Model Building, Control Design and Practical Implementation of a High Precision, High Dynamical MEMS Acceleration Sensor

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ABSTRACT

This paper presents the whole process of building up a high precision, high dynamical MEMS acceleration sensor. The first samples have achieved a resolution of better than 500 $\mu$g and a bandwidth of more than 200 Hz. The sensor fabrication technology is shortly covered in the paper. A theoretical model is built from the physical principles of the complete sensor system, consisting of the MEMS sensor, the charge amplifier and the PWM driver for the sensor element. The mathematical modeling also covers problems during startup. A reduced order model of the entire system is used to design a robust control with the Mixed-Sensitivity $\mathcal{H}_\infty$-Approach. Since the system has an unstable pole, imposed by the electrostatic field and time delay, caused by A/D-D/A conversation delay and DSP computing time, limitations for the control design are given. The theoretical model might be inaccurate or lacks of completeness, because the parameters for the theoretical model building vary from sample to sample or might be not known. A new identification scheme for open or closed-loop operation is deployed to obtain directly from the samples the parameters of the mechanical system and the voltage dependent gains. The focus of this paper is the complete system development and identification process including practical tests in a DSP TI-TMS320C3000 environment.

Keywords: Acceleration Sensor, Model Building, $\mathcal{H}_\infty$-Control, Mixed-Sensitivity Approach, Identification

1. INTRODUCTION

MEMS (micro-electro-mechanical systems) play an important role in the realization of sensor/actor systems. MEMS are small, very compact, have a simple and robust layout. They are producable in large quantities and thus very inexpensive compared to precision engineering systems. Another advantage for MEMS is the technology, which can directly be applied from the micro electronics and hence, makes the integration of the electronics possible. There are, however, some limitations such as packaging, MEMS mechanical limitation, as well as the strong nonlinearity of the electrostatic field component and the nonlinearity of fluid damping, which makes it difficult to control the system. Therefore, an intensive analysis of the open and closed-loop system may be necessary.

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Control systems were widely applied to MEMS to improve the system behavior. Observer based state feedback with integral action [1] found its application for microactuators. A sigma-delta converter [2] was used to control an accelerometer. Model reference adaptive control systems (MRAC) were successfully applied to micro-mirrors [3] and gyroscopes [4, 5], as well as a model reference based neural network [6] on an accelerometer. A phase lead-lag controller in combination with a second-order low-pass filter [7] and further control schemes like prediction and dead-beat control [8] are successfully attached to micro-mirror systems.

From control theory is known, that special requirements are needed – MRAC cannot be applied to non-minimum phase systems, the pole-placement control needs a full state measurement or a state observer, and all applied control schemes are developed in time domain and do not consider robustness explicitly. Therefore, the $H_{\infty}$ control design is introduced.

System identification is a well-established field, but the number of applications to identify MEMS are very minor. The Newton method [9] is used to identify the nonlinear model of a MEMS optical switch, a neural network (NN) approach [10] is successfully applied to an accelerometer.

![Figure 1. Block diagram of the system](image)

It is clear, that the performance index for a nonlinear identification model is difficult to describe and the NN approach just gives a black-box model. Both identification methods need large numerical computation and also a large amount of data, and the nonlinear optimization might just detect a local optimum. Thus, a two-stage identification algorithm is introduced, which completely relies on polynomial data-fitting and which just needs two data sets to fully describe the nonlinear system, depicted in Fig. 1.

2. DESIGN

The sensor was fabricated in bulk-micromachining technology and consists of a glass-silicon-glass sandwich structure. The seismic mass, shown in Fig. 2, is suspended at two vertical torsion beams. The ditches on the mass surface reduce noticeably the air damping in the gap without influencing the electrostatic field too much. The SiO$_2$-spacers at the edges of the seismic mass prevent the electrical contact with the outer electrodes and also the sticking during the bonding process.

Fig. 3(b) shows the whole chip configuration. The sensor cover consists of glass, where the Aluminum electrodes are sputtered on the glass wafer. In contrast to silicon covers, which could be used in the same way, this configuration minimizes the parasitic capacities, arising mainly from the sensor frame. This reduces the electronic effort of the electronic detection circuit noticeably. Drill holes in the glass cover provide the electrical contact and the equalization to ambient pressure. Ambient pressure guarantees equal heating of silicon and glass wafers during the bonding process, and thus the deformation of the components will be minimized. Ambient pressure also guarantees approximate equal dynamics for all sensor samples.

3. MODEL BUILDING

The electrical actuated acceleration sensor can generally be described as a rotational mechanical spring-mass system

$$[-\omega^2 J + j\omega \{D + D_4(\omega, \varphi(\omega))\} + K + K_4(\omega, \varphi(\omega))] \varphi(\omega) = M_{\text{ext}}(\omega) + M_{\text{el}}(\varphi(\omega), u(\omega))$$

(1)
Figure 2. SEM micrograph of the seismic mass

(a) Photograph

(b) Schematic configuration

Figure 3. The sensor chip

with the moment of inertia $J$, the damping $D$ and spring constant $K$, consisting of constant mechanical and frequency dependent squeeze-film parts, the electrostatic moment $M_{el}$, the rotation angle $\varphi$ and the disturbance, the mechanical moment $M_{ext}$. The acceleration sensor has several mechanical modes in general, where the higher modes usually appear in higher frequency areas. Just only the movement of the first mode with the eigenfrequency $\frac{1}{2\pi}\sqrt{\frac{K}{J}}$ is controllable and detectable and is considered in this paper.

The mechanical constants can be derived from the mechanical laws, where the moment of inertia $J$ and mechanical spring constant $K$ are given by

$$J = \frac{1}{12}m\left(4a_m^2 + d_m^2\right); \quad K = 2\frac{GI_t}{b_b},$$

where

$$I_t = \frac{1}{3}\left(1 - \frac{0.630}{n} + \frac{0.052}{n^3}\right)a_b^3d_b, \quad n = \frac{d_b}{a_b} \geq 1 \quad (2)$$

with the mass $m$ and its dimension $a_m \times b_m \times d_m$ (width \times length \times thickness), the beam with its dimension $a_b \times b_b \times d_b$, the elastic shear modulus of silicon $G$ and torsion moment of inertia for a rectangle $I_t$. 

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3.1. Squeeze-Film Effect

Fluid damping plays an important role in micro systems, where the gas in the gap is used as the damping element. Because of the compressibility of air, this damping element produces frequency dependent damping and spring parts.

General flow problems can be described with the Navier-Stokes Equations. A special simplification, the Reynolds Lubrication Equation [11], describes the pressure distribution in a small air gap between two moving plates. A simple, linear dependence can be derived with a Taylor approximation of the pressure \( p = p_0 + \delta p \), the plate distance \( d = d_0 + \phi(x - c) \) and further neglecting high order terms. The resulting equation can be solved with separating the variables, which was done in [12] for parallel moving rectangular plates. The procedure shall now be extended to tilting plates with a variational rotation axis. The linearized Reynolds Equation reduces in the case of laminar flow and Cartesian coordinates and its normalization to

\[
\frac{\partial^2 \phi}{\partial \chi^2} + \frac{\partial^2 \phi}{\partial \gamma^2} - \sigma \frac{\partial \phi}{\partial t} = \sigma \frac{\partial \eta}{\partial t} \quad (3)
\]

with \( \phi = \frac{\delta p}{p_0} \); \( \chi = \frac{X}{a} \); \( \gamma = \frac{Y}{a} \)

and \( \eta = \frac{\alpha \phi}{a_0} (X - \zeta) = \eta(X - \zeta) \) with \( \zeta = \frac{c}{a} \)

and the squeeze number \( \sigma = \frac{12 \mu}{p_0 a_0^2} \)

introduced in [12]. If a step excitation signal at \( t = 0 \) is applied, the right hand side of equation (3) is zero for a time \( t > 0 \).

This results into the linear Differential Equation

\[
\frac{\partial^2 \phi}{\partial \chi^2} + \frac{\partial^2 \phi}{\partial \gamma^2} = \sigma \frac{\partial \phi}{\partial t} \quad . (4)
\]

The separation approach \( \phi(\chi, \gamma, t) = \Psi(\chi) \Gamma(\gamma) T(t) \) gives the solution

\[
\Psi'' + \mu \Psi = 0 \quad \implies \quad \Psi(\chi) = A_\chi \cos(\sqrt{\mu} \chi) + B_\chi \sin(\sqrt{\mu} \chi) \quad (5a)
\]

\[
\Gamma'' + \vartheta \Gamma = 0 \quad \implies \quad \Gamma(\gamma) = A_\gamma \cos(\sqrt{\vartheta} \gamma) + B_\gamma \sin(\sqrt{\vartheta} \gamma) \quad (5b)
\]

\[
T' + \frac{\lambda}{\sigma} T = 0 \quad \implies \quad T(t) = C_t \exp\left(-\frac{\lambda}{\sigma} t\right) ; \quad \lambda = \mu + \vartheta \quad . (5c)
\]

The solution of (4) must fulfill the boundary conditions of the pressure

\[
\phi(\chi = \pm \frac{1}{2} \gamma, t) = \phi(\chi = \zeta, \gamma, t) = 0 ; \quad \phi(\chi, \gamma = \pm \frac{\beta}{2}, t) = 0 \quad \text{with} \quad \beta = \frac{b}{a} , (6)
\]

where the surface edges and the variable rotation axis are at ambient pressure level. The eigenvalues therefore must solve

\[
\sqrt{\mu} = n \pi , \quad n \in \{ x \mid x \in \mathbb{N} \land x \neq 0 \} ; \quad \sqrt{\vartheta} = \frac{m \pi}{\beta} , \quad m \in \{ x \mid x \in \mathbb{N} \land x \neq 2 \} ; \quad (7a)
\]

\[
\lambda = \mu + \vartheta = \pi^2 \left( n^2 + \frac{(2m - 1)^2}{\beta^2} \right) , \quad m, n \in \{ x \mid x \in \mathbb{N} \land x \neq 0 \} , \quad (7b)
\]

where for odd \( n \) the sinus part \( B_\chi \) and for even \( n \) the cosine part \( A_\chi \) must vanish. The base solution yields to

\[
\phi(\chi, \gamma, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( A_{m,n} \cos(n \pi \chi) + B_{m,n} \sin(n \pi \chi) \right) \cos\left( \frac{2m - 1}{\beta} \pi \right) \cos\left( \frac{2m - 1}{\beta} \right) e^{-\left( n^2 + \frac{(2m - 1)^2}{\beta^2} \right) \frac{\lambda}{\sigma} t} . (8)
\]
The unknown parameters in the solution must now be calculated from the conditions at start time. The pressure at time \( t = 0 \) must satisfy the boundary conditions (6) and the start condition (9).

Therefore, the pressure distribution in X direction is satisfied with a rectangular wave overlaid with a sawtooth wave (Fig. 5).

The boundary condition in Y direction is fulfilled with a rectangular wave. It must be noted, that for regular cases the boundary condition \( \phi(\chi = \zeta, \gamma, t) = 0 \) can be fulfilled just approximately for finite terms \( m, n \).

With incorporating the Fourier series for both coordinates, the pressure distribution of the normalized pressure (8) can be derived to (11).

The squeeze moment, calculated by integration over the whole plate area, results in (12).

The Laplace transform of the pressure distribution (13) and step function gives the transfer function of the squeeze-film moment to the angle change (14).

An equivalent circuit model (Fig. 6) can be used to describe the behavior of the spring-mass system. The system is
converted via force-current analogy, described in Tab. [1] where the squeeze-film part alters to an infinite sum of series connections of a resistance and inductance [13]. From Eq. (14) the squeeze-film components can be calculated to

\[ R_{s m n} = \frac{(2m-1)^2 n^2 d_0 \pi^4}{8 a^3 b p_0 \sigma (1 + (1 - 1)^n + 4 \zeta^2 (1 - (1)^n))} \left( n^2 + \frac{(2m-1)^2}{\beta^2} \right) \quad (15a) \]

\[ L_{s m n} = \frac{(2m-1)^2 n^2 d_0 \pi^4}{8 a^3 b p_0 (1 + (1 - 1)^n + 4 \zeta^2 (1 - (1)^n))}, \quad m, n \in \{x \mid x \in \mathbb{N} \land x \neq 0\} \quad (15b) \]

It is to note, that \( n \) must stop evenly for normal cases, since (8) consists in general of an even and odd part. The Eqs. (15) reduce for the special case of \( \zeta = -\frac{1}{2} \) to

\[ R_{s m n} = \frac{m^2 n^2 d_0 \pi^4}{16 a^3 b p_0 \sigma} \left( n^2 + \frac{m^2}{\beta^2} \right), \quad L_{s m n} = \frac{m^2 n^2 d_0 \pi^4}{16 a^3 b p_0}, \quad m \in \{1, 3, 5, 7, \ldots\}, \quad n \in \{1, 2, 3, 4, \ldots\} \quad (16) \]

The transfer function of the acceleration sensor can be described in state-space form, found from the equivalent circuit model. The simple form of the state-space description allows one to easily extend the transfer function to higher orders

\[
\begin{bmatrix}
    u_C \\
    i_L \\
    i_{i,1}^{a} \\
    i_{i,1}^{b} \\
    \vdots \\
    i_{m,n}^{a} \\
    i_{m,n}^{b}
\end{bmatrix} =
\begin{bmatrix}
    -\frac{1}{C_R} & 0 & 0 & \cdots & 0 & 0 \\
    -\frac{1}{L_1} & 0 & 0 & \cdots & 0 & 0 \\
    -\frac{1}{L_{1,1}^{a}} & 0 & 0 & \cdots & 0 & 0 \\
    -\frac{1}{L_{1,1}^{b}} & 0 & 0 & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    -\frac{1}{L_{m,n}^{a}} & 0 & 0 & \cdots & 0 & 0 \\
    -\frac{1}{L_{m,n}^{b}} & 0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
    u_C \\
    i_L \\
    i_{i,1}^{a} \\
    i_{i,1}^{b} \\
    \vdots \\
    i_{m,n}^{a} \\
    i_{m,n}^{b}
\end{bmatrix} +
\begin{bmatrix}
    0 \\
    0 \\
    0 \\
    0 \\
    \vdots \\
    0 \\
    0
\end{bmatrix}
\end{bmatrix}
\]

\[
\varphi(t) = \begin{bmatrix} 0 & L & 0 & \cdots & 0 & 0 \end{bmatrix}^T [u_C, i_L, i_{i,1}^{a}, i_{i,1}^{b}, \ldots, i_{m,n}^{a}, i_{m,n}^{b}]^T,
\]

where the two branches with the indices \( a \) and \( b \) represent the squeeze-film effect on the upper and lower surface of the seismic mass. An one-term approximation of the squeeze-film damping parts is sufficiently accurate for most cases. The damping parts at the top and bottom side are about the same size, because the seismic mass is controlled in zero position.

The model for the control design reduces further to

\[
G_{mech} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix}
    \frac{1}{L_1} & -\frac{1}{C} & -\frac{1}{C} & 0 & 0 \\
    0 & \frac{1}{L_1} & 0 & 0 & 0 \\
    0 & 0 & \frac{1}{L_{1,1}^{a}} & 0 & 0 \\
    0 & 0 & 0 & \frac{1}{L_{1,1}^{b}} & 0 \\
    0 & 0 & 0 & 0 & \frac{1}{L_{m,n}^{a}} \\
\end{bmatrix}
\]

with the values

\[
R_s = \frac{1}{2} R_{1,1}^{a} = \frac{1}{2} R_{1,1}^{b}, \quad L_s = \frac{1}{2} L_{1,1}^{a} = \frac{1}{2} L_{1,1}^{b} \quad (18)
\]

This approximation will be also used for the control design, since the order of the model directly determines the order of the controller.

3.2. The Detection Unit

The Charge Amplifier (Fig. [7]) has two modes of operation. In the first mode, the switch is closed and the amplifier realizes a voltage follower. In this stage the middle electrode of the sensor is biased and the capacitor \( C_f \) in the feedback loop is discharged.
\[ u_a(t) = \frac{-V Q(t)}{C_p + (1 + V) C_f} \approx -\frac{Q(t)}{C_f} \] (19a)

\[ Q(t) = C_1(t) u_{C_1}(t) + C_2(t) u_{C_2}(t) \]
\[ = u_b (C_1(t) - C_2(t)) = u_b \Delta C(t) \] . (19b)

\[ u_b = \frac{u_a}{\Delta C(x)} \approx -\frac{u_b}{C_f} \] . (20)

It is to note, that the simplification in (19a) is only valid for a high open loop gain \( V \) of the amplifier and if the parasitic capacitance is small compared to the feedback capacitance \( C_p \ll C_f \).

The difference capacitance of the sensor can be calculated from the single capacitance of a rotating plate in case of a small deflection angle \( \varphi \). The integration over the plate area, divided into infinite small stripes, results into

\[ C(\varphi) = \int_0^a \frac{\varepsilon b}{d_0 + x \sin \varphi} \, dx = \frac{\varepsilon b}{\sin \varphi} \ln \left( 1 + \frac{a \sin \varphi}{d_0} \right) \approx \frac{\varepsilon b}{d_0} \sum_{n=1}^{\infty} \frac{(-\varphi)^{n-1} a^n}{n d_0^n} \] . (21)

The difference capacity of the lower and upper plate of the sensor yields with (21) into the term

\[ \Delta C(\varphi) = C(-\varphi) - C(\varphi) \approx \frac{\varepsilon b}{d_0} \sum_{n=1}^{\infty} \frac{\varphi^{2n-1} a^{2n}}{n d_0^{2n}} \] . (22)

The detection sensitivity at the working point \( \varphi = 0 \) can be easily get from the infinite sum (22)

\[ k_c = \frac{d \Delta C(\varphi)}{d \varphi} \bigg|_{\varphi=0} = \frac{\varepsilon a^2 b}{d_0^3} \] . (23)

### 3.3. The Electromechanical Moment

The electromechanical moment is mainly responsible for the strong nonlinear behavior of the system. It results in the electrostatic spring softening effect and also in a change in the open loop gain of the system. Thus, the electromechanical moment can be divided into two gain parts, the inner feedback part \( \frac{\partial M_e}{\partial \varphi} \), which is responsible for the spring softening and in the gain part \( \frac{\partial M_e}{\partial u} \) in the control loop.

The electrostatic moment can be easily derived from the energy conservation rule between the electrical and mechanical energy. It is further supposed, that the outer electrodes will be at the potential \( \pm \frac{u_b}{2} \) and the seismic mass at the control signal \( u(t) \). The total moment derives to

\[ M_e(\varphi, u) = M(\varphi, u + \frac{u_b}{2}) - M(-\varphi, u - \frac{u_b}{2}) \approx \frac{\varepsilon b}{8 \varphi^3} \sum_{n=1}^{\infty} \frac{(-\varphi)^n(u - 1)}{n d_0^n} \left[ (-\varphi)^n(u_b - 2u)^2 - \varphi^n(u_b + 2u)^2 \right] . \] (24)
The derivatives for the time-variant values $\varphi(t)$ and $u(t)$ are in zero position

$$k_\varphi = \frac{\partial M_{\varphi}}{\partial \varphi} \bigg|_{\varphi=0} = \varepsilon a^2 b \frac{d^6}{6d_0^3} (4a^2 + u_b^2) \ , \quad k_u = \frac{\partial M_{\varphi}}{\partial u} \bigg|_{\varphi=0} = -\varepsilon a^2 b u_b \ ,$$

where the voltage $u^2$ can be seen as the quadratic mean value of the control voltage.

### 3.4. The PWM-Driver

The PWM driver governs the position of the seismic mass with changing the duty-cycle of the PWM. It also realizes the charge flow in the detection unit out of or into the sensor electrode in every period. The two switching states in a period $T$ can be modeled independently as two linear systems, where at time $t_0$ the states are switched.

The state-space averaged modeling approach \cite{14} will be used to model the behavior of the PWM driven system. The state-space descriptions of the two states are

$$\dot{x}_+ = A x_+ + B u \ , \ \forall t \in [0, t_0] \ , \quad (26a)$$
$$\dot{x}_- = A x_- - B u \ , \ \forall t \in [t_0, T] \ , \quad (26b)$$

where the vector $x_+$ is the state vector of positive operating voltage and $x_-$ the vector of negative operating voltage. The further constraints during the switching stages

$$x_+(t = t_0) = x_-(t = t_0) \ , \quad x_+(t = 0) = x_-(t = T)$$

must be satisfied in static operation. One can solve both systems \eqref{26} with the Equation of Motion

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t - \tau) Bu(\tau) \, d\tau \ , \quad \Phi(t) = e^{At} \ , \quad \Psi(t) = A^{-1} [\Phi(t) - I]$$

under the conditions \eqref{27} and further approximating the result with the Taylor-series of the matrix exponential function. One gets the static turning points

$$x_-(T, d) \approx -A^{-1} [(2d - 1)I + dd'T A] Bu \ , \quad x_+(t_0, d) \approx -A^{-1} [(2d - 1)I - dd'T A] Bu \ , \quad (29)$$

where $d' = 1 - d$ is the complementary duty cycle. Eqs. \eqref{29} tend to an averaged state vector for high PWM switching frequencies

$$\ddot{x}(d) = -A^{-1} (2d - 1) Bu \ , \quad u = \frac{u_b}{2}$$

The derivation in the working point solves into the expected amplification

$$k_{PWM} = \frac{u_b}{2} \frac{d}{dd} (2d - 1) = u_b \ .$$

*For PWM drive is $u^2 = \frac{u_b^2}{d}$, for analog drive is $u^2 = 0$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure8}
\caption{Definition of the two switched PWM intervals}
\end{figure}
The Startup: The rectangular wave of the PWM driver can be modeled as a sum of sinusoids plus a DC part, resulting from the Fourier transform. The fundamental oscillation of the rectangular wave is \( \omega_{\text{pwm}} = \frac{2\pi}{T} \). This injection of sinusoids at the start up time gives rise to the conclusion, that the system shows an impulse response, even if the duty cycle is \( d = 0 \).

This behavior in the control loop might be undesirable for systems with a high PWM voltage or low PWM frequency, where the amplification of the fundamental wave is \( u^2 \). The behavior should be avoided by a proper compensation. If the control signal is unlimited, the compensation can be done by an additive negative impulse response. But it should also be possible to use an appropriate duty cycle at the startup time to cancel the disturbance. Thus, it has to be found an optimal duty cycle \( d_{\text{opt}} \) for the first period. The switch point at the end of the first period can be found with (26)

\[
x_1(T, d) = \Phi(d'T)x_{10} - \Psi(d'T)Bu, \quad x_{10} = \Psi(d'T)Bu
\]

\[
= [I + \Phi(T) - 2\Phi(d'T)]A^{-1}Bu \approx (2d - 1)TBu.
\]

The goal is to reach the static turning points (29) after the first period. The performance function, which has to be minimized, can be found to

\[
J = \frac{1}{2} \bar{x}^T \bar{x} \implies \bar{x} = x_1(T, d) - x_\perp(T, d = \frac{1}{2})
\]

\[
dJ \over dd = \left( \frac{dx_1(T, d)}{dd} \right)^T \bar{x} = 0.
\]

One can find the vector norm ratio for \( d \), if the approximation (32) and the result of \( x_\perp(T, d = \frac{1}{2}) \) is inserted into (34) and solved for \( d \)

\[
d(T) = \frac{1}{2} \frac{u^TB^T [TA - 4I]^{-1} [TA - 3I] Bu}{u^TB^T Bu} \quad \text{(35a)}
\]

\[
d(T) \leq \frac{1}{2} \frac{[TA - 4I]^{-1} [TA - 3I]}{2} \quad \text{(35b)}
\]

Eq. (35a) can be further simplified, if \( T \) tends to zero. The system matrix \( A \) is in that case ineffective and the ratio reduces to

\[
d_{\text{opt}} = \lim_{T \to 0} d(T) = \frac{3}{8} = 0.375 \quad \text{(36)}
\]

In the practical implementation, the control could be directly initialized with the value. Thus, at start up time the values will be directly available from the controller.

3.5. LTI System Model

As already noted, the system contains of an inner loop part with positive feedback and the outer loop parts, which contain all other system gains. Combining Eqs. (20,22,25,31) results in a linear time-invariant (LTI) system for the control design

\[
\dot{x} = \begin{pmatrix} A + k_p BC \end{pmatrix} x + B k_{\text{pwm}} u, \quad y = k_x C x.
\]

4. CONTROL DESIGN

The S/KS/GS/T-Standard-Design Problem (Mixed-Sensitivity-Approach) is used for the control design. In this design, the transfer function matrix \( N \) will be minimized with the \( \infty \)-Norm.
\[ N(K) = \begin{bmatrix} W^*_w S W^*_w & -W^*_w S G W^*_d \\ W^*_y K S W^*_w & -W^*_y T W^*_d \\ W^*_y T W^*_w & W^*_y S G W^*_d \end{bmatrix} \] (38)

Figure 9. Block description of the S/KS/GS/T standard problem

The \( H_\infty \) problem is the minimization of the transfer function matrix \( N \)

\[ \min_K \| N(K) \|_\infty \]

over all stable and proper controllers \( K \). Fig. 9 depicts the general control loop, where \( y \) is the plant output, \( w \) the reference signal, \( u \) the control signal, \( d \) the plant input disturbance, \( e \) the error signal and \( z \) the performance signals, which have to be minimized.

In our case, the main objective is the loop-shaping of the sensitivity transfer function \( S = [I + GK]^{-1} \). Thus, the disturbance rejection and the closed loop system bandwidth of \( S(j\omega) \) will be optimized. A constant bound is put on the control signal to further avoid actuator saturation. The realization of the weighting scheme is done with proper weighting factors \( W_x \) at the inputs and outputs of the block \( P(s) \).

**Limitation of the Bandwidth:** The early work [15] gives insight into the fundamental limits, which restrict the achievable performance of a control system. A limitation in the closed loop performance can be expected, since the system is unstable, because of the spring softening effect. The unstable system pole gives a lower bound on the closed loop system bandwidth. The system might also contain phase shifter systems (delay elements), which give an upper bound on the closed loop bandwidth.

It is shown in literature [16], that at least a ratio of two must be present between the gain crossover frequency\(^1\) and the unstable pole \( p_u \), and between the unstable zero \( z_u \) and \( \omega_c \) to reach acceptable robustness for the control system. Thus, a system might be badly robust, if the unstable pole is too far located in the right-hand plane (RHP), because of a highly undamped mechanical system or high sensor operating voltage, or if too much time delay is added into the loop, because of a low sampling time, low computing time or high conversion delay.

In [17] was suggested, that the best gain crossover frequency is the geometric mean of the unstable pole and zero. This might be the best proposal, if the unstable pole and unstable zero are closely located together. Otherwise, the best suggestion is a ratio between \( \omega_c \) and \( p_u \) of greater than two.

5. IDENTIFICATION

Since the system is highly unstable at nominal operating voltage, a direct system identification in an open loop fashion way is not possible. A two-stage identification to circumvent this problem was introduced in [18], where the system was identified at a very low operating voltage and in the second stage fine-tuned at nominal operating voltage.

The identification at very low operating voltage might be impossible for some circuit configuration or other reasons. Or the mechanical resonance frequency is not known or inaccurate, resulting in an incorrect extended model. All this gives rise to a more detailed examination of the voltage influence for the system and also could prove the theoretical aspects.

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\(^1\)The gain crossover frequency \( \omega_c \) is defined as the frequency, where \( |G(j\omega_c)K(j\omega_c)| \) first crosses 1 form above.
Apart from the two undamped imaginary poles from the spring-mass system, the squeeze-film effect introduces a real pole \( p_1 \) and a real zero \( z_1 \). The real pole is dominant as long as the system is not under-damped from small pressure rates or a large gap between the electrode and seismic mass. In the low frequency range the plant

\[
G_{\text{mech}}(s) = k_p \frac{\prod_{i=1}^{m} (s + z_i)}{\prod_{j=1}^{n} (s + p_j)}, \quad m = 1, \; n = 3
\]

(39)
can be reduced to a simple PT\(_1\)-system with neglecting the non-dominant imaginary poles \( p_2, p_3 \) and zero \( z_1 \). The resulting electromechanical model yields into

\[
\tilde{G}_{\text{mech}}(s) = \frac{k \tilde{G}_{\text{mech}}(s)}{1 - k_e G_{\text{mech}}(s)} = \frac{k \tilde{k}_p}{s + p_1 - k_e k_p}; \quad \tilde{G}_{\text{mech}}(s) = \frac{k_p}{s + p_1}, \quad k = k_u k_a k_c k_{PWM}.
\]

(40)
The pole of the electromechanical model is located at

\[
p_{1,u}(u_b) = -p_1 + \tilde{k}_p k_e
\]

(41)
The real pole moves into the right-half plane (RHP) with increasing influence of the electrostatic field. The influence of the operating voltage \( u_b \) on \( k_e \) is quadratic \((25)\). Thus, the pole moves on a parabola with the parameter \( u_b \). The imaginary and real parts of poles of higher order systems also move on a "parabola like" convex function, as long as the mapping of the poles \( p_{1,u}(u_b) = f(p_1; p_1 \ldots p_{i-1}, p_{i+1} \ldots p_n, z_1 \ldots z_m, u_b) \) do not reach the union point, and thus changes the space \( \mathbb{C} \) or \( \mathbb{R} \) of the poles (Fig. 10(a)). The global optimum of the convex function is the position of the mechanical poles.

This pole shift behavior gives the chance to approximately detect the mechanical pole location in the area around the mechanical poles with a parabola (Fig. 10(b)). This approximation offers another advantage – The parabola is fully described with just two points. The behavior also explains the two-stage identification algorithm – The pole shift is very low in the low operating voltage region, because the gradient around the vertex of the parabola is low. The higher order electrostatic system including the outer gain \( \tilde{k} \) will be defined in a monic polynomial configuration

\[
G_{\text{mech}}(s, u_b) = \tilde{k} k_p \frac{\prod_{i=1}^{m} (s + z_i)}{\prod_{j=1}^{n} (s + p_{2,u}(u_b))} = \tilde{k} k_p \frac{\prod_{i=1}^{m} (s + z_i)}{\prod_{j=1}^{n} (s + p_j) - k_p k_e \prod_{i=1}^{m} (s + z_i)}, \quad n > m
\]

(42)

Figure 10. Pole and zero locations of a theoretical example in dependence on the operating voltage

\[
\begin{align*}
\text{(a) Root locus} & & \text{(b) Pole location} \\
\end{align*}
\]

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\[
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\]

(42)
where the static gain $\hat{k}_p$ is dependent on the pole and zero location and orders $m$ and $n$. From (42) can also be seen, that the independence of the static gain from $k_p$ is only valid for a strictly proper \textit{mechanical System} $G_{\text{mech}}(s)$. The identification algorithm to fully represent the mechanical system can now be described as follows:

\section*{Algorithm 1} The pole movement around the zero operating voltage can be approximately described with a parabola of second order

\[ p_{j,i}(u_b) \approx \left( \varphi_{jnr} + \varphi_{jzi} \right)^{u_b^2} + j \left( \varphi_{jnr} + \varphi_{jzi} \right)^{u_b^2}. \]

where the vertex of the parabola defines the approximate pole location of the mechanical poles $\varphi_{jnr} + j\varphi_{jzi}$. The value of the mechanical plant gain $k_p$ can be approximately calculated from the dominant real pole to

\[ k_p \approx \hat{k}_p = \frac{\varphi_{jzi}u_b^2}{k_p}. \]

The system forward gain $k_p \hat{k}$ is the static gain of $G_{\text{mech}}$ divided by the quotient of the products of the pole and zero locations

\[ k_p \hat{k} = G_{\text{mech}}(s = 0, u_b) \prod_{i=1}^{p} p_{j,i}(u_b) \prod_{i=1}^{z} \frac{1}{z_i}. \]

The dependence of the gain $k_p \hat{k}$ is described with a cubic parabola in $u_b$

\[ k_p \hat{k} = \varphi_{jzi}u_b^3. \]

It is to note, that finding the system poles and zeros in $S$-domain is not a trivial task. Identification routines generally find a discrete-time system model. The mapping from $S$ to $Z$-domain usually increases the order of the zeros, depending on the transformation. Apart from the system order, the ZOH transformation fails for real negative poles and does not provide a simple mapping for the zeros.

\section*{6. RESULTS}

The identification and control design routine have been successfully applied to several samples. Fig. [11(a)] shows the high order\footnote{A system $G(s)$ is strictly proper, if $G(s) \rightarrow 0$ as $s \rightarrow \infty$ and thus $n > m$.} identified system with the ARX (Autoregressive Model with exogenous Input) model structure. This predictor guarantees stability for the direct system identification \footnote{A numerator and denominator order of 10 was used for the identification model.} and is very simple to implement. The bode diagram shows no dependences in the high frequency region and therefore, it can indeed be modeled as a simple $PT_1$-system. Fig. [11(b)] shows the influence of the electrostatic field on the dominant pole location, where the parabola fit shows a very good agreement to the measured values. Because of the circuit configuration, the measurement had to be stopped at $u_b = 10$ V.

Fig. [12(a)] shows the system forward gain, and the parabola fit shows also here a good concordance. The system DC-gain, evaluated from the two fits, is shown in Fig. [12(b)]. The DC-gain shows a good conformity with the measured values too, even the fit is not ideal. This inaccuracy might arise from the two fully decoupled identification steps and a possible, too large operating voltage interval.

The result of the applied control\footnote{The predictor form of the system is $\hat{y}(t|\Theta) = H^{-1}(z, \Theta)G(z, \Theta)u(t) + [1 - H^{-1}(z, \Theta)] y(t)$, where $G$ defines the to be identified model and $H$ the noise filter. The transfer functions for the ARX structure are $G(z, \Theta) = \frac{H(z)}{\Phi(z)}$ and $H(z, \Theta) = \frac{\Phi(z)}{\Phi(z)}$, where the equal monic denominators in $G$ and $H$ cancel the (unstable) poles and therefore guarantee stability for the predictor. The predictor reduces to a linear regression model, which can be simply solved with the least-square method.} compared to the simulation shows Fig. [13]. It can be seen, that the practical results show a very good conformance with the simulated results. But the results also show some still neglected problems. The nonlinear simulation model shows differences in the output (Fig. [13(b,d)]) for both, the DC-gain and dynamics, even if the

\footnote{The weights $W_w = 1, W_u^{-1} = \frac{1}{2}$ and $W_e = \frac{M_0^{-1}}{\pi \omega_b}$ were applied for the $H_\infty$-minimization, where $M_0 = 2, \omega_b = 2\pi200$ Hz and $e \rightarrow 0$.}
applied disturbance signal has a very low amplitude. One other problem is the slow disturbance rejection at low operating voltage. The open-loop poles reappear after mirroring into the left-half plane (LHP) as closed-loop poles [20], which is not propitious for a good design and may be avoided with other weights in the control design. The slow disturbance rejection also rises the nonlinear dynamical system behavior which can be seen in Fig. 13(a,c).

**The rectangular disturbance signal at the input of the controller has an amplitude of 50 mV and a period of $T = 4096 \ T_s$, where the sample time $T_s = \frac{4096}{60 \ MHz}$.**

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Figure 11. Influence of the electrostatic field

(a) Bode diagrams of the identified system

(b) Pole location of the dominant pole

Figure 12. Influence of the electrostatic field on the DC-gain

(a) Evaluated voltage dependent system forward gain

(b) Electrostatic system DC-gain
7. CONCLUSIONS

Analytical methods have been used to describe the nonlinear accelerometer system. A reduced order LTI system for the control design was generated at the zero operating point. The $\mathcal{H}_\infty$-control design guarantees enough robustness to control the nonlinear accelerometer system. Limitations to define the system bandwidth were given explicitly.

A new identification algorithm was introduced to approximately define the mechanical system and the system open loop gain, which is used to get the system model at nominal operating voltage. The algorithm makes it possible to fully describe the electrostatic system with just only two data sets at two different operating voltages. The operating voltages can be set individually.

The identification and control routines have been successfully applied to several sample systems. The identification routine also proves the analytical model of the electrostatic system and its dependence on the operating voltage.
References