) it follows

$$\begin{aligned}\mathbf{E}\{\widehat{\mathbf{z}}\} &= \left(\mathbf{I} - \widehat{\mathbf{A}}^{-1}\mathbf{E}\{\mathbf{C}\}\,\eta + \widehat{\mathbf{A}}^{-1}\mathbf{E}\left\{\mathbf{C}\widehat{\mathbf{A}}^{-1}\mathbf{C}\right\}\eta^2\right)\,\mathbf{x} + o(\eta^2) \\ &= \left(\mathbf{I} + \widehat{\mathbf{A}}^{-1}\mathbf{E}\left\{\mathbf{C}\widehat{\mathbf{A}}^{-1}\mathbf{C}\right\}\eta^2\right)\,\mathbf{x} + o(\eta^2)\end{aligned}$$

since $\mathbf{E}\{\mathbf{C}\} = \mathbf{0}$. This proves (3.7).

The proof of Eq. (3.8) uses the relation $\mathbf{cov}(\widehat{\mathbf{z}}, \widehat{\mathbf{z}}) = \mathbf{E}\{(\widehat{\mathbf{z}} - \mathbf{E}\{\widehat{\mathbf{z}}\})(\widehat{\mathbf{z}} - \mathbf{E}\{\widehat{\mathbf{z}}\})^*\}$ and the expansion

$$\begin{aligned}\widehat{\mathbf{z}} - \mathbf{E}\{\widehat{\mathbf{z}}\} &= \left(\mathbf{I} - \widehat{\mathbf{A}}^{-1}\mathbf{C}\eta + o(\eta)\right)\mathbf{x} - (\mathbf{x} + o(\eta)) \\ &= -\widehat{\mathbf{A}}^{-1}\mathbf{C}\,\mathbf{x}\,\eta + o(\eta)\end{aligned}$$

which follows from (3.7) and (3.9). It results

$$\mathbf{cov}(\widehat{\mathbf{z}}, \widehat{\mathbf{z}}) = \widehat{\mathbf{A}}^{-1}\mathbf{E}\{\mathbf{C}\,\mathbf{x}\,\mathbf{x}^*\mathbf{C}^*\}\,\widehat{\mathbf{A}}^{*^{-1}}\eta^2 + o(\eta^2).$$

□

Remark 3.5 With the help of the above theorem there can be given an answer (in an approximative sense) to the so-called averaging problem which consists in the computation of the difference between the average of the stationary solution $\mathbf{E}\{\mathbf{z}(t)\}$ and the stationary solution $\mathbf{x}$ of the averaged equation (3.5). While for systems with a non-random linear operator both quantities coincide this is in general not the case for systems containing random operators or nonlinearities. It holds

$$\mathbf{E}\{\mathbf{z}(t)\} - \mathbf{x} = \mathbf{E}\{\widehat{\mathbf{z}}\} - \mathbf{x} = \widehat{\mathbf{A}}^{-1}\mathbf{E}\left\{\mathbf{C}\widehat{\mathbf{A}}^{-1}\mathbf{C}\right\}\eta^2\,\mathbf{x} + o(\eta^2).$$

3.2.2 Correlation function of $\widetilde{\mathbf{z}}$

The evaluation of Formula (3.3) for the correlation function of $\mathbf{z}$ requires the computation of the correlation function of $\widetilde{\mathbf{z}}$ which is the stationary solution of system (3.1). To find an expansion of the latter correlation function in powers of η the function $\widetilde{\mathbf{z}}$ is represented as a power series

$$\widetilde{\mathbf{z}}(t,\omega) = \sum_{p=0}^{\infty} {}^{p}\boldsymbol{\zeta}(t,\omega)\,\eta^p \tag{3.10}$$

with respect to the parameter η. The coefficients ${}^{0}\boldsymbol{\zeta}$, ${}^{1}\boldsymbol{\zeta}$,... can be found by substituting series (3.10) into Eq. (3.1) and equating the coefficients of the powers of η. First, this procedure is carried out formally. A verification of the results is given afterwards.

The substitution of series (3.10) into Eq. (3.1) gives

$$\sum_{p=0}^{\infty} {}^{p}\dot{\boldsymbol{\zeta}}(t,\omega)\,\eta^p = (\widehat{\mathbf{A}} + \eta\mathbf{C}(\omega)) \sum_{p=0}^{\infty} {}^{p}\boldsymbol{\zeta}(t,\omega)\,\eta^p + \widetilde{\mathbf{f}}(t,\omega).$$

For the coefficients ${}^{p}\boldsymbol{\zeta}$ it results an infinite sequence of linear first-order systems

$$\begin{aligned} {}^{0}\dot{\boldsymbol{\zeta}} &= \widehat{\mathbf{A}}\ {}^{0}\boldsymbol{\zeta} + \widetilde{\mathbf{f}} \\ {}^{p}\dot{\boldsymbol{\zeta}} &= \widehat{\mathbf{A}}\ {}^{p}\boldsymbol{\zeta} + \mathbf{C}\ {}^{p-1}\boldsymbol{\zeta}, \quad p \geq 1. \end{aligned}$$

According to Lemma 2.1 for the above linear systems with the non-random matrix $\widehat{\mathbf{A}}$ stationary solutions can be found recursively as follows

$$\begin{aligned} {}^{0}\boldsymbol{\zeta}(t,\omega) &= \int_0^{\infty} e^{\widehat{\mathbf{A}}u}\,\widetilde{\mathbf{f}}(t-u,\omega)du \\ {}^{p}\boldsymbol{\zeta}(t,\omega) &= \int_0^{\infty} e^{\widehat{\mathbf{A}}u}\,\mathbf{C}(\omega)\ {}^{p-1}\boldsymbol{\zeta}(t-u,\omega)du, \quad p \geq 1, \end{aligned} \tag{3.11}$$

which imply the following explicit representation of ${}^{p}\boldsymbol{\zeta}$ in terms of ${}^{0}\boldsymbol{\zeta}$

$${}^{p}\boldsymbol{\zeta}(t,\omega) = \int\limits_{\mathbb{R}_+^p} \prod_{k=1}^{p} \left(e^{\widehat{\mathbf{A}}u_k}\mathbf{C}(\omega)\right) {}^{0}\boldsymbol{\zeta}(t-u_1-\ldots-u_p,\omega)du_1\ldots du_p, \quad p \geq 1.$$

After the investigation of the coefficients of the perturbation series conditions for the convergence of series (3.10) will be determined. We use the assumption on the stability of $\mathbf{A}$ which implies the stability of $\widehat{\mathbf{A}}$, i.e., it possesses eigenvalues with strictly negative real parts, only. Then positive real numbers λ_0 and v_0 exist such that $\left\| e^{\widehat{\mathbf{A}}u} \right\| \leq v_0\, e^{-\lambda_0 u}$ for all $u \geq 0$. Further, it will be used that the random matrix $\mathbf{C}$ is a.s. bounded and that the function $\widetilde{\mathbf{f}}$ is pathwise continuous on $\mathbb{R}$. As an additional condition we impose the pathwise boundedness of $\widetilde{\mathbf{f}}$.

Theorem 3.6
Let the following assumptions be fulfilled

1. *the matrix* $\widehat{\mathbf{A}}$ *is stable, there exist positive real numbers* λ_0 *and* v_0 *such that* $\left\| e^{\widehat{\mathbf{A}}u} \right\| \le v_0\, e^{-\lambda_0 u}$ *for* $u \ge 0$,

2. $\widetilde{\mathbf{f}}$ *is stationary, with a.s. continuous and bounded paths, there exists a positive random variable* $f_0(\omega)$ *such that* $\left\| \widetilde{\mathbf{f}}(t,\omega) \right\| \le f_0(\omega)$, $\forall\, t \in \mathbb{R}$, *a.s.,*

3. *there exists a positive real number* c_0 *such that* $\|\mathbf{C}(\omega)\| \le c_0$ *a.s. .*

Then the series $\sum\limits_{p=0}^{\infty} {}^{p}\boldsymbol{\zeta}(t)\, \eta^p$ *with coefficients* ${}^{p}\boldsymbol{\zeta}(t)$ *given in (3.11) converges almost surely with respect to* ω *and uniformly with respect to* $t \in \mathbb{R}$ *for* $\eta < \eta_P := \frac{\lambda_0}{v_0 c_0}$.

Proof.
In a first step by means of mathematical induction it is proven that $\|{}^{q}\boldsymbol{\zeta}(t)\|$ is bounded by

$$\|{}^{q}\boldsymbol{\zeta}(t)\| \le \frac{v_0 f_0}{\lambda_0} \left(\frac{v_0 c_0}{\lambda_0} \right)^q \quad \forall\, t \in \mathbb{R},\ q = 0, 1, \ldots\ , a.s.. \tag{3.12}$$

We start with $q = 0$ where it holds ${}^{0}\boldsymbol{\zeta}(t) = \int_0^\infty e^{\widehat{\mathbf{A}}u}\, \widetilde{\mathbf{f}}(t-u)du$ and

$$\|{}^{0}\boldsymbol{\zeta}(t)\| \;\le\; \int_0^\infty \left\| e^{\widehat{\mathbf{A}}u} \right\| \left\| \widetilde{\mathbf{f}}(t-u) \right\| du \le f_0 \int_0^\infty \left\| e^{\widehat{\mathbf{A}}u} \right\| du \quad \forall t \in \mathbb{R}, a.s.. \tag{3.13}$$

Using assumption 1 it follows

$$\int_0^\infty \left\| e^{\widehat{\mathbf{A}}u} \right\| du \;\le\; \int_0^\infty v_0\, e^{-\lambda_0 u}\, du = \frac{v_0}{\lambda_0}. \tag{3.14}$$

Applying inequalities (3.13) and (3.14) it results

$$\|{}^{0}\boldsymbol{\zeta}(t)\| \le \frac{v_0 f_0}{\lambda_0} = \frac{v_0 f_0}{\lambda_0} \left(\frac{v_0 c_0}{\lambda_0} \right)^0 \quad \forall\, t \in \mathbb{R}, a.s..$$

Now assuming the assertion (3.12) is valid for $q \le p$ the assertion for $q = p+1$ will be proven. For ${}^{p+1}\boldsymbol{\zeta}(t)$, it follows

$$\left\|{}^{p+1}\boldsymbol{\zeta}(t)\right\| \;=\; \left\| \int_0^\infty e^{\widehat{\mathbf{A}}u} \mathbf{C}\, {}^{p}\boldsymbol{\zeta}(t-u)\, du \right\| \le \int_0^\infty \left\| e^{\widehat{\mathbf{A}}u} \right\| \|\mathbf{C}\|\ \|{}^{p}\boldsymbol{\zeta}(t-u)\|\ du, \tag{3.15}$$

$\forall t \in \mathbb{R}$, a.s.. Using relation (3.12) for ${}^{p}\boldsymbol{\zeta}$, relation (3.14) and $\|\mathbf{C}\| \le\ c_0$ it follows

$$\left\|{}^{p+1}\boldsymbol{\zeta}(t)\right\| \;\le\; \frac{v_0}{\lambda_0} \cdot c_0 \cdot \frac{v_0 f_0}{\lambda_0} \left(\frac{v_0 c_0}{\lambda_0} \right)^p = \frac{v_0 f_0}{\lambda_0} \left(\frac{v_0 c_0}{\lambda_0} \right)^{p+1} \quad \forall t \in \mathbb{R}, a.s.$$

and the assertion (3.12) is proven.

From inequality (3.12) it results for the perturbation series

$$\left\| \sum_{p=0}^{\infty} {}^{p}\boldsymbol{\zeta}(t)\, \eta^p \right\| \leq \sum_{p=0}^{\infty} \|{}^{p}\boldsymbol{\zeta}(t)\|\ \eta^p \leq \frac{v_0 f_0}{\lambda_0} \sum_{p=0}^{\infty} \left(\frac{v_0 c_0 \eta}{\lambda_0} \right)^p \qquad \forall t \in \mathbb{R}, a.s..$$

Since the majorizing series converges for $\frac{v_0 c_0 \eta}{\lambda_0} < 1$, a sufficient condition for the convergence of perturbation series (3.10) is $\eta < \eta_P = \frac{\lambda_0}{v_0 c_0}$. □

Remark 3.7 If in addition to the stability of the matrix $\widehat{\mathbf{A}}$ (see assumption 1 in Theorem 3.6) the diagonalizability of $\widehat{\mathbf{A}}$ is supposed as an additional technical condition then the positive numbers λ_0 and v_0 can be further specified. Let there exists a representation $\widehat{\mathbf{A}} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}$ with a matrix $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, ..., \lambda_n)$, containing the eigenvalues of $\widehat{\mathbf{A}}$ on its diagonal and a regular matrix $\mathbf{V}$. Because of the stability of $\widehat{\mathbf{A}}$ the eigenvalues satisfy the relation $\mathrm{Re}\,[\lambda_i] < 0$, $i = 1, ..., n$.

The above representation of $\widehat{\mathbf{A}}$ yields

$$\left\| e^{\widehat{\mathbf{A}}u} \right\| = \left\| \mathbf{V}\, e^{\boldsymbol{\Lambda} u}\, \mathbf{V}^{-1} \right\| \leq \|\mathbf{V}\| \left\|\mathbf{V}^{-1}\right\| \left\|e^{\boldsymbol{\Lambda} u}\right\|.$$

It holds $\mathrm{cond}(\mathbf{V}) = \|\mathbf{V}\|\,\|\mathbf{V}^{-1}\|$ where cond(.) denotes the condition number of a matrix. If moreover the matrix norm is set to be column-sum, spectral or row-sum norm denoted by $\|.\|_1$, $\|.\|_2$ or $\|.\|_\infty$, respectively, then it holds for the diagonal matrix $e^{\boldsymbol{\Lambda} u}$

$$\left\|e^{\boldsymbol{\Lambda} u}\right\|_{1,2,\infty} = \max_i \left|e^{\lambda_i u}\right| = e^{\max\limits_i \mathrm{Re}[\lambda_i]\, u} = e^{-\min\limits_i |\mathrm{Re}[\lambda_i]|\, u}.$$

Hence the positive numbers λ_0 and v_0 involved in the inequality $\left\| e^{\widehat{\mathbf{A}}u} \right\|_{1,2,\infty} \leq v_0\, e^{-\lambda_0 u}$, for all $u \in \mathbb{R}$, can be chosen as

$$\lambda_0 = \min_i |\mathrm{Re}\,[\lambda_i]| \quad \text{ and } \quad v_0 = \mathrm{cond}_{1,2,\infty}(\mathbf{V}).$$

For other matrix norms the number v_0 has to be modified. If the matrix $\mathbf{V}$ can be chosen as a unitary matrix then in case of the spectral norm it holds $v_0 = \mathrm{cond}_2(\mathbf{V}) = \|\mathbf{V}\|_2 \left\|\mathbf{V}^{-1}\right\|_2 = 1$.

Assumption 2 of Theorem 3.6 on the pathwise boundedness of the random process $\widetilde{\mathbf{f}}$ excludes e.g. Gaussian processes from the consideration. On the other hand the positive random variable $f_0(\omega)$ which bounds $\left\|\widetilde{\mathbf{f}}(t,\omega)\right\|$ is not involved in the definition of the bound η_P for the radius of convergence. Next we show that assumption 2 can be relaxed by requiring the boundedness of $\widetilde{\mathbf{f}}(t)$ in the mean-square sense. This property is fulfilled for stationary mean-square continuous random processes. Then it is possible to prove a similar convergence statement in the mean-square sense (see Theorem 3.8 below). The class of mean-square bounded processes contains the pathwise bounded processes but also Gaussian processes.

Let $\mathcal{Q} = \mathcal{Q}(\Omega, \mathcal{G}, \mathbf{P}; \mathbb{C}^n)$ be the space of random vectors with values in $\mathbb{C}^n$ and finite second-order moments equipped with the the norm

$$\|\boldsymbol{\xi}\|_{\mathcal{Q}} = (\mathbf{E}\left\{\|\boldsymbol{\xi}\|^2\right\})^{\frac{1}{2}},$$

where $\boldsymbol{\xi} \in \mathcal{Q}$ is a random vector and $\|.\|$ denotes some norm in $\mathbb{C}^n$. The space $\mathcal{Q}$ is known to be complete. For the convergence of a sequence $(\boldsymbol{\xi}_m)$ in the mean-square sense it is necessary and sufficient that $(\boldsymbol{\xi}_m)$ is fundamental. Moreover if $\|\boldsymbol{\xi}_m\|_{\mathcal{Q}} \leq c_m$ with $c_m \in \mathbb{R}$, $\forall m \in \mathbb{N}$ and $\sum\limits_m c_m < \infty$ then $\sum\limits_m \boldsymbol{\xi}_m$ converges in the mean-square sense.

Now we formulate the announced convergence theorem.

Theorem 3.8
Let the following assumptions be fulfilled

1. *the matrix* $\widehat{\mathbf{A}}$ *is stable, there exist positive real numbers* λ_0 *and* v_0 *such that* $\left\|e^{\widehat{\mathbf{A}}u}\right\| \leq v_0\, e^{-\lambda_0 u}$ *for* $u \geq 0$,
2. $\widetilde{\mathbf{f}}$ *is stationary, mean-square and pathwise continuous on* $\mathbb{R}$ *and there exists a positive real number* f_0 *such that* $\left\|\widetilde{\mathbf{f}}(t)\right\|_{\mathcal{Q}} = f_0$, $\forall\, t \in \mathbb{R}$,
3. *there exists a positive real number* c_0 *such that* $\|\mathbf{C}(\omega)\| \leq c_0$ *a.s. .*

Then the series $\sum\limits_{p=0}^{\infty} {}^p\boldsymbol{\zeta}(t)\,\eta^p$ *with coefficients* ${}^p\boldsymbol{\zeta}(t)$ *given in (3.11) converges in the mean-square sense with respect to* ω *and uniformly with respect to* $t \in \mathbb{R}$ *for* $\eta < \eta_P := \frac{\lambda_0}{v_0 c_0}$.

Proof.
As in the proof of Theorem 3.6 we use the mathematical induction to prove that $\|{}^q\boldsymbol{\zeta}(t)\|_{\mathcal{Q}}$ is bounded by

$$\|{}^q\boldsymbol{\zeta}(t)\|_{\mathcal{Q}} \leq \frac{v_0 f_0}{\lambda_0}\left(\frac{v_0 c_0}{\lambda_0}\right)^q \quad \forall\, t \in \mathbb{R},\ q = 0, 1, \dots\ . \tag{3.16}$$

For a non-random matrix $\mathbf{M}$ and a random vector $\boldsymbol{\xi} \in \mathcal{Q}$ there holds the relation

$$\|\mathbf{M}\boldsymbol{\xi}\|_{\mathcal{Q}}^2 = \mathbf{E}\left\{\|\mathbf{M}\boldsymbol{\xi}\|^2\right\} \leq \mathbf{E}\left\{\|\mathbf{M}\|^2\|\boldsymbol{\xi}\|^2\right\} = \|\mathbf{M}\|^2\,\mathbf{E}\left\{\|\boldsymbol{\xi}\|^2\right\} = \|\mathbf{M}\|^2\|\boldsymbol{\xi}\|_{\mathcal{Q}}^2,$$

hence we have $\|\mathbf{M}\boldsymbol{\xi}\|_{\mathcal{Q}} \leq \|\mathbf{M}\|\,\|\boldsymbol{\xi}\|_{\mathcal{Q}}$. For $q = 0$ the above relation yields

$$\left\|{}^0\boldsymbol{\zeta}(t)\right\|_{\mathcal{Q}} \ \leq\ \int_0^\infty \left\|e^{\widehat{\mathbf{A}}u}\right\|\,\left\|\widetilde{\mathbf{f}}(t-u)\right\|_{\mathcal{Q}} du = f_0 \int_0^\infty \left\|e^{\widehat{\mathbf{A}}u}\right\| du \quad \forall t \in \mathbb{R}.$$

Applying inequality (3.14) to the integral on the right hand side it results

$$\left\| {}^0\boldsymbol{\zeta}(t) \right\|_{\mathcal{Q}} \leq \frac{v_0 f_0}{\lambda_0} = \frac{v_0 f_0}{\lambda_0} \left(\frac{v_0 c_0}{\lambda_0} \right)^0 \qquad \forall\, t \in \mathbb{R}.$$

Now assuming the assertion (3.12) is valid for $q \leq p$ the assertion for $q = p+1$ will be proven. For ${}^{p+1}\boldsymbol{\zeta}(t)$, it follows

$$\begin{aligned}
\left\| {}^{p+1}\boldsymbol{\zeta}(t) \right\|_{\mathcal{Q}} &= \left\| \int_0^\infty e^{\widehat{\mathbf{A}}u} \mathbf{C}\, {}^p\boldsymbol{\zeta}(t-u)\, du \right\|_{\mathcal{Q}} \leq \int_0^\infty \left\| e^{\widehat{\mathbf{A}}u} \mathbf{C}\, {}^p\boldsymbol{\zeta}(t-u) \right\|_{\mathcal{Q}} du \\
&\leq \int_0^\infty \left\| e^{\widehat{\mathbf{A}}u} \right\| \left\| \mathbf{C}\, {}^p\boldsymbol{\zeta}(t-u) \right\|_{\mathcal{Q}} du \\
&\leq \int_0^\infty \left\| e^{\widehat{\mathbf{A}}u} \right\| \left(\mathbf{E}\left\{ \|\mathbf{C}\|^2 \left\| {}^p\boldsymbol{\zeta}(t-u) \right\|^2 \right\} \right)^{\frac{1}{2}} du \\
&\leq \int_0^\infty \left\| e^{\widehat{\mathbf{A}}u} \right\| c_0 \left\| {}^p\boldsymbol{\zeta}(t-u) \right\|_{\mathcal{Q}} du \qquad \forall t \in \mathbb{R}.
\end{aligned}$$

Using relation (3.16) for ${}^p\boldsymbol{\zeta}$ and relation (3.14) it follows

$$\left\| {}^{p+1}\boldsymbol{\zeta}(t) \right\|_{\mathcal{Q}} \leq \frac{v_0}{\lambda_0} \cdot c_0 \cdot \frac{v_0 f_0}{\lambda_0} \left(\frac{v_0 c_0}{\lambda_0} \right)^p = \frac{v_0 f_0}{\lambda_0} \left(\frac{v_0 c_0}{\lambda_0} \right)^{p+1} \qquad \forall t \in \mathbb{R}$$

and the assertion (3.16) is proven.

From inequality (3.16) it results for the perturbation series

$$\left\| \sum_{p=0}^\infty {}^p\boldsymbol{\zeta}(t)\, \eta^p \right\|_{\mathcal{Q}} \leq \sum_{p=0}^\infty \left\| {}^p\boldsymbol{\zeta}(t) \right\|_{\mathcal{Q}} \eta^p \leq \frac{v_0 f_0}{\lambda_0} \sum_{p=0}^\infty \left(\frac{v_0 c_0 \eta}{\lambda_0} \right)^p \qquad \forall t \in \mathbb{R}.$$

Since the majorizing series converges for $\frac{v_0 c_0 \eta}{\lambda_0} < 1$, a sufficient condition for the mean-square convergence of perturbation series (3.10) is $\eta < \eta_P = \frac{\lambda_0}{v_0 c_0}$. □

The a.s. convergence of the series $\boldsymbol{\zeta}(t) = \sum_{p=0}^{\infty} {}^p\boldsymbol{\zeta}(t)\, \eta^p$ for $\eta < \eta_P$ which follows from Theorem 3.6 is the key result of the proof of the following theorem.

Theorem 3.9
Let $\widetilde{\mathbf{f}}(t,\omega)$ be a stationary and an a.s. pathwise as well as mean-square continuous random function. Further, let $\mathbf{A}(\omega) = \widehat{\mathbf{A}} + \eta\mathbf{C}(\omega)$ be a random matrix which is independent of $\widetilde{\mathbf{f}}$ and $\widehat{\mathbf{A}} = \mathbf{E}\{\mathbf{A}\}$ is assumed to be a stable matrix. Finally, let the assumptions of Theorem 3.6 be fulfilled and let $\eta_S > 0$ be such that for $\eta < \eta_S$ the matrix $\mathbf{A}(\omega)$ is a.s. stable.

Then for $\eta < \eta_S$ Equation (3.1)

$$\dot{\widetilde{\mathbf{z}}}(t,\omega) = \mathbf{A}(\omega)\widetilde{\mathbf{z}}(t,\omega) + \widetilde{\mathbf{f}}(t,\omega)$$

possesses the unique stationary solution

$$\widetilde{\mathbf{z}}(t,\omega) = \int_0^\infty e^{\mathbf{A}(\omega)u}\,\widetilde{\mathbf{f}}(t-u,\omega)du$$

which admits for $\eta < \min\{\eta_S, \eta_P\}$ a representation as perturbation series $\widetilde{\mathbf{z}}(t) = \sum\limits_{p=0}^{\infty} {}^p\boldsymbol{\zeta}(t)\,\eta^p$ with coefficients ${}^p\boldsymbol{\zeta}$ given in (3.11).

Proof.
First, we prove that for $\eta < \eta_P$ the perturbation series $\boldsymbol{\zeta}(t) = \sum\limits_{p=0}^{\infty} {}^p\boldsymbol{\zeta}(t)\,\eta^p$ is a stationary process and stationarily related to $\widetilde{\mathbf{f}}$. Following the lines of the proof of Theorem 3 in our paper [9] by means of mathematical induction it can be deduced that for all $N = 0, 1, \ldots$ the processes $\widetilde{\mathbf{f}}, {}^0\boldsymbol{\zeta}, \ldots, {}^N\boldsymbol{\zeta}$ as well as the processes $\widetilde{\mathbf{f}}$ and $\sum\limits_{p=0}^{N} {}^p\boldsymbol{\zeta}\,\eta^p$ are stationarily related.

From Theorem 3.6 it is known that for $\eta < \eta_P$ the series $\sum\limits_{p=0}^{N} {}^p\boldsymbol{\zeta}(t)\,\eta^p$ converges for $N \to \infty$ almost surely with respect to ω and uniformly with respect to $t \in \mathbb{R}$. Consequently, the limit $\boldsymbol{\zeta}(t) = \sum\limits_{p=0}^{\infty} {}^p\boldsymbol{\zeta}(t)\,\eta^p$ is stationarily related to $\widetilde{\mathbf{f}}$.

Now, it suffices to prove that $\boldsymbol{\zeta}$ satisfies Equation (3.1) for $\eta < \min\{\eta_S, \eta_P\}$ since due to Theorem 2.2, Eq. (3.1) possesses a unique stationary solution. Therefore, if $\boldsymbol{\zeta}$ satisfies (3.1) then it coincides with the stationary solution $\widetilde{\mathbf{z}}$ given above.

In order to show that $\boldsymbol{\zeta}$ satisfies Eq. (3.1) it is first proven that the representation $\dot{\boldsymbol{\zeta}}(t) = \sum\limits_{p=0}^{\infty} {}^p\dot{\boldsymbol{\zeta}}(t)\,\eta^p$ is valid for $\eta < \eta_P$. To this end the uniform convergence of the formal differentiated series $\frac{d}{dt}\big(\sum\limits_{p=0}^{\infty} {}^p\boldsymbol{\zeta}(t)\,\eta^p\big)$ for $\eta < \min\{\eta_S, \eta_P\}$ is checked. Using representation (3.11) of the coefficients ${}^p\boldsymbol{\zeta}$ formal differentiation leads to

$$\begin{aligned}
\frac{d}{dt}\bigg(\sum_{p=0}^{\infty} {}^p\boldsymbol{\zeta}(t)\,\eta^p\bigg) &= \sum_{p=0}^{\infty} {}^p\dot{\boldsymbol{\zeta}}(t)\,\eta^p \\
&= \widehat{\mathbf{A}}\,{}^0\boldsymbol{\zeta} + \widetilde{\mathbf{f}} + \sum_{p=1}^{\infty}\Big(\widehat{\mathbf{A}}\,{}^p\boldsymbol{\zeta}(t) + \mathbf{C}\,{}^{p-1}\boldsymbol{\zeta}(t)\Big)\,\eta^p \\
&= (\widehat{\mathbf{A}} + \eta\mathbf{C})\sum_{p=0}^{\infty} {}^p\boldsymbol{\zeta}(t)\eta^p + \widetilde{\mathbf{f}}(t). \qquad (3.17)
\end{aligned}$$

The uniform convergence of the series on the right hand side for $\eta < \min\{\eta_S, \eta_P\}$ follows immediately from Theorem 3.6. Moreover, from the above relation it follows $\dot{\boldsymbol{\zeta}}(t) = \mathbf{A}\boldsymbol{\zeta} + \widetilde{\mathbf{f}}$, i.e., $\boldsymbol{\zeta}$ satisfies Eq. (3.1) for $\eta < \min\{\eta_S, \eta_P\}$. □

The series expansion of $\widetilde{\mathbf{z}}$ given in Theorem 3.9 can be used to find expansions of the correlation function of $\widetilde{\mathbf{z}}$ in powers of η. Using $\widetilde{\mathbf{z}} = {}^0\boldsymbol{\zeta} + {}^1\boldsymbol{\zeta}\eta + {}^2\boldsymbol{\zeta}\eta^2 + o(\eta^2)$ leads to the following result.

Theorem 3.10
Let $\mathbf{E}\{\|A^{-2}\|\} < \infty$. Then under the assumptions of Theorem 3.9 the correlation function of $\widetilde{\mathbf{z}}(t) = \int_0^\infty e^{\mathbf{A}u}\,\widetilde{\mathbf{f}}(t-u)\,du$ for $\eta < \eta_S$ possesses the expansion for $\eta \downarrow 0$

$$\mathbf{R}_{\widetilde{\mathbf{z}}\widetilde{\mathbf{z}}}(\tau) = \mathbf{R}_{{}^0\boldsymbol{\zeta}{}^0\boldsymbol{\zeta}}(\tau) + (\mathbf{R}_{{}^2\boldsymbol{\zeta}{}^0\boldsymbol{\zeta}}(\tau) + \mathbf{R}_{{}^1\boldsymbol{\zeta}{}^1\boldsymbol{\zeta}}(\tau) + \mathbf{R}_{{}^0\boldsymbol{\zeta}{}^2\boldsymbol{\zeta}}(\tau))\eta^2 + o(\eta^2),$$

uniformly for all $\tau \in \mathbb{R}$ where

$$\begin{aligned}
\mathbf{R}_{{}^2\boldsymbol{\zeta}{}^0\boldsymbol{\zeta}}(\tau) &= \int_0^\infty\!\!\int_0^\infty e^{\widehat{\mathbf{A}}u_1}\mathbf{E}\left\{\mathbf{C}e^{\widehat{\mathbf{A}}u_2}\mathbf{C}\right\}\mathbf{R}_{{}^0\boldsymbol{\zeta}{}^0\boldsymbol{\zeta}}(\tau+u_1+u_2)\,du_1du_2,\\
\mathbf{R}_{{}^1\boldsymbol{\zeta}{}^1\boldsymbol{\zeta}}(\tau) &= \int_0^\infty\!\!\int_0^\infty e^{\widehat{\mathbf{A}}u_1}\mathbf{E}\left\{\mathbf{C}\,\mathbf{R}_{{}^0\boldsymbol{\zeta}{}^0\boldsymbol{\zeta}}(\tau+u_1-u_2)\,\mathbf{C}^*\right\}e^{\widehat{\mathbf{A}}^*u_2}\,du_1du_2\\
\text{and}\quad \mathbf{R}_{{}^0\boldsymbol{\zeta}{}^2\boldsymbol{\zeta}}(\tau) &= \mathbf{R}^*_{{}^2\boldsymbol{\zeta}{}^0\boldsymbol{\zeta}}(-\tau).
\end{aligned}$$

Proof.
The assumption $\mathbf{E}\{\|A^{-2}\|\} < \infty$ ensures that the correlation function $\mathbf{R}_{\widetilde{\mathbf{z}}\widetilde{\mathbf{z}}}(\tau)$ exists and is well-defined.

From the perturbation series representation of $\widetilde{\mathbf{z}}$ given in Theorem 3.9 it follows $\widetilde{\mathbf{z}}(t) = {}^0\boldsymbol{\zeta}(t) + {}^1\boldsymbol{\zeta}(t)\eta + {}^2\boldsymbol{\zeta}(t)\eta^2 + o(\eta^2)$ uniformly for all $t \in \mathbb{R}$ and

$$\begin{aligned}
\mathbf{R}_{\widetilde{\mathbf{z}}\widetilde{\mathbf{z}}}(\tau) &= \mathbf{R}_{{}^0\boldsymbol{\zeta}{}^0\boldsymbol{\zeta}}(\tau) + \left(\mathbf{R}_{{}^1\boldsymbol{\zeta}{}^0\boldsymbol{\zeta}}(\tau) + \mathbf{R}_{{}^0\boldsymbol{\zeta}{}^1\boldsymbol{\zeta}}(\tau)\right)\eta\\
&\quad + \left(\mathbf{R}_{{}^2\boldsymbol{\zeta}{}^0\boldsymbol{\zeta}}(\tau) + \mathbf{R}_{{}^1\boldsymbol{\zeta}{}^1\boldsymbol{\zeta}}(\tau) + \mathbf{R}_{{}^0\boldsymbol{\zeta}{}^2\boldsymbol{\zeta}}(\tau)\right)\eta^2 + o(\eta^2) \qquad (3.18)
\end{aligned}$$

uniformly for all $\tau \in \mathbb{R}$. Recalling representations (3.11) for ${}^0\boldsymbol{\zeta}$, ${}^1\boldsymbol{\zeta}$ and ${}^2\boldsymbol{\zeta}$, i.e.,

$$\begin{aligned}
{}^0\boldsymbol{\zeta}(t) &= \int_0^\infty e^{\widehat{\mathbf{A}}u}\,\widetilde{\mathbf{f}}(t-u)\,du,\\
{}^1\boldsymbol{\zeta}(t) &= \int_0^\infty e^{\widehat{\mathbf{A}}u}\,\mathbf{C}\,{}^0\boldsymbol{\zeta}(t-u)\,du \quad \text{and}\\
{}^2\boldsymbol{\zeta}(t) &= \int_0^\infty e^{\widehat{\mathbf{A}}u}\,\mathbf{C}\,{}^1\boldsymbol{\zeta}(t-u)\,du = \int_0^\infty\!\!\int_0^\infty e^{\widehat{\mathbf{A}}u_1}\,\mathbf{C}\,e^{\widehat{\mathbf{A}}u_2}\,\mathbf{C}\,{}^0\boldsymbol{\zeta}(t-u_1-u_2)\,du_1du_2,
\end{aligned}$$

and using the independence of $\mathbf{C}$ and $\widetilde{\mathbf{f}}$, $\mathbf{E}\{\mathbf{C}\} = \mathbf{0}$ and $\mathbf{E}\left\{\widetilde{\mathbf{f}}(t)\right\} = \mathbf{0}$ it follows

$$\begin{aligned}
\mathbf{E}\left\{{}^0\boldsymbol{\zeta}(t)\right\} &= \int_0^\infty e^{\widehat{\mathbf{A}}u}\,\mathbf{E}\left\{\widetilde{\mathbf{f}}(t-u)\right\}du = \mathbf{0},\\
\mathbf{E}\left\{{}^1\boldsymbol{\zeta}(t)\right\} &= \int_0^\infty e^{\widehat{\mathbf{A}}u}\,\mathbf{E}\{\mathbf{C}\}\;\mathbf{E}\left\{{}^0\boldsymbol{\zeta}(t-u)\right\}du = \mathbf{0} \quad \text{and}\\
\mathbf{E}\left\{{}^2\boldsymbol{\zeta}(t)\right\} &= \int_0^\infty\!\!\int_0^\infty e^{\widehat{\mathbf{A}}u_1}\,\mathbf{E}\left\{\mathbf{C}\,e^{\widehat{\mathbf{A}}u_2}\,\mathbf{C}\right\}\;\mathbf{E}\left\{{}^0\boldsymbol{\zeta}(t-u_1-u_2)\right\}du_1du_2 = \mathbf{0}.
\end{aligned}$$

Then for the correlation functions involved in (3.18) in the coefficient of η it can be derived

$$\begin{aligned}\mathbf{R}_{{}^1\zeta{}^0\zeta}(\tau) &= \mathbf{E}\left\{{}^1\boldsymbol{\zeta}(t){}^0\boldsymbol{\zeta}^*(t+\tau)\right\} = \mathbf{E}\left\{\int_0^\infty e^{\widehat{\mathbf{A}}u}\,\mathbf{C}\,{}^0\boldsymbol{\zeta}(t-u)\,du\,{}^0\boldsymbol{\zeta}^*(t+\tau)\right\}\\ &= \int_0^\infty e^{\widehat{\mathbf{A}}u}\,\mathbf{E}\left\{\mathbf{C}\right\}\,\mathbf{E}\left\{{}^0\boldsymbol{\zeta}(t-u)\,{}^0\boldsymbol{\zeta}^*(t+\tau)\right\}\,du = \mathbf{0}\end{aligned}$$

and analogously $\mathbf{R}_{{}^0\zeta{}^1\zeta}(\tau) = \mathbf{0}$ while for the correlation functions involved in the coefficient of η^2 it holds

$$\begin{aligned}\mathbf{R}_{{}^2\zeta{}^0\zeta}(\tau) &= \mathbf{E}\left\{{}^2\boldsymbol{\zeta}(t){}^0\boldsymbol{\zeta}^*(t+\tau)\right\}\\ &= \mathbf{E}\left\{\int_0^\infty\int_0^\infty e^{\widehat{\mathbf{A}}u_1}\,\mathbf{C}\,e^{\widehat{\mathbf{A}}u_2}\,\mathbf{C}\,{}^0\boldsymbol{\zeta}(t-u_1-u_2)\,du_1du_2\,{}^0\boldsymbol{\zeta}^*(t+\tau)\right\}\\ &= \int_0^\infty\int_0^\infty e^{\widehat{\mathbf{A}}u_1}\,\mathbf{E}\left\{\mathbf{C}\,e^{\widehat{\mathbf{A}}u_2}\,\mathbf{C}\right\}\,\mathbf{E}\left\{{}^0\boldsymbol{\zeta}(t-u_1-u_2)\,{}^0\boldsymbol{\zeta}^*(t+\tau)\right\}\,du_1du_2\\ &= \int_0^\infty\int_0^\infty e^{\widehat{\mathbf{A}}u_1}\,\mathbf{E}\left\{\mathbf{C}\,e^{\widehat{\mathbf{A}}u_2}\,\mathbf{C}\right\}\,\mathbf{R}_{{}^0\zeta{}^0\zeta}(\tau+u_1+u_2)\,du_1du_2\end{aligned}$$

and

$$\begin{aligned}\mathbf{R}_{{}^1\zeta{}^1\zeta}(\tau) &= \mathbf{E}\left\{{}^1\boldsymbol{\zeta}(t){}^1\boldsymbol{\zeta}^*(t+\tau)\right\}\\ &= \mathbf{E}\left\{\int_0^\infty e^{\widehat{\mathbf{A}}u}\,\mathbf{C}\,{}^0\boldsymbol{\zeta}(t-u)\,du\left(\int_0^\infty e^{\widehat{\mathbf{A}}u}\,\mathbf{C}\,{}^0\boldsymbol{\zeta}(t+\tau-u)\,du\right)^*\right\}\\ &= \int_0^\infty\int_0^\infty e^{\widehat{\mathbf{A}}u_1}\,\mathbf{E}\left\{\mathbf{C}\,{}^0\boldsymbol{\zeta}(t-u_1)\,{}^0\boldsymbol{\zeta}^*(t+\tau-u_2)\mathbf{C}^*\right\}e^{\widehat{\mathbf{A}}^*u_2}\;du_1du_2.\end{aligned}$$

Since for $t_1, t_2 \in \mathbb{R}$

$$\begin{aligned}\mathbf{E}\left\{\mathbf{C}\,{}^0\boldsymbol{\zeta}(t_1)\,{}^0\boldsymbol{\zeta}^*(t_2)\,\mathbf{C}^*\right\} &= \mathbf{E}\,\mathbf{E}\left\{\mathbf{C}\,{}^0\boldsymbol{\zeta}(t_1)\,{}^0\boldsymbol{\zeta}^*(t_2)\,\mathbf{C}^*\middle|\mathbf{C}\right\}\\ &= \mathbf{E}\left\{\mathbf{C}\,\mathbf{E}\left\{{}^0\boldsymbol{\zeta}(t_1)\,{}^0\boldsymbol{\zeta}^*(t_2)\middle|\mathbf{C}\right\}\,\mathbf{C}^*\right\}\\ &= \mathbf{E}\left\{\mathbf{C}\,\mathbf{E}\left\{{}^0\boldsymbol{\zeta}(t_1)\,{}^0\boldsymbol{\zeta}^*(t_2)\right\}\,\mathbf{C}^*\right\}\\ &= \mathbf{E}\left\{\mathbf{C}\,\mathbf{R}_{{}^0\zeta{}^0\zeta}(t_2-t_1)\,\mathbf{C}^*\right\}\end{aligned}$$

it follows

$$\mathbf{R}_{{}^1\zeta{}^1\zeta}(\tau) = \int_0^\infty\int_0^\infty e^{\widehat{\mathbf{A}}u_1}\,\mathbf{E}\left\{\mathbf{C}\,\mathbf{R}_{{}^0\zeta{}^0\zeta}(\tau+u_1-u_2)\mathbf{C}^*\right\}e^{\widehat{\mathbf{A}}^*u_2}\;du_1du_2.$$

□

Remark 3.11 If the distribution of the random matrix $\mathbf{C}$ is symmetric w.r.t. 0 then it can be shown, that there vanish all terms of the expansion for the correlation function corresponding to odd powers of η. In this case the remainder term in the above theorem is of order $o(\eta^3)$ instead of $o(\eta^2)$.

The above theorem allows an approximate computation of the correlation function of $\widetilde{\mathbf{z}}$ in terms of the second-order moments of the random matrix $\mathbf{C}$ and the correlation function of ${}^0\boldsymbol{\zeta}$ which is the stationary solution of the linear system ${}^0\dot{\boldsymbol{\zeta}} = \widehat{\mathbf{A}}{}^0\boldsymbol{\zeta} + \widetilde{\mathbf{f}}$. Here, the system matrix $\widehat{\mathbf{A}}$ is non-random and standard procedures for the computation of the correlation function of ${}^0\boldsymbol{\zeta}$ can be applied.

The correlation function of ${}^0\boldsymbol{\zeta}$ coincides with the correlation function of the stationary solution $\mathbf{y}$ of system (3.4) since the excitation terms differ only by the constant vector $\widehat{\mathbf{f}}$.

4 Computation of expansion terms

This section deals with procedures for an efficient computation of the expansion terms for the mean and the correlation function of the stationary solution $\mathbf{z}$. Based on the decomposition $\mathbf{z}(t,\omega) = \widehat{\mathbf{z}} + \widetilde{\mathbf{z}}(t,\omega)$ Theorem 3.2 shows that $\mathbf{E}\mathbf{z}(t) = \mathbf{E}\,\widehat{\mathbf{z}}$ and $\mathbf{R}_{\mathbf{zz}}(\tau) = \mathbf{cov}(\widehat{\mathbf{z}},\widehat{\mathbf{z}}) + \mathbf{R}_{\widetilde{\mathbf{z}}\widetilde{\mathbf{z}}}(\tau)$. For the moments on the right hand sides Theorems 3.4 and 3.10 provide the leading and the first non-zero correction terms of expansions in powers of η where the representation $\mathbf{A}(\omega) = \widehat{\mathbf{A}} + \eta\mathbf{C}(\omega)$ has been used.

While the correction terms of the expansions for $\mathbf{E}\,\widehat{\mathbf{z}}$ and $\mathbf{cov}(\widehat{\mathbf{z}},\widehat{\mathbf{z}})$ allow a straightforward computation as linear combinations of the covariances of the entries of $\mathbf{C}$ the situation in the case of $\mathbf{R}_{\widetilde{\mathbf{z}}\widetilde{\mathbf{z}}}(\tau)$ is more complicated. Here, the computation of the correction term requires the evaluation of double integrals containing the covariances of $\mathbf{C}$, the correlation function of $\widetilde{\mathbf{f}}$ and matrix exponentials of the form $e^{\widehat{\mathbf{A}}u}$.

A numerically efficient computation of these double integrals is possible for the special case of a diagonal matrix $\widehat{\mathbf{A}}$, i.e., $\widehat{\mathbf{A}} = \boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$, then it holds $e^{\widehat{\mathbf{A}}u} = \mathrm{diag}(e^{\lambda_1 u},\ldots,e^{\lambda_n u})$. In the general case Eq. (1.1) can be transformed into a system with a diagonal matrix provided $\widehat{\mathbf{A}}$ is diagonalizable. Using the substitution $\mathbf{z} = \mathbf{V}\mathbf{z}'$ where $\mathbf{V}$ is such that $\widehat{\mathbf{A}} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}$ from Eq. (1.1) it follows that $\mathbf{z}'$ satisfies

$$\begin{aligned} \dot{\mathbf{z}}' &= \mathbf{V}^{-1}(\widehat{\mathbf{A}} + \eta\mathbf{C})\mathbf{V}\mathbf{z}' + \mathbf{V}^{-1}\mathbf{f} \\ &= (\boldsymbol{\Lambda} + \eta\mathbf{C}')\mathbf{z}' + \mathbf{f}', \end{aligned} \tag{4.1}$$

where $\mathbf{C}' = \mathbf{V}^{-1}\mathbf{C}\mathbf{V}$ and $\mathbf{f}' = \mathbf{V}^{-1}\mathbf{f}$. The moments of $\mathbf{C}'$ and $\mathbf{f}'$ are obtained from the corresponding moments of $\mathbf{C}$ and $\mathbf{f}$, it holds

$$\begin{aligned} \mathbf{E}\mathbf{f}' &= \mathbf{V}^{-1}\,\mathbf{E}\mathbf{f}, \\ \mathbf{R}_{\mathbf{f}'\mathbf{f}'}(\tau) &= \mathbf{V}^{-1}\mathbf{R}_{\mathbf{ff}}(\tau)\mathbf{V}^{*^{-1}}, \\ \mathbf{E}\mathbf{C}' &= \mathbf{0} \quad \text{and} \\ \mathbf{E}\left\{\mathbf{C}'(\mathbf{C}')^*\right\} &= \mathbf{V}^{-1}\mathbf{E}\left\{\mathbf{C}\mathbf{V}\mathbf{V}^*\mathbf{C}^*\right\}\mathbf{V}^{*^{-1}} \end{aligned}$$

while the moments of $\mathbf{z}$ can be found from the moments of $\mathbf{z}'$ by

$$\mathbf{E}\mathbf{z} = \mathbf{V}\,\mathbf{E}\mathbf{z}' \quad \text{and} \quad \mathbf{R}_{\mathbf{zz}}(\tau) = \mathbf{V}\,\mathbf{R}_{\mathbf{z}'\mathbf{z}'}(\tau)\,\mathbf{V}^*.$$

Therefore the subsequent computation of expansion terms can be restricted to the case of a diagonal matrix $\widehat{\mathbf{A}}$.

Corollary 4.1
Let the assumptions of Theorem 2.2 applied to Eq. (4.1)

$$\dot{\mathbf{z}} = (\mathbf{\Lambda} + \eta \mathbf{C})\mathbf{z} + \mathbf{f},$$

be fulfilled. Then for $\eta < \min\{\eta_S, \eta_M\}$ *there hold the following expansions for* $\eta \downarrow 0$

$$\begin{aligned} \mathbf{E}\mathbf{z} &= \mathbf{E}\widehat{\mathbf{z}} = \Big(\mathbf{I} + \Big\{ \frac{1}{\lambda_i} \sum_{k=1}^{n} \frac{1}{\lambda_k} \mathbf{E}\left\{C_{ik}C_{kj}\right\} \Big\}_{ij} \eta^2 \Big)\mathbf{x} + o(\eta^2) \\ \textit{and} \quad \mathbf{cov}(\widehat{\mathbf{z}}, \widehat{\mathbf{z}}) &= \Big\{ \frac{1}{\lambda_i \overline{\lambda}_j} \sum_{k,l=1}^{n} \mathbf{E}\left\{C_{ik}\overline{C}_{jl}\right\} x_k \overline{x}_l \Big\}_{ij} \eta^2 + o(\eta^2), \end{aligned}$$

where $\mathbf{x} = -\mathbf{\Lambda}^{-1}\,\widehat{\mathbf{f}}$.

Proof.
Applying Theorem 3.4 to Eq. (4.1) it follows

$$\begin{aligned} \mathbf{E}\mathbf{z} = \mathbf{E}\widehat{\mathbf{z}} &= \left(\mathbf{I} + \mathbf{\Lambda}^{-1}\,\mathbf{E}\left\{\mathbf{C}\mathbf{\Lambda}^{-1}\mathbf{C}\right\}\eta^2\right)\mathbf{x} + o(\eta^2) \\ \text{and} \quad \mathbf{cov}(\widehat{\mathbf{z}}, \widehat{\mathbf{z}}) &= \mathbf{\Lambda}^{-1}\mathbf{E}\left\{\mathbf{C}\,\mathbf{x}\,\mathbf{x}^*\mathbf{C}^*\right\}\mathbf{\Lambda}^{*^{-1}}\eta^2 + o(\eta^2), \end{aligned}$$

from which the assertions follow immediately. □

For the correlation function of $\widetilde{\mathbf{z}}$ Theorem 3.10 gives for $\eta < \eta_S$ and $\tau \in \mathbb{R}$ the expansion for $\eta \downarrow 0$

$$\begin{aligned} \mathbf{R}_{\widetilde{\mathbf{z}}\widetilde{\mathbf{z}}}(\tau) &= \mathbf{R}_{0\boldsymbol{\zeta}0\boldsymbol{\zeta}}(\tau) + (\mathbf{R}_{2\boldsymbol{\zeta}0\boldsymbol{\zeta}}(\tau) + \mathbf{R}_{1\boldsymbol{\zeta}1\boldsymbol{\zeta}}(\tau) + \mathbf{R}_{0\boldsymbol{\zeta}2\boldsymbol{\zeta}}(\tau))\eta^2 + o(\eta^2), \\ \text{where} \quad \mathbf{R}_{2\boldsymbol{\zeta}0\boldsymbol{\zeta}}(\tau) &= \int_0^\infty \int_0^\infty e^{\mathbf{\Lambda}u_1}\mathbf{E}\left\{\mathbf{C}e^{\mathbf{\Lambda}u_2}\mathbf{C}\right\}\mathbf{R}_{0\boldsymbol{\zeta}0\boldsymbol{\zeta}}(\tau + u_1 + u_2)\,du_1 du_2, \\ \mathbf{R}_{1\boldsymbol{\zeta}1\boldsymbol{\zeta}}(\tau) &= \int_0^\infty \int_0^\infty e^{\mathbf{\Lambda}u_1}\mathbf{E}\left\{\mathbf{C}\,\mathbf{R}_{0\boldsymbol{\zeta}0\boldsymbol{\zeta}}(\tau + u_1 - u_2)\,\mathbf{C}^*\right\}e^{\mathbf{\Lambda}^*u_2}\,du_1 du_2 \\ \text{and} \quad \mathbf{R}_{0\boldsymbol{\zeta}2\boldsymbol{\zeta}}(\tau) &= \mathbf{R}^*_{2\boldsymbol{\zeta}0\boldsymbol{\zeta}}(-\tau). \end{aligned}$$

For the explicit computation of the expansion terms it is necessary to prescribe a certain form of the correlation function $\mathbf{R}_{0\boldsymbol{\zeta}0\boldsymbol{\zeta}}(\tau)$. Here it is assumed that it holds

$$R_{0\boldsymbol{\zeta}0\boldsymbol{\zeta}}(\tau) = \sum_{r=1}^{m} \boldsymbol{\rho}_r(\tau) \quad \text{with} \quad \boldsymbol{\rho}_r(\tau) = \begin{cases} \mathbf{Q}_r\,e^{\mathbf{\Pi}_r\tau} & \text{for } \tau \geq 0 \\ e^{-\mathbf{\Pi}_r^*\tau}\mathbf{Q}_r^* & \text{for } \tau < 0 \end{cases} \tag{4.2}$$

where $m \in \mathbb{N}$ and for $r = 1, \ldots, m$ the $\mathbf{Q}_r$ are some non-negative definite and Hermitian $n \times n$-matrices and $\mathbf{\Pi}_r = \operatorname{diag}(\pi_{r1}, \ldots, \pi_{rn})$ with $\operatorname{Re}[\pi_{ri}] < 0$.

This type of correlation function arises e.g. if the correlation function of $\widetilde{\mathbf{f}}$ is chosen to be $\mathbf{R}_{\widetilde{\mathbf{f}}\widetilde{\mathbf{f}}}(\tau) = \mathbf{L}\, e^{\mathbf{\Gamma}\tau},\ \tau \geq 0$, where $\mathbf{L}$ is some non-negative definite and Hermitian matrix and $\mathbf{\Gamma} = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$ with $\mathrm{Re}\,[\gamma_i] < 0$, $i = 1\ldots, n$. Then for $\tau \geq 0$ it can be derived

$$\mathbf{R}_{{}^0\boldsymbol{\zeta}{}^0\boldsymbol{\zeta}}(\tau) = \mathbf{Q}_1 e^{\mathbf{\Lambda}^*\tau} + \mathbf{Q}_2 e^{\mathbf{\Gamma}\tau},$$

with some matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ depending on $\mathbf{L}, \mathbf{\Lambda}$ and $\mathbf{\Gamma}$ whose sum $\mathbf{Q}_1 + \mathbf{Q}_2$ forms the covariance matrix of ${}^0\boldsymbol{\zeta}$.

It is noticed that for a δ-correlated excitation $\widetilde{\mathbf{f}}$ with the correlation function $\mathbf{R}_{\widetilde{\mathbf{f}}\widetilde{\mathbf{f}}}(\tau) = \mathbf{L}\,\delta(\tau)$ one obtains for $\tau \geq 0$

$$\mathbf{R}_{{}^0\boldsymbol{\zeta}{}^0\boldsymbol{\zeta}}(\tau) \quad = \quad \mathbf{Q}\, e^{\mathbf{\Lambda}^*\tau} \quad \text{with} \qquad \mathbf{Q} = \left\{ \frac{-L_{ij}}{\lambda_i + \overline{\lambda}_j} \right\}_{i,j=1,\ldots,n}$$

and for a weakly correlated excitation the above exponential type correlation function arises in the terms of the expansion of $\mathbf{R}_{{}^0\boldsymbol{\zeta}\boldsymbol{\zeta}}(\tau)$ in powers of the correlation length (see [17] and [7, 8]).

The subsequent evaluations of the entries of the matrix-valued functions $\mathbf{R}_{{}^2\boldsymbol{\zeta}{}^0\boldsymbol{\zeta}}(\tau)$, $\mathbf{R}_{{}^1\boldsymbol{\zeta}{}^1\boldsymbol{\zeta}}(\tau)$ and $\mathbf{R}_{{}^0\boldsymbol{\zeta}{}^2\boldsymbol{\zeta}}(\tau)$ are given for $\tau \geq 0$. For negative τ the property $\mathbf{R}_{\boldsymbol{\xi}_1\boldsymbol{\xi}_2}(\tau) = \mathbf{R}^*_{\boldsymbol{\xi}_2\boldsymbol{\xi}_1}(-\tau)$ can be used.

It holds for $i, j = 1, \ldots, n$

$$\begin{aligned}
R_{{}^2\zeta_i{}^0\zeta_j}(\tau) &= \int_0^\infty \int_0^\infty \sum_{l=1}^n e^{\lambda_i u_1} \left\{\mathbf{E}\left\{\mathbf{C} e^{\mathbf{\Lambda} u_2}\mathbf{C}\right\}\right\}_{il} \sum_{r=1}^m \rho_{rlj}(\tau + u_1 + u_2,)\, du_1 du_2 \\
&= \int_0^\infty \int_0^\infty \sum_{l=1}^n e^{\lambda_i u_1} \sum_{k=1}^n \mathbf{E}\left\{C_{ik}C_{kl}\right\} e^{\lambda_k u_2} \sum_{r=1}^m \rho_{rlj}(\tau + u_1 + u_2)\, du_1 du_2 \\
&= \sum_{r=1}^m \sum_{k,l=1}^n \mathbf{E}\left\{C_{ik}C_{kl}\right\} J_{1,rijkl}(\tau), \\
\text{where} \quad J_{1,rijkl}(\tau) &= \int_0^\infty \int_0^\infty e^{\lambda_i u_1 + \lambda_k u_2}\ \rho_{rlj}(\tau + u_1 + u_2)\, du_1 du_2, \\
R_{{}^0\zeta_i{}^2\zeta_j}(\tau) &= \overline{R_{{}^2\zeta_j{}^0\zeta_i}(-\tau)} = \sum_{r=1}^m \sum_{k,l=1}^n \mathbf{E}\left\{\overline{C}_{jk}\overline{C}_{kl}\right\} \overline{J_{1,rjikl}(-\tau)}
\end{aligned}$$

$$\begin{aligned}
\text{and } R_{{}^1\zeta_i{}^1\zeta_j}(\tau) &= \int_0^\infty\int_0^\infty e^{\lambda_i u_1}\left[\mathbf{E}\left\{\mathbf{C}\sum_{r=1}^{m}\boldsymbol{\rho}_r(\tau+u_1-u_2)\mathbf{C}^*\right\}\right]_{ij} e^{\overline{\lambda}_j u_2}\,du_1du_2 \\
&= \int_0^\infty\int_0^\infty e^{\lambda_i u_1}\sum_{k,l=1}^{n}\mathbf{E}\left\{C_{ik}\overline{C}_{jl}\right\}\sum_{r=1}^{m}\rho_{rkl}(\tau+u_1-u_2)\, e^{\overline{\lambda}_j u_2}du_1du_2 \\
&= \sum_{r=1}^{m}\sum_{k,l=1}^{n}\mathbf{E}\left\{C_{ik}\overline{C}_{jl}\right\} J_{2,rijkl}(\tau), \\
\text{where}\quad J_{2,rijkl}(\tau) &= \int_0^\infty\int_0^\infty e^{\lambda_i u_1+\overline{\lambda}_j u_2}\ \rho_{rkl}(\tau+u_1-u_2)\,du_1du_2.
\end{aligned}$$

Using that the entries of $\boldsymbol{\rho}_r(\tau)=\{\rho_{rij}(\tau)\}_{i,j=1,\ldots,n}$ given in (4.2) can be represented as

$$\rho_{rij}(\tau)=\begin{cases} Q_{rij}\ e^{\pi_{rj}\tau} & \text{for } \tau\geq 0 \\ \overline{Q}_{rji}\ e^{-\overline{\pi}_{ri}\tau} & \text{for } \tau<0 \end{cases}.$$

the terms J_1 and J_2 can be computed explicitly. First, J_1 is evaluated, it holds

$$\begin{aligned}
J_{1,rijkl}(\tau) &= \int_0^\infty\int_0^\infty e^{\lambda_i u_1+\lambda_k u_2}\ \rho_{rlj}(\tau+u_1+u_2)\,du_1du_2 \\
&= Q_{rlj}\int_0^\infty\int_0^\infty e^{\lambda_i u_1+\lambda_k u_2+\pi_{rj}(\tau+u_1+u_2)}\,du_1du_2 \\
&= Q_{rlj}\ e^{\pi_{rj}\tau}\int_0^\infty e^{(\lambda_i+\pi_{rj})u_1}\,du_1\int_0^\infty e^{(\lambda_k+\pi_{rj})u_2}\,du_2, \\
J_{1,rijkl}(\tau) &= \frac{Q_{rlj}}{(\lambda_i+\pi_{rj})(\lambda_k+\pi_{rj})}\ e^{\pi_{rj}\tau}. \qquad (4.3)
\end{aligned}$$

Here, the stability of $\boldsymbol{\Lambda}$, i.e., $\mathrm{Re}\,[\lambda_i]<0$, $\mathrm{Re}\,[\pi r i_i]<0$ for all r,i and the property $\tau+u_1+u_2\geq 0$ for $\tau\geq 0$ has been used.

Next, $J_{1,rijkl}(-\tau)$ is evaluated. In this case it is necessary to split the integral according to the sign of the argument of ρ. The substitution $v = u_1$ and $w = -\tau + u_1 + u_2$ leads to

$$\begin{aligned}
J_{1,rijkl}(-\tau) &= \int_0^\infty \int_{v-\tau}^\infty e^{\lambda_i v + \lambda_k(\tau + w - v)}\ \rho_{rlj}(w)\, dw\, dv \\
&= e^{\lambda_k \tau} \int_0^\infty e^{(\lambda_i - \lambda_k)v} \int_{v-\tau}^\infty e^{\lambda_k w}\ \rho_{rlj}(w)\, dw\, dv \\
&= e^{\lambda_k \tau}(I_{1,rijkl}(\tau) + I_{2,rijkl}(\tau) + I_{3,rijkl}(\tau))
\end{aligned}$$

where

$$\begin{aligned}
I_{1,rijkl}(\tau) &= \int_0^\tau e^{(\lambda_i - \lambda_k)v} \int_{v-\tau}^0 e^{\lambda_k w}\ \rho_{rlj}(w)\, dw\, dv \\
I_{2,rijkl}(\tau) &= \int_0^\tau e^{(\lambda_i - \lambda_k)v} \int_0^\infty e^{\lambda_k w}\ \rho_{rlj}(w)\, dw\, dv \\
I_{3,rijkl}(\tau) &= \int_\tau^\infty e^{(\lambda_i - \lambda_k)v} \int_{v-\tau}^\infty e^{\lambda_k w}\ \rho_{rlj}(w)\, dw\, dv.
\end{aligned}$$

For I_1 it follows by using the substitution $w' = -w$ and $\rho_{rlj}(-w) = \overline{\rho_{rjl}(w)} = \overline{Q}_{rjl}\ e^{\overline{\pi}_{rl}\, w}$

$$\begin{aligned}
I_{1,rijkl}(\tau) &= \int_0^\tau e^{(\lambda_i - \lambda_k)v} \int_0^{\tau - v} e^{-\lambda_k w'}\ \overline{\rho_{rjl}(w')}\, dw'\, dv \\
&= \overline{Q}_{rjl} \int_0^\tau e^{(\lambda_i - \lambda_k)v} \int_0^{\tau - v} e^{(\overline{\pi}_{rl} - \lambda_k)w'}\, dw'\, dv \\
&= \overline{Q}_{rjl} \int_0^\tau e^{(\lambda_i - \lambda_k)v} G(\tau - v, \overline{\pi}_{rl}, \lambda_k)\, dv
\end{aligned}$$

with

$$G(s, \alpha, \beta) = \int_0^s e^{(\alpha - \beta)v}\, dv = \begin{cases} \frac{1}{\alpha - \beta}\left(e^{(\alpha-\beta)s} - 1\right) & \text{for } \alpha \neq \beta \\ s & \text{for } \alpha = \beta. \end{cases} \tag{4.4}$$

For $\lambda_k \neq \overline{\pi}_{rl}$ then it holds

$$\begin{aligned}
I_{1,rijkl}(\tau) &= \frac{\overline{Q}_{rjl}}{\overline{\pi}_{rl} - \lambda_k} \int_0^\tau e^{(\lambda_i - \lambda_k)v} \left(e^{(\overline{\pi}_{rl} - \lambda_k)(\tau - v)} - 1\right) dv \\
&= \frac{\overline{Q}_{rjl}}{\overline{\pi}_{rl} - \lambda_k} \left[e^{(\overline{\pi}_{rl} - \lambda_k)\tau} \int_0^\tau e^{(\lambda_i - \overline{\pi}_{rl})v}\, dv - \int_0^\tau e^{(\lambda_i - \lambda_k)v}\, dv\right] \\
&= \frac{\overline{Q}_{rjl}}{\overline{\pi}_{rl} - \lambda_k} \left[e^{(\overline{\pi}_{rl} - \lambda_k)\tau} G(\tau, \lambda_i, \overline{\pi}_{rl}) - G(\tau, \lambda_i, \lambda_k)\right].
\end{aligned}$$

In case of $\lambda_k = \overline{\pi}_{rl}$ it follows $I_{1,rijkl}(\tau) = \overline{Q}_{rjl} \int\limits_0^\tau e^{(\lambda_i-\lambda_k)v}(\tau - v)\, dv$.

If moreover $\lambda_k = \lambda_i$ then it holds $I_{1,rijkl}(\tau) = \overline{Q}_{rjl} \int\limits_0^\tau (\tau - v)\, dv = \overline{Q}_{rjl}\, \frac{\tau^2}{2}$ while for $\lambda_k \neq \lambda_i$ it follows

$$\begin{aligned}
I_{1,rijkl}(\tau) = \overline{Q}_{rjl}\Big[\frac{\tau}{\lambda_i-\lambda_k}\big(e^{(\lambda_i-\lambda_k)\tau}-1\big) - \int\limits_0^\tau e^{(\lambda_i-\lambda_k)v}\; v\, dv\Big] \\
= \overline{Q}_{rjl}\Big[\frac{\tau}{\lambda_i-\lambda_k}\big(e^{(\lambda_i-\lambda_k)\tau}-1\big) - \frac{e^{(\lambda_i-\lambda_k)v}}{\lambda_i-\lambda_k}\, v\Big|_0^\tau + \frac{1}{\lambda_i-\lambda_k}\int\limits_0^\tau e^{(\lambda_i-\lambda_k)v}\, dv\Big] \\
= \frac{\overline{Q}_{rjl}}{\lambda_i-\lambda_k}\Big[\tau\big(e^{(\lambda_i-\lambda_k)\tau}-1\big) - e^{(\lambda_i-\lambda_k)\tau}\,\tau + \frac{1}{\lambda_i-\lambda_k}\big(e^{(\lambda_i-\lambda_k)\tau}-1\big)\Big] \\
= \frac{\overline{Q}_{rjl}}{\lambda_i-\lambda_k}(G(\tau,\lambda_i,\lambda_k)-\tau).
\end{aligned}$$

Summarizing, the integral I_1 can be represented as

$$I_{1,rijkl}(\tau) = \begin{cases} \frac{\overline{Q}_{rjl}}{\overline{\pi}_{rl}-\lambda_k}\big(e^{(\overline{\pi}_{rl}-\lambda_k)\tau} G(\tau,\lambda_i,\overline{\pi}_{rl}) - G(\tau,\lambda_i,\lambda_k)\big) & \text{for } \lambda_k \neq \overline{\pi}_{rl} \\ \frac{\overline{Q}_{rjl}}{\lambda_i-\lambda_k}(G(\tau,\lambda_i,\lambda_k)-\tau) & \text{for } \lambda_k = \overline{\pi}_{rl},\ \lambda_k \neq \lambda_i \\ \overline{Q}_{rjl}\,\frac{\tau^2}{2} & \text{for } \lambda_k = \overline{\pi}_{rl} = \lambda_i \end{cases} . \tag{4.5}$$

The integral I_2 can be evaluated as follows

$$\begin{aligned}
I_{2,rijkl}(\tau) &= \int\limits_0^\tau e^{(\lambda_i-\lambda_k)v}\int\limits_0^\infty e^{\lambda_k w}\; \rho_{rlj}(w)\, dw\, dv \\
&= Q_{rlj}\int\limits_0^\tau e^{(\lambda_i-\lambda_k)v}\int\limits_0^\infty e^{(\lambda_k+\pi_{rj})w}\, dw\, dv = \frac{-Q_{rlj}}{\lambda_k+\pi_{rj}}\; G(\tau,\lambda_i,\lambda_k)
\end{aligned}$$

and for I_3 one obtaines

$$\begin{aligned}
I_{3,rijkl}(\tau) = \int\limits_\tau^\infty e^{(\lambda_i-\lambda_k)v}\int\limits_{v-\tau}^\infty e^{\lambda_k w}\; \rho_{rlj}(w)\, dw\, dv \\
= Q_{rlj}\int\limits_\tau^\infty e^{(\lambda_i-\lambda_k)v}\int\limits_{v-\tau}^\infty e^{(\lambda_k+\pi_{rj})w}\, dw\, dv \\
= \frac{-Q_{rlj}}{\lambda_k+\pi_{rj}}\int\limits_\tau^\infty e^{(\lambda_i-\lambda_k)v}e^{(\lambda_k+\pi_{rj})(v-\tau)}\, dv = \frac{-Q_{rlj}e^{-(\lambda_k+\pi_{rj})\tau}}{\lambda_k+\pi_{rj}}\int\limits_\tau^\infty e^{(\lambda_i+\pi_{rj})v}\, dv \\
= \frac{Q_{rlj}}{(\lambda_k+\pi_{rj})(\lambda_i+\pi_{rj})}\; e^{(\lambda_i-\lambda_k)\tau}.
\end{aligned}$$

Summarizing, the integral $J_{1,rijkl}(-\tau)$ is given by

$$\begin{aligned} J_{1,rijkl}(-\tau) &= e^{\lambda_k\tau}(I_{1,rijkl}(\tau)+I_{2,rijkl}(\tau)+I_{3,rijkl}(\tau)) \\ &= e^{\lambda_k\tau}\Big(I_{1,rijkl}(\tau)+\frac{Q_{rlj}}{\lambda_k+\pi_{rj}}\Big(\frac{e^{(\lambda_i-\lambda_k)\tau}}{\lambda_i+\pi_{rj}}-G(\tau,\lambda_i,\lambda_k)\Big)\Big), \end{aligned} \tag{4.6}$$

where I_1 is defined in (4.5) and G in (4.4).

Finally, $J_{2,rijkl}(\tau)$ is evaluated, it holds

$$J_{2,rijkl}(\tau)=\int_0^\infty\int_0^\infty e^{\lambda_i u_1+\overline{\lambda}_j u_2}\,\rho_{rkl}(\tau+u_1-u_2)\,du_1du_2.$$

As in the case of the evaluation of J_1 for negative τ it is necessary to split the integral according to the sign of the argument of ρ. The substitution $v=u_1$ and $w=\tau+u_1-u_2$ leads to

$$\begin{aligned} J_{2,rijkl}(\tau) &= \int_0^\infty\int_{-\infty}^{v+\tau} e^{\lambda_i v+\overline{\lambda}_j(\tau+v-w)}\,\rho_{rkl}(w)\,dw\,dv \\ &= e^{\overline{\lambda}_j\tau}\int_0^\infty e^{(\lambda_i+\overline{\lambda}_j)v}\int_{-\infty}^{v+\tau} e^{-\overline{\lambda}_j w}\,\rho_{rkl}(w)\,dw\,dv \\ &= e^{\overline{\lambda}_j\tau}(I_{4,rijkl}(\tau)+I_{5,rijkl}(\tau)) \\ \text{where}\quad I_{4,rijkl}(\tau) &= \int_0^\infty e^{(\lambda_i+\overline{\lambda}_j)v}\int_{-\infty}^{0} e^{-\overline{\lambda}_j w}\,\rho_{rkl}(w)\,dw\,dv \\ I_{5,rijkl}(\tau) &= \int_0^\infty e^{(\lambda_i+\overline{\lambda}_j)v}\int_{0}^{v+\tau} e^{-\overline{\lambda}_j w}\,\rho_{rkl}(w)\,dw\,dv. \end{aligned}$$

Using $\rho_{rkl}(w)=\overline{\rho_{rlk}(-w)}=\overline{Q}_{rlk}\,e^{-\overline{\pi}_{rk}w}$ for $w<0$ it follows

$$\begin{aligned} \int_{-\infty}^{0} e^{-\overline{\lambda}_j w}\,\rho_{rkl}(w)\,dw &= \overline{Q}_{rlk}\int_{-\infty}^{0} e^{-(\overline{\lambda}_j+\overline{\pi}_{rk})w}\,dw=\frac{-\overline{Q}_{rlk}}{\overline{\lambda}_j+\overline{\pi}_{rk}} \\ \text{and}\quad I_{4,rijkl}(\tau) &= \int_0^\infty e^{(\lambda_i+\overline{\lambda}_j)v}\int_{-\infty}^{0} e^{-\overline{\lambda}_j w}\,\rho_{rkl}(w)\,dw\,dv \\ &= \frac{\overline{Q}_{rlk}}{(\lambda_i+\overline{\lambda}_j)(\overline{\pi}_{rk}+\overline{\lambda}_j)}. \end{aligned}$$

For the evaluation of I_5 the substitution of $\rho_{rkl}(w) = Q_{rkl}\, e^{\pi_{rl} w}$ for $w \geq 0$ yields

$$\begin{aligned}\int\limits_0^{v+\tau} e^{-\overline{\lambda}_j w}\ \rho_{rkl}(w)\, dw &= Q_{rkl} \int\limits_0^{v+\tau} e^{(\pi_{rl}-\overline{\lambda}_j)w}\, dw \\ &= \begin{cases} Q_{rkl} \frac{1}{\pi_{rl}-\overline{\lambda}_j} \big(e^{(\pi_{rl}-\overline{\lambda}_j)(v+\tau)} - 1 \big) & \text{for } \pi_{rl} \neq \overline{\lambda}_j \\ Q_{rkl}(v+\tau) & \text{for } \pi_{rl} = \overline{\lambda}_j. \end{cases}\end{aligned}$$

For $\pi_{rl} \neq \overline{\lambda}_j$ then it can be derived

$$\begin{aligned} I_{5,rijkl}(\tau) &:= \int\limits_0^{\infty} e^{(\lambda_i+\overline{\lambda}_j)v} \int\limits_0^{v+\tau} e^{-\overline{\lambda}_j w}\ \rho_{rkl}(w)\, dw\, dv \\ &= \frac{Q_{rkl}}{\pi_{rl}-\overline{\lambda}_j} \Big[e^{(\pi_{rl}-\overline{\lambda}_j)\tau} \int\limits_0^{\infty} e^{(\lambda_i+\pi_{rl})v}\, dv - \int\limits_0^{\infty} e^{(\lambda_i+\overline{\lambda}_j)v}\, dv \Big] \\ &= \frac{Q_{rkl}}{\pi_{rl}-\overline{\lambda}_j} \Big[\frac{1}{\lambda_i+\overline{\lambda}_j} - \frac{1}{\lambda_i+\pi_{rl}}\ e^{(\pi_{rl}-\overline{\lambda}_j)\tau} \Big] \end{aligned}$$

while for $\pi_{rl} = \overline{\lambda}_j$ it holds

$$\begin{aligned} I_{5,rijkl}(\tau) &= Q_{rkl} \int\limits_0^{\infty} e^{(\lambda_i+\overline{\lambda}_j)v} (v+\tau)\, dv \\ &= Q_{rkl} \left(\int\limits_0^{\infty} e^{(\lambda_i+\overline{\lambda}_j)v}\, v\, dv - \frac{\tau}{\lambda_i+\overline{\lambda}_j} \right) \\ &= Q_{rkl} \Big(\frac{1}{(\lambda_i+\overline{\lambda}_j)^2} - \frac{\tau}{\lambda_i+\overline{\lambda}_j} \Big) = \frac{Q_{rkl}}{\lambda_i+\overline{\lambda}_j} \Big(\frac{1}{\lambda_i+\overline{\lambda}_j} - \tau \Big). \end{aligned}$$

Summarizing it follows

$$\begin{aligned} J_{2,rijkl}(\tau) &= e^{\overline{\lambda}_j\tau} (I_{4,rijkl}(\tau) + I_{5,rijkl}(\tau)) \\ &= e^{\overline{\lambda}_j\tau} \Big(\frac{\overline{Q}_{rlk}}{(\lambda_i+\overline{\lambda}_j)(\overline{\pi}_{rk}+\overline{\lambda}_j)} + I_{5,rijkl}(\tau) \Big) \qquad (4.7) \\ \text{with} \quad I_{5,rijkl}(\tau) &= \begin{cases} \frac{Q_{rkl}}{\pi_{rl}-\overline{\lambda}_j} \Big(\frac{1}{\lambda_i+\overline{\lambda}_j} - \frac{1}{\lambda_i+\pi_{rl}}\ e^{(\pi_{rl}-\overline{\lambda}_j)\tau} \Big) & \text{for } \pi_{rl} \neq \overline{\lambda}_j \\ \frac{Q_{rkl}}{\lambda_i+\overline{\lambda}_j} \Big(\frac{1}{\lambda_i+\overline{\lambda}_j} - \tau \Big) & \text{for } \pi_{rl} = \overline{\lambda}_j \end{cases}. \end{aligned}$$

Corollary 4.2
Let the assumptions of Theorem 3.10 applied to

$$\dot{\widetilde{\mathbf{z}}} = (\mathbf{\Lambda} + \eta \mathbf{C})\widetilde{\mathbf{z}} + \widetilde{\mathbf{f}},$$

be fulfilled and $\mathbf{R}_{{}^0\boldsymbol{\zeta}{}^0\boldsymbol{\zeta}}(\tau)$ *be of the form (4.2). Then the correlation function of the stationary solution* $\widetilde{\mathbf{z}}$ *for* $\eta < \eta_S$ *and* $\tau \geq 0,\ i,j = 1,\ldots,n$*, possesses the expansion for* $\eta \downarrow 0$

$$\begin{aligned} R_{\widetilde{z}_i\widetilde{z}_j}(\tau) &= \sum_{r=1}^{m}\Big[Q_{rij}\, e^{\pi_{rj}\tau} + \eta^2 \sum_{k,l=1}^{n}\Big(\mathbf{E}\left\{C_{ik}C_{kl}\right\} J_{1,rijkl}(\tau) \\ &\quad + \mathbf{E}\left\{\overline{C}_{jk}\overline{C}_{kl}\right\} \overline{J_{1,rjikl}(-\tau)} + \mathbf{E}\left\{C_{ik}\overline{C}_{jl}\right\} J_{2,rijkl}(\tau)\Big)\Big] + o(\eta^2), \end{aligned}$$

where J_1 *and* J_2 *are given in (4.3), (4.6) and (4.7).*

5 Numerical results

In this section the results of the preceding sections are applied to the special case of a real and scalar equation (1.1) which arises for $n = 1$ and real-valued random parameters. Thus $\mathbf{A}(\omega) = a(\omega)$ is a real-valued random variable and the excitation $f(t,\omega)$ is a scalar real-valued random process. As before the random parameters are decomposed, i.e.,

$$a(\omega) = \widehat{a} + \eta c(\omega) \quad \text{and} \quad f(t,\omega) = \widehat{f} + \widetilde{f}(t,\omega),$$

with $\mathbf{E}a = \widehat{a}$, $\mathbf{E}f(t) = \widehat{f}$, a centered random variable $c(\omega)$ and a centered process $\widetilde{f}(t,\omega)$. Then Eq. (1.1) reads as

$$\dot{z}(t,\omega) = (\widehat{a} + \eta c(\omega))\, z(t,\omega) + f(t,\omega). \tag{5.1}$$

The random variable c is assumed to be a.s. bounded, i.e., $|c(\omega)| \leq c_0$ where for convenience it is set $c_0 = 1$. The stability condition for $\mathbf{A}(\omega) = a(\omega) = \widehat{a} + \eta c(\omega)$ is satisfied for $a(\omega) < 0$ a.s., i.e., for $\eta < \eta_S = \frac{-\widehat{a}}{c_0} = -\widehat{a}$. In this case Eq. (5.1) possesses the unique stationary solution

$$\begin{aligned} z(t,\omega) &= \int_0^\infty e^{a(\omega)u}\, f(t-u,\omega)\,du = \widehat{z}(\omega) + \widetilde{z}(t,\omega) \\ \text{with} \quad \widehat{z}(\omega) &= \int_0^\infty e^{a(\omega)u}\, \widehat{f}\,du = -a^{-1}(\omega)\,\widehat{f} \\ \text{and} \quad \widetilde{z}(t,\omega) &= \int_0^\infty e^{a(\omega)u}\, \widetilde{f}(t-u,\omega)\,du. \end{aligned}$$

Due to Theorem 3.2 the mean and the correlation function of z are given by

$$\begin{aligned} \mathbf{E}\left\{z(t)\right\} &= \mathbf{E}\left\{\widehat{z}\right\} = -\mathbf{E}\left\{a^{-1}\right\}\widehat{f} \\ \text{and} \quad R_{zz}(\tau) &= \mathbf{D^2}\,\widehat{z} + R_{\widetilde{z}\widetilde{z}}(\tau), \end{aligned} \tag{5.2}$$

where $\mathbf{D^2}\,\widehat{z} = \mathbf{cov}(\widehat{z},\widehat{z})$ denotes the variance of $\widehat{z}$.

In the considered special case it is possible to compute these moments explicitly in terms of the mean and the correlation function of f and the distribution of c. In order to perform numerical experiments which compare these exact moments with approximations from the perturbation approach it is assumed that c is uniformly distributed on $[-1,1]$, then a is uniformly distributed on $[\widehat{a}-\eta, \widehat{a}+\eta]$ and we have the following probability density functions

$$p_c(s) = \begin{cases} \frac{1}{2} & \text{for } s \in [-1,1] \\ 0 & \text{else} \end{cases} \quad \text{and} \quad p_a(s) = \begin{cases} \frac{1}{2\eta} & \text{for } s \in [\widehat{a}-\eta, \widehat{a}+\eta] \\ 0 & \text{else} \end{cases}.$$

Moreover it is assumed that f possesses the exponentially decaying correlation function

$$R_{ff}(\tau) = R_{\widetilde{f}\widetilde{f}}(\tau) = \sigma^2 \, e^{\gamma|\tau|}, \quad \sigma > 0, \gamma < 2\widehat{a},$$

where σ denotes the standard deviation of f and γ is a parameter describing the decay of the correlation function of f. It is noted that we impose instead of $\gamma < 0$ the stronger condition $\gamma < 2\widehat{a}$ in order to simplify the subsequent calculations. This condition ensures that γ does not coincide with the parameter $a \in [\widehat{a}-\eta, \widehat{a}+\eta] \subset (2\widehat{a}, 0)$ since $\eta < \eta_S = -\widehat{a}$.

First- and second-order moments of $\widehat{z}$ exist if $\mathbf{E}\left\{a^{-2}\right\} < \infty$. This condition is fulfilled for $\eta < \eta_M = -\widehat{a} = \eta_S$ since

$$\mathbf{E}\left\{a^{-2}\right\} = \int_{\widehat{a}-\eta}^{\widehat{a}+\eta} \frac{1}{s^2}\, p_a(s) ds \leq \int_{\widehat{a}-\eta}^{\widehat{a}+\eta} \frac{1}{(\widehat{a}+\eta)^2}\, p_a(s) ds = \frac{1}{(\widehat{a}+\eta)^2}.$$

If a is uniformly distributed on $[\widehat{a}-\eta, \widehat{a}+\eta]$ for $\eta < \eta_M$ it holds

$$\begin{aligned} \mathbf{E}\widehat{z} &= -\mathbf{E}\left\{a^{-1}\right\} \widehat{f} = -\widehat{f} \int_{\widehat{a}-\eta}^{\widehat{a}+\eta} \frac{1}{s}\, p_a(s) ds = -\frac{\widehat{f}}{2\eta} \int_{\widehat{a}-\eta}^{\widehat{a}+\eta} \frac{1}{s} ds \\ &= \frac{\widehat{f}}{2\eta} \ln \frac{\widehat{a}-\eta}{\widehat{a}+\eta}, \end{aligned} \tag{5.3}$$

$$\begin{aligned} \mathbf{E}\widehat{z}^2 &= \widehat{f}^2 \int_{\widehat{a}-\eta}^{\widehat{a}+\eta} \frac{1}{s^2}\, p_a(s) ds = \frac{\widehat{f}^2}{2\eta} \int_{\widehat{a}-\eta}^{\widehat{a}+\eta} \frac{1}{s^2} ds \\ &= \frac{\widehat{f}^2}{2\eta} \Big(\frac{1}{\widehat{a}-\eta} - \frac{1}{\widehat{a}+\eta} \Big) = \frac{\widehat{f}^2}{\widehat{a}^2 - \eta^2}, \end{aligned}$$

$$\text{and} \quad \mathbf{D^2}\,\widehat{z} = \mathbf{E}\widehat{z}^2 - (\mathbf{E}\widehat{z})^2 = \widehat{f}^2 \Big(\frac{1}{\widehat{a}^2-\eta^2} - \frac{1}{4\eta^2} \ln^2 \frac{\widehat{a}-\eta}{\widehat{a}+\eta} \Big). \tag{5.4}$$

For the correlation function of $\widetilde{z}$ it holds $R_{\widetilde{z}\widetilde{z}}(\tau) = R_{\widetilde{z}\widetilde{z}}(-\tau)$ and for $\tau \geq 0$ it can be evaluated as follows

$$\begin{aligned} R_{\widetilde{z}\widetilde{z}}(\tau) &= \mathbf{E}\left\{\widetilde{z}(t)\widetilde{z}(t+\tau)\right\} = \mathbf{E}\left\{ \int_0^\infty \int_0^\infty e^{au_1} \widetilde{f}(t-u_1) e^{au_2} \widetilde{f}(t+\tau-u_2)\, du_1\, du_2 \right\} \\ &= \int_0^\infty \int_0^\infty \mathbf{E}\left\{e^{a(u_1+u_2)}\right\} R_{\widetilde{f}\widetilde{f}}(\tau+u_1-u_2)\, du_1\, du_2, \end{aligned}$$

where the independence of a and f has been applied. Using the substitution $v = u_1,\ w = \tau + u_1 - u_2$ and the assumption of an exponential decaying correlation function, i.e., $R_{\tilde{f}\tilde{f}}(\tau) = \sigma^2 e^{\gamma|\tau|}$, it follows

$$\begin{aligned} R_{\tilde{z}\tilde{z}}(\tau) &= \int_0^\infty \int_{-\infty}^{v+\tau} \mathbf{E}\left\{e^{a(\tau+2v-w)}\right\} R_{\tilde{f}\tilde{f}}(w)\, dw\, dv \\ &= \int_0^\infty \int_{-\infty}^{v+\tau} \int_{\widehat{a}-\eta}^{\widehat{a}+\eta} p_a(x)\, e^{x(\tau+2v-w)}\, \sigma^2 e^{\gamma|w|}\, dx\, dw\, dv \\ &= \sigma^2 \int_{\widehat{a}-\eta}^{\widehat{a}+\eta} p_a(x) e^{x\tau} J(x) dx \end{aligned}$$

$$\begin{aligned} \text{where} \quad J(x) &:= \int_0^\infty e^{2xv} \left[\int_{-\infty}^0 e^{-(\gamma+x)w}\, dw + \int_0^{v+\tau} e^{(\gamma-x)w}\, dw \right] dv \\ &= \int_0^\infty e^{2xv} \left[\frac{-1}{\gamma+x} + \frac{1}{\gamma-x}\left(e^{(\gamma-x)(v+\tau)} - 1\right)\right] dv \\ &= \int_0^\infty e^{2xv} \left[\frac{-2\gamma}{\gamma^2-x^2} + \frac{e^{(\gamma-x)\tau}}{\gamma-x} e^{(\gamma-x)v}\right] dv \\ &= \frac{\gamma}{x(\gamma^2-x^2)} - \frac{e^{(\gamma-x)\tau}}{\gamma^2-x^2}. \end{aligned}$$

Hence

$$R_{\tilde{z}\tilde{z}}(\tau) = \sigma^2 \int_{\widehat{a}-\eta}^{\widehat{a}+\eta} p_a(x) \left[\frac{\gamma}{x(\gamma^2-x^2)} e^{x\tau} - \frac{1}{\gamma^2-x^2} e^{\gamma\tau} \right] dx.$$

If additionally a is uniformly distributed on $[\widehat{a}-\eta, \widehat{a}+\eta]$ we have

$$\begin{aligned} R_{\tilde{z}\tilde{z}}(\tau) &= \frac{\sigma^2}{2\eta}\left[\gamma \int_{\widehat{a}-\eta}^{\widehat{a}+\eta} \frac{1}{x(\gamma^2-x^2)} e^{x\tau}\, dx - \int_{\widehat{a}-\eta}^{\widehat{a}+\eta} \frac{1}{\gamma^2-x^2} e^{\gamma\tau}\, dx \right] \\ &= \frac{\sigma^2}{2\eta}\left[\gamma I_1(\tau) - e^{\gamma\tau} I_2\right] \qquad (5.5) \end{aligned}$$

$$\text{where} \quad I_1(\tau) := \int_{\widehat{a}-\eta}^{\widehat{a}+\eta} \frac{e^{x\tau}}{x(\gamma^2-x^2)}\, dx \quad \text{and} \quad I_2 := \int_{\widehat{a}-\eta}^{\widehat{a}+\eta} \frac{1}{\gamma^2-x^2}\, dx.$$

The integrand of the integral $I_1(\tau)$ is bounded and continuous in the domain of integration $[\widehat{a}-\eta, \widehat{a}+\eta]$, since $\gamma < 2\widehat{a}$ and $\eta < \eta_S = -\widehat{a}$ has been supposed. So the integral is well-defined and can be computed using

$$I_1(\tau) \quad = \quad H(\tau, \widehat{a}+\eta) - H(\tau, \widehat{a}-\eta)$$

with the primitive function $H(\tau,z) := \int^z \frac{e^{x\tau}}{x(\gamma^2-x^2)}\, dx$. For the evaluation of $H(\tau,z)$ we use partial fraction decomposition and obtain

$$\begin{aligned} H(\tau,z) = \int^z \frac{e^{x\tau}}{x(\gamma^2-x^2)}\, dx \quad &= \quad \frac{1}{\gamma^2} \int^z e^{x\tau} \left(\frac{1}{x} - \frac{1}{2(x-\gamma)} - \frac{1}{2(x+\gamma)} \right) dx \\ &= \quad \frac{1}{\gamma^2} \left[H_1(\tau,z,0) - \frac{1}{2} H_1(\tau,z,-\gamma) - \frac{1}{2} H_1(\tau,z,+\gamma) \right] \\ \text{where} \quad H_1(\tau,z,b) \quad &:= \quad \int^z \frac{e^{x\tau}}{x+b}\, dx, \qquad b \in \mathbb{R}. \end{aligned}$$

For the sake of shorter notation we suppress the integration constants in the subsequent evaluation of the indefinite integrals contained in H and H_1. For $\tau = 0$ we have $H_1(\tau,z,b) = \ln|z+b|$ while for $\tau > 0$ this integral can be expressed in terms of the so-called exponential integral (see Abramowitz, Stegun [1], p.56)

$$E_1(z) := \int_z^\infty \frac{e^{-u}}{u}\, du = \int_{-\infty}^{-z} \frac{e^v}{v}\, dv \qquad \text{for } z \in \mathbb{R} \setminus \{0\}.$$

Thereby for $z < 0$ the integral is understood as a Cauchy principal value integral.

The substitution $v = \tau(x+b)$ yields

$$\begin{aligned} H_1(\tau,z,b) \quad &= \quad \int^{\tau(z+b)} \frac{e^{v-b\tau}}{v}\, dv = e^{-b\tau} \int^{\tau(z+b)} \frac{e^v}{v}\, dv \\ &= \quad e^{-b\tau} E_1(-\tau(z+b)) \end{aligned}$$

and we obtain

$$H(\tau,z) = \begin{cases} \frac{1}{\gamma^2} \left(\ln|z| - \frac{1}{2} \ln|z-\gamma| - \frac{1}{2} \ln|z+\gamma| \right), & \tau = 0 \\ \frac{1}{\gamma^2} \left(E_1(-\tau z) - \frac{e^{\gamma\tau}}{2} E_1(-\tau(z-\gamma)) - \frac{e^{-\gamma\tau}}{2} E_1(-\tau(z+\gamma)) \right), & \tau > 0 \end{cases} \tag{5.6}$$

For the integral I_2 one gets

$$\begin{aligned} I_2 &= \int_{\widehat{a}-\eta}^{\widehat{a}+\eta} \frac{1}{\gamma^2 - x^2}\, dx = \frac{1}{2\gamma} \int_{\widehat{a}-\eta}^{\widehat{a}+\eta} \left(\frac{1}{\gamma + x} + \frac{1}{\gamma - x} \right) dx \\ &= \frac{1}{2\gamma} \Big[\ln|\gamma + x| - \ln|\gamma - x| \Big]_{\widehat{a}-\eta}^{\widehat{a}+\eta} \\ &= \frac{1}{2\gamma} \ln \left| \frac{\gamma + (\widehat{a} + \eta)}{\gamma - (\widehat{a} + \eta)}\ \frac{\gamma - (\widehat{a} - \eta)}{\gamma + (\widehat{a} - \eta)} \right| \\ &= \frac{1}{2\gamma} \ln \frac{(\gamma + \eta)^2 - \widehat{a}^2}{(\gamma - \eta)^2 - \widehat{a}^2}. \end{aligned}$$

Substituting the representations for $I_1(\tau)$ and I_2 into (5.5) it yields

$$R_{\widetilde{z}\,\widetilde{z}}(\tau) = \frac{\sigma^2}{2\eta} \left[\gamma(H(\tau, \widehat{a} + \eta) - H(\tau, \widehat{a} - \eta)) - \frac{e^{\gamma\tau}}{2\gamma} \ln \frac{(\gamma + \eta)^2 - \widehat{a}^2}{(\gamma - \eta)^2 - \widehat{a}^2} \right]$$

where H is given in (5.6).

Lebesgue's Theorem on dominated convergence ensures that $\lim_{\tau \downarrow 0} R_{\widetilde{z}\,\widetilde{z}}(\tau) = R_{\widetilde{z}\,\widetilde{z}}(0)$.

Finally the correlation function of $z = \widehat{z} + \widetilde{z}$ can be derived using relation (5.2) which yields

$$R_{zz}(\tau) = \mathbf{D^2}\widehat{z} + R_{\widetilde{z}\,\widetilde{z}}(\tau)$$

where $\mathbf{D^2}\widehat{z}$ is given in (5.4).

After the exact computation of the mean, the variance and the correlation function of z approximations resulting from the perturbation approach presented in Section 3.2 are derived. For the approximation of $\mathbf{E}\,\widehat{z}$ and $\mathbf{D^2}\,\widehat{z}$ a Neumann series expansion of $\mathbf{A}^{-1}(\omega) = a^{-1}(\omega) = (\widehat{a} + \eta c(\omega))^{-1}$ has been used. This series is convergent for

$$\eta < \eta_N = \sup\{\eta > 0 : \quad \eta < \left\| \widehat{\mathbf{A}}^{-1} \mathbf{C}(\omega) \right\|^{-1} = |\widehat{a}^{-1} c(\omega)|^{-1} = |\widehat{a}||c(\omega)|^{-1} \text{ a.s.}\}.$$

Since $|c|$ is assumed to be a.s. bounded by $c_0 = 1$ it holds $\eta_N = |\widehat{a}| = \eta_S = \eta_M$.

The expansions of the moments of $\widehat{z}$ given in Theorem 3.4 applied to the present scalar case read as

$$\begin{aligned} \mathbf{E}z = \mathbf{E}\,\widehat{z} &= \left(1 + \frac{\mathbf{E}c^2}{\widehat{a}^2}\,\eta^2\right) x + o(\eta^2) \\ \text{and} \quad \mathbf{D^2}\,\widehat{z} &= \frac{\mathbf{E}c^2}{\widehat{a}^2} x^2\, \eta^2 + o(\eta^2) \end{aligned}$$

where $x = -\frac{\widehat{f}}{\widehat{a}}$.

If additionally c is uniformly distributed on $[-1, 1]$ then $\mathbf{E}c^2 = \frac{1}{3}$ and

$$\mathbf{E}\,\widehat{z} = \Big(1+\frac{1}{3\widehat{a}^2}\,\eta^2\Big)\frac{\widehat{f}}{-\widehat{a}} + o(\eta^2)$$
$$\mathbf{D^2}\,\widehat{z} = \frac{1}{3\widehat{a}^4}\,\widehat{f}^2\,\eta^2 + o(\eta^2).$$

It can easily be checked that these expansions coincide with the corresponding Taylor expansions of the exact results given in (5.3) and (5.4) in powers of η.

For the correlation function $R_{\widetilde{z}\widetilde{z}}(\tau)$ Theorem 3.10 provides an expansion for $\eta < \eta_P = \frac{\lambda_0}{v_0 c_0}$ with $\lambda_0 = -\widehat{a},\ v_0 = 1$ and $c_0 = 1$, i.e., $\eta_P = -\widehat{a} = \eta_s = \eta_M = \eta_N$. It can be observed that in this example all bounds $\eta_.$ coincide.

In the present scalar case Eqs. (1.1) and (4.1) coincide and for the assumed exponentially decaying correlation of f the correlation function $R_{{}^0\zeta\,{}^0\zeta}(\tau)$ of the solution of the unperturbed equation is for $\tau \geq 0$

$$R_{{}^0\zeta\,{}^0\zeta}(\tau) = \frac{\sigma^2}{\gamma^2-\widehat{a}^2}\Big(\frac{\gamma}{\widehat{a}}e^{\widehat{a}\tau} - e^{\gamma\tau}\Big) = q_1 e^{\pi_1\tau} + q_2 e^{\pi_2\tau}$$
$$\text{where}\qquad \pi_1 = \widehat{a},\quad q_1 = \frac{\sigma^2}{\gamma^2-\widehat{a}^2}\,\frac{\gamma}{\widehat{a}}$$
$$\pi_2 = \gamma,\quad q_2 = -\frac{\sigma^2}{\gamma^2-\widehat{a}^2}.$$

In view of the prescribed form of the correlation function $R_{{}^0\zeta\,{}^0\zeta}(\tau)$ given in Eq. (4.2) we can set $m = 2,\ \mathbf{Q}_1 = q_1,\ \mathbf{Q}_2 = q_2,\ \mathbf{\Pi}_1 = \pi_1 = \widehat{a}$ and $\mathbf{\Pi}_2 = \pi_2 = \gamma$. Moreover it is $n = 1$ and $\Lambda = \widehat{a}$, so we can apply Corollary 4.2 to give an expansion in powers of η for the correlation function $R_{\widetilde{z}\widetilde{z}}(\tau)$. Thereby we suppress the indices i, j, k, l which in our case take the value 1, only. It follows for $\tau \geq 0$

$$R_{\widetilde{z}\widetilde{z}}(\tau) = \sum_{r=1}^{2}\big[q_r e^{\pi_r\tau} + \eta^2\mathbf{E}c^2(J_{1,r}(\tau) + J_{1,r}(-\tau) + J_{2,r}(\tau))\big] + o(\eta^2). \tag{5.7}$$

For $J_{1,r}(\tau)$ Eq. (4.3) gives

$$J_{1,1}(\tau) = \frac{q_1}{4\widehat{a}^2}e^{\widehat{a}\tau}\quad\text{and}\quad J_{1,2}(\tau) = \frac{q_2}{(\widehat{a}+\gamma)^2}e^{\gamma\tau}.$$

$J_{1,r}(-\tau)$ can be evaluated using Eq. (4.6), it yields

$$J_{1,1}(-\tau) = q_1 e^{\widehat{a}\tau}\left(\frac{\tau^2}{2} - \frac{\tau}{2\widehat{a}} + \frac{1}{4\widehat{a}^2}\right)$$
$$J_{1,2}(-\tau) = -q_2 e^{\widehat{a}\tau}\frac{2\gamma}{\gamma^2-\widehat{a}^2}\left(\tau + \frac{2\widehat{a}}{\gamma^2-\widehat{a}^2}\right) + q_2 e^{\gamma\tau}\frac{1}{(\widehat{a}-\gamma)^2}$$

while from Eq. (4.7) it follows for $J_{2,r}(\tau)$

$$J_{2,1}(\tau) = q_1 e^{\widehat{a}\tau}\left(\frac{1}{2\widehat{a}^2} - \frac{\tau}{2\widehat{a}}\right)$$
$$J_{2,2}(\tau) = q_2 e^{\widehat{a}\tau}\frac{\gamma}{\widehat{a}(\gamma^2-\widehat{a}^2)} - q_2 e^{\gamma\tau}\frac{1}{\gamma^2-\widehat{a}^2}.$$

system	$\widehat{a} = -1$
perturbation	c uniformly distributed on $[-1, 1]$
	$\eta \in [0, 1)$
external excitation	
mean	$\widehat{f} = 1$
standard deviation	$\sigma = \nu \widehat{f},\ \nu \geq 0$
correlation function	$R_{ff}(\tau) = \sigma^2 e^{\gamma\lvert\tau\rvert}$
decay parameter	$\gamma = -3$

Table 5.1: Parameter settings

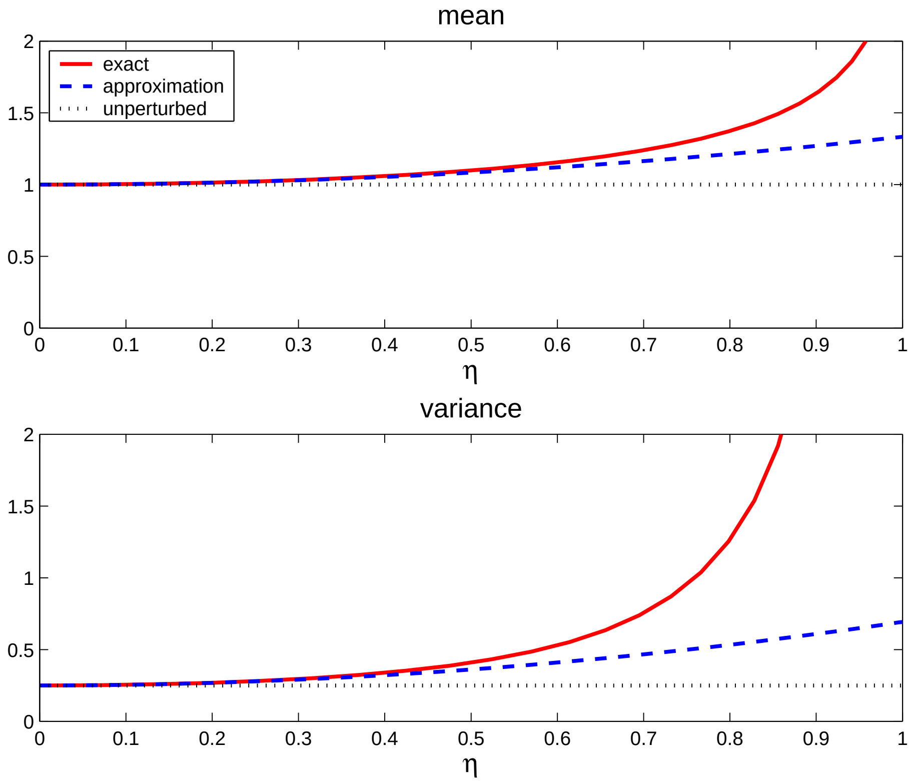

Figure 5.1: Exact (solid) and approximate (dashed) mean and variance of z, $\nu = 1$

Substituting the above terms into (5.7) we get the following expansion for the correlation function of $\widetilde{z}$

$$\begin{aligned} R_{\widetilde{z}\widetilde{z}}(\tau) &= e^{\widehat{a}\tau}\left\{q_1+\eta^2\mathbf{E}c^2\left[q_1\left(\frac{\tau^2}{2}-\frac{\tau}{\widehat{a}}+\frac{1}{\widehat{a}^2}\right)-\frac{q_2\gamma}{\gamma^2-\widehat{a}^2}\left(2\tau+\frac{5\widehat{a}^2-\gamma^2}{\widehat{a}(\gamma^2-\widehat{a}^2)}\right)\right]\right\} \\ &\quad +e^{\widehat{\gamma}\tau}\left\{q_2+\eta^2\mathbf{E}c^2q_2\frac{3\widehat{a}^2+\gamma^2}{(\gamma^2-\widehat{a}^2)^2}\right\}+o(\eta^2) \\ &= \frac{\sigma^2}{\gamma^2-\widehat{a}^2}\left(\frac{\gamma}{\widehat{a}}e^{\widehat{a}\tau}-e^{\gamma\tau}\right) \\ &\quad +\eta^2\mathbf{E}c^2\frac{\sigma^2}{\gamma^2-\widehat{a}^2}\left[e^{\widehat{a}\tau}\left\{\frac{\gamma}{\widehat{a}}\left(\frac{\tau^2}{2}-\frac{\tau}{\widehat{a}}+\frac{1}{\widehat{a}^2}\right)+\frac{\gamma}{\gamma^2-\widehat{a}^2}\left(2\tau+\frac{5\widehat{a}^2-\gamma^2}{\widehat{a}(\gamma^2-\widehat{a}^2)}\right)\right\}\right. \\ &\qquad \left. -e^{\widehat{\gamma}\tau}\frac{3\widehat{a}^2+\gamma^2}{(\gamma^2-\widehat{a}^2)^2}\right]+o(\eta^2). \end{aligned}$$

If c is uniformly distributed on $[-1,1]$ then $\mathbf{E}c^2=\frac{1}{3}$ can be substituted in the above formula.

Table 5.1 shows the parameters for the subsequent numerical experiments. It is noted that the standard deviation σ of the excitation is set proportional to the mean excitation $\widehat{f}$, i.e., $\sigma=\nu\widehat{f}$, with $\nu\geq 0$.

It holds $\eta_S=\eta_M=\eta_N=\eta_P=-\widehat{a}=1$, i.e., for $\eta\in[0,1)$ we have stability and convergence of the perturbation series.

Remark 5.1 For an optimal view of the subsequent figures we recommend a colored hardcopy or the electronic version of this paper which is available at http://archiv.tu-chemnitz.de/pub/2005/ or http://www.fh-wickau.de/∼raw.

Figure 5.1 plots for $\nu=1$ the mean $\mathbf{E}z$ and the variance $\mathbf{D^2}z$ versus the perturbation parameter η. Moreover the corresponding quantities of the unperturbed equation ($\eta=0$) are drawn. The approximations found from the perturbation approach compare well with the exact results for $\eta<0.5$. For larger values of η the approximations are less accurate and they are not suited to describe the limiting behaviour for $\eta\to\eta_S=1$. Here the mean as well as the variance of z tends to infinity while approximations grow quadratically and remain finite.

Figure 5.2 plots for $\nu=0.5$ (left) and $\nu=2$ (right) the exact total variance $\mathbf{D^2}z=\mathbf{D^2}\widehat{z}+\mathbf{D^2}\widetilde{z}$ (red, solid) together with the two contributions $\mathbf{D^2}\widehat{z}$ (blue, dashed) and $\mathbf{D^2}\widetilde{z}$ (cyan, dotted) versus η. Since $\widehat{z}$ does not depend on $\widetilde{f}$, the variance of $\widehat{z}$ is not influenced by ν while the variance of $\widetilde{z}$ is proportional to ν^2. It can be seen, that for small η the contribution of $\mathbf{D^2}\widetilde{z}$ dominates the contribution of $\mathbf{D^2}\widehat{z}$ while for large η a reversed situation can be observed.

This phenomenon is also visualized in the two lower plots which show the relative contribution of the variance of $\widehat{z}$ to the total variance, i.e., $\frac{\mathbf{D^2}\widehat{z}}{\mathbf{D^2}z}=\frac{\mathbf{D^2}\widehat{z}}{\mathbf{D^2}\widehat{z}+\mathbf{D^2}\widetilde{z}}$. Thereby the solid line corresponds to the exact values and the dashed line to the approximations.

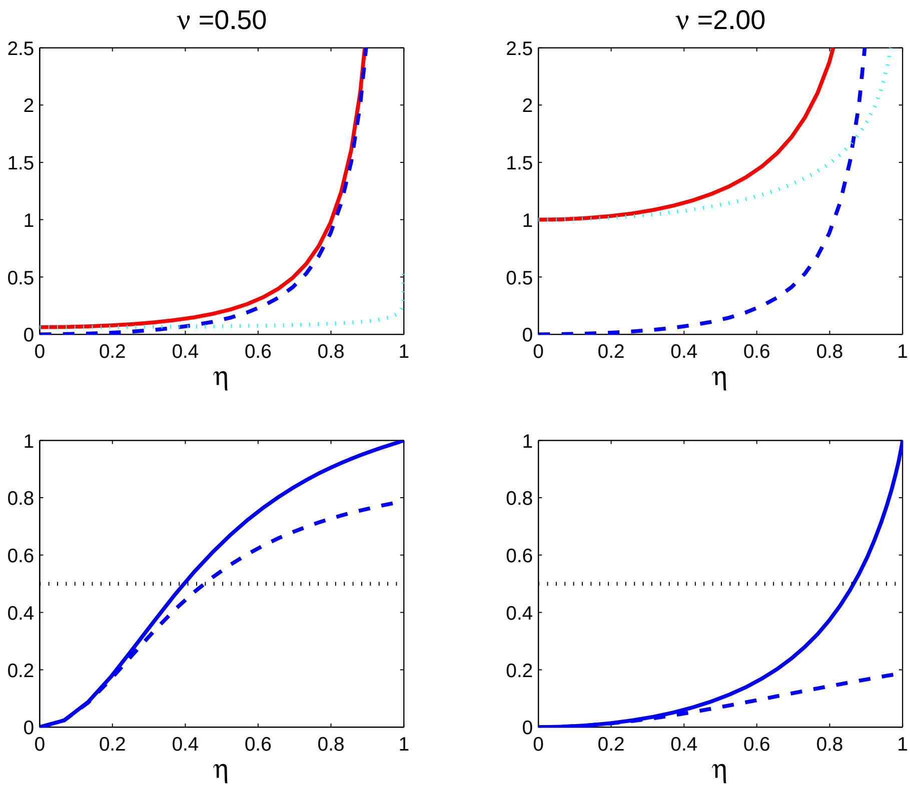

Figure 5.2: Upper plots: Exact total variance $\mathbf{D^2}z = \mathbf{D^2}\widehat{z} + \mathbf{D^2}\widetilde{z}$ (red, solid) and contributions $\mathbf{D^2}\widehat{z}$ (blue, dashed) and $\mathbf{D^2}\widetilde{z}$ (cyan, dotted),
lower plots: relative contribution $\frac{\mathbf{D^2}\widehat{z}}{\mathbf{D^2}z}$, exact (solid), approximation (dashed),
left plots $\nu = 0.5$, right plots $\nu = 2$

The graphs indicate that the approximations are able to reproduce the correct relative contribution only for $\eta < 0.3$.

The different values of ν can be interpreted as small ($\nu = 0.5$) and large ($\nu = 2$) random fluctuations $\widetilde{f}$ of the external excitation around the mean $\widehat{f}$. While for $\nu = 0.5$ the variance $\mathbf{D^2}\widetilde{z}$ is the dominating term if $\eta < 0.4$ for $\nu = 2$ this is the case even for larger values of the perturbation parameter up to $\eta \approx 0.85$.

Figure 5.3 shows the correlation functions $R_{\widetilde{z}\widetilde{z}}(\tau)$ for various values of η and compares the exact correlation functions with the approximations resulting from the expansions in powers of the perturbation parameter η and the results for an unperturbed equation ($\eta = 0$). Again a high accuracy of the approximations for $\eta < 0.5$ and growing deviations for larger values of η can be observed.

The correlation function of z is the sum of $R_{\widetilde{z}\widetilde{z}}(\tau)$ studied above and $\mathbf{D^2}\widehat{z}$ which does not depend on τ. While $R_{\widetilde{z}\widetilde{z}}(\tau)$ tends to zero for $\tau \to \infty$ we have

$$R_{zz}(\tau) = R_{\widetilde{z}\widetilde{z}}(\tau) + \mathbf{D^2}\widehat{z} \to \mathbf{D^2}\widehat{z} > 0 \quad \text{for } \tau \to \infty.$$

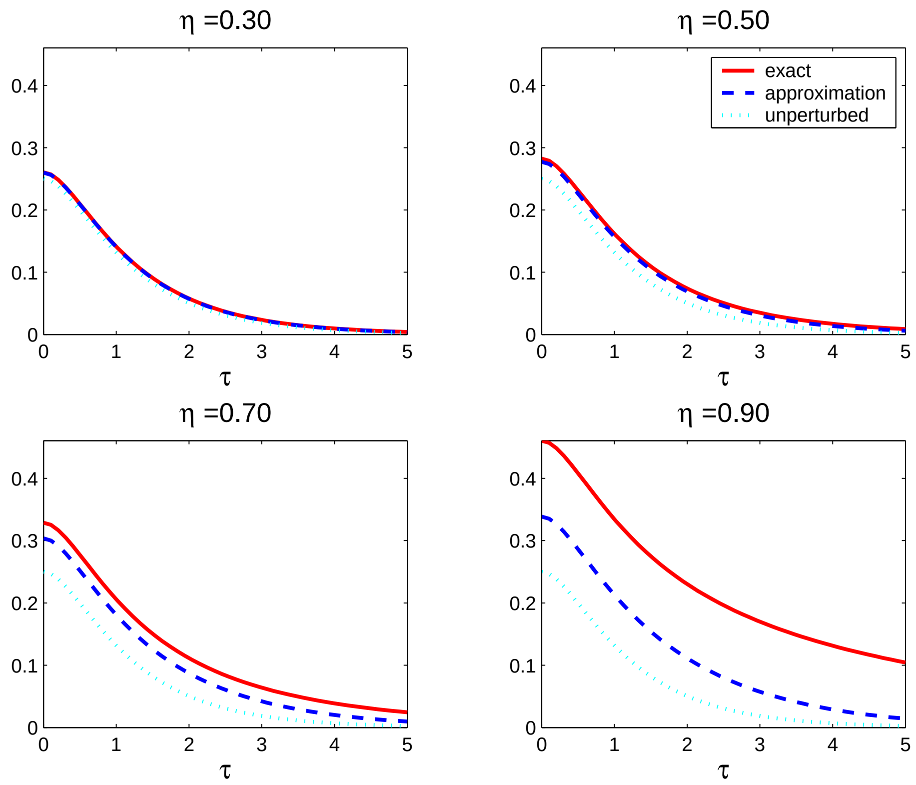

Figure 5.3: Exact (red, solid) and approximate (blue, dashed) correlation functions $R_{\widetilde{z}\widetilde{z}}(\tau)$ for $\nu = 1$ and various values of η. The cyan dotted lines correspond to the unperturbed case ($\eta = 0$)

Figure 5.4 shows normalized correlation functions of z defined by

$$R^N(\tau) = \frac{R_{zz}(\tau)}{R_{zz}(0)} = \frac{R_{\widetilde{z}\widetilde{z}}(\tau) + \mathbf{D^2}\widehat{z}}{\mathbf{D^2}\widetilde{z} + \mathbf{D^2}\widehat{z}}$$

for various values of ν. So one can compare the influence of the variances of the external excitation f which is given by $\mathbf{D^2} f = \nu^2 \widehat{f}^2$. It holds $R^N(\tau) \longrightarrow \frac{\mathbf{D^2}\widehat{z}}{\mathbf{D^2}\widetilde{z} + \mathbf{D^2}\widehat{z}} = 1 - \frac{\mathbf{D^2}\widetilde{z}}{\mathbf{D^2}z}$ for $\tau \longrightarrow \infty$.

The plot indicates that for small ν, i.e., for small fluctuations of the external excitation around its mean $\widehat{f}$ the normalized correlation function of z is only slightly decreasing, remains positive and tends for $\tau \longrightarrow \infty$ to a larger limit than in case of larger values of ν. That means there is a large positive correlation between the values of $z(t_1)$ and $z(t_2)$ even for large distances $|t_2 - t_1|$.

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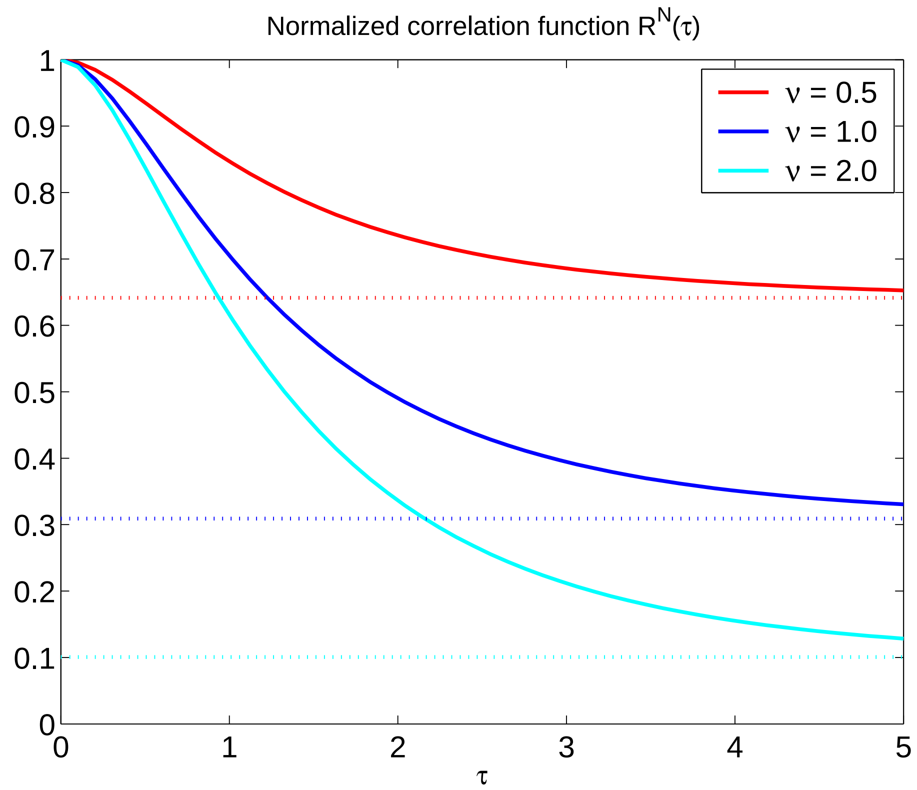

Figure 5.4: Normalized correlation functions $R^N(\tau) = \frac{R_{zz}(\tau)}{R_{zz}(0)}$ for $\eta = 0.5$, $\nu = 0.5$ (red), $\nu = 1$ (blue), $\nu = 2$ (cyan) and limits for $\tau \to \infty$ (dotted)

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