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Preface

The results of this dissertation refer to the geometry of Minkowski spaces, i.e., of finite dimensional normed linear spaces. Also in view of very recent developments, this field can be located at the intersection of Finsler Geometry, Banach Space Theory, and Convex Geometry, but it is also closely related to Distance Geometry and Abstract Convexity. Moreover, the main part of the obtained results belongs to the Discrete Geometry of normed planes, where the use of numerical methods is new in this field. Having chosen such an approach, we follow the modern trend that numerical computations are based on exact arithmetics. Within these computations semi-algebraic subsets of the real numbers occur frequently as basic mathematical objects.

More precisely, the results obtained here can be classified to belong to the following topics: Discrete and Convex Geometry (MR 52), including the theory of polytopes, finite dimensional Banach Space Theory (MR 46Bxx), and Foundations of (non-Euclidean) Geometries (MR 51 and MR 53).

Many specialists in the field of Discrete Geometry know the problem of classifying 2-distance sets. We present such a classification, which can be considered as the main result of the dissertation. Due to several talks of myself at international conferences, these experts are waiting with large interest for this complete classification of 2-distance sets. It occurs here for the first time in printed form. Other parts of the dissertation refer to characterizations of inner product spaces by geometric properties of the space and related subjects.

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Introduction

Within Chapter 1 we collect a lot of definitions and basic results which are well known and standard. For basic notions, such as real linear vector space, we give exact definitions as far as appropriate. We will not state all their well known properties, although later on these might be used.

In the following Chapter 2 we consider some possibilities to generalize the well known concepts of measuring angles and bisecting angles of the Euclidean plane to Minkowski planes. Then we focus on all Minkowski planes with the property that two such possibilities yield exactly the same angular bisector for every angle. This yields characterizations of inner product spaces, of Radon planes and of equiframed planes. These results and parts of this chapter are contained in my papers [14, 15].

Then we consider higher dimensional Minkowski spaces with the same property. Using well known results from convex geometry, the two-dimensional result generalizes easily to higher dimensions in case of Euclidean and Radon planes. A new similar result for equiframed planes is obtained in Chapter 3 which is also published in [13].

In Chapter 4 we introduce the theory of embedding a given finite metric space into a suitable Minkowski space of given dimension $d$, or into a given Minkowski space with polytopal unit ball. The results of this chapter are needed in Chapter 6 for $d = 2$. But also not depending on this, the theory developed here has many applications in view of practical embedding problems.

Chapter 5 is another preparatory part for Chapter 6 also to use the results from Chapter 4. Here we introduce terminology and algorithms for solving families of linear systems of equations and inequalities. We emphasize the use of certificates to decouple the algorithms which should find a solution of a family of parametrized linear systems from a simpler procedure, which verifies that this solution is correct.

Finally, Chapter 6 gives a complete classification of 2-distance sets in Minkowski planes.

The appendix contains the bibliography, an index, another index of symbols with the corresponding terms and introducing page numbers, and the theses.
Chapter 1

Prerequisites

1.1 Basics

We denote the field of real numbers by \( \mathbb{R} \), the field of rational numbers by \( \mathbb{Q} \), the ring of integers by \( \mathbb{Z} \), the set of positive integers by \( \mathbb{N} \) and the set of non-negative integers by \( \mathbb{N}^0 \). We use the standard symbols for the set operations union \( A \cup B \), intersection \( A \cap B \) and complement \( \overline{A} \) of the sets \( A \) and \( B \). \( A \) is a subset of \( B \), \( A \subseteq B \) and \( B \supseteq A \), if every element of \( A \) is also an element of \( B \), thus \( A \subseteq B \) holds true. \( A \subset B \) means that \( A \subset B \) and \( A \neq B \). The notation \( A \cup B \) of disjoint union is an expression for the union set \( A \cup B \) and also expresses the relation \( A \cap B = \emptyset \). This symbol is also used for more than two sets. We use the abbreviation \( \mathbb{N}_n := \{1, 2, \ldots, n\} \) for \( n \in \mathbb{N} \).

Furthermore, we denote the closed interval \( \{ x \in \mathbb{R} : a \leq x \leq b \} \) by \([a, b] \) \( (a, b) \in \mathbb{R} \), the open interval \( \{ x \in \mathbb{R} : a < x < b \} \) by \((a, b) \) \( a \in \mathbb{R} \) and \((b, a) \) \( b \in \mathbb{R} \). The half open intervals are denoted by \([a, b) \) \{ \( [a, b) \) \ and \( (a, b] \) \{ \( (a, b] \) \}. The set of positive real numbers is denoted by \( \mathbb{R}^+ := (0, \infty) \), the set of non-negative real numbers is denoted by \( \mathbb{R}^{\geq 0} := [0, \infty) \).

By \( f : A \to B \) we denote a function (or map) which maps exactly one \( b \in B \) to each \( a \in A \), also denoted by \( f(a) = b \) or \( f : a \mapsto b \). The set of all such functions is denoted by \( B^A \). For some subset \( X \subset A \) we denote by \( f(X) := \{ f(x) : x \in X \} \) the image (set) of \( X \) under \( f \), and for \( Y \subset B \) we denote by \( f^{-1}(Y) := \{ a \in A : f(a) \in Y \} \) the inverse image of \( Y \) under \( f \). The function \( f \) is called bijective (\( f \) is a bijection), if for each \( b \in B \) there is exactly one \( a \in A \) with \( f(a) = b \), i.e., if \( |f^{-1}\{\{b\}\}| = 1 \) for all \( b \in B \). We denote the cardinality of \( X \) by |\( X | \). This is for finite sets \( X \) the number of elements in \( X \), otherwise some symbolic value \( \infty \) which is greater than any natural number. For two functions \( f : A \to B \) and \( g : B \to C \) we define its concatenation \( g \circ f : A \to C \) by \( a \mapsto g(f(a)) \). A sequence in \( A \) is a function \( f : \mathbb{N} \to A \) (called finite sequence) or a function \( f : \mathbb{N} \to A \) (called infinite sequence). For the sequence \( f : \mathbb{N} \to A \) we also write \( (f_1, f_2, \ldots, f_n) \) or \( (f_i)_{i \in \mathbb{N}_n} \). In this sense we identify \( A^{\mathbb{N}_n} \) with \( A^n \). Analogously for \( f : \mathbb{N} \to A \) we write \( (f_1, f_2, \ldots) \) and \( (f_i)_{i \in \mathbb{N}} \). In some circumstances, if the meaning is clear from the context, we use a mixed notation, \((i, J, k)\) for example with \( i, k \in A, J \in A^l \), which is formally in \( A \times A^l \times A \), to denote the corresponding sequence \((i, J_1, \ldots, J_k, k)\) in \( A^{l+2} \).

We denote the power set of \( X \) by \( \mathcal{P}(X) := \{ Y : Y \subset X \} \) and the \( n \)-power set of \( X \) by \( \left(\begin{array}{c} X \\ n \end{array}\right) := \{ Y \in \mathcal{P}(X) : |Y| = n \} \). \( \mathbb{P}_n := \mathbb{N}^2 \setminus \{ (i, i) : i \in \mathbb{N}_n \} \) is the set of all pairs of distinct integers from 1 to \( n \). We denote the sign of a real number \( \lambda \) by \( \text{sign}(\lambda) \) and the absolute value of \( \lambda \) by |\( \lambda \|.

1.2 Metric spaces

From a certain classical point of view, geometry is the science of measuring distances. Therefore the metric space is a very basic and general geometric object.

**Definition 1.1** A pair \((X, \rho)\) consisting of a set \( X \) and a function \( \rho : X \times X \to \mathbb{R} \) is called a metric space with distance function (metric) \( \rho \) if and only if
Definition 1.2 An embedding $f$ of the metric space $(X, \rho)$ into the metric space $(\tilde{X}, \tilde{\rho})$ is a map $f : X \rightarrow \tilde{X}$ satisfying $\tilde{\rho}(f(x), f(y)) = \rho(x, y)$ for all $x, y \in X$.

Definition 1.3 We call a function $\phi : X \rightarrow \tilde{X}$ an isometry between the two metric spaces $(X, \rho)$ and $(\tilde{X}, \tilde{\rho})$ if it is a bijective embedding of $(X, \rho)$ into $(\tilde{X}, \tilde{\rho})$. If there is any isometry between $(X, \rho)$ and $(\tilde{X}, \tilde{\rho})$, then these two metric spaces are called isometric.

Definition 1.4 A subspace of a metric space $(X, \rho)$ is any metric space $(\tilde{X}, \tilde{\rho}|_{\tilde{X} \times \tilde{X}})$ where $\tilde{X} \subset X$ and $\tilde{\rho}|_{\tilde{X} \times \tilde{X}}$ denotes the restriction of the function $\rho$ to arguments consisting of two elements from $\tilde{X}$.

By $U_\varepsilon(\delta) := \{x \in X : \rho(x, \delta) \leq \varepsilon\}$ we denote a (closed) $\varepsilon$-neighborhood of $\delta \in X$ belonging to some metric space $(X, \rho)$ for the real number $\varepsilon > 0$.

1.3 Topological notation

- A set $A \subset X$ is called open (in $(X, \rho)$) if for each $a \in A$ there is some $\varepsilon > 0$ with $U_\varepsilon(a) \subset A$.
- A set $A \subset X$ is called closed (in $(X, \rho)$) if its complement $X \setminus A$ is open in $(X, \rho)$.
- The interior of $A$, $\text{int} A$, is the largest open set contained in $A \subset X$, $\text{int} A = \{a \in A : \exists \varepsilon > 0 : U_\varepsilon(a) \subset A\}$.
- The (topological) closure $\text{cl} A$ of a set $A \subset X$ is the set $\text{cl} A := X \setminus \text{int}(X \setminus A)$.
- The boundary $\partial A$ of $A \subset X$ is the set $\partial A := \text{cl} A \setminus \text{int} A$.

1.4 Vector spaces

The linear vector space is a model of some simple algebraic structure, well suited as background space for doing geometry.

Definition 1.5 A real linear vector space $(\mathbb{V}, \cdot, +)$ is a set $\mathbb{V}$ together with two binary operations $\cdot : \mathbb{R} \times \mathbb{V} \rightarrow \mathbb{V}$, $(\lambda, \mathbf{f}) \mapsto \lambda \mathbf{f}$ (scalar multiple) and $+: \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{V}$, $(\mathbf{f}, \mathbf{g}) \mapsto \mathbf{f} + \mathbf{g}$ (addition) satisfying the following axioms:

- $\mathbf{f} + \mathbf{g} = \mathbf{g} + \mathbf{f}$ for all $\mathbf{f}, \mathbf{g} \in \mathbb{V}$
- $(\mathbf{f} + \mathbf{g}) + \mathbf{h} = \mathbf{f} + (\mathbf{g} + \mathbf{h})$ for all $\mathbf{f}, \mathbf{g}, \mathbf{h} \in \mathbb{V}$
- there is one origin, $\mathbf{0} \in \mathbb{V}$,
- $\mathbf{f} + \mathbf{0} = \mathbf{f}$ for all $\mathbf{f} \in \mathbb{V}$
- for each $\mathbf{f} \in \mathbb{V}$ there is one vector $-\mathbf{f} := \mathbf{g} \in \mathbb{V}$ with $\mathbf{f} + \mathbf{g} = \mathbf{0}$

(i.e., $(\mathbb{V}, +)$ forms an Abelian group)

- $1 \mathbf{f} = \mathbf{f}$ for all $\mathbf{f} \in \mathbb{V}$
- $\lambda (\mu \mathbf{f}) = (\lambda \mu) \mathbf{f}$ for all $\mathbf{f} \in \mathbb{V}$ and $\lambda, \mu \in \mathbb{R}$
- $(\lambda + \mu) \mathbf{f} = \lambda \mathbf{f} + \mu \mathbf{f}$ for all $\mathbf{f} \in \mathbb{V}$ and $\lambda, \mu \in \mathbb{R}$
- $\lambda (\mathbf{f} + \mathbf{g}) = \lambda \mathbf{f} + \lambda \mathbf{g}$ for all $\mathbf{f}, \mathbf{g} \in \mathbb{V}$ and $\lambda \in \mathbb{R}$
(i.e., the scalar multiple fits together with the addition operation). The elements of \( V \) are called **vector** or **point** of \( V \), and we denote them by lowercase Euler Fraktur letters.

Note that we identify \( V \) with \((V, \cdot, +)\) whenever it is appropriate.

**Definition 1.6** A system \( (\xi_i)_{i \in I} \) of vectors \( \xi_i \) (\( i \in I \)) of a real linear vector space \( V \) is called **linearly independent**, if there is no corresponding finite system \( (\alpha_i)_{i \in I'} \), \( \emptyset \neq I' \subset I \), of real nonzero numbers \( 0 \neq \alpha_i \in \mathbb{R} \) (for \( i \in I' \)) with \( \sum_{i \in I'} \alpha_i \xi_i = 0 \). Otherwise this system is called **linearly dependent**.

**Definition 1.7** A **linear subspace** \((V', \cdot|_{V'}, +|_{V'})\) of a linear real vector space \((V, \cdot, +)\) is a subset \( V' \subset V \) which is itself a real linear vector space with the same, restricted, binary operations, i.e., if it holds \( \forall \lambda \in \mathbb{R}, \phi, \eta \in V' \) that \( \lambda \phi \in V' \) and \( \phi + \eta \in V' \).

**Definition 1.8** An **affine subspace** \((V', \cdot', +')\) of a real linear vector space \((V, \cdot, +)\) is any real linear vector space which is a translate of a linear subspace \( \operatorname{aff} V \) of \( V \). More precisely, for any linear subspace \( \operatorname{aff} V \) of \( V \) and \( \alpha \in \mathbb{R} \) we define \( V' := \operatorname{aff} V + \alpha \cdot' := \alpha \cdot \) (new origin), \( \lambda \cdot' \xi := \alpha' + \lambda \cdot (\xi - \alpha') \) and \( \xi +' \eta := \alpha' + (\xi - \alpha') + (\eta - \alpha') = \xi + \eta - \alpha \), and then we call \((V', \cdot', +')\) an affine subspace of \( V \).

Note that \( X \subset V \) is an affine subspace of \( V \) if and only if \( \lambda \xi + (1 - \lambda) \eta \in X \) for all \( \xi, \eta \in X \), \( \lambda \in \mathbb{R} \).

**Definition 1.9** For a system \( (\xi_i)_{i \in I} \) of vectors \( \xi_i \) (\( i \in I \neq \emptyset \)) of a real linear vector space \( V \) the **linear subspace spanned by** \( (\xi_i)_{i \in I} \), \( \operatorname{lin} \{ \xi_i : i \in I \} \), also called the **linear hull** of \( (\xi_i)_{i \in I} \) or of the set \( \{ \xi_i : i \in I \} \), is the smallest linear subspace of \( V \) containing all vectors \( \xi_i \), \( i \in I \). More precisely, we have

\[
\operatorname{lin} \{ \xi_i : i \in I \} = \left\{ \sum_{i \in I'} \alpha_i \xi_i : I' \subset I, |I'| < \infty, \alpha_i \in \mathbb{R} \forall i \in I' \right\}.
\]

**Definition 1.10** For a system \( (\xi_i)_{i \in I} \) of vectors \( \xi_i \) (\( i \in I \neq \emptyset \)) of a real linear vector space \( V \) the **affine hull** \( \operatorname{aff} \{ \xi_i : i \in I \} \), of \( (\xi_i)_{i \in I} \) or of the set \( \{ \xi_i : i \in I \} \), is the smallest affine subspace of \( V \) containing all vectors \( \xi_i \), \( i \in I \). More precisely, we have

\[
\operatorname{aff} \{ \xi_i : i \in I \} = \left\{ \sum_{i \in I'} \alpha_i \xi_i : I' \subset I, |I'| < \infty, \alpha_i \in \mathbb{R} \forall i \in I', \sum_{i \in I'} \alpha_i = 1 \right\}.
\]

Note that this set can be regarded as affine subspace of \( V \) after fixing some \( \alpha' \in \operatorname{aff} \{ \xi_i : i \in I \} \) as origin, as described in Definition 1.8.

**Definition 1.11** A system \( (\xi_i)_{i \in I} \) of vectors \( \xi_i \) (\( i \in I \)) of a real linear vector space \( V \) forms a **basis** of \( V \), if it is linearly independent and linearly spans \( V = \operatorname{lin} \{ \xi_i : i \in I \} \). In this case we call the cardinality \( |I| \) of \( I \) the **dimension** of \( V \), \( \dim V \), and if \( \dim V < \infty \), then \( V \) is called a **finite dimensional real linear vector space**. For any set \( U \subset V \) we denote its (affine) **dimension** by \( \dim U := \dim \operatorname{aff} U \). Note that this quantity is independent on the choice of the origin of \( \operatorname{aff} U \), since \( \operatorname{aff} U \) is implicitly considered as affine subspace of \( V \) for which we already defined the dimension.

**Definition 1.12** We call a set \( S \subset V \) **affinely independent**, if for all \( \xi \in S \) we have that \( \xi \notin \operatorname{aff} (S \setminus \{ \xi \}) \). Otherwise \( S \) is called affinely dependent.

The finite subset \( S \subset V \) is affinely independent if and only if \( \dim \operatorname{aff} S = |S| - 1 \).

It is a fundamental result that each finite dimensional real linear vector space \( V \) is isomorphic to \( \mathbb{R}^d \), where \( d = \dim V \). This means that there is a bijection \( \phi : \mathbb{R}^d \to V \) satisfying \( \phi(\lambda \xi) = \lambda \phi(\xi) \) for all \( \lambda \in \mathbb{R} \) and \( \xi \in \mathbb{R}^d \) and \( \phi(\xi + \eta) = \phi(\xi) + \phi(\eta) \) for all \( \xi, \eta \in \mathbb{R}^d \), always using the operations of the corresponding vector space. Thus we know all finite dimensional real linear vector spaces if we know all the spaces \( \mathbb{R}^d \), \( d = 0, 1, \ldots, d \) is the set of all \( d \)-tuples \( (\xi_1, \ldots, \xi_d) \) of \( d \) real numbers \( \xi_i, 1 \leq i \leq d \).

Addition and the scalar multiple in \( \mathbb{R}^d \) is simply done for each component via \( \lambda (\xi_1, \ldots, \xi_d) = (\lambda \xi_1, \ldots, \lambda \xi_d) \) and \( (\xi_1, \ldots, \xi_d) + (\eta_1, \ldots, \eta_d) = (\xi_1 + \eta_1, \ldots, \xi_d + \eta_d) \). We also use the notation of
column-vectors,
\[
\mathbf{r} = \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix} := a_1 \mathbf{b}_1 + \cdots + a_d \mathbf{b}_d,
\]

where \( B := (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_d) \) is some fixed basis of \( \mathbb{R}^d \) and \( a_1, \ldots, a_d \) are called the coordinates of \( \mathbf{r} \) in the basis \( B \). For the standard basis \((e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1))\), the coordinates \( a_i \) of \( \mathbf{r} \) coincide with the components \( r_i \) of \( \mathbf{r} = (r_1, \ldots, r_d) \). We denote the transposed of a matrix \( M \) or vector \( \mathbf{r} \) by \( M^\top \) or \( \mathbf{r}^\top \) (that is, interchanging rows and columns of \( M \) or \( \mathbf{r} \)). We denote by \( M_{i,i} \) the \( i \)-th row of \( M \) and by \( M_{i,j} \) the \( j \)-th column of \( M \).

For some vector \( \mathbf{r} \in X^d \) (\( X = \mathbb{R} \) or \( X = \mathbb{Z}[X_1, \ldots, X_k] \) for example), or sequence \( \mathbf{r} \) or \( d \)-tuple \( \mathbf{r} \) (which are all identified) and each \( i \in \mathbb{N}_d \) we denote by \( r_i \) the \( i \)-th component of \( \mathbf{r} \) as long as we did not introduce another notation. Sometimes we will explicitly use the exception \( \mathbf{r} = (r_0, \ldots, r_d) \in X^{d+1} \).

1.4.1 Algebraic notations

For subsets \( A, B \) of \( \mathbb{R}^d \), vectors \( \mathbf{r} \in \mathbb{R}^d \) and scalars \( \lambda \in \mathbb{R} \) we use the common algebraic notations
\[
A + B := \{ a + b : a \in A, b \in B \} \quad \text{(Minkowski sum),}
\lambda A := \{ \lambda a : a \in A \}
\]
and its special cases
\[
-A := (-1)A, \quad A - B := A + (-B), \quad A + \mathbf{r} := A + \{ \mathbf{r} \}, \quad A - \mathbf{r} := A + \{-\mathbf{r}\}
\]
and so on. Multiplication of vectors by a scalar can be generalized to sets: if \( A \subset \mathbb{R}, X \subset \mathbb{R}^n \), then \( A \cdot X := AX := \{ a\mathbf{x} : a \in A, \mathbf{x} \in X \} \), with the special case \( A\mathbf{r} := A\{\mathbf{r}\} \) for \( \mathbf{r} \in \mathbb{R}^n \).

1.4.2 The Euclidean metric

The most natural way to define a metric in \( \mathbb{R}^d \) is to use the Pythagorean theorem, which gives the Euclidean metric \( g_e(\mathbf{r}, \mathbf{η}) := \|\mathbf{r} - \mathbf{η}\|_2 \) via the Euclidean norm \( \|(x_1, \ldots, x_d)\|_2 := \sqrt{x_1^2 + \cdots + x_d^2} \).

We denote the Euclidean \( d \)-dimensional space by \( \mathbb{E}^d := (\mathbb{R}^d, g_e) \).

1.4.3 Topological notions

We consider all topological notions of \( \mathbb{R}^d \) (open and closed sets, interior, closure and boundary) as usually defined in the metric space \( \mathbb{E}^d \). It is well known that all \( d \)-dimensional normed spaces (see below) induce the same topology in \( \mathbb{R}^d \).

- The relative boundary \( \text{rel bd} \ A \subset \mathbb{R}^d \) is the boundary of \( A \) with respect to the affine hull \( H := \text{aff} \ A \) of \( A \), i.e., in the metric space \( (H, g_e(\cdot, \cdot)|_H) \) we have \( \text{rel bd} \mathbb{E}^d A := \partial_H (\text{rel bd} \mathbb{E}^d A) \).
- A set \( A \subset \mathbb{R}^d \) is called bounded, if there is some \( c \in \mathbb{R} \) with \( g_e(\mathbf{o}, \mathbf{r}) < c \) for all \( \mathbf{r} \in A \).
- A set \( A \subset \mathbb{R}^d \) is called compact, if it is closed and bounded.

1.4.4 Affine geometric notions

The straight segment joining \( \mathbf{r} \) and \( \mathbf{η} \) is denoted by \( \mathbb{S} := \{ \lambda \mathbf{r} + (1 - \lambda) \mathbf{η} : \lambda \in [0, 1] \} \), the straight line through \( \mathbf{r} \) and \( \mathbf{η} \) by \( \mathbb{L} := \{ (1 - \lambda) \mathbf{r} + \lambda \mathbf{η} : \lambda \in \mathbb{R} \} \), and the ray with starting point \( \mathbf{r} \) passing through \( \mathbf{η} \) by \( \mathbb{R} := \{ (1 - \lambda) \mathbf{r} + \lambda \mathbf{η} : \lambda \geq 0 \} \). Small Greek letters denote real numbers or functions.

A subset \( A \subset \mathbb{R}^d \) is called convex, if for all \( a, b \in A \) also the segment \( \overline{ab} \) belongs to \( A \). The convex hull, \( \text{conv} \ A \), of some subset \( A \subset \mathbb{R}^d \) is the smallest convex set in \( \mathbb{R}^d \) containing \( A \),
\[
\text{conv} \ A = \{ \alpha_1 a_1 + \cdots + \alpha_k a_k : \alpha_1, \ldots, \alpha_k \geq 0, \ a_1, \ldots, a_k \in A, \ \alpha_1 + \cdots + \alpha_k = 1 \}.
\]

We call each compact, convex subset of \( \mathbb{R}^d \) with interior points a convex body.

We say that two sets \( A, B \subset \mathbb{R}^d \) are homothetic if there is some \( \lambda > 0 \) and \( \mathbf{r} \in \mathbb{R}^d \) with \( B = \lambda A + \mathbf{r} \).
1.4.5 Linear geometric notions

A convex cone is a convex set $C \subseteq \mathbb{R}^d$ which contains for every vector $x \in C$ also the ray $[0, x) \subset C$. The convex conical hull, cone $A$, of a set $A \subseteq \mathbb{R}^d$ is the smallest convex cone containing $A$, 

$$\text{cone } A = \{ \alpha_1 a_1 + \cdots + \alpha_k a_k : a_1, \ldots, a_k \in A, \alpha_1, \ldots, \alpha_k \geq 0 \}.$$

A set $B \subseteq \mathbb{R}^d$ which is symmetric with respect to the origin $0$, i.e., $B = -B$, is called centered.

1.5 Polynomials

Definition 1.13 Let $R$ be some commutative ring with identity (e.g., $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Q}$). A function $f : R \to R$ is called mono-variate polynomial function over $R$, if $f(x) = \sum_{i=0}^{m} a_i x^i = a_0 + a_1 x + \cdots + a_m x^m$. Note that in this context we define $x^0 := 1$, even if $x = 0$. We further assume that the characteristic of $R$ is 0, i.e., adding a finite number of summands 1 will never yield the sum 0. Then we can identify polynomial functions with abstract polynomials over $R$, which are represented by a sequence of coefficients $f := (a_0, a_1, \ldots, a_m) \in R[X]$, $a_0, \ldots, a_m \in R$. A contracted polynomial satisfies $a_m \neq 0$. $m$ is called in this case the degree of $f$, and $a_m$ is called its leading coefficient. The trivial polynomial 0 is the empty coefficient sequence, its corresponding polynomial function is identically zero. The degree of 0 is defined as $-\infty$. Each other polynomial has a contracted representation defining its degree.

The set $R[X]$ of all (abstract) mono-variate polynomials is itself a ring, using the pointwise addition and multiplication operations of the corresponding polynomial functions.

Bivariate and polyvariate polynomials $p \in R[X_1, \ldots, X_n]$ over $R$ can be defined recursively as mono-variate polynomials over mono-variate or polyvariate polynomials, i.e., over $R[X_1, \ldots, X_{n-1}]$. They can be identified with functions $f : R^n \to R, \mathbf{r} \mapsto \sum_{i \in \mathbb{N}^n} a_i \mathbf{r}^i$, where for $i \in \mathbb{N}^n$ and $\mathbf{r} \in R^n$ we use the notation $\mathbf{r}^i := \Pi_{j=1}^n r_j^{i_j}$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called homogeneous (of degree $h$), if $f(\lambda \mathbf{r}) = \lambda^h f(\mathbf{r})$ for each $\mathbf{r} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$. It is called positive homogeneous (of degree $h$), if $f(\lambda \mathbf{r}) = |\lambda|^h f(\mathbf{r})$ for each $\mathbf{r} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.

The degree of a polyvariate polynomial $f \in R[X_1, \ldots, X_n]$, $f(\mathbf{r}) = \sum_{i \in \mathbb{N}^n} a_i \mathbf{r}^i$, is $\deg f = \max_{i \in \mathbb{N}^n, a_i \neq 0} \sum_{j=1}^n i_j$.

Note that $f \in R[X_1, \ldots, X_n]$ is homogeneous of degree $h$ if $\sum_{j=1}^n i_j = h$ for each $i \in \mathbb{N}^n$ with $a_i \neq 0$.

Evaluating a polynomial $f \in R[X_1, \ldots, X_n]$ at some point $p \in \mathbb{R}^n$ simply means to determine the function value $f(p)$ of $f$ at $X_1 = p_1, \ldots, X_n = p_n$. For container objects $C$ consisting of more polynomials (e.g., sequences, vectors, matrices, sets) we denote by $C(p)$ the corresponding object where each polynomial is replaced by its evaluated value at $p$.

1.6 Algebraic numbers

The set $\mathcal{A}$ of real algebraic numbers consists of all real roots of mono-variate polynomials $a \in \mathbb{Z}[X]$, $a \neq 0$.

For each real algebraic number $a$ there is a unique representation as either a rational number $a = \frac{p}{q}$ or otherwise as the $n$-smallest real root of an irreducible mono-variate polynomial $f \in \mathbb{Z}[X]$ of degree at least 2. This means that $\{ x \in \mathbb{R} : f(x) = 0 \} = \{ r_1, \ldots, r_k \}$ with $r_1 < r_2 < \cdots < r_k$. Then we write $a = \text{RootOf}(n, f) := r_n$, with $1 \leq n \leq k$. The pair $(p, q)$ and the polynomial $f$ are only unique up to multiplication by a nonzero integer. But there is a unique reduced version if we additionally require that $q > 0$ or that the leading coefficient of $f$ is positive, respectively.
1.7 Linear functions

**Definition 1.14** A function \( a : \mathbb{R}^n \to \mathbb{R}^m \) is called an affine linear function (or simply affine map), if there is some matrix \( A \in \mathbb{R}^{m \times n} \) and vector \( b \in \mathbb{R}^m \) with \( a(x) = Ax + b \) for all \( x \in \mathbb{R}^n \). If \( b = 0 \), then the map \( a \) is even called a linear function.

The rank \( r \) of a matrix \( A \in \mathbb{R}^{m \times n} \) is the largest number of rows (and equivalently, of columns) of \( A \) which are linearly independent. We denote this value by \( \text{rank} A \).

1.8 Minkowski geometry

The geometry of finite dimensional real linear normed spaces (finite dimensional real Banach spaces) is a generalization of Euclidean geometry. It describes the local geometry of Finsler spaces. A basic reference on this so-called Minkowski geometry is [36].

**Definition 1.15** We say that a function \( \|\cdot\| : V \to \mathbb{R} \) is a norm of the real linear vector space \( V \) if it satisfies the following properties:

- \( \|x\| \geq 0 \) for all \( x \in V \).
- \( \|x\| = 0 \) if and only if \( x = 0 \).
- \( \|\lambda x\| = |\lambda| \|x\| \) for all \( x \in V \) and \( \lambda \in \mathbb{R} \) (positive homogeneity)
- \( \|x + y\| \leq \|x\| + \|y\| \) for all \( x, y \in V \) (triangle inequality)

**Definition 1.16** For a fixed norm \( \|\cdot\| \) (see Definition 1.15) in the finite dimensional real linear vector space \( V \) we call the metric space \((V, \rho : (x, y) \mapsto \|x - y\|)\) a Minkowski space.

If \((V, \rho)\) is a Minkowski space, then its corresponding norm is determined by \( \|x\| = \varrho(x, 0) \), where \( \varrho \) is the origin of \( V \).

Since each finite dimensional real linear vector space \( V \) is isomorphic to \( \mathbb{R}^d \), \( d = \dim V \), every Minkowski space \((V, \rho)\) is isometric to some Minkowski space \((\mathbb{R}^d, \rho_0)\). In fact, if \( \varphi : \mathbb{R}^d \to V \) satisfying \( \varphi(\lambda x) = \lambda \varphi(x) \) and \( \varphi(x + y) = \varphi(x) + \varphi(y) \) for all \( \lambda \in \mathbb{R} \) and \( x, y \in \mathbb{R}^d \), we just have to take \( \varrho(x, y) := \rho_0(\varphi(x), \varphi(y)) \). We denote Minkowski spaces \((\mathbb{R}^d, \rho)\) by \( \mathbb{M}^d \), but identify \( \mathbb{M}^d \) with \( \mathbb{R}^d \) whenever it is used in the context as a set. In general, without loss of generality we can restrict our investigations to Minkowski spaces \( \mathbb{M}^d \).

The unit ball \( B \) of a Minkowski space \( \mathbb{M}^d \) is defined by

\[
B = B(\mathbb{M}^d) := \{ x \in \mathbb{M}^d : \|x\| \leq 1 \}.
\]

The boundary of \( B \) is also called the unit sphere. For \( d = 2 \) we additionally call \( \mathbb{M}^2 \) a Minkowski plane, \( B \) the unit disc and \( \partial B \) the unit circle.

Furthermore, given a centered convex body \( B \in \mathbb{R}^d \), there is exactly one \( d \)-dimensional Minkowski space \( \mathbb{M}^d = \mathbb{M}^d(B) \) with unit ball \( B = B(\mathbb{M}^d(B)) \), see [36]. Whenever necessary, we denote the corresponding norm by \( \|\cdot\|_B := \|\cdot\| = \inf\{ \lambda \in [0, \infty) : x \in \lambda B \} \).

We call a Minkowski space Euclidean if it is isometric to the Euclidean \( d \)-dimensional space \( \mathbb{E}^d \) for some \( d = 0, 1, \ldots, \). We note that \( \mathbb{M}^d(B) \) is Euclidean if and only if \( B \) is an ellipsoid.

The length of the segment \( \overline{ab} \) of \( \mathbb{M}^d \) is denoted by \( \|\overline{ab}\| := \|a - b\| \), which coincides with the distance between the points \( a \) and \( b \).

For a vector \( x \in \mathbb{M}^d \), \( x \neq 0 \), we denote by \( \tilde{x} = (x) := \frac{1}{\|x\|} x \) the normalization of \( x \). The distance of \( \epsilon \in \mathbb{M}^d \) to a set \( M \subset \mathbb{M}^d \) is denoted by \( g(\epsilon, M) := \inf_{m \in M} g(\epsilon, m) \).

Of particular interest is the following Mazur-Ulam theorem, see for example [36].

**Theorem 1.17 (Mazur-Ulam theorem)** Each isometry \( \phi : \mathbb{M}^d \to \mathbb{M}^d \), \( d \geq 1 \), between two Minkowski spaces \( \mathbb{M}^d \) and \( \mathbb{M}^d \) is necessarily an affine linear map.
Definition 1.18 A (linear) subspace of a Minkowski space \((\mathcal{V}, \varrho)\) is any Minkowski space \((X, \varrho|_{X \times X})\), where \(X\) is a linear subspace of \(\mathcal{V}\).

An affine subspace of the Minkowski space \((\mathcal{V}, \varrho)\) is any Minkowski space \((X, \varrho|_{X \times X})\), where \(X\) is an affine subspace of the real linear vector space \(\mathcal{V}\) (see Definition 1.8).

1.8.1 Duality

In Minkowski geometry, there is some duality between a Minkowski plane \(M\) and a second Minkowski plane \(I\), where the convex body \(I\) is the isoperimetrix of the plane \(M\).

Definition 1.19 For a set \(X \subseteq \mathbb{R}^d\) we define the polar (reciprocal) set as the set \(\{ \eta \in \mathbb{R}^d : \langle x, \eta \rangle \leq 1 \ \forall x \in X \}\). Here \(\langle x, \eta \rangle\) denotes the scalar product of \(x = (x_1, \ldots, x_d)\) and \(\eta = (\eta_1, \ldots, \eta_d)\) in \(\mathbb{R}^d\), \(\langle x, \eta \rangle := x_1\eta_1 + x_2\eta_2 + \cdots + x_d\eta_d\).

The polar reciprocal curve of some closed convex centered curve \(C \subset \mathbb{R}^2\) is the boundary of the polar reciprocal of \(C\).

Definition 1.20 The isoperimetrix \(I\) of a Minkowski plane \(M\) is the 90 degree rotated image of the polar reciprocal of \(B\).

Note that this definition is not invariant under linear transformations. Definition 1.20 is strongly related to the underlying Euclidean space with the notion of scalar product and the rotation by a fixed angle. But considering \(I\) for two different Euclidean background spaces (more precisely, this means the Euclidean space in the image of some linear map), they differ only by some scaling, which can be easier seen by

\[
I = \{ \eta \in \mathbb{R}^2 : \det[x, \eta] \leq 1 \ \forall x \in B \}
\]

(taken from [25]). Other sources compensate for this fact by introducing some appropriate scaling factor.

The name ‘isoperimetrix’ comes from the fact that \(I\) and all its homothetic copies are the solutions of the isoperimetric problem in \(M\): which convex bodies have maximal area if the length of the boundary (measured in \(M\), of course) is fixed?

1.9 Birkhoff orthogonality

The following definition of normality is due to Birkhoff [5].

Definition 1.21 We say that \(x\) is normal to \(\eta\), denoted by \(x \perp \eta\), if \(\|x\| \leq \|x + \lambda \eta\|\) for all \(\lambda \in \mathbb{R}\).

Note that \(x \perp \eta\) holds if and only if \(x = 0\) or \(\eta = 0\) or the line through \(x\) in direction \(\eta\) is contained in some supporting hyperplane of \(\|x\|\ B\).

The property \(x \perp \eta\) is really two-dimensional, i.e., it holds in \(M\) if and only if it holds in the corresponding linear subspace \(\text{lin}(x, \eta)\) of \(M\), which is of dimension at most 2.

Definition 1.22 Assume we are given two linearly independent unit vectors \(x, \eta\) of \(M\) with unit ball \(B\). Then we denote by \(C(x, \eta)\) the oriented planar curve which is the intersection of \(\partial B\) and the plane \(P(x, \eta) := \text{lin}(x, \eta)\) so that \(x, \eta, -x, -\eta\) are ordered in this way along \(C(x, \eta)\). Furthermore, \([\cdot, \cdot]_{(x, \eta)}\) denotes the bilinear skew-symmetric form on \(P(x, \eta)\) defined as the determinant of the matrix containing the coordinates of two vectors in \(P(x, \eta)\) in the base \((x, \eta)\). We omit the index when it is clear.

\[
[a_1 x + a_2 \eta, a_1 x + a_2 \eta] := [a_1 x + a_2 \eta, a_1 x + a_2 \eta]_{(x, \eta)}
\]

\[
:= \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}
\]
Note that $[\cdot, \cdot]_{(\mathbf{f}, \mathbf{g})}$ is the uniquely defined bilinear skew-symmetric form on $P(\mathbf{f}, \mathbf{g})$ with $[\mathbf{f}, \mathbf{g}]_{(\mathbf{f}, \mathbf{g})} = 1$, since all these forms are equal up to some multiple. Geometrically, $|\mathbf{a}, \mathbf{b}|_{(\mathbf{f}, \mathbf{g})}$ is the ratio of the area of the triangles $\Delta \mathbf{aob}$ and $\Delta \mathbf{cgn}$, and the sign of $|\mathbf{a}, \mathbf{b}|_{(\mathbf{f}, \mathbf{g})}$ determines the orientation from $\mathbf{a}$ to $\mathbf{b}$, compared to $\mathbf{f}$ and $\mathbf{g}$.

**Lemma 1.23** For two linearly independent unit vectors $\mathbf{f}, \mathbf{g} \in M^2$ the condition $\mathbf{f} \perp \mathbf{g}$ is equivalent to $[\mathbf{f}, \mathbf{g}]_{(\mathbf{f}, \mathbf{g})} \leq 1$ for all $\mathbf{e} \in C(\mathbf{f}, \mathbf{g})$. More generally, $\mathbf{u} \perp \mathbf{v}$ holds for $\mathbf{u}, \mathbf{v} \in C(\mathbf{f}, \mathbf{g})$ if and only if $|\mathbf{u}, \mathbf{v}| \leq |\mathbf{u}, \mathbf{v}|$ for all $\mathbf{e} \in C(\mathbf{f}, \mathbf{g})$.

### 1.10 The functional $\beta$

**Definition 1.24** For $\mathbf{a}, \mathbf{b} \in M^2$ we denote by $\det[\mathbf{a}, \mathbf{b}]$ the determinant of the matrix whose columns are formed by the coordinates of $\mathbf{a}$ and $\mathbf{b}$ in a fixed basis of $\mathbb{R}^2$. For every unit vector $\mathbf{a}$ we denote by $\beta(\mathbf{a}) := \max_{|\mathbf{b}| = 1} \det[\mathbf{a}, \mathbf{b}]$ the maximal value of $\det[\mathbf{a}, \cdot]$ among all unit vectors.

We note that a maximal value exists since by $\det[\mathbf{a}, \cdot]$ the compact unit circle is mapped continuously to real numbers. Furthermore, $\frac{1}{2}\beta(\mathbf{a})$ is the maximal area of a triangle with edge $\overline{\mathbf{a} \mathbf{b}}$ and which is completely inside the unit disc.

**Lemma 1.25** We have $\phi(\mathbf{f}, (\mathbf{f}, \mathbf{g})) \cdot \beta(\mathbf{g}) = |\det[\mathbf{f}, \mathbf{g}]|$ for all vectors $\mathbf{f} \in M^2$ and $\mathbf{g} \in \partial B$.

**Proof** We have $\phi(\mathbf{f}, (\mathbf{f}, \mathbf{g})) = \min\{ \phi(\mathbf{f}, \mathbf{f} \lambda) : \lambda \in \mathbb{R} \}$. So we can assume

$$\rho := \phi(\mathbf{f}, (\mathbf{f}, \mathbf{g})) = \phi(\mathbf{f}, \lambda^* \mathbf{g}) \leq \phi(\mathbf{f}, \mathbf{g})$$

for some fixed $\lambda^* \in \mathbb{R}$ and all $\lambda \in \mathbb{R}$. We have $\rho = 0$ only in the case $\mathbf{f} = \lambda^* \mathbf{g}$, when also $\phi(\mathbf{f}, (\mathbf{f}, \mathbf{g})) \cdot \beta(\mathbf{g}) = 0 = |\det[\mathbf{f}, \mathbf{g}]|$ holds. So we can assume that $\rho > 0$ and $\mathbf{f}, \mathbf{g}$ are non-collinear.

By $\mathbf{g}^*$ we denote a unit vector satisfying $\beta(\mathbf{g}) = \det[\mathbf{g}, \mathbf{g}^*] \geq \det[\mathbf{g}, \mathbf{f}]$ (1.2) for all unit vectors $\mathbf{f}$.

First we use (1.2) with $\mathbf{f} := \pm \frac{\lambda^* \mathbf{g}}{\rho}$ (since $||\mathbf{f}|| = 1$) to get

$$\beta(\mathbf{g}) \geq \frac{+1}{\rho} \det[\mathbf{g}, \mathbf{f} - \lambda^* \mathbf{g}] = \frac{\pm 1}{\rho} \det[\mathbf{g}, \mathbf{f}] .$$

Thus $\beta(\mathbf{g}) \geq \frac{|\det[\mathbf{g}, \mathbf{f}]|}{\rho}$ and

$$\rho \cdot \beta(\mathbf{g}) \geq |\det[\mathbf{f}, \mathbf{g}]| .$$

(1.3)

Since $\mathbf{f}, \mathbf{g}, \mathbf{h} \in M^2$ are non-collinear, we can represent $\mathbf{g}^*$ as $\xi \mathbf{f} + \eta \mathbf{g}$ with $\xi, \eta \in \mathbb{R}$. Now we have that $\beta(\mathbf{g}) = \det[\mathbf{g}, \mathbf{g}^*] = \det[\mathbf{g}, \xi \mathbf{f} + \eta \mathbf{g}] = \xi \det[\mathbf{g}, \mathbf{f}] > 0$, and so we also have $\xi \neq 0$. Therefore we can use inequality (1.1) with $\lambda := -\frac{\eta}{\xi}$:

$$\rho \leq \phi(\mathbf{f}, \mathbf{g}) = \left\| \mathbf{f} + \frac{\mathbf{g}}{\xi} \right\| = \frac{\|\xi \mathbf{f} + \eta \mathbf{g}\|}{|\xi|} = \frac{1}{|\xi|} .$$

This gives

$$\rho \cdot \beta(\mathbf{g}) = \rho \cdot \xi \det[\mathbf{f}, \mathbf{g}] = \rho \cdot |\xi| \cdot |\det[\mathbf{f}, \mathbf{g}]| \leq |\det[\mathbf{f}, \mathbf{g}]| ,$$

completing our proof in view of (1.3). \qed

**Lemma 1.26** For unit vectors $\mathbf{f}, \mathbf{g} \in M^2$ the condition $\mathbf{f} \perp \mathbf{g}$ is equivalent to the condition $|\det[\mathbf{f}, \mathbf{g}]| = \beta(\mathbf{g})$. 

We leave the proof to the reader. \[ \square \]

### 1.11 Parametrization of the unit circle

**Notation 1.27** In the Minkowski plane \( M^2 \) we denote by

\[ u : [0, U) \rightarrow \partial B \]

a parametrization of the unit circle \( \partial B \) by arc length in the positive orientation. This means that \( u \) maps the interval \( [0, U) \) bijectively and continuously to \( \partial B \) such that the length (measured in the norm of \( M^2 \)) \( \lambda_1 (u([0, U])) \) equals \( s \) for all \( s \in [0, U) \) and the set \( B \) is locally to the left of this curve \( u \). Thus we have \( U = \lambda_1(\partial B) \). Obviously, we have exactly one such parametrization for each starting point \( x = u(0) = u(U) \in \partial B \).

### 1.12 Tangent vectors

We say that \( \mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n \) are ordered along the closed oriented curve \( C \), denoted by \( \langle \mathbf{f}_1, \ldots, \mathbf{f}_n \rangle_C \), if and only if the curve passes through these points in this order within one cycle along \( C \). More precisely, if \( C \) is parametrized by \( v : [0, T] \rightarrow C \), we have for \( t_1 \leq t_2 \leq \cdots \leq t_n \leq t_1 + T \) that \( \langle v(t_1), \ldots, v(t_n) \rangle_C \), taking \( t_j \) modulo \( T \) \( v(t) := v(t - T) \) if \( t > T \), \( v(t) := v(t + T) \) if \( t < 0 \), and only sequences possessing such a representation have this property.

**Definition 1.28** For an oriented closed convex centered curve \( C \) we assign to each vector \( \mathbf{z} \in C \) the right tangent vector \( \mathbf{z}_r^C \in C \) and the left tangent vector \( \mathbf{z}_l^C \in C \) as the last and first vector \( v \), respectively, on \( C \) between \( \mathbf{z} \) and \( -\mathbf{z} \), \( \langle \mathbf{z}, \mathbf{v}, -\mathbf{z} \rangle_C \), satisfying \( \mathbf{z} \not\prec v \) in the Minkowski plane with unit circle \( C \).

This definition is correct since the set of such vectors \( v \) is compact. The last and first vector are taken with respect to the following total order on the arc \( A_3 := \{ \mathbf{v} \in C : \langle \mathbf{z}, \mathbf{v}, -\mathbf{z} \rangle_C \}, \mathbf{v}_1 \leq \mathbf{v}_2 \) iff \( \langle \mathbf{z}, \mathbf{v}_1, \mathbf{v}_2 \rangle_C \) holds. The notation ‘left’ and ‘right’ are taken from [29], where the following analytic description was used to define the right tangent vector of the parametrization \( u : [0, U] \rightarrow \partial B \) by arc length of the unit circle as another function \( u'_r : [0, U) \rightarrow \mathbb{R}^2, u'_r(t) := \lim_{h \downarrow 0} \frac{u(t + h) - u(t)}{h} \).

**Lemma 1.29** The two notions \( \mathbf{z}_r^C \) and \( u'_r \) for right tangent vectors are equivalent in the following sense: if \( C = \partial B \) has the parametrization \( u : [0, U] \rightarrow \partial B \) by arc length and \( \mathbf{z} = u(t) \) for \( t \in [0, U) \), then we have \( \mathbf{z}_r^C = u'_r(t) \).

We leave the proof to the reader.

**Corollary 1.30** For unit vectors \( \mathbf{u}, \mathbf{v} \in M^d \) we have that \( \mathbf{u} \not\prec \mathbf{v} \) if and only if \( \langle \mathbf{u}_r^C, \mathbf{v}, \mathbf{u}_l^C \rangle_C \) is true for the oriented curve \( C := C(u, v) \) (see Definition 1.22).

**Notation 1.31** In a Minkowski plane \( M^2 \) with parametrization \( u \) of \( \partial B \) by arc length and fixed basis of \( \mathbb{R}^2 \) defining the determinant, we define for each \( t \in [0, U) \) the value \( \alpha(t) := \det[u(t), u'_r(t)] \).

In more general Minkowski spaces \( M^d \), we generalize this function to arbitrary distinct unit vectors \( \mathbf{x}, \mathbf{y} \in M^d \):

\[
\alpha(\mathbf{x}, \mathbf{y}) := [\mathbf{x}, \mathbf{x}_r^C]_{\mathbf{y}}(\mathbf{y}) \tag{1.4}
\]

### 1.13 Radon curves

Radon curves were introduced by Radon [32], see also [36] Chapter 4] and [28].
Definition 1.32 The boundary of the unit disc of a Minkowski plane $\mathbb{M}^2$ (and its homothetic copies) is called a Radon curve (and $\mathbb{M}^2$ a Radon plane) if and only if the relation of normality is symmetric, i.e., if and only if
\[ x \vdash y \iff y \vdash x \] for any $x, y \in \mathbb{M}^2 \setminus \{0\}$.

Proposition 1.33 The following conditions are equivalent for a centered closed convex curve $C = \partial B \subset \mathbb{R}^2$:

1. $C$ is a Radon curve (the induced normed plane $\mathbb{M}^2(B)$ is Radon),
2. $B$ is homothetic to the isoperimetrix $I$ of $\mathbb{M}^2(B)$,
3. $|\det[u, v]|$ is constant for all $u \in C$ and all $v \in C$ with $u \vdash v$.

In [26] we can find the $1 \iff 3$ characterization of Radon curves (p. 308, Proposition 3, Conditions 1 and 3).

There is a fundamental result due to Blaschke [7], Birkhoff [5] and James [25] (first complete proofs without additional requirements), see also Thompson [36].

Theorem 1.34 The relation of normality in a Minkowski space $\mathbb{M}^d$, $d \geq 3$, is symmetric, $x \vdash y \iff y \vdash x$ for any $x, y \in \mathbb{M}^d$, if and only if $\mathbb{M}^d$ is Euclidean.

In other terms this is exactly the following

Theorem 1.35 If all 2-dimensional linear subspaces of a Minkowski space $\mathbb{M}^d$, $d \geq 3$, are Radon, then $\mathbb{M}^d$ is Euclidean.

Theorem 1.36 A centered convex body $B$ in the space $\mathbb{R}^d$ with $d \geq 3$ all whose boundaries of sections with 2-dimensional linear subspaces are Radon curves, is an ellipsoid.

A simple consequence is the following fact.

Theorem 1.37 A Minkowski space $\mathbb{M}^d$, $d \geq 2$, is Euclidean if and only if each 2-dimensional subspace of $\mathbb{M}^d$ is Euclidean.

1.14 Equiframed bodies

Pełczyński and Szarek introduced in [30] the notion of equiframed convex bodies, see also [26].

Definition 1.38 A convex body $B \subset \mathbb{R}^d$ is called equiframed, if each point in its boundary belongs to the boundary of some parallelepiped of minimal volume containing $B$. An equiframed curve is the boundary of a two-dimensional equiframed convex body, i.e., a convex closed curve in the plane that is touched at each of its points by some circumscribed parallelogram of smallest area.

A Minkowski plane $\mathbb{M}^2$ is called equiframed, if its unit ball $B$ is equiframed.

In two-dimensional space, this notion is strongly related to that of a Radon curve. If the boundary of a planar convex body $B$ is a Radon curve, then $B$ is equiframed. Conversely, if the planar centered convex body $B$ is equiframed and its boundary curve $C$ is smooth or $B$ is strictly convex, then $C$ is also a Radon curve. Thus equiframed curves are more general than Radon curves.

There is a characterization similar to 1. $\iff 3$. in Proposition 1.33 taken from [26] p.308, Proposition 2, 1. $\iff 3$.

Lemma 1.39 A compact planar convex centered body $B \subset \mathbb{M}^2$ is equiframed if and only if for all $u \in C := \partial B$ the value $[u, u_C]$ is constant. Here $[\cdot, \cdot]$ denotes any bilinear skew-symmetric form in $\mathbb{R}^2$. 

1.15 Area

As in the Euclidean case, there is a nice formula for the area of a triangle in a Minkowski plane, see [2, Theorem 7]:

**Proposition 1.40 (Averkov)** For the Euclidean area measure $a$ of a triangle $\triangle abc$ we have the formula

$$a = \frac{1}{2} h \tilde{g},$$

where $h$ is the height $g(c, \langle a, b \rangle)$ in $\mathbb{M}^2(B)$ of the triangle with respect to the side $\langle a, b \rangle$, and $\tilde{g} = g(a, b)$ is the length of $\overline{ab}$ measured in $\mathbb{M}^2(I)$ with the isoperimetriz $I$ of $\mathbb{M}^2(B)$ as unit ball.

1.16 Classification

What do we mean with “classification” (of 2-distance sets, for example)? We have to classify some complicated set $S$, which may be defined as the set of all equivalence classes of some equivalence relation (e.g., $\equiv_s$) in some given set (e.g., $\mathcal{C}_2$, the set of all 2-distance sets).

The most important requirement for the classification is: We want to know the size (number of elements, cardinality, dimension) of $S$. Second we want to describe each element of $S$ by simple mathematical terms. These are

- one of finitely many categories, described by a number $n \in \mathbb{N}_m$,
- one of countable many categories, described by a number $n \in \mathbb{N}$ or $z \in \mathbb{Z}$,
- real parameters $p \in \mathbb{R}$ or even geometric points $p \in \mathbb{R}^d$,
- real functions $f : A \rightarrow \mathbb{R}$, where $A \subset \mathbb{R}$ is an interval, and real sequences $s : \mathbb{N} \rightarrow \mathbb{R}$,
- combinations thereof as finite tuples or unions.

The next requirement is that if not all of the values of these mathematical objects are used to describe $S$, then these restrictions should have a simple structure:

- real numbers $p$ should be restricted only to intervals,
- $p \in \mathbb{R}^d$ can be restricted to Euclidean balls and polyhedra, possibly also to the relative interior or boundary of such sets,
- sequences can be restricted by simple properties such as monotony, boundedness, bounds, summability,
- real functions may be restricted by simple properties such as monotony, boundedness, upper and lower bounds, convexity, smoothness (continuity, differentiability), integrability.

Thus, what we understand by a classification of $S$ is a bijection $\phi$ between $S$ and a set $C$ formed by the above rules. Additional requirements for a good classification are

- to be, in some sense, continuous regarding the real parameters as far as possible,
- there is an explicit construction of $\phi(s) \in C$ for all $s \in S$ (not necessarily in an algorithmical way, but by explicit definitions; using a representative if $S$ is a set of equivalence classes),
- there is an explicit construction of (a representative of) the inverse $\phi^{-1}(c) \in S$ for all $c \in C$. 
1.17 Polyhedra and polytopes

Definition 1.41 Every set of the form
\[ H(a, b) := \{ \mathbf{x} \in \mathbb{R}^d : a^\top \mathbf{x} \leq b \} , \]
where \( a \in \mathbb{R}^d \setminus \{ 0 \} \) and \( b \in \mathbb{R} \), is called a (closed) (affine) half-space of \( \mathbb{R}^d \).

Definition 1.42 A set \( P \subset \mathbb{R}^d \) is called a (convex) polytope, if there are finitely many points \( \mathbf{f}_1, \ldots, \mathbf{f}_n \in \mathbb{R}^d \) with \( P = \text{conv} \{ \mathbf{f}_1, \ldots, \mathbf{f}_n \} \). Each intersection of finitely many closed half-spaces of \( \mathbb{R}^d \) is called a (convex) polyhedron. \( P \) is called a polyhedral cone, if it admits a representation \( P = \text{cone} \{ \mathbf{f}_1, \ldots, \mathbf{f}_n \} \) with \( \mathbf{f}_1, \ldots, \mathbf{f}_n \in \mathbb{R}^d \).

A Minkowski space \( \mathbb{M}^d \) is called polytopal, if its unit ball is a convex polytope.

There are several other characterizations of these notions:

Theorem 1.43 \( P \subset \mathbb{R}^d \) is a polytope if and only if it is a bounded polyhedron. A convex cone \( P \) is a polyhedral cone if and only if it is a polyhedron. \( P \) is a polyhedral cone if and only if it is a finite intersection of half-spaces \( H(a_i, 0) \) with \( a_i \in \mathbb{R}^d \).

From the book of Schrijver [33] we take for example the following.

Theorem 1.44 (Decomposition Theorem) A set \( P \) of vectors in Euclidean space is a polyhedron, if and only if \( P = Q + C \) for some polytope \( Q \) and some polyhedral cone \( C \). If \( P = Q + C \), then necessarily \( C = \{ \mathbf{y} : \mathbf{f} + \mathbf{y} \in P \forall \mathbf{f} \in P \} \).

We shall say that the polyhedron \( P \) is generated by the points \( \mathbf{f}_1, \ldots, \mathbf{f}_m \) and by the directions \( \mathbf{y}_1, \ldots, \mathbf{y}_t \) if
\[ P = \text{conv} \{ \mathbf{f}_1, \ldots, \mathbf{f}_m \} + \text{cone} \{ \mathbf{y}_1, \ldots, \mathbf{y}_t \} . \]

In this case we call \( \text{charcone} P := \text{cone} \{ \mathbf{y}_1, \ldots, \mathbf{y}_t \} \) the characteristic cone of \( P \) and \( \text{charcone} P \cap -(\text{charcone} P) \) the lineality space of \( P \), see also [39]. The polyhedron \( P \) is called pointed, if it has a trivial lineality space of dimension zero. Note that only pointed polyhedra \( P \) admit a unique representation \( P = Q + C \) with a minimal (with respect to the number \( m \) of generators) polytope \( Q \) and a polyhedral cone \( C \). In general each minimal \( Q \) contains exactly one vector of each (inclusion) minimal face (see below) of \( P \). To simplify some statements we also consider \( \emptyset \) as pointed polyhedron. A face \( F \) of \( P \) is the intersection of \( P \) with a supporting hyperplane and additionally the two trivial faces \( F = \emptyset \) and \( F = P \). All faces of \( P \) form a lattice, called the face-lattice of \( P \). A facet of \( P \) is a inclusion maximal face \( F \neq P \), i.e., a face of \( P \) with \( \text{dim} F = \text{dim} P - 1 \). The points \( \mathbf{f}_1, \ldots, \mathbf{f}_m \) are also called vertices of \( P \) if \( \{ \mathbf{f}_1 \} \) is a face of \( P \). An extremal ray of a convex cone is a face of dimension 1. All vectors \( \mathbf{f}_1, \ldots, \mathbf{f}_m \) and \( \mathbf{y}_1, \ldots, \mathbf{y}_t \) are called generators of \( P \).

Theorem 1.45 (Caratheodory’s Theorem) If \( X \subseteq \mathbb{R}^d \) and \( x \in \text{conv}(X) \), then \( x \in \text{conv} \{ x_0, \ldots, x_d \} \) for certain vectors \( x_0, \ldots, x_d \) in \( X \).

1.18 Tasks, systems, and algorithm

Definition 1.46 A (finite) system of equations and inequalities in \( \mathbb{R}^d \), or a \( d \)-dimensional system of equations and inequalities, is a triple \( S = (E, W, S) \) of finite sets of functions \( f : \mathbb{R}^d \to \mathbb{R} \). So \( f \in \mathbb{R}^{E \times \mathbb{R}^d} \), and \( S \in \mathcal{P} (\mathbb{R}^{E \times \mathbb{R}^d})^3 \). All functions \( f \in E \cup W \cup S \) are called restrictions of \( S \).

We say that \( \mathbf{x} \in \mathbb{R}^d \) is a solution vector of \( S \) if
- \( f(\mathbf{x}) = 0 \) for all \( f \in E \),
- \( f(\mathbf{x}) \geq 0 \) for all \( f \in W \), and
- \( f(\mathbf{x}) > 0 \) for all \( f \in S \).
S is called admissible if it has at least one solution vector. Its solution set \( L = L(S) \) is the set of all solution vectors of \( S \). \( d = \dim S \) is called the dimension of \( S \).

\( S \) is called homogeneous linear system if each restriction is a linear function. It is called inhomogeneous linear system if each restriction is an affine linear function.

A system with \( d \) variables, \( e \) equations, \( w \) non-strict (weak) inequalities and \( s \) strict inequalities is a \( d \)-dimensional system \((E,W,S)\) of equations and inequalities with \( |E| = e \), \( |W| = w \) and \( |S| = s \).

Two systems \( S_1 \) and \( S_2 \) in \( \mathbb{R}^d \) are called equivalent if they have the same solution set. We call two systems \( S_1 \) in \( \mathbb{R}^{d_1} \) and \( S_2 \) in \( \mathbb{R}^{d_2} \) equivalent with respect to \( \phi \), if we know a function \( \phi : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2} \) such that for the – not necessarily known – solution sets \( L_1 \) and \( L_2 \) we can assure that \( \phi' : L_1 \to L_2, x \mapsto \phi(x) \) is bijective.

A system \( S_1 = (E_1, W_1, S_1) \) is called a subsystem of \( S = (E, W, S) \) if \( E_1 \subset E \), \( W_1 \subset W \) and \( S_1 \subset S \).

**Task 1.47 (Solving a system \( S \))** Given a system \( S \), give a classification\(^1\) of all solution vectors of \( S \).

**Task 1.48 (Admissibility of a system \( S \))** Given a system \( S \), decide whether or not \( S \) is admissible.

**Task 1.49 (Find a solution of the system \( S \))** Given a system \( S \), find a solution vector \( x \in \mathbb{R}^d \) or decide that \( S \) is not admissible.

The following comes from the theory of real closed fields, see [35, 4, 8].

**Theorem 1.50** The admissibility of a system \( S \) of equations and inequalities (cf. Task 1.48) is algorithmically decidable if all restrictions of \( S \) are polyvariate polynomials in \( \mathbb{Z}[X_1, \ldots, X_d] \).

We also use the complexity notation for upper bounds \( f(n) \in O(g(n)) \), if there is some constant \( c \in \mathbb{R} \) with \( f(n) \leq cg(n) \) for all \( n \in \mathbb{N} \).

\(^{1}\) see Section 1.16
Chapter 2

Angular measures and bisectors

In a Minkowski plane $\mathbb{M}^2$, there is no natural definition of a unique angular measure as in the Euclidean plane. In fact, there are several possibilities of defining such a measure. We will study the measures $\mu_a$ and $\mu_l$ which are proportional to the area and to the arc length of the corresponding sector of the unit circle, respectively, as well as a class of further measures satisfying certain axioms.

For each angular measure we can define a corresponding angular bisector which halves that sector. As long as this bisector is defined by an angular measure, there is a one-to-one correspondence between angular measures and angular bisectors. But generalizations of geometric properties of Euclidean angular bisectors yield definitions of angular bisectors in $\mathbb{M}^2$ which are independent of an angular measure.

By means of angular bisectors in normed linear spaces various deep characterizations of special Minkowski spaces can be obtained, cf. the survey [27, §4]. We will give further such characterization theorems.

We prove that a Minkowski plane is Euclidean if and only if Busemann’s or Glogovskij’s definitions of angular bisectors coincide with a bisector defined by an angular measure in the sense of Brass. In addition, bisectors defined by the area measure coincide with bisectors defined by the circumference (arc length) measure if and only if the unit circle is an equiframed curve. We prove that a Minkowski plane is a Radon plane if and only if Busemann’s and Glogovskij’s definitions of angular bisectors coincide.

2.1 Some definitions

Inspired by Brass [9, Definition on page 207] we define the following angular measure for Minkowski planes.

Definition 2.1 An angular measure of the Minkowski plane $\mathbb{M}^2$ (in the sense of Brass) is a (non-negative and $\sigma$-additive) measure $\mu$ on the Borel sets of the unit circle $\partial B$ which has the following properties:

1. $\mu$ is normed, i.e., $\mu(\partial B) = 2\pi$,
2. $\mu$ is centrally symmetric, i.e., for $X \subset \partial B$ we have $\mu(X) = \mu(-X)$,
3. for each point $p \in \partial B$ we have $\mu(\{p\}) = 0$,
4. every arc $A$ on $\partial B$ with distinct endpoints $a \neq b$ has a positive measure $\mu(A) > 0$.

The fourth property was not demanded by Brass but is necessary for uniqueness when defining an angular bisector. We generalize this notion to angular measures $\mu$ of Minkowski spaces $\mathbb{M}^d$, $d \geq 2$, in the sense that $\mu$ maps all Borel sets of (unit) circles $\partial B \cap P$, where $P \ni o$ is a 2-plane, to nonnegative real numbers, and $\mu$ restricted to (Borel sets of) $\partial B \cap P$ is an angular measure in the sense of Definition 2.1. Note that all these planar angular measures can be independently chosen to construct a higher dimensional angular measure.
Example 2.2 We can normalize the arc length (denoted by \( \lambda_1(X) \)) of a subset \( X \subset \partial B \cap P \), measured with the metric of \( \mathbb{M}^d \), to get an angular measure \( \mu_1 \).

\[
\mu_1(X) = \frac{2\pi}{\lambda_1(\partial B \cap P)} \lambda_1(X)
\]

Example 2.3 We can normalize the area of the corresponding sector \([0,1] \cdot X\) of the unit disc \( B \cap P \) to define the angular measure \( \mu_a \).

\[
\mu_a(X) = \frac{2\pi}{\lambda_2(B \cap P)} \lambda_2([0,1] \cdot X) \quad X \subset \partial B \cap P
\]

Note that the 2-dimensional Lebesgue-measure \( \lambda_2 \) is unique in each 2-plane \( P \) up to some multiple having no influence to the value of \( \mu_a(X) \).

For our purposes an angle of \( \mathbb{M}^2 \) is a closed convex subset \( T \) of \( \mathbb{M}^2 \) whose boundary \( \partial T \) is the union of two rays \( r_1, r_2 \) not on a line (called the sides) with common endpoint, called the apex of the angle. The two limit cases of a single ray and of a half-plane (“straight angle”) are not called angles for simplicity. An angle of \( \mathbb{M}^d, d \geq 2 \), is any angle in an (affine) 2-plane of \( \mathbb{M}^d \). Thus the angle \( T \) is uniquely determined by its sides \( r_1, r_2 \) (which must have the same apex and affinely span a 2-plane) and vice versa, and we denote it by \( \angle (r_1 r_2) = T = \text{conv}(r_1 \cup r_2) \). Furthermore, we use the notation \( \angle b a c := \angle ([a, b] [a, c]) \). An angular bisector of an angle \( T \) is a ray \( r \) such that there are two angles \( T_1, T_2 \) with \( T_1 \cup T_2 = T \) and \( \text{rel bd} T_1 \cap \text{rel bd} T_2 = r \). In this case we say that \( r \) divides \( T \) in \( T_1 \) and \( T_2 \). A system of angular bisectors is a function \( A \) mapping each angle \( T \) of \( \mathbb{M}^d \) to a corresponding angular bisector \( r = A(T) \). The normalized representation \( \hat{A} \) of \( A \) is the function \( \hat{A} : (\xi, \eta) \mapsto r \in \partial B, \text{where } r \in A(\angle \xi \eta) \text{ for } \xi, \eta \in \partial B \) with \( \xi \neq \pm \eta \). We can reconstruct a system \( A \) of angular bisectors by its normalized representation \( \hat{A} \) in the following way: \( A(\angle b a c) := \{ a, a + \hat{A}(b - a, c - a) \} \).

Given an angular measure \( \mu \) we can measure every angle in an obvious manner: \( \mu(\angle b a c) := \mu(\angle (b - a)\hat{c} \cap \partial B) \).

Definition 2.4 Given an angular measure \( \mu \) of \( \mathbb{M}^d, d \geq 2 \), the system of angular bisectors such that \( A_\mu(T) \) divides \( T \) into \( T_1 \) and \( T_2 \) with \( \mu(T_1) = \mu(T_2) = \frac{1}{2} \mu(T) \) is called the system of \( \mu \)-bisectors, and \( A_\mu(T) \) is the \( \mu \)-bisector of \( T \).

There is exactly one \( \mu \)-bisector for every angle of \( \mathbb{M}^d \). (The uniqueness follows from 4. in Definition 2.1; the existence from 3. there.)

Following Busemann [11], we give

Definition 2.5 The system \( A_B = A_{B, \mathbb{M}^d} \) of angular bisectors of \( \mathbb{M}^d, d \geq 2 \), given by \( \hat{A}_B(a, b) = a + b \) for \( a, b \in \partial B, a + b \neq \hat{c} \), is called the system of Busemann angular bisectors, and \( A_B(T) \) is said to be the Busemann angular bisector of \( T \).

The following definition is due to Glogovskij [12].

Definition 2.6 The system \( A_G = A_{G, \mathbb{M}^d} \) of angular bisectors of \( \mathbb{M}^d \) given by \( A_G(\angle (r_1 r_2)) = \{ c \in \angle (r_1 r_2) : g(c, \text{aff } r_1) = g(c, \text{aff } r_2) \} \), i.e., the set of all points of the angle which are equidistant to the lines carrying the sides, is called the system of Glogovskij angular bisectors, and \( A_G(T) \) is said to be the Glogovskij angular bisector of \( T \).

### 2.2 Properties of angular bisectors

In a Minkowski plane \( \mathbb{M}^2 \), Definition 2.5 yields a bisector \( [a, c] \) satisfying Property 2.4 below. This property was used by Busemann [11] to define an angular bisector in a more general sense.

Property 2.7 Given an angle \( T \) with apex \( p \) and sides \( a, b \). The angular bisector \( c \) of \( T \) has the Busemann bisector property if and only if for every segment \( \overline{ap} \) joining a point \( r \) from \( a \setminus \{ p \} \) with
one point \( y \) from \( b \setminus \{ p \} \) the ray \( c \) divides \( \overrightarrow{py} \) in the ratio of the lengths \( \|\overrightarrow{px}\| \) and \( \|\overrightarrow{py}\| \), i.e., for \( \{ z \} := \overrightarrow{px} \cap c \) one has
\[
\frac{\|\overrightarrow{xz}\|}{\|\overrightarrow{zy}\|} = \frac{\|\overrightarrow{px}\|}{\|\overrightarrow{py}\|}.
\] (2.1)

Also another property of Euclidean angular bisectors is obvious, namely

Property 2.8 The distance of any point \( p \) from the angular bisector \([0, c]\) to the two straight lines carrying the sides \([0, a]\) and \([0, b]\) is the same.

This motivated Definition 2.6 of angular bisectors.

Lemma 2.9 An angular bisector \( c \) of \( T \) has the Busemann bisector property (Property 2.7) in \( M^2 \) if and only if it is the Busemann angular bisector of \( T \), \( c = \text{AB}(T) \).

Lemma 2.10 The set \( \text{AG}(\angle a0b) \) equals the ray \([0, c]\) for arbitrary nonzero vectors \( c \in \text{AG}(\angle a0b) \).

2.3 Results

The first two results are already published in the Journal of Geometry under the title “A new characterization of Radon curves via angular bisectors” [14].

Theorem 2.11 We have \( \text{AG}(T) = \text{AB}(T) \) for all angles \( T \) of a Minkowski plane if and only if the plane is Radon.

In view of Theorem 1.35 for Minkowski spaces \( M^d \), \( d \geq 3 \), this statement yields the

Corollary 2.12 A Minkowski space \( M^d \), \( d \geq 3 \), is Euclidean if and only if the systems of Busemann angular bisectors and of Glogovskij angular bisectors coincide.

The remaining results of this section are published in the Canadian Mathematical Bulletin under the title “Angle measures and bisectors in Minkowski planes” [15].

Theorem 2.13 In a Minkowski plane \( M^2 \), the area angular measures and arc length angular measures coincide (for each angle), \( \mu_a = \mu_l \), if and only if \( M^2 \) is equiframed.

The question for the coincidence of \( \mu_a \) and \( \mu_l \) was posed by Helfenstein [23] in 1959, and two years later he himself gave a wrong answer, see [24]. In his solution Helfenstein assumed continuous differentiability of the radial function, yielding a restriction of the characterized class of unit circles from equiframed curves to Radon curves (which are more specific). But it is not true that the general case follows from the differentiable case via limits. To see that this restriction is too strong, one might consider the \( l_\infty \)-norm where \( \partial B \) is the square. Then we have \( \mu_a = \mu_l \), but \( \partial B \) is equiframed and not a Radon curve.

Now we characterize Minkowski spaces in which two of the introduced systems of angular bisectors \( \text{AB}, \text{AG}, \text{A}_\mu \) are equal.

The following proposition is an easy consequence of standard arguments from analysis.

Proposition 2.14 For two angular measures \( \mu_1, \mu_2 \) of \( M^2 \) we have \( \text{A}_{\mu_1} = \text{A}_{\mu_2} \) if and only if \( \mu_1 = \mu_2 \).

Theorem 2.15 In a Minkowski space \( M^d \), \( d \geq 2 \), we have \( \text{A}_\mu = \text{A}_\mu \) for an angular measure \( \mu \) if and only if \( M^d \) is Euclidean and \( \mu \) denotes the Euclidean standard angular measure (which is \( \mu_l = \mu_a \)).

Theorem 2.16 In a Minkowski space \( M^d \), \( d \geq 2 \), we have \( \text{AG} = \text{A}_\mu \) for an angular measure \( \mu \) if and only if \( M^d \) is Euclidean and \( \mu \) denotes its standard angular measure.
The two Theorems 2.15 and 2.16 are really equivalent in view of the following Lemma. It says that the Glogovskij angular bisector coincides with the Busemann angular bisector in the plane $M^2(I)$ with the isoperimetrix $I$ of the plane $M^2 = M^2(B)$ as unit ball. Now the isoperimetrix of the introduced Minkowski plane $M^2(I)$ is homothetic to the unit ball $B$ of $M^2$. Thus this lemma also holds if we interchange the roles of $I$ and $B$, i.e., if we interchange the roles of $A_B$ and $A_G$: $A_{G,M^2(I)} = A_{B,M^2(B)}$.

**Lemma 2.17** For the two Minkowski planes $M^2(B)$ with isoperimetrix $I$ and $M^2(I)$ with unit ball $I$ we have

$$A_{B,M^2(I)} = A_{G,M^2(B)}.$$  

### 2.4 Proofs

#### 2.4.1 Proofs concerning the properties of Busemann and Glogovskij angular bisector

**Proof of Lemma 2.9** First we show that the ray $[o, c] = A_B (\angle aob)$ from Definition 2.7 has the Busemann bisector property (Property 2.7). To see this, we only have to compare condition (2.1) for $z$ to real numbers, we have

$$\lambda (\triangle 0x3) = \lambda (\triangle 0x3).$$

But since $a, b$ and $z$ are lying on a line, we have $g_e(x, y) = g_e(x, z)$ with $g_e(x, a) = 1 = g_e(x, a)$ and $g_e(x, b) = 1 = g_e(x, b)$ for the rays $[a, c]$ and $[b, c]$, respectively, we get $g_e(x, a) = g_e(x, b)$ and $g_e(y, a) = g_e(y, b)$. Thus equation (2.1) holds, and $A_B (\angle aob)$ has the Busemann bisector property.

If the angular bisector $[o, c]$ of $\angle aob$ (with $||a|| = ||b|| = 1$) has Property 2.7 then, with $z := a$, $\eta := b$ and the corresponding intersection $z \in \overline{a} \cap [o, c]$, we have $g_e(x, y) = \frac{||a||}{||b||} = \frac{1}{1} = 1$. Thus $z = \frac{1}{2}(a + b)$ is the midpoint of $\overline{a}$. Consequently, $a + b$ belongs to $[o, c]$ and $[o, c]$ equals $A_B (\angle aob)$.

**Proof of Lemma 2.10** We have to show that the set $A_G (\angle aob)$ equals the ray $[o, c]$ for an arbitrary vector $c \in A_G (\angle aob) \setminus \{0\}$.

Considering the continuous function $f(y) := g(y, \langle o, a \rangle) - g(y, \langle o, b \rangle)$ mapping the segment $\overline{ab}$ to real numbers, we have

$$f(a) = g(a, \langle o, a \rangle) - g(a, \langle o, b \rangle) = -g(a, \langle o, b \rangle) < 0$$

and

$$f(b) = g(b, \langle o, a \rangle) - g(b, \langle o, b \rangle) = 0.$$  

Thus there is some $c \in \overline{ab}$ with $f(c) = 0$, and so $c \in A_G (\angle aob)$. Furthermore, $f$ is strictly monotone ($f((1 - \lambda)a + \lambda b)$ increases linearly in $\lambda$), and therefore the point $c$ is unique. Obviously, with $z \in A_G (\angle aob)$ and $\mu \geq 0$ we also have $\mu z \in A_G (\angle aob)$ (since $\mu \in \text{cone}(a, b)$
and \( g(\mu, (\phi, \mathbf{a})) = \mu g(\phi, (\phi, \mathbf{a})) = g(\mu, (\phi, \mathbf{b})) \). As a consequence, \( A_G(\angle \mathbf{a} \mathbf{b}) \supset [\phi, \mathbf{c}] \), but also \( A_G(\angle \mathbf{a} \mathbf{b}) \subset [\phi, \mathbf{c}] \). Let \( \phi' \in A_G(\angle \mathbf{a} \mathbf{b}) \) for a nonzero \( \phi' \). Then \( [\phi, \phi'] \subset A_G(\angle \mathbf{a} \mathbf{b}) \), as proven above. Due to the uniqueness of \( \phi \), \([\phi, \phi'] \) must intersect \( \mathbf{a} \mathbf{b} \) in \( \phi \). Thus we have \( \phi' \in [\phi, \mathbf{c}] \). \( \Box 

2.4.2 \text{ Parametrization of the unit circle} \)

For the proofs of our results we need some more technique. We start with an analog of the parametrization of the unit circle \( \partial B \) by arc length (see Notation 1.27) for angular measures.

**Notation 2.18** For an angular measure \( \mu \) and a parametrization \( u : [0, U] \to \partial B \) of the unit circle we define the angle function \( w = w_{u, \mu} : [0, U] \to [0, 2\pi] \) by

\[
w(s) = w_{u, \mu}(s) = \mu([0, s]) .
\]

The angle function \( w_{\mu} \) is strictly monotone increasing (since \( w_{\mu}(t + dt) = w_{\mu}(t) + \mu([t, t + dt]) \)). \( w_{\mu}(t) + \mu([t, t + dt]) \geq w_{\mu}(t) \) for \( 0 < dt < U - t \), see Definition 2.1 and, because of 3. in Definition 2.1 continuous. Thus it is a bijection (from \( [0, 2\pi] \) to \([0, U] \)). Additionally, the following relations hold:

\[
w(0) = 0, \\
w(U) = \mu(\partial B) = 2\pi, \quad \text{and} \\
w \left( s + \frac{1}{2} U \right) = w(s) + \mu \left( u \left( [s, s + \frac{1}{2} U] \right) \right) = \pi + w(s) \quad \text{for } 0 \leq s \leq \frac{1}{2} U .
\]

(Note that \( u \left( [s, s + \frac{1}{2} U] \right) \cup -u \left( [s, s + \frac{1}{2} U] \right) = \partial B \), and the intersection has empty measure.) Furthermore, we see that the angle function is bijective.

**Definition 2.19** Let \( u : [0, U] \to \partial B \) be a parametrization of the unit circle by arc length in the positive orientation, \( \mu \) be an angular measure and \( w = w_{\mu} \) be the corresponding angle function. Then the parametrization of \( \partial B \) by the angle is the function

\[
m = m_{u, \mu} : [0, 2\pi] \to \partial B, \quad \phi \mapsto u(w_{u, \mu}^{-1}(\phi)),
\]

where \( w_{u, \mu}^{-1} : [0, 2\pi] \to [0, U] \) denotes the inverse to the angle function \( w = w_{u, \mu} \). The extended parametrization of \( \partial B \) by the angle is the function

\[
m = m_{u, \mu} : \mathbb{R} \to \partial B, \quad \phi \mapsto u(w_{u, \mu}^{-1}(\phi \mod 2\pi)) .
\]

**Corollary 2.20** The \( \mu \)-bisector corresponding to the angular measure \( \mu \) can be expressed very simple in the following form:

\[
m \left( \frac{\phi_1 + \phi_2}{2} \right) = \hat{A}_\mu(m(\phi_1), m(\phi_2)) \in A_\mu(\angle m(\phi_1) \circ m(\phi_2)) \quad \forall \phi_1, \phi_2 : |\phi_1 - \phi_2| < \pi
\]

**Proof of Theorem 2.13** Let \( u : [0, U] \to \partial B \) be a parametrization of \( \partial B \) by arc length in positive orientation (Notation 1.27). For \( 0 \leq t_1 < t_2 < U \) and \( t_2 - t_1 < \frac{U}{2} \) we have that

\[
\mu_t(\angle u(t_1) u(t_2)) = 2\pi \frac{t_2 - t_1}{U} = \int_{t=t_1}^{t_2} \frac{2\pi}{\lambda_2(B)} dt .
\]

Using \( \alpha(\cdot) \) from Notation 1.31 this yields

\[
\mu_t(\angle u(t_1) u(t_2)) = \int_{t=t_1}^{t_2} \frac{2\pi}{\lambda_2(B)} \frac{1}{\alpha(t)} dt = \int_{t=t_1}^{t_2} \frac{2\pi \alpha(t)}{\lambda_2(B)} dt .
\]

Thus we have \( \mu_t \equiv \mu_a \) if and only if \( \frac{2\pi}{\alpha(t)} = \frac{\pi \alpha(t)}{\lambda_2(B)} \forall t \in [0, U] \), and if and only if \( \alpha(t) = \frac{2\lambda_2(B)}{U} \forall t \in [0, U] \) (note that \( \alpha(t) \) is almost everywhere continuous). Thus we have that \( \alpha(t) \) is constant for \( \mu_t \equiv \mu_a \), and therefore (Lemma 1.39) \( B \) is equiframed.

If now \( B \) is equiframed, then we have \( \alpha(t) = c \) for some constant \( c \), and so \( \mu_a(\partial B) = 2\pi = \int_{t=0}^{U} \frac{\pi \alpha(t)}{\lambda_2(B)} dt = U \frac{\pi c}{\lambda_2(B)} \) as well as \( \alpha(t) = c = \frac{2\lambda_2(B)}{U} \), yielding \( \mu_t \equiv \mu_a \). \( \Box \)
2.4.3 The equivalence of Busemann and Glogovskij angular bisectors

Proposition 2.21 For non-collinear rays $[\mathbf{o}, \mathbf{a}], [\mathbf{o}, \mathbf{b}]$ with unit vectors $\mathbf{a}$ and $\mathbf{b}$ the following conditions are equivalent:

1. $A_G(\angle \mathbf{o} \mathbf{a} \mathbf{b}) = A_B(\angle \mathbf{o} \mathbf{a} \mathbf{b})$, and
2. $\beta(\mathbf{a}) = \beta(\mathbf{b})$ (according to Definition 1.24).

Proof Since $\mathbf{o} \neq \mathbf{a} + \mathbf{b} \in A_B(\angle \mathbf{o} \mathbf{a} \mathbf{b})$, we have that $A_G(\angle \mathbf{o} \mathbf{a} \mathbf{b}) = A_B(\angle \mathbf{o} \mathbf{a} \mathbf{b})$ if and only if $\mathbf{a} + \mathbf{b} \in A_G(\angle \mathbf{o} \mathbf{a} \mathbf{b})$, thus if and only if $\varrho(\mathbf{a} + \mathbf{b}, (\mathbf{o}, \mathbf{a})) = \varrho(\mathbf{a} + \mathbf{b}, (\mathbf{o}, \mathbf{b}))$ (since $\mathbf{a} + \mathbf{b} \in \text{cone}(\mathbf{a}, \mathbf{b})$). Since $\mathbf{o}, \mathbf{a}$ and $\mathbf{b}$ are non-collinear we conclude that $\varrho(\mathbf{a} + \mathbf{b}, (\mathbf{o}, \mathbf{a})) \neq 0$ as well as $\varrho(\mathbf{a} + \mathbf{b}, (\mathbf{o}, \mathbf{b})) \neq 0$. Further equivalent conditions to $A_G(\angle \mathbf{o} \mathbf{a} \mathbf{b}) = A_B(\angle \mathbf{o} \mathbf{a} \mathbf{b})$ are

$$\frac{|\det[\mathbf{a}, \mathbf{b}]|}{\varrho(\mathbf{a} + \mathbf{b}, (\mathbf{o}, \mathbf{a}))} = \frac{|\det[\mathbf{a}, \mathbf{b}]|}{\varrho(\mathbf{a} + \mathbf{b}, (\mathbf{o}, \mathbf{b}))}$$

and (since $\det[\mathbf{a}, \mathbf{a}] = \det[\mathbf{b}, \mathbf{b}] = 0$)

$$\frac{|\det[\mathbf{b} + \mathbf{a}, \mathbf{a}]|}{\varrho(\mathbf{a} + \mathbf{b}, (\mathbf{o}, \mathbf{a}))} = \frac{|\det[\mathbf{a} + \mathbf{b}, \mathbf{a}]|}{\varrho(\mathbf{a} + \mathbf{b}, (\mathbf{o}, \mathbf{b}))}.$$

Lemma 1.25 now states the essential equivalent condition $\beta(\mathbf{a}) = \beta(\mathbf{b})$. □

From this proposition we can conclude that in every Minkowski plane $\mathbb{M}^2$ there are such unit vectors $\mathbf{a}, \mathbf{b}$ with $A_G(\angle \mathbf{o} \mathbf{a} \mathbf{b}) = A_B(\angle \mathbf{o} \mathbf{a} \mathbf{b})$ since the function $\beta(\cdot)$ is continuous. But in general there are also rays $[\mathbf{o}, \mathbf{a}], [\mathbf{o}, \mathbf{b}]$ with $A_G(\angle \mathbf{o} \mathbf{a} \mathbf{b}) \neq A_B(\angle \mathbf{o} \mathbf{a} \mathbf{b})$.

Now we can prove Theorem 2.11. We have $A_G(T) = A_B(T)$ for all angles $T$ of a Minkowski plane if and only if the plane is Radon.

Proof of Theorem 2.11 By Proposition 2.21 we have that $A_G(\angle \mathbf{o} \mathbf{a} \mathbf{b}) = A_B(\angle \mathbf{o} \mathbf{a} \mathbf{b})$ for all non-collinear rays $[\mathbf{o}, \mathbf{a}], [\mathbf{o}, \mathbf{b}]$ of a plane $\mathbb{M}^2$ if and only if the function $\beta(\cdot)$ is constant, i.e., $\beta(\mathbf{r}) = \beta$ for all $\mathbf{r} \in \partial B$.

Let us first assume that the function $\beta(\cdot)$ is constant. Assume further that $\mathbf{r} \mapsto \mathbf{r} \mathbf{r}$ for $\mathbf{r}, \mathbf{r} \neq \mathbf{o}$. Without loss of generality we can assume that $\|\mathbf{r}\| = \|\mathbf{r}\| = 1$ (otherwise consider vectors $\mathbf{r} : = \frac{1}{\|\mathbf{r}\|}\mathbf{r}$ and $\mathbf{r} : = \frac{1}{\|\mathbf{r}\|}\mathbf{r}$, where $\mathbf{r} \mapsto \mathbf{r} \mathbf{r}$ if and only if $\mathbf{r} \mapsto \mathbf{r} \mathbf{r}$). Lemma 1.26 yields $|\det[\mathbf{r}, \mathbf{r}]| = \beta(\mathbf{r})$. Since $\beta(\cdot)$ is constant, we also have $|\det[\mathbf{r}, \mathbf{r}]| = \beta(\mathbf{r})$ and, again by Lemma 1.26, $\mathbf{r} \mapsto \mathbf{r}$. This shows that the boundary of the unit disc is a Radon curve, see Definition 1.32.

For the other direction let us assume that the boundary of the unit disc is a Radon curve. For $\|\mathbf{r}\| = \|\mathbf{r}\| = 1$, Proposition 1.33 yields $|\det[\mathbf{r}, \mathbf{r}]| = c$ (c is constant) for $\mathbf{r} \mapsto \mathbf{r}$. Obviously there is such a $\mathbf{r}$ for any given unit vector $\mathbf{r}$. Lemma 1.26 now yields

$$\beta(\mathbf{r}) = |\det[\mathbf{r}, \mathbf{r}]| = |\det[\mathbf{r}, \mathbf{r}]| = c,$$

i.e., $\beta(\cdot)$ is constant. Consequently we have $A_G(\angle \mathbf{o} \mathbf{a} \mathbf{b}) = A_B(\angle \mathbf{o} \mathbf{a} \mathbf{b})$ for all non-collinear rays $[\mathbf{o}, \mathbf{a}], [\mathbf{o}, \mathbf{b}]$. □

2.4.4 Extensions of systems of angular bisectors for straight angles

Definition 2.22 For a system $A$ of angular bisectors of $\mathbb{M}^2$ with normalized representation $\hat{A}$ we define for every half-plane $H$ and unit vector $\mathbf{a} \in \partial B$ with $\partial H = \langle -\mathbf{a}, \mathbf{a} \rangle$ the inner limit of $\hat{A}$ with fixed side $[\mathbf{o}, \mathbf{a}]$, if it exists, by

$$\hat{A}_\mathbf{a}(H) := \lim_{b \to \mathbf{a}, H} \hat{A}(\mathbf{a}, \mathbf{b}).$$

Definition 2.23 For a system $A$ of angular bisectors of $\mathbb{M}^2$ we define the following binary relation in the set of nonzero vectors $\mathbf{r}, \mathbf{r} \in \mathbb{M}^2 \setminus \{\mathbf{o}\}$: $\mathbf{r}$ is $A$-normal to $\mathbf{r}$ if and only if there is a half-plane $H$ with $\partial H = \langle -\mathbf{r}, \mathbf{r} \rangle$ and $\mathbf{r} \in \{\mathbf{o}, \hat{A}_\mathbf{a}(H)\}$. In this case we write $\mathbf{r} \leftrightarrow A \mathbf{r}$.
We want to show that the equivalence of the Busemann bisectors with the corresponding \( \mu \)-bisector (for some fixed angular measure \( \mu \)) for all angles of \( \mathbb{M}^2 \) implies that \( \mathbb{M}^2 \) is Euclidean. In a first step, we study the inner limit of \( \overline{A_B} \) with a fixed side and show that for the equivalence \( \overline{A_B} = A_\mu \) the plane is Radon (Lemma 2.28 states the relevant condition). The proof of Theorem 2.15 will use induction to show that \( \mathbb{M}^2 \) is Euclidean.

**Notation 2.25** Using the orientation of a parametrization \( u : [0, U] \to \partial B \) of \( \partial B \), we define the half-plane \( H_a = H_{u,a} \) spanned by \( a := u(t) \) for some \( t \in [0, U] \) as the half-plane with boundary \( \langle -a, a \rangle \) also containing the arc \( u([t, t + \frac{1}{2}U]) \) (mod \( U \)).

**Lemma 2.26** Let \( u \) be a parametrization of \( \partial B \) by arc length, and \( a := u(t) \), \( t \in [0, U] \), be an arbitrary unit vector. Then for the inner limit of \( \overline{A_B} \) of \( H_a \) with fixed sides \( [0, a] \) and \( [0, -a] \), respectively, we have

\[
(\overline{A_B})_a(H_a) = u'_-(t) \quad \text{and} \quad (\overline{A_B})_{-a}(H_a) = u'_+(t)
\]

See also Figure 2.2.

**Proof** Let us assume that \( a = u(t) \). Then by Definition 2.22 and Definition 2.3, we have

\[
(\overline{A_B})_{-a}(H_a) = \lim_{\epsilon \downarrow 0} \overline{A_B} \left( u(t + \epsilon), u(t + \frac{1}{2}U) \right) = \lim_{\epsilon \downarrow 0} (u(t + \epsilon) - u(t))^{-1} \lim_{\epsilon \downarrow 0} \frac{u(t + \epsilon) - u(t)}{\epsilon} = u'_+(t).
\]

Here we use the fact that \( \frac{\|u(t+\epsilon) - u(t)\|}{\epsilon} \to 1 \) for \( \epsilon \downarrow 0 \). Analogously we have

\[
(\overline{A_B})_a(H_a) = \lim_{\epsilon \downarrow 0} \overline{A_B} \left( u(t), u(t + \frac{1}{2}U - \epsilon) \right) = \lim_{\epsilon \downarrow 0} \overline{A_B} \left( u(t), -u(t - \epsilon) \right) = \lim_{\epsilon \downarrow 0} (u(t) - u(t - \epsilon))^{-1} \lim_{\epsilon \downarrow 0} \frac{u(t) - u(t - \epsilon)}{-\epsilon} = u'_-(t).
\]

**Lemma 2.27** Let \( \mu \) be an angular measure of \( \mathbb{M}^2 \), \( H \) be a half-plane and \( a \in \partial H \) be a unit vector with \( \partial H = \langle -a, a \rangle \). Then for the inner limit of \( \overline{A_\mu} \) of \( H \) with fixed sides \( [0, a] \) and \( [0, -a] \), respectively, we have

\[
(\overline{A_\mu})_a(H) = (\overline{A_\mu})_{-a}(H).
\]

**Proof** This statement is trivial, since there is exactly one \( b \in H \cap \partial B \) with \( \mu(\angle a0b) = \mu(\angle (-a)0b) = \frac{\pi}{2} \), which coincides with both inner limits above.
CHAPTER 2. ANGULAR MEASURES AND BISECTORS

Lemma 2.28 If \( A_B = A_\mu \), then \( u \vdash v \) holds if and only if \( u \vdash_{A_\mu} v \) holds.

**Proof** If we have \( A_B = A_\mu \), Lemma 2.26 and Lemma 2.27 imply that for all \( t \in [0, U) \) and corresponding \( a := u(t) \) and respective half-plane \( H_a = H_{u,a} \) (Notation 2.25) we have

\[
u'(t) = (\dot{A}_B) - a(H_a) = (\dot{\mu}) - a(H_a) = (\dot{\mu})a(H_a) = (\dot{A}_B)a(H_a) = u'(t).
\]

Therefore the unit circle \( \partial B \) has only regular points. Thus \( u(t) \vdash v \) is equivalent to \( v = \lambda u'(t) \) for some (nonzero) \( \lambda \in \mathbb{R} \). Furthermore, this is equivalent to \( [0, v] = \pm A_\mu(H_a) = A_\mu(\pm H_a) \) as well as \( u(t) = a \vdash_{A_\mu} v \). By scaling, this result extends to all nonzero vectors \( u \in \mathbb{M}^2 \).

**Proof of Theorem 2.15** By construction, in any Euclidean space \( \mathbb{E}^d, d \geq 2 \), the Busemann angular bisector of any angle coincides with the \( \mu \)-bisector, where \( \mu \) denotes the standard angular measure in \( \mathbb{E}^d \).

For the other direction we can restrict ourselves to the planar case \( d = 2 \) by Theorem 1.37. So we can assume that in a given Minkowski plane \( \mathbb{M}^2 \) we have \( A_B = A_\mu \) for some fixed angular measure \( \mu \). We can assume that \( u : [0, U] \to \partial B \) is a parametrization of the unit circle by arc length in positive orientation (Notation 1.31). We denote the parametrization of \( \partial B \) by the angle \( \mu \) by \( m := m_{u,\mu} \) (Definition 2.19).

Since the \( A_\mu \)-normality is symmetric (we have \( u \vdash_{A_\mu} v \) if and only if \( \mu(\angle uv) = \frac{\pi}{2} \)), by Lemma 2.26 also the normality \( \vdash \) is symmetric; thus \( \partial B \) is a Radon curve. Therefore \( \partial B \) is an equiframed curve, and we have that the function \( \alpha(t) \) (Notation 1.31) is constant (Lemma 1.39).

Let us now define a Euclidean background metric \( g_{\mathbb{E}}(\cdot, \cdot) \) by using \( x := m(0) \) and \( y : = m(\frac{\pi}{2}) \) as orthogonal unit vectors. We will show that for every nonnegative \( k, l \in \mathbb{Z} \) we have for \( \phi = \frac{l}{2^k} \pi \) that \( m(\phi) \) coincides with the Euclidean unit vector \( m_\mathbb{E}(\phi) := \cos(\phi) x + \sin(\phi) y \). This vector \( m_\mathbb{E}(\phi) \) is obtained by a Euclidean rotation from \( x \) by the angle \( \phi \) in positive orientation. Since both functions \( m(\cdot) \) and \( m_\mathbb{E}(\cdot) \) are continuous, it follows that they coincide. Thus our metric is the Euclidean one.

Using the natural area measure according to the Euclidean background metric, we have \( \alpha(t) = \alpha(0) = 1 \) for all \( 0 \leq t < U \).

We show inductively for \( k = 0, 1, \ldots \) that for all \( 0 \leq l \leq 2^{k+1} \) the vector \( m(\frac{2l}{2^k} \pi) \in \partial B \) coincides with \( m_\mathbb{E}(\frac{2l}{2^k} \pi) \).

For \( k = 0 \) we trivially have \( m(0) = x = \cos(0) x + \sin(0) y = m_\mathbb{E}(0) \), and \( m(\frac{\pi}{2}) = y = \cos(\frac{\pi}{2}) x + \sin(\frac{\pi}{2}) y = m_\mathbb{E}(\frac{\pi}{2}) \). By properties of the function \( m \). For \( k = 1 \) we additionally have \( m(\frac{\pi}{4}) = \frac{x}{2} = m_\mathbb{E}(\frac{\pi}{4}) \), and \( m(\frac{3\pi}{4}) = -\frac{y}{2} = m_\mathbb{E}(\frac{3\pi}{4}) \).

Now assume that for \( k \geq 2 \) we have proved that \( m(\phi) = m_\mathbb{E}(\phi) \) for all \( \phi = \frac{l}{2^k} \pi = \frac{2l}{2^k} \pi \) with \( l = 0, 1, 2, \ldots, 2^k \). We will show that for \( 0 \leq l < 2^k \) and \( \phi = \frac{2l+1}{2^k} \pi \) the equation \( m(\phi) = m_\mathbb{E}(\phi) \) follows.

By Corollary 2.20 and by \( A_{B,M^2} = A_{\mu,M^2} \), Definition 2.5 our induction hypothesis and the definition of \( m_\mathbb{E}(\cdot) \) we have for \( 0 \leq l < 2^k \) that

\[
m(\frac{2l+1}{2^k} \pi) \in A_\mu \left( \left\{ m\left( \frac{l}{2^{k-1}} \pi \right) \right\} \right) \cup m\left( \frac{l+1}{2^{k-1}} \pi \right) = A_B \left( \left\{ m\left( \frac{l}{2^{k-1}} \pi \right) \right\} \right) = \left\{ m\left( \frac{l}{2^{k-1}} \pi \right) + m\left( \frac{l+1}{2^{k-1}} \pi \right) = \left\{ m\left( \frac{2l+1}{2^k} \pi \right) \right\} \right.
\]

This means that for \( \phi = \frac{2l+1}{2^k} \pi \) the vectors \( m(\phi) \) and \( m_\mathbb{E}(\phi) \) have the same direction and \( \| m(\phi) \|_2 = \| m_\mathbb{E}(\phi) \|_2 \), see also Figure 2.3.

Next we show that for \( 0 < l < 2^k \)

\[
\left\| m\left( \frac{2l-1}{2^k} \pi \right) \right\|_2 = \left\| m\left( \frac{2l+1}{2^k} \pi \right) \right\|_2
\]
have the same Euclidean length. The figure shows why it is not possible that
also using the metric of $M$.

Here we use the facts that $m(\frac{t+1}{2\pi} \pi) = m(\frac{t}{2\pi} \pi)$, $m(\frac{2t+1}{2\pi} \pi) = m(\frac{2t}{2\pi} \pi)$, and $m(\frac{t-1}{2\pi} \pi) = m(\frac{t}{2\pi} \pi)$.

Figure 2.3: Proof of Theorem 2.15: For $\phi = \frac{2k+1}{2} \pi$ the vectors $m(\phi)$ and $m_2(\phi)$ have the same direction ($k = 3, l = 0$).

Figure 2.4: Proof of Theorem 2.15: For $\phi_1 := \frac{2k-1}{2} \pi$ and $\phi_2 := \frac{2k+1}{2} \pi$ the vectors $m(\phi_1)$ and $m(\phi_2)$ have the same Euclidean length. The figure shows why it is not possible that $\|m(\phi_1)\|_2 \neq \|m(\phi_2)\|_2$ for $k = 3, l = 3$: then also $l(m(\phi_1) C) \neq l(C m(\phi_2))$.

holds, giving $\|m(\frac{t}{2\pi} \pi)\|_2 = c$ for all odd $l$ with $0 < l < 2^{k+1}$. We use the abbreviations $\phi_1 := \frac{2k-1}{2} \pi$, $\phi_2 := \frac{2k+1}{2} \pi$ and $\phi := \frac{l}{2\pi} \pi = \frac{2k+1}{2} \pi$. Using Property 2.7, we get with $\{C\} := \{o, m(\phi)\} \cap m(\phi_1) m(\phi_2)$ that

$$1 = \frac{\rho(m(\phi_1), o)}{\rho(m(\phi_2), o)} = \frac{\rho(m(\phi_1), C)}{\rho(C, m(\phi_2))} = \frac{\rho_c(m(\phi_1), C)}{\rho_c(C, m(\phi_2))} = \frac{\rho_c(m(\phi_1), o)}{\rho_c(m(\phi_2), o)} = \frac{\|m(\phi_1)\|_2}{\|m(\phi_2)\|_2}$$

Here we use the facts that $\{o, m(\phi)\}$ is the Euclidean angular bisector of the two rays $\{o, m(\phi_1)\} = \{o, m(\phi_1)\}$ and $\{o, m(\phi_2)\} = \{o, m(\phi_2)\}$, thus satisfies Property 2.7 using the Euclidean metric and also using the metric of $M^2$, see Figure 2.4.

Thus we know that $m(\phi) = c \cdot m_2(\phi)$ for all $\phi = \frac{l}{2\pi} \pi$ with odd $l$ and with common constant $c$.

Finally, for $u := m(\frac{1}{2\pi} \pi)$ and $v := m(\frac{1+2^{k-1}}{2\pi} \pi)$ the relation $u \dashv_{A_c} v$ holds by Corollary 2.24. By Lemma 2.27, $u \dashv v$ follows. There is some $t \in [0, U]$ (precisely $t = w_0^{-1}(\frac{1}{2\pi} \pi)$) with $u = u(t)$, thus we have by $\|v\| = 1$ that $u'(t) = sv$ with $s \in \{1, -1\}$. Obviously, $u$ and $v$ are also orthogonal in the Euclidean sense. Thus $1 = \alpha(t) = \det[u(t), u'(t)] = \det[u, sv] = s \det[c m_2(\frac{1}{2\pi} \pi), c m_2(\frac{1+2^{k-1}}{2\pi} \pi)] = sc^2$. We conclude that $s = 1$ and $c = 1$, thus $m(\phi) = c \cdot m_2(\phi)$ for all $\phi = \frac{1}{2\pi} \pi$ with $l = 0, \ldots, 2^{k+1}$. Hence our induction argument is complete, showing that $m(\phi) = m_2(\phi)$ for all $0 \leq \phi \leq 2\pi$, since $m$ and $m_2$ are both continuous functions.

Thus the Minkowskian metric coincides with the introduced Euclidean metric, and $\mu$ coincides with the standard angular measure in this Euclidean plane.
2.4.6 The equivalence of Glogovskij’s definition with that of a $\mu$-bisectors

First we show that the Glogovskij angular bisector in a Minkowski plane is really a Busemann angular bisector in the dual plane, i.e., $A_{B, M^2(I)} = A_{G, M^2(B)}$ where $I$ is the isoperimetrix in $M^2(B)$.

**Proof of Lemma 2.17** Let us consider two linearly independent vectors $a$ and $b$ which have unit length in $M^2(I)$, i.e.,

$$\|a\|_I = \|b\|_I = 1.$$ 

Note that we regard $M^2(B)$ as our primary Minkowski plane, simply denoted by $M^2$ and its metric by $\varrho(\cdot, \cdot)$. We have to show that the two angular bisectors of the angle $T = \angle \alpha \beta$, namely

$$A_{G, M^2}(T) = \{ \epsilon \in T : \varrho(\epsilon, \langle \alpha, a \rangle) = \varrho(\epsilon, \langle \beta, b \rangle) \}$$

and

$$A_{B, M^2(I)}(T) = [\alpha, \alpha + b],$$

coincide. By Lemma 2.10 it suffices to show that

$$\varrho(a + b, \langle \alpha, a \rangle) = \varrho(a + b, \langle \beta, b \rangle). \tag{2.2}$$

The parallelogram with vertices $\alpha, \alpha, b$ and $a + b$ contains two triangles $T_a$ and $T_b$ with vertices $\alpha, a, a + b$ and $\beta, b, a + b$, respectively, which both have the same area (see also Figure 2.5).

Applying Proposition 1.40 to the triangles $T_a$ and $T_b$ and sides $\langle \alpha, a \rangle$ and $\langle \beta, b \rangle$, respectively, we get

$$v_a = h_a \tilde{g}_a = v_b = h_b \tilde{g}_b,$$

where $\tilde{g}_a = \|a\|_I = 1 = \|b\|_I = \tilde{g}_b$. Thus the Minkowskian heights $h_a$ and $h_b$ are equal which, in fact, is equation (2.2).

**Proof of Theorem 2.16** As in the Proof of Theorem 2.15 we only have to show that $A_{G, M^2} = A_{\mu, M^2}$ implies that the Minkowski plane $M^2$ is Euclidean.

We denote by $I$ the isoperimetrix of $M^2 = M^2(B)$ and transform the angular measure $\mu$ of $M^2$ into an angular measure $\mu'$ of $M^2(I)$ such that for any angle $T$ we have that $\mu(T) = \mu'(T)$. Using Lemma 2.17 we conclude from the assumption $A_{G, M^2} = A_{\mu, M^2}$ that $A_{B, M^2(I)} = A_{G, M^2} = A_{\mu, M^2} = A_{\mu', M^2(I)}$. By Theorem 2.15 this implies that $M^2(I)$ is Euclidean and $\mu'$ its standard angular measure. Thus $M^2$ is Euclidean, and $\mu$ is its standard angular measure.

2.5 Summary

We considered three different kinds for angular bisectors in a Minkowski space.

The first two types of bisectors (Busemann’s and Glogovskij’s angular bisectors) are uniquely determined by the metric. The third definition involves an angular measure as parameter.

We answered the question when two of these approaches coincide for the whole space. Table 2.1 summarizes the planar results, where $\mu_1$ and $\mu_2$ denote arbitrary angular measures and $\mu_a$ and $\mu_l$ are the angular measures induced by area and arc length.
### Table 2.1: Characterization of equivalences for angular bisectors in Minkowski planes

<table>
<thead>
<tr>
<th>$=$</th>
<th>$A_G$</th>
<th>$A_{\mu_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_B$</td>
<td>$\mathbb{M}^2$ is Radon, see Theorem 2.11</td>
<td>$\mathbb{M}^2$ is Euclidean, see Theorem 2.15</td>
</tr>
<tr>
<td>$A_{\mu_1}$</td>
<td>$\mu_1 = \mu = \mu_1$, see Theorem 2.16</td>
<td>$\mu_1 = \mu_2$, see Proposition 2.14 for $\mu_1 = \mu_2$, $\mathbb{M}^2$ has an equiframed unit circle, see Theorem 2.13</td>
</tr>
</tbody>
</table>

Table 2.1 summarizes the results for spaces of dimension $d \geq 3$, also including the result of the following chapter.

### Table 2.2: Characterization of equivalences for angular bisectors in Minkowski spaces $\mathbb{M}^d$, $d \geq 3$

<table>
<thead>
<tr>
<th>$=$</th>
<th>$A_G$</th>
<th>$A_{\mu_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_B$</td>
<td>$\mathbb{M}^d$ is Euclidean, see Corollary 2.12</td>
<td>$\mathbb{M}^d$ is Euclidean, see Theorem 2.15</td>
</tr>
<tr>
<td>$A_{\mu_1}$</td>
<td>$\mu_1 = \mu_1 = \mu_1$, see Theorem 2.16</td>
<td>$\mu_1 = \mu_2$, see Proposition 2.14 for $\mu_1 = \mu_2$, $\mathbb{M}^d$ is Euclidean, see Chapter 3</td>
</tr>
</tbody>
</table>
Chapter 3

About convex bodies with equiframed two-dimensional sections

The results of the following chapter are contained in the paper entitled “Convex bodies with equiframed two-dimensional sections”, which is accepted for publication by the journal “Archiv der Mathematik” [13].

3.1 The statement

It is well known that Minkowski spaces $\mathbb{M}^d$, $d \geq 3$, all whose 2-subspaces are Radon, are Euclidean, see Theorem [1.35]. This chapter contains the proof of a slight strengthening of this statement for equiframed 2-subspaces: We show that a compact, convex, centered body in $\mathbb{R}^d$, $d \geq 3$, all whose two-dimensional sections through the origin are equiframed bodies, is an ellipsoid.

Theorem 3.1 Every $d$-dimensional Minkowski space $\mathbb{M}^d$, $d \geq 3$, all whose 2-dimensional subspaces are equiframed, is Euclidean.

We remark that due to Definition [1.18] we mean linear subspaces $L$ of $\mathbb{M}^d$ which are required to be equiframed. These subspaces have $L \cap B$ as unit ball. But Theorem 3.1 has an equivalent meaning if we consider all affine subspaces of $\mathbb{M}^d$ of dimension 2. Such a 2-plane $P$ is isometric to the corresponding parallel linear subspace $L$ of $\mathbb{M}^d$. In general the unit ball of $P$ is not the intersection $P \cap B$ but a translate of $L \cap B$.

The proof of Theorem 3.1 is distributed over the Sections 3.2 to 3.5. Section 3.2 describes the framework of the proof by contradiction, based on the fundamental Theorem 1.35. Then, in Section 3.3, we derive the main geometric properties of planes which are equiframed but not Radon. Using topological reasoning, in Section 3.4 we establish the corresponding local properties of the three-dimensional unit balls if they would exist. Finally, in Section 3.5, this proof is finished by looking globally at this assumed unit ball.

In the remaining two sections of this chapter we consider the construction developed within this proof from another point of view and an application of Theorem 3.1 respectively.

3.2 Indirect approach

For the proof we assume that the 3-dimensional Minkowski space $\mathbb{M}^3$ has the property that every 2-subspace $P$ is equiframed. Furthermore, we assume that at least one 2-subspace $P$ is not Radon, otherwise the claim of Theorem 3.1 follows by Theorem 1.35. We will show that this is not possible, i.e., that such spaces do not exist. We only need to consider the case $d = 3$ since for $d > 3$ the consideration of one 3-subspace containing $P$ will yield the same contradiction.
CHAPTER 3. CONVEX BODIES WITH EQUIFRAMED 2-DIMENSIONAL SECTIONS

**Definition 3.2** We say that $\mathbf{r}$ and $\mathbf{\eta}$ are **symmetrically normal** in the Minkowski space $\mathbb{M}^d = \mathbb{M}^d(B)$, denoted by $\mathbf{r} \perp \mathbf{\eta}$, if $\mathbf{r} + \mathbf{\eta}$ and $\mathbf{\eta} - \mathbf{r}$ holds.

We note that $\mathbf{r} \perp \mathbf{\eta}$ holds if and only if $C(\mathbf{r}, \mathbf{\eta})$ is contained in the parallelogram $\text{conv}\{\mathbf{r} + \mathbf{\eta}, \mathbf{r} - \mathbf{\eta}, -\mathbf{r} - \mathbf{\eta}, -\mathbf{r} + \mathbf{\eta}\}$.

To benefit from the properties of $B \cap \tilde{P}$ we need the two characterizations in Proposition [1.32] and Lemma [1.39]. We will have a closer look at the function $\alpha(\mathbf{r}, \mathbf{\eta}) = [\mathbf{r}, \mathbf{r}_+^{(\mathbf{r}, \mathbf{\eta})}]_{(\mathbf{r}, \mathbf{\eta})}$ for linearly independent vectors $\mathbf{r}, \mathbf{\eta} \in \partial B$, already introduced in Notation [1.31].

### 3.3 Planar considerations

The proof of Theorem [3.1] is based on the following property of equiframed curves which are not Radon curves, see also Figure 3.1.

**Lemma 3.3** For any two symmetrically normal vectors $\mathbf{b}, \mathbf{p} \in C = \partial B$ on the boundary of an equiframed planar convex body $B (\mathbf{b} \perp \mathbf{p})$ with $\alpha(\mathbf{b}, \mathbf{p}) < 1$, there are four points $\mathbf{a}, \mathbf{c}, \mathbf{e}, \mathbf{d} \in C$ such that $C$ contains the four segments $\overline{\mathbf{a}\mathbf{b}} \cup \overline{\mathbf{b}\mathbf{c}} \cup \overline{\mathbf{p}\mathbf{e}} \cup \overline{\mathbf{e}\mathbf{d}} \subset C$ and $(\mathbf{a}, \mathbf{b}) \parallel (\mathbf{d}, \mathbf{e}) \parallel \mathbf{a}, (\mathbf{e}, \mathbf{p}) \parallel \mathbf{c}$ and $(\mathbf{e}, \mathbf{d}) \parallel \mathbf{b}$, see also Figure 3.1.

\[\begin{array}{c}
\text{Figure 3.1: Segments contained in an equiframed curve for } \alpha(\mathbf{b}, \mathbf{p}) < 1 \text{ (Lemma 3.3)}
\end{array}\]

**Proof** To $C$ we give the orientation of $C(\mathbf{b}, \mathbf{p})$ such that $(\mathbf{b}, \mathbf{p}, -\mathbf{p}, -\mathbf{b})_C$ holds. We define

\[\mathbf{a} := -\mathbf{p}_C, \quad \mathbf{c} := -\mathbf{p}_C^+, \quad \mathbf{d} := \mathbf{b}_C, \quad \mathbf{e} := \mathbf{b}_C^+.
\]

Thus we have immediately that $\mathbf{p} \perp \mathbf{a}$, $\mathbf{p} \perp \mathbf{c}$, $\mathbf{b} \perp \mathbf{d}$ and $\mathbf{b} \perp \mathbf{c}$. Furthermore, with $\alpha := \alpha(\mathbf{b}, \mathbf{p})$ and $[\cdot, \cdot] := [\cdot, \cdot]_{(\mathbf{b}, \mathbf{p})}$ we have by Lemma 1.39 that $[\mathbf{a}, \mathbf{d}] = [\mathbf{b}, \mathbf{e}] = [\mathbf{p}, -\mathbf{a}] = [\mathbf{p}, -\mathbf{c}] = \alpha$. This yields $|\mathbf{p}, \mathbf{a} - \mathbf{c}| = |\mathbf{b}, \mathbf{d} - \mathbf{e}| = 0$, which means $(\mathbf{a}, \mathbf{c}) \parallel \mathbf{p}$ and $(\mathbf{d}, \mathbf{e}) \parallel \mathbf{b}$.

Furthermore, by Corollary 1.39 we have the ordering $(\mathbf{b}, \mathbf{b}_C, \mathbf{p}, \mathbf{b}_C^+)$ (since $\mathbf{p} \perp \mathbf{p}$ and $C = C(\mathbf{b}, \mathbf{p}))$. Especially we have $(\mathbf{a}, \mathbf{p}, -\mathbf{b})_C$ and thus the same for the tangent vectors $(\mathbf{d}_C, \mathbf{p}_C, \mathbf{p}_C^+, -\mathbf{b}_C)_C$, which implies, together with Corollary 1.30 (since $\mathbf{p} \perp \mathbf{p}$ and $C = C(\mathbf{b}, \mathbf{p})$), that $(\mathbf{d}_C, \mathbf{p}_C, -\mathbf{b}, \mathbf{p}_C^+, -\mathbf{d})_C$ and finally that $(\mathbf{d}_C, \mathbf{d}_C, -\mathbf{a}, -\mathbf{b}, -\mathbf{d})_C$. By Lemma 1.39 we have $[\mathbf{a}, \mathbf{d}] = \alpha = [\mathbf{b}, \mathbf{d}] = [\mathbf{b}, -\mathbf{b}]$ and $[\mathbf{d}, -\mathbf{d}] = 0$. By convexity, $|\mathbf{b}, -\mathbf{r}| \geq \alpha$ follows for all $\mathbf{r}$ between $-\mathbf{d}_C$ and $\mathbf{b}$, in particular for all $\mathbf{r}$ with $(\mathbf{a}, \mathbf{r})_C$. Since $\mathbf{b} \perp \mathbf{d}$, we get that $[\mathbf{d}, -\mathbf{r}] = [\mathbf{r}, \mathbf{d}] \leq [\mathbf{b}, \mathbf{d}] = \alpha$, thus $[\mathbf{a}, \mathbf{d}] = [\mathbf{r}, \mathbf{d}] = \alpha$ for $(\mathbf{a}, \mathbf{r})_C$. This immediately yields $\overline{\mathbf{a}\mathbf{b}} \subset C$ and $\mathbf{a} \perp \mathbf{b}$. By analogous reasoning we get $\overline{\mathbf{b}\mathbf{c}} \subset C$, $\overline{\mathbf{p}\mathbf{e}} \subset C$ and $\overline{\mathbf{e}\mathbf{d}} \subset C$, as well as $(\mathbf{b}, \mathbf{c}) \parallel \mathbf{e}$ and $(\mathbf{p}, \mathbf{d}) \parallel \mathbf{a}$, and our proof is complete. \(\square\)

The preliminaries of Lemma 3.3 can also be replaced in the following way:

**Lemma 3.4** If for two vectors $\mathbf{b}, \mathbf{p} \in \partial B$ of an equiframed two-dimensional convex body $B$ the inequality $\alpha(\mathbf{b}, \mathbf{p}) < 1$ holds, and $\mathbf{b}, \mathbf{p}$ are non-regular points of $C := \partial B$ (i.e., $\mathbf{b}_C^+ \neq \mathbf{b}_C^-$ and $\mathbf{p}_C^+ \neq \mathbf{p}_C^-$, where $C$ is arbitrarily oriented), then they are symmetrically normal, i.e., $\mathbf{b} \perp \mathbf{p}$.

**Proof** As in the proof of Lemma 3.3 we choose the orientation $C = C(\mathbf{b}, \mathbf{p})$ and define $\mathbf{a} := -\mathbf{p}_C$, $\mathbf{c} := -\mathbf{p}_C^+$, $\mathbf{d} := \mathbf{b}_C$, $\mathbf{e} := \mathbf{b}_C^+$ and $[\cdot, \cdot] := [\cdot, \cdot]_{(\mathbf{b}, \mathbf{p})}$. Since $0 = [\mathbf{b}, \mathbf{b}] < [\mathbf{b}, \mathbf{d}] = [\mathbf{b}, \mathbf{c}] = \alpha(\mathbf{b}, \mathbf{p}) <$
CHAPTER 3. CONVEX BODIES WITH EQUIFRAMED 2-DIMENSIONAL SECTIONS

1 = [b, p], d ≠ c, (b, d, c, −d)_{C}, and by the convexity of B, we get that p lies between d and c, i.e., (b^C, p, b^C_{C}), thus b ⊥ p. Analogously, we get that b lies between a and c, and that p ⊥ b. □

The next lemma assures that close to \( \beta \) the effect of Lemma 3.3 will not vanish. Thus the given inequalities need not be best possible.

Lemma 3.5 With \( \alpha := \alpha(b, p) \) we have for the six points a, b, c, d, p, ε in Lemma 3.3 that

\[
\begin{align*}
|a\beta| & \geq (1 - \alpha)|\beta\beta|, \\
|b\gamma| & \geq (1 - \alpha)|\gamma\gamma|, \\
|a\gamma| & \geq (1 - \alpha)|\gamma\alpha|, \\
|a\gamma| & \geq (1 - \alpha)|\alpha\alpha|.
\end{align*}
\]

Proof Since \( a\beta \not\parallel d \), we have \( b - a = \lambda d \) for some real number \( \lambda \neq 0 \). Since \( (a, b) \) is a tangent to \( C \) at \( b \), we get \( |[d, \gamma]| = |[b, a]| = \alpha \) for all \( \gamma \in C \), in particular \( |[d, \gamma]| \leq \alpha \). Now we have that \( \lambda |\alpha| \geq |\lambda[d, \gamma]| = |[b - a, \gamma]| = |[b, \gamma] - [a, \gamma]| = 1 - \alpha \). Thus \( |\beta\gamma| = |\lambda| |\beta\gamma| \geq \frac{1 - \lambda}{\lambda} |\beta\gamma| \geq (1 - \alpha)|\beta\beta| \), which proves our first inequality. The other three inequalities can be proven with analogous arguments.

Again we consider the plane \( \hat{P} \). Since \( \partial B := \partial B \cap \hat{P} \) is not a Radon curve, there are points \( \tilde{b}, \tilde{p} \) on this curve which satisfy the preliminaries of Lemma 3.3. \( \tilde{b} \perp \tilde{p} \) and \( \alpha(\tilde{b}, \tilde{p}) < 1 \). For example, the pair \( (\tilde{b}, \tilde{p}) \) on \( \partial B \times \partial B \) can be arbitrarily chosen such that \( [\tilde{b}, \tilde{p}] \) is maximal for some fixed skew-symmetric form \([·, ·]\) in \( \hat{P} \) (and thus \(|[\tilde{b}, \tilde{p}]|\) is maximal for every skew-symmetric form in \( \hat{P} \)).

Then \( \tilde{b} \perp \tilde{p} \) is clear (for arbitrary \( \partial B \)) and we have to show that \( 1 = [\tilde{b}, \tilde{p}](\tilde{b}, \tilde{p}) > \alpha(\tilde{b}, \tilde{p}) =: \alpha \). We abbreviate \([·, ·](\tilde{b}, \tilde{p})\) by \([·, ·]\) and have that \( \alpha = |[u, u^{\partial B}| = [u, u^{\partial B}] \) for all \( u \in \partial B \) (with orientation \( \partial B := C(\tilde{b}, \tilde{p}) \)). By convexity and Lemma 1.3 we have \(|[u, v]| \geq \alpha \) for all \( u, v \in \partial B \) with \( u \perp v \), thus in particular \( 1 = [\tilde{b}, \tilde{p}] \geq \alpha \). Now assume that we had \( \alpha = 1 \) and that \( u \perp v \) for some \( u, v \in \partial B \). Since \( [\tilde{b}, \tilde{p}] \) was chosen as maximum, we have \( \alpha = [\tilde{b}, \tilde{p}] \geq |[u, v]| \geq \alpha \), hence \(|[u, v]| = \alpha \). Now Proposition 1.3 states that \( \partial B \) is a Radon curve. Since this contradicts our preliminaries, we get that \( \alpha < 1 \) as wanted.

3.4 Local 3-dimensional extensions of the planar properties

The next important fact is that \( \alpha(b, p) \) depends continuously on \( b \) and \( p \) (whenever \( b \) and \( p \) are linearly independent). This can be seen by the formula

\[
\alpha(b, p) = \frac{\text{minimal area of an parallellogram containing } C(b, p)}{2(\text{area of the parallellogram conv} \{\pm b, \pm p\})},
\]

which holds for every area measure in the corresponding planes. With the usual area measure defined by the Euclidean length, both the nominator and the denominator are continuous and not zero.

By continuity, for a neighborhood \( U := U_{\varepsilon}(\tilde{b}) \times U_{\varepsilon}(\tilde{p}) \) of \( (\tilde{b}, \tilde{p}) \), for some \( \varepsilon > 0 \), we still have that \( \alpha(b, p) \leq \tilde{\alpha} := \frac{1 + \alpha(b, \tilde{p})}{2} < 1 \) for all \( (b, p) \in U \).

Next we construct pairs \( (b, p) \) in \( U \cap (\partial \hat{B})^2 \) such that \( b \perp p \) still holds.

For this we rotate the plane \( \hat{P} \) a little bit around \( \tilde{p} \), with angle \( \gamma \), say, and get a new plane \( \hat{P}_\gamma \) and a new curve \( C_\gamma = \partial B \cap \hat{P}_\gamma \). We are looking for vectors \( b(\gamma) \in C_\gamma \) such that \( b(\gamma) \perp \tilde{p} \) if \( |\gamma| \) is small enough.

For this we can define \( b(\gamma) \) as a vector in \( C_\gamma \) such that \( b(\gamma) \perp \tilde{p} \) and \( b(\gamma) \) is closer to \( \tilde{b} \) than to \( -\tilde{b} \). In general, \( \gamma \perp \tilde{p} \) and \( \gamma \in C_\gamma \) is satisfied by all points \( \gamma \in \gamma_1 \gamma_1 \cup -\gamma_2 \gamma_2 \) for some (in most cases degenerate) segment \( \gamma_1 \gamma_2 \) in \( C_\gamma \). For small \( |\gamma| \) we choose \( b(\gamma) := \pm \frac{1}{2}(\gamma_1 + \gamma_2) \) with the appropriate sign such that \( \|b(\gamma)\| \leq \|b(-b(\gamma))\| \). (It is unimportant how to choose \( b(\gamma) \) in case these distances are equal, since this will not occur if \( |\gamma| \) is sufficiently small.)

Lemma 3.6 Assume that each planar section of \( B \subset \mathbb{R}^3 \) is equiframed, \( u, v \in \partial B \) and \( u \perp v \) as well as \( \alpha(u, v) < 1 \). Then there is a neighborhood \( V \) of \( v \) such that \( u \perp \gamma \) for all \( \gamma \in V \).
Proof Let $S_+$ and $S_-$ be supporting planes of $B$ at $u$ parallel to $u^C(u,v)$ and $u^C(u,v)$, respectively. Let $\bar{v}$ be a vector with the same direction as the line $S_+ \cap S_-$, i.e., both planes $S_+ = u + R u^C(u,v) + R \bar{v}$ and $S_- = u + R u^C(u,v) + R \bar{v}$ support $B$ at $u$. Consequently, for each vector $\tilde{\bar{v}}$ between $u^C(u,v)$ and $u^C(u,v)$ on $C(u,v)$ the plane $u + R \tilde{\bar{v}} + R \bar{v}$ supports $B$ at $u$. Thus for each vector $\bar{v} = \tilde{\bar{v}} + \lambda \bar{v}$, with $(u^C(u,v), \tilde{\bar{v}}, u^C(u,v))$ and $\lambda \in \mathbb{R}$, the line $u + R \bar{v} = u + R (\tilde{\bar{v}} + \lambda \bar{v})$ is contained in some supporting hyperplane of $B$, and hence $u + \bar{v}$ holds.

Obviously, there is some $\varepsilon > 0$ such that all $\bar{v} \in U_\varepsilon(u)$ can be represented in the required form since $(u^C(u,v), \bar{v}, u^C(u,v))$ (by $u + \bar{v}$) and $\bar{v}$ is also different from $u^C(u,v)$ and $u^C(u,v)$ (since $\alpha(u,v) = [u,u^C(u,v)](u,v) = [u,u^C(u,v)](u,v) < 1 = [u,u^C(u,v)](u,v)$).

By Lemma 3.3 it is clear that for small $|\gamma|$ we also have $\bar{v} \not\perp b(\gamma)$, since $b(\gamma)$ is continuous in $\gamma = 0$ with $\bar{b} = b(0)$. In fact, $b(\gamma)$ is everywhere continuous where it is “unique”, i.e., where the corresponding segment $\bar{a}_\gamma \bar{b}_\gamma$ degenerates to $\{\bar{b}\} = \{\bar{a}\}$.

Thus we defined a local extension of $\bar{b}$ to $b(\gamma)$ ($-\varepsilon < \gamma < \varepsilon$ for some $\varepsilon > 0$) such that still $b(\gamma) \perp \bar{p}$ and $\alpha(b(\gamma), \bar{p}) \leq \bar{b} \not\perp 1$ for all $|\gamma| < \varepsilon$. By Lemma 3.3 with respect to $(b_1(\gamma), p_1(\gamma)) := (b(\gamma), \bar{p})$ there are points $a_1(\gamma) = c_1(\gamma)$, $a_2(\gamma) = c_2(\gamma)$ with $a_1(\gamma) b_1(\gamma) \cup b_1(\gamma)c_1(\gamma) \cup c_1(\gamma)p_1(\gamma) \cup c_1(\gamma)b_1(\gamma) \subset \partial B \cap P_\gamma$. In the same way we can extend $\bar{p}$ to $p(\delta)$, $-\varepsilon < \delta < \varepsilon$ (we make $\delta$ smaller, if necessary), with $\bar{b} \perp \bar{p}(\delta)$ and $\alpha(b(\gamma), p(\delta)) \leq \bar{b} \not\perp 1$ for all $|\delta| < \varepsilon$. Applying Lemma 3.3 in the plane $Q_\delta := P(b(\gamma), p(\delta))$ we get points $a_2(\delta), c_2(\delta), c_2(\delta)$ and $c_2(\delta)$ and have $a_2(\delta) b_2(\delta) \cup b_2(\delta)c_2(\delta) \cup c_2(\delta)p_2(\delta) \cup p_2(\delta)c_2(\delta) \subset \partial B \cap Q_\delta$.

Next the fact that close to $b(\gamma)$ the surface $\partial B$ is linear with respect to planar sections is extended. This surface is really locally planar and $b(\gamma)$ is locally straight.

Lemma 3.7 Assume that $\partial B$ is the boundary of a convex body $B$ in $\mathbb{R}^3$ which is linear in a neighborhood $U_\varepsilon(b)$ of $b \in \partial B$ in the following way:

- There are two continuous families of planes $P_\gamma (\gamma_1 \leq \gamma \leq \gamma_2)$ and $Q_\delta (\delta_1 \leq \delta \leq \delta_2)$ such that $P_\gamma \cap \partial B$ as well as $Q_\delta \cap \partial B$ consist of two straight segments within $U_\varepsilon(b)$ for all $\gamma, \delta$:

  $P_\gamma \cap \partial B \cap U_\varepsilon(b) = \overline{\bar{a}(\gamma)b(\gamma) \cup b(\gamma)c(\gamma)}$,

  $Q_\delta \cap \partial B \cap U_\varepsilon(b) = \overline{\bar{a}(\delta)b(\delta) \cup b(\delta)c(\delta)}$.

For $\gamma_1 \leq \gamma \leq \gamma_2$ as well as $\delta_1 \leq \delta \leq \delta_2$ these representations are not degenerate, i.e., the corresponding segments have positive lengths and their union is not a segment itself.
\[
\hat{a}(\gamma), \hat{b}(\gamma), \hat{c}(\gamma), \hat{d}(\delta), \text{ and } \hat{c}(\delta) \text{ are continuous functions mapping } \gamma \in [\gamma_1, \gamma_2] \text{ and } \delta \in [\delta_1, \delta_2] \\
\text{injectively to the boundary } R := \partial B \cap \partial U_\varepsilon(b) \text{ (except for the function } b) \text{ or a curve } S \subset \partial B \cap U_\varepsilon(b) \text{ (for } b).}
\]

Furthermore, all planes \( P_\varepsilon \) contain some fixed line having no points within \( U_\varepsilon(b) \), and all planes \( Q_\delta \) contain some fixed line containing \( b \).

There is one plane belonging to both families, \( P_{\gamma^*} = Q_{\delta^*}, \hat{a}(\gamma^*) = \hat{a}(\delta^*), \hat{b}(\gamma^*) = b \text{ and } \hat{c}(\gamma^*) = \hat{c}(\delta^*), \) with \( \gamma_1 < \gamma^* < \gamma_2 \) and \( \delta_1 < \delta^* < \delta_2 \).

There are some \( \gamma_3, \gamma_4 \in \mathbb{R} \) with \( \gamma_1 \leq \gamma_3 < \gamma^* < \gamma_4 \leq \gamma_2 \) such that for all \( \gamma_3 \leq \gamma \leq \gamma_4 \) we have that \( P_{\gamma} \cap \partial B \cap U_\varepsilon(b) = \hat{a}(\gamma)\hat{b}(\gamma) \cup \hat{b}(\gamma)\hat{c}(\gamma) \) has nonempty intersection with both \( Q_{\delta_1} \) and \( Q_{\delta_2} \).

Then the surface \( \partial B \) is locally planar and the curve \( S \) is locally straight: for \( \gamma \in [\gamma_3, \gamma_4] \) we have \( \hat{b}(\gamma) \in \hat{b}(\gamma_3)\hat{b} \cup \hat{b}(\gamma_4) \), and the planar quadrilaterals with vertices \( \{b, \hat{b}(\gamma_3), \hat{a}(\gamma_3), \hat{a}(\gamma^*)\}, \{b, \hat{b}(\gamma_4), \hat{a}(\gamma_4), \hat{a}(\gamma^*)\}, \{b, \hat{b}(\gamma_3), \hat{c}(\gamma_3), \hat{c}(\gamma^*)\}, \) and \( \{b, \hat{b}(\gamma_4), \hat{c}(\gamma_4), \hat{c}(\gamma^*)\} \) belong to \( \partial B \).

**Proof** For \( \gamma_5 := \frac{1}{2}(\gamma^* + \gamma_3) \) the segment \( \hat{a}(\gamma_5)\hat{b}(\gamma_5) \) has intersection with \( Q_{\delta_1} \) or \( Q_{\delta_2} \). Without loss of generality we may say that \( I := \hat{a}(\gamma_5)\hat{b}(\gamma_5) \cap \hat{a}(\delta_1)\hat{b} \) contains one point which is not an endpoint of the two segments.

Since \( B \) is convex, \( \partial B \) must contain the planar quadrilateral with vertices \( \hat{a}(\gamma_5), \hat{a}(\delta_1), \hat{b}(\gamma_5) \) and \( b \). Now for all \( \gamma \) strictly between \( \gamma_3 \) and \( \gamma^* \) the segment \( \hat{a}(\gamma)\hat{b}(\gamma) \) has a small segment in common with this quadrilateral and thus lies in the same plane. Since \( \partial B \) is compact, both \( \hat{a}(\gamma_3)\hat{b}(\gamma_3) \) as well as \( \hat{a}(\gamma^*)\hat{b}(\gamma^*) \) belong to this plane, and so does the quadrilateral with vertices \( b, \hat{b}(\gamma_3), \hat{a}(\gamma_3), \hat{a}(\gamma^*) \).

Analogous reasoning yields that the other three quadrilaterals belong to \( \partial B \), too. By the preliminaries, the two quadrilaterals with vertices \( \{b, \hat{b}(\gamma_3), \hat{a}(\gamma_3), \hat{a}(\gamma^*)\} \) and \( \{b, \hat{b}(\gamma_4), \hat{c}(\gamma_3), \hat{c}(\gamma^*)\} \) are not contained in one plane, thus the intersection of the two carrying planes is the line through \( b \) and \( \hat{b}(\gamma_3) \). By the above arguments, \( \hat{b}(\gamma) \) must belong to this intersection for all \( \gamma \) between \( \gamma^* \) and \( \gamma_3 \).

By analogy, for \( \gamma \) between \( \gamma^* \) and \( \gamma_4 \) the points \( \hat{b}(\gamma) \) belong to \( bb(\gamma_4) \). \( \square \)

We can apply Lemma 3.7 with respect to \( b \) and the planes \( P_\varepsilon \) and \( Q_\delta \). By Lemma 3.5 for all \( |\gamma| < \varepsilon \) and \( |\delta| < \varepsilon \) the segments \( a_1(\gamma)b_1(\gamma), b_1(\gamma)c_1(\gamma), a_2(\delta)b_2(\delta), \) and \( b_2(\delta)c_2(\delta) \) have length at least \( (1 - \alpha)l \), where \( l := \min \|b\| : b \in \partial B \) is positive and depends only on \( B \). Now let \( \varepsilon_1 > 0 \) be small enough such that \( 2\varepsilon_1 \leq (1 - \alpha)l \) and \( \varepsilon_1 \leq \|\hat{b}(\pm \varepsilon)\| \). This ensures that \( P_\varepsilon \) (for all \( \gamma \) with \( \varepsilon < \gamma_1 \leq \gamma_2 < \varepsilon_1 \), for appropriate \( \gamma_1 < 0 < \gamma_2 \) as well as \( Q_\delta \) (for all \( \varepsilon > 0 \)) \( \delta_1 \leq \delta \leq \delta_2 := \varepsilon \)) intersect \( U_\varepsilon(b) \cap \partial B \) in two segments, and the preliminaries of Lemma 3.7 are fulfilled (with \( \varepsilon = \varepsilon_1 \) and \( \gamma^* = \delta^* = 0 \)), i.e., the planar quadrilaterals with vertices \( \{b, b_1(\gamma_3), a_1(\gamma_3), a_1(0)\}, \{b, b_1(\gamma_4), a_1(\gamma_4), a_1(0)\}, \{b, b_1(\gamma_3), c_1(\gamma_3), c_1(0)\}, \{b, b_1(\gamma_4), c_1(\gamma_4), c_1(0)\} \) belong to \( \partial B \) for some \( \gamma_3 < 0 \leq \gamma_4 \). Note that \( \hat{a}(\gamma) \) will not be the same as \( a_1(\gamma) \), but belongs to the segment \( a_1(\gamma)b_1(\gamma) \). The same holds true for \( \hat{c}(\gamma) \in \hat{b}(1)(c_1(\gamma)) \). Thus the results from Lemma 3.7 easily generalize to the extended quadrilaterals as stated above.

As a next step we show that in our situation, resulting from the application of Lemma 3.7 with respect to \( b \), \( P_\varepsilon \), and \( Q_\delta \), even the quadrilaterals \( \{b, b_1(\gamma_3), a_1(\gamma_3), a_1(0)\} \) and \( \{b, b_1(\gamma_4), a_1(\gamma_4), a_1(0)\} \) are coplanar and belong to \( \partial B \).

An important fact is that Lemma 3.5 also applies to \( b = b(\gamma) \) and \( p = p(\delta) \) for all \( \gamma, \delta \in [-\varepsilon, \varepsilon] \), i.e., for \( |\gamma|, |\delta| \) small enough we have \( b(\gamma) \perp p(\delta) \) (as we will show now) and \( \alpha(b(\gamma), p(\delta)) < \alpha < 1 \) (as we already know). The application of Lemma 3.7 as described above yields that \( b(\gamma) \) belongs to an edge of \( \partial B \) for \( |\gamma| \) small enough. Thus \( b(\gamma) \) is a non-regular point in each planar section of \( \partial B \) through \( b(\gamma) \) which does not contain the edge. Applying again Lemma 3.7 with interchanged roles of \( b \) and \( p \) and the corresponding objects, we get that \( p(\delta) \), for \( |\delta| \) small enough, is a non-regular point in each planar section through \( p \) and \( p(\delta) \) which does not contain the two edges formed by \( p(\delta) \) for \( \delta > 0 \) or for \( \delta < 0 \). Consequently, for \( |\gamma|, |\delta| \) small enough we have that
the planar section of $\partial \tilde{B}$ through $\mathbf{o}$, $\mathbf{b}(\gamma)$ and $\mathbf{p}(\delta)$ has non-regular points $\mathbf{b}(\gamma)$ and $\mathbf{p}(\delta)$. Applying Lemma 3.3 we get $\mathbf{b}(\gamma) \perp \mathbf{p}(\delta)$.

Now there is a plane through $\mathbf{o}$, $\mathbf{p}(\varepsilon)$ and some appropriate $\mathbf{b}(\gamma)$ ($|\gamma| < \varepsilon$) which intersects the interior of $\mathbf{a}_1(0)\mathbf{b}$. By Lemma 3.3 (and again the same lower bound for the lengths of the segments) this plane intersects $U_{\varepsilon,1}(\mathbf{b}) \cap \partial \tilde{B}$ in two segments whose endpoints are not in the interior of $\mathbf{a}_1(0)\mathbf{b}$. Consequently, the planes carrying $\{\mathbf{b}, \mathbf{b}_1(\gamma_3), \mathbf{a}_1(\gamma_3), \mathbf{a}_1(0)\}$ and $\{\mathbf{b}, \mathbf{b}_1(\gamma_4), \mathbf{a}_1(\gamma_4), \mathbf{a}_1(0)\}$ must be the same.

By symmetry reasons the same is true for $\text{conv}\{\mathbf{b}, \mathbf{b}_1(\gamma_3), \mathbf{c}_1(\gamma_3), \mathbf{c}_1(0)\}$ and $\text{conv}\{\mathbf{b}, \mathbf{b}_1(\gamma_4), \mathbf{c}_1(\gamma_4), \mathbf{c}_1(0)\}$, thus showing that $\mathbf{b}$ belongs to the interior of an edge of $\tilde{B}$. Summarizing, we have

**Corollary 3.8** For each vector $\mathbf{b} \in \partial \tilde{B}$ such that for some $\mathbf{p} \in \partial \tilde{B}$ we have $\mathbf{b} \perp \mathbf{p}$ and $\alpha(\mathbf{b}, \mathbf{p}) < 1$, there is a neighborhood $U_{\varepsilon}(\mathbf{b})$ of $\mathbf{b}$ such that $U_{\varepsilon}(\mathbf{b}) \cap \partial \tilde{B}$ consists of two planar pieces intersecting in some straight segment containing $\mathbf{b}$.

### 3.5 Global 3-dimensional properties of $\partial \tilde{B}$

Now, in view of Corollary 3.8 the local behavior of $\partial \tilde{B}$ near $\tilde{b}$ and $\tilde{p}$ is well understood; it is determined by the direction of the edge (besides $\tilde{b} \cap \tilde{p}$, of course). Next we will be concerned with the relation between these edges near $\tilde{p}$, $\tilde{b}$ as well as the behavior near $\mathbf{a}_1(0)$, $\mathbf{c}_1(0)$, $\mathbf{d}_1(0)$ and $\mathbf{c}_1(0)$.

For our pair $\{\tilde{b}, \tilde{p}\}$ the two planes carrying the planar parts of $\partial \tilde{B}$ near $\tilde{b}$ are denoted by $T_{ba}$ and $T_{bc}$, according to where the points $\mathbf{a}_1(0)$ and $\mathbf{c}_1(0)$ belong to. $T_{pd}$ and $T_{pe}$ are the planes which describe $\partial \tilde{B}$ near $\tilde{p}$ in the same way. If we translate these planes to the origin, we get planes $O_{ba} := T_{ba} - T_{ba}$, $O_{bc} := T_{bc} - T_{bc}$, $O_{pd} := T_{pd} - T_{pd}$ and $O_{pe} := T_{pe} - T_{pe}$.

For all $|\gamma|, |\delta| < \varepsilon$ we have that $\mathbf{b}(\gamma) \in T_{ba} \cap T_{bc} =: L_b$ and $\mathbf{p}(\delta) \in T_{pd} \cap T_{pe} =: L_p$.

Considering values $\lambda$ with $|\lambda| < \varepsilon$, we get for $i \in \{1, 2\}$ that $\mathbf{a}_i(\lambda) \parallel \mathbf{p}(\lambda)\mathbf{b}(\lambda) \subset T_{pd}$, thus $\mathbf{a}_i(\lambda) \in O_{pd}$. Since also $\mathbf{a}_i(\lambda) \in T_{ba}$, we get that $\mathbf{a}_i(\lambda) \cap O_{pd} =: L_a$, which is a line. Analogously we have that $\mathbf{c}_i(\lambda) \in T_{bc} \cap O_{bc} =: L_c$, $\mathbf{d}_i(\lambda) \in T_{pd} \cap O_{ba} =: L_d$ and $\mathbf{c}_i(\lambda) \in T_{pe} \cap O_{bc} =: L_e$.

Next we show that all these lines are parallel. Let $P_a, P_b, P_c, P_d, P_e, P_p$ denote the planes containing the origin and $L_a, L_b, \ldots, L_p$, respectively. Since $\mathbf{a}_i(\lambda) \in O_{pd}$ and so forth, we get that $P_a = O_{pd}, P_c = O_{pe}, P_d = O_{ba}$, and $P_c = O_{bc}$. The line $L_b = T_{ba} \cap T_{bc}$ is parallel to $P_d \cap P_e$, and the line $L_p$ is parallel to $P_a \cap P_e$.

Now we define the plane $T_{ac} = L_a + \mathbb{R}\tilde{p}$. Then for all $|\gamma| < \varepsilon$ we have that $(\mathbf{a}_1(\gamma), \mathbf{c}_1(\gamma)) \parallel \mathbf{p}_1(\gamma) = \tilde{p}$ and $\mathbf{a}_1(\gamma) \in L_a \subset T_{ac}$, thus also $\mathbf{c}_1(\gamma) \in T_{ac}$. Consequently, $L_a, L_c \subset T_{ac} \parallel P_p$. In the same way we get a plane $T_{de}$ with $L_d, L_e \subset T_{de} \parallel P_b$.

We will show that $L_a, L_b, L_c, L_d, L_p, L_e \parallel L := P_d \cap P_e$. First $L_b = T_{ab} \cap T_{bc} \parallel P_d \cap P_e = L$, and second $L_d = T_{bd} \cap T_{bc} \parallel P_d \cap P_b = L$, since $L \subset P_d$ by construction, and $L \parallel L_b \subset P_b$ with $\mathbf{o} \in L \cap P_b$, thus $L \subset P_b$; analogously $L_e = P_e \cap T_{de} \parallel L$. But now also $L_p = T_{dp} \cap T_{ep} \parallel L$, $L_a = T_{ab} \cap T_{ac} \parallel T_{ab} \cap P_p \parallel L$ as well as $L_c = T_{bc} \cap T_{ac} \parallel L$. Thus all six lines $L_a, L_b, L_c, L_d, L_e, L_p$ are parallel to $L$.

The next step is to show that $\alpha(\mathbf{b}(\gamma), \mathbf{p}(\delta)) = \alpha(\tilde{b}, \tilde{p})$ for $|\gamma|, |\delta| \leq \varepsilon$, i.e., $\alpha$ is locally constant. In fact, after parallel projection of $P(\mathbf{b}(\gamma), \mathbf{p}(\delta))$ along $L$ onto $P$, all images of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$ and $\mathbf{p}$ are identical to the corresponding ones in $P$. Since for vectors $\mathbf{b}, \mathbf{p}, \mathbf{r}$ from Lemma 3.3 and any skew-symmetric form $[\cdot, \cdot]$, the expression $\alpha(\mathbf{b}, \mathbf{p}) = [\mathbf{b}, \mathbf{b}_s(\mathbf{b}, \mathbf{p})]/[\mathbf{b}, \mathbf{p}] = [\mathbf{b}, \mathbf{c}]/[\mathbf{b}, \mathbf{p}]$ is invariant under linear transformations, we get the relation $\alpha(\mathbf{b}(\gamma), \mathbf{p}(\delta)) = \alpha(\tilde{b}, \tilde{p})$.

Now we consider the largest possible extension of the edge containing $\mathbf{b}(\cdot)$, i.e., the set $S := \{\mathbf{x} \in L_a \cap \partial \tilde{B} : \mathbf{r} \perp \tilde{p}, \alpha(\mathbf{x}, \tilde{p}) = \alpha(\tilde{b}, \tilde{p})\}$. We already know that $S$ is open in $L$, since the above arguments in $\{\mathbf{b}, \tilde{p}\}$ also hold for $(\mathbf{x}, \tilde{p})$ if $\mathbf{x} \in S$. But $S$ is also closed: With $\bar{\mathbf{x}} \in S$ and $\mathbf{r} = \lim_{t \to \infty} \mathbf{x}_t$, we obviously get $\mathbf{r} \in L_a \cap \partial \tilde{B}$, $\mathbf{r} \perp \tilde{p}$ and $\alpha(\mathbf{r}, \tilde{p}) = \alpha(\tilde{b}, \tilde{p})$. But this is a contradiction. Only $\emptyset$ and $L$ are open in $L$ and closed at the same time. Both sets are not possible for $S$ since it is, as
subset of \( \partial \tilde{B} \), bounded and not empty.

This contradiction completes the proof of Theorem 3.1.

### 3.6 Interpretation of the contradiction

The results from the previous section also suggest that besides ellipsoids there are convex, centered subsets of \( \mathbb{R}^3 \) which have the property that “nearly” all planar sections are equiframed; but they have to be unbounded (and, therefore, are not convex bodies). Namely, for an arbitrary equiframed planar convex body \( B_2 \) and any vector \( \mathbf{r} \) not parallel to the affine hull of \( B_2 \) consider the unbounded cylinder \( B_2 + \text{lin}(\mathbf{r}) \). Each planar section of \( B_2 + \text{lin}(\mathbf{r}) \) is a linear image of \( B_2 \) or (if the intersecting plane is parallel to \( \mathbf{r} \)) simply an unbounded strip as Minkowski sum of a segment and \( \text{lin}(\mathbf{r}) \). Thus each bounded planar section of \( B_2 + \text{lin}(\mathbf{r}) \) is equiframed.

### 3.7 Application to the results for angular measures

We complete this chapter with an application of Theorem 3.1.

Theorem 2.13 says that in the planar case the angular measures \( \mu_a \) and \( \mu_l \) coincide in \( \mathbb{M}^2 \) if and only if \( \mathbb{M}^2 \) is equiframed.

Thus the question arises what higher dimensional Minkowski spaces have the property, that for every 2-subspace these measures coincide? In fact, this was the reason for the author to study the material in this chapter.

Combining Theorem 3.1 and Theorem 2.13 we easily get

**Corollary 3.9** Each \( d \)-dimensional Minkowski space \( \mathbb{M}^d \), \( d \geq 3 \), with the property that for every 2-subspace the angular measures \( \mu_a \) and \( \mu_l \) coincide, is necessarily Euclidean.

Finally we should mention that the class of equiframed convex bodies in higher dimensions is much richer than simply consisting of ellipsoids, see [30, Section 6]. Thus the property that every 2-dimensional section is equiframed is very restrictive, not holding for most equiframed convex bodies in dimensions \( d \geq 3 \).
Chapter 4

Conditions that a metric space can be embedded into a Minkowski space

In this chapter we study embedding tasks (see Definition 1.2).

4.1 Tasks

The main topic of this chapter is represented by the following decision problems.

**Task 4.1 (General Decision Problem on Embedding)** \( (\mathbb{N}_n, \rho) \text{ in } M^d \) Decide whether or not there is an embedding of a given metric space \((\mathbb{N}_n, \rho), n \geq 2\), into a suitable Minkowski space \(M^d\) of given dimension \(d \geq 1\).

**Task 4.2 (Special Decision Problem on Embedding)** \((\mathbb{N}_n, \rho) \text{ in } M^d \) Decide whether or not there is an embedding of a given metric space \((\mathbb{N}_n, \rho), n \geq 2\), into the given Minkowski space \(M^d\) with \(d \geq 1\).

Even more difficult is the task to determine all possible embeddings.

**Task 4.3 (General \(d\)-Embeddings of \(\rho\))** Describe all possible embeddings of a given metric space \((\mathbb{N}_n, \rho), n \geq 2\), into a suitable Minkowski space \(M^d\) of given dimension \(d \geq 1\).

**Task 4.4 (\(M^d\)-Embeddings of \(\rho\))** Describe all possible embeddings of a given metric space \((\mathbb{N}_n, \rho), n \geq 2\), into the given Minkowski space \(M^d\) with \(d \geq 1\).

4.2 Transformation of the embedding task

We will transform Task 4.1 and Task 4.3 into the more analytic task of determining admissibility and the task of solving (respectively) a finite number of \(m = m(d, n)\)-dimensional homogeneous linear systems with extra polygonal equations of degree 2. Definition 4.5 states precisely what we mean by that class of systems. In Theorem 4.18 we will specify the concrete systems resulting from the embedding task.

For the special case of polytopal Minkowski spaces, the Tasks 4.2 and 4.4 are both transformed to the Task 1.47 of solving one linear inhomogeneous system in \(\mathbb{R}^{d(n-1)}\) and combinatorial evaluation of the solution set. The corresponding fact is stated in Theorem 4.19.
4.2.1 Embedding into a suitable Minkowski space

**Definition 4.5** A system \((E_1 \cup E_p, W, S)\) of equations and inequalities in \(\mathbb{R}^d\) is called homogeneous linear system with extra polynomial equations of degree \(m\), if \((E_1, W, S)\) is a homogeneous linear system and each \(f \in E_p\) is a homogeneous polynomial in \(d\) variables whose degree is at most \(m\).

**Definition 4.6** We say that a set \(U \subset \mathbb{R}^d\) is in weak convex position if

\[ U \subset \text{rel bd}(\text{conv } U). \]

For any family of objects \(o_i, i \in M \subset \mathbb{N}\), and a set \(\{i_1, \ldots, i_m\} = I \subset M\) with \(i_1 < i_2 < \cdots < i_m\) we use the notation \((o_{i_1}, o_{i_2}, \ldots, o_{i_m})\) for the \(m\)-tuple \((o_{i_1}, o_{i_2}, \ldots, o_{i_m})\). Such \(m\)-tuples can be used to denote the \(m\) arguments to a function: \(V((o_i)_{i \in I}) = V(o_{i_1}, \ldots, o_{i_m})\) if \(m = k\). Then we can write \((4.2)\) more precisely as

\[ V((o_i)_{i \in I}) \leq \sum_{j \in I} V((o_i)_{i \in I \setminus \{j\}}). \]

If in particular \(k = d\), then we can replace \(V(\ldots)\) by the absolute value of the determinant, \(|\det(\ldots)|\) in \((4.2)\). For example, for \(k = d = 2\) we get

\[ |\det(a, b)| \leq |\det(a, c)| + |\det(b, c)|, \tag{4.3} \]

which must hold for all \(a, b, c \in U\).

For the simple case \(k = 1\) the system \((4.2)\) becomes \(\|ba\| = \|ob\|\) for all \(a, b \in U\) (where \(\|\cdot\|\) is an arbitrary norm).

Furthermore we mention that the condition \(k = \dim U\) is important in Theorem \((4.7)\). For \(k > \dim U\) the inequalities \((4.2)\) are trivially satisfied as \(0 \leq 0\), but \(U\) is in general not in weak convex position! Thus the inequalities \((4.2)\) are necessary for \(U\) to be in weak convex position if \(k \geq \dim U\). The inequalities \((4.2)\) are sufficient for \(U\) to be in weak convex position if \(k = \dim U\).

Since an affine linear bijection preserves the property to be in weak convex position, we can transform each instance \(U \subset \mathbb{R}^d\) with \(\dim U < d\) to another instance \(U' \subset \mathbb{R}^k\) of full dimension \(\dim U' = k\).

Before we combine the results of Theorem \((4.7)\) and Theorem \((4.8)\) to get systems of equalities and inequalities representing the general embedding problem, we summarize our results in an algorithm.

**Algorithm 4.9** Input: A function \(f_e : N_n \to \mathbb{R}^d\), and a metric space \((N_n, \rho)\)

Output: Yes/No, whether or not \(f_e\) is an embedding of \((N_n, \rho)\) into a suitable Minkowski space \(M^d\)

1. We first construct the set \(U \subset \mathbb{R}^d\) by \((4.1)\).
2. Then we determine the dimension \( k = \dim U = \dim(f_n(N_n)) \) as the rank of the matrix 
\[ [f_n(1)f_n(2) \cdots f_n(n)]. \]

3. To calculate all the values of the volumes in (4.2) we construct some linear function \( a \) which projects \( \text{lin} U \) injectively onto \( \mathbb{R}^{\dim U} \). For some constant \( c > 0 \) we then have
\[ V((x_i)_{i \in N_n}) = c|\det((a(x_i))_{i \in N_n})| \tag{4.4} \]
for all \( x_1, \ldots, x_k \in \text{lin} U \).

4. So we can check whether or not (4.2) holds for all \((k + 1)\)-tuples \((x_1, \ldots, x_{k+1})\) of vectors from \( U \) by using (4.4). The result of this test does not depend on \( c \).

This algorithm is the starting point to transform the general embedding task into the admissibility or solution task of an analytical system of equations and inequalities. For each \( k = 1, 2, \ldots, d \) we get a system \( \text{SysEm}(\rho, k) \) of equations and inequalities in \( \mathbb{R}^{n_k} \) for the case that there is a really \( k \)-dimensional embedding \( f_e \) of \((N_n, \rho)\) into \( \mathbb{R}^k \). Note that we identify the set of functions \( f_e : N_n \to \mathbb{R}^k \), which is formally \((\mathbb{R}^k)^{N_n}\), with the set \((\mathbb{R}^k)^n\) and with the vector space \( \mathbb{R}^{n_k} \). All restrictions of these systems are positive homogeneous functions of degree \( k \), as linear combinations of absolute values of polynomials.

\[
\begin{align*}
w^I_{\rho,k}(f_e) & := \frac{|\det((f_e(I_{1,1}) - f_e(I_{1,2})))_{i \in N_k}|}{\Pi_{I \in N_k} \rho(I_{1,1}, I_{1,2})} \\
& \quad + \sum_{j \in N_k} \frac{|\det((f_e(I_{1,1}) - f_e(I_{1,2})))_{i \in N_{k+1} \setminus (j)}|}{\Pi_{I \in N_{k+1} \setminus (j)} \rho(I_{1,1}, I_{1,2})} \geq 0, \\
s_{n,k}(f_e) & := \sum_{I \in (P_n)^k} |\det((f_e(I_{1,1}) - f_e(I_{1,2})))_{i \in N_k})| > 0, \\
\text{SysEm}(\rho, k) & := (\emptyset, \{ w^I_{\rho,k} : I \in (P_n)^{k+1}, \{s_{n,k}\}\}) \tag{4.9}
\end{align*}
\]

Note that (4.6) is equivalent to (4.2) for \( U \) defined by (4.1) in view of (4.3). (4.8) is just one possibility to ensure that \( \dim U = k \).

**Corollary 4.10** The metric space \((N_n, \rho)\), \( n \geq 2 \), can be embedded into a suitable Minkowski space \( \mathbb{M}^d \) of fixed dimension \( d \geq 1 \) if and only if for at least one \( k \in N_d \) the system \( \text{SysEm}(\rho, k) \) is admissible. The set of all \( d \)-dimensional embeddings \( f_e : N_n \to \mathbb{R}^d \), i.e., satisfying \( \dim \text{aff} f_e(N_n) = d \), is exactly the solution set of \( \text{SysEm}(\rho, d) \).

The first part of Corollary 4.10 can be strengthened a little bit.

**Proposition 4.11** The metric space \((N_n, \rho)\), \( n \geq d+1 \), can be embedded into a suitable Minkowski space \( \mathbb{M}^d \) of dimension \( d \geq 1 \) if and only if the system \( \text{SysEm}(\rho, d) \) is admissible.

Note that \( \text{SysEm}(\rho, k) \) is not a homogeneous linear system with extra polynomial equations because of the absolute value functions.

We will transform this system into other systems

- which have only homogeneous polynomial restrictions,
- which have only linear inequality restrictions, i.e., non-linear restrictions are only allowed as equations, and
- whose solutions represent equivalence classes of affinely equivalent embeddings up to scaling. More precisely, if we apply a reversible affine transformation to \( f_e \), then both embeddings correspond to the same solution of the analytical system up to scalar multiplication by a nonzero real number.
For the first point, we replace all terms $|T|$ by the term $s_T \cdot T$. If we can assure that $s_T = \text{sign} T$, then $|T| = s_T T$ and the restrictions stay the same. The corresponding numbers $s_T \in \{-1, 0, 1\}$ are introduced as parameter. The condition $s_T = \text{sign} T$ is equivalent to $s_T T > 0$ if $s_T \neq 0$, and to $T = 0$ if $s_T = 0$. For the function set $F$ we denote by $F_{A \rightarrow B}$ the set of functions defined by expressions which are modified expressions\(^\dagger\) of functions in $F$ by replacing each occurrence of $A$ by $B$.

Lemma 4.14 The system $\mathcal{S} = (E, W, S)$ is admissible if and only if at least one of the replaced systems $\mathcal{S}_{T+} := (E_{|T|+T}, W_{|T|+T}, S_{|T|+T} \cup \{T\})$, $\mathcal{S}_{T-} := (E_{|T|-T}, W_{|T|-T}, S_{|T|-T} \cup \{-T\})$, and $\mathcal{S}_{T0} := (E_{|T|0}, W_{|T|0}, S_{|T|0})$ is admissible. Note that we identify the expression $T$ with the function evaluating this expression. The solution set is the union of the three pairwise disjoint solution sets of the replaced systems, $L(S) = L(S_{T+}) \cup L(S_{T-}) \cup L(S_{T0})$.

For the system $\text{SysEm}(\rho, k)$ we have to apply Lemma 4.12 several (up to $(n(n-1))^d$) times, yielding finitely many systems with only homogeneous polynomial restrictions.

Finally, we introduce new variables to get systems which are primary linear. The new variables are for $I := ((x_1, y_1), (x_2, y_2), \ldots, (x_d, y_d)) \in (\mathbb{N}_n)^d$ the real numbers

$$b(I) = b^I(I) := \det((f_i(I_1, 1) - f_i(I_2)))_{i \in \mathbb{N}_d} = \det(f_i(x_1) - f_i(y_1), \ldots, f_i(x_d) - f_i(y_d)). \quad (4.10)$$

These variables are invariant under affine transformations of $f_i$ up to some common factor $c \in \mathbb{R} \setminus \{0\}$: for the affine map $a : \mathbb{R}^d \rightarrow \mathbb{R}^d$ we have $b^{a \circ f_i} = cb^I$. From the structure of $\text{SysEm}(\rho, d)$ it follows that we can express all its restriction by the new variables $b$.

We use the notation $s(I) := s_b(I)$. We summarize the last two transformations and some trivial simplifications by the linear system $\text{SysEmDConv}(\rho, d, s)$ of equations and inequalities in $\mathbb{R}^{(\mathbb{N}_n)^d}$, identified with $\mathbb{R}^{1+\mathfrak{r}}$:

$$0 \leq w^I_{\rho, d, s}(b) := \rho(I_{d+1, 1}, I_{d+1, 2}) s((I_i)_{i \in \mathbb{N}_d}) b((I_i)_{i \in \mathbb{N}_d}) + \sum_{j \in \mathbb{N}_d} \rho(I_{j, 1}, I_{j, 2}) s((I_i)_{i \in \mathbb{N}_d \setminus \{j\}}) b((I_i)_{i \in \mathbb{N}_d \setminus \{j\}}), \quad (4.11)$$

$$0 < s^I_{n, d, s}(b) := s(I) b(I) \quad \text{if } s(I) \neq 0, \quad (4.12)$$

$$0 = e^I_{n, d}(b) := b(I) \quad \text{if } s(I) = 0, \quad (4.13)$$

$$E_{\text{conv}}(d, s) := \{ e^I_{n, d} : I \in (\mathbb{N}_n)^d, s(I) = 0 \}, \quad (4.14)$$

$$W_{\text{conv}}(\rho, d, s) := \{ w^I_{\rho, d, s} : I \in (\mathbb{P}_n)^{d+1} \}, \quad (4.15)$$

$$S_{\text{conv}}(d, s) := \{ s^I_{n, d, s} : I \in (\mathbb{N}_n)^d, s(I) \neq 0 \}, \quad (4.16)$$

$$\text{SysEmDConv}(\rho, d, s) := (E_{\text{conv}}(d, s), W_{\text{conv}}(d, s, \rho), S_{\text{conv}}(d, s)) \ . \quad (4.17)$$

The system $\text{SysEmDConv}(\rho, d, s)$ describes a necessary condition in $b$ that an embedding $f_\rho : (\mathbb{N}_n, \rho) \rightarrow \mathbb{M}^d$ for a suitable space $\mathbb{M}^d$ exists (then with $b = b^I$) with fixed $s(I) = \text{sign} b^I(I)$ for all $I \in (\mathbb{N}_n)^d$.

Corollary 4.13 If there is a full-dimensional embedding $f_\rho$ from $(\mathbb{N}_n, \rho)$ into a suitable Minkowski space $\mathbb{M}^d$ then at least one system $\text{SysEmDConv}(\rho, d, s)$ with nontrivial $s : (\mathbb{N}_n)^d \rightarrow \{0, \pm 1\}$ is admissible.

If we want to avoid strict inequalities, we can use the following

Lemma 4.14 The system $\mathcal{S} = (E, W, S)$ all of whose restrictions are homogeneous or positive homogeneous (not necessarily all of the same degree) is admissible if and only if the modified system $\mathcal{S}_{\text{weak}} := (E, W \cup \{ f \mapsto f(y) - 1 : f \in S \}, \emptyset)$ is admissible. Indeed,

$$L(\mathcal{S}_{\text{weak}}) \subseteq L(\mathcal{S}) = \mathbb{R}^+ L(\mathcal{S}_{\text{weak}}) . \quad (4.18)$$

\(^\dagger\)The considered expression for a function must be fixed and will be clear from the context.
To obtain a system in $b = b^f$ which is also sufficient for the existence of an embedding $f_e$, we add the consequences of the following properties of determinants:

\[
0 = \det(b, \ldots, a, \ldots) + \det(a, \ldots, b, \ldots),
\]
\[
\det(\lambda a + \mu b, \ldots) = \lambda \det(a, \ldots) + \mu \det(b, \ldots),
\]
\[
0 = \sum_{j=1}^{d+1} (-1)^j \det((\mathbf{x}_j)_{i\in\mathbb{N}_{d+1}\setminus\{j\}}) \det(\mathbf{x}_j, (\mathbf{n}_i)_{i\in\mathbb{N}_{d-1}}).
\]

(4.20) is the well known Grassmann identity from Grassmann, which writes for $d = 2$

\[
0 = \det(\mathbf{x}_2, \mathbf{x}_3) \det(\mathbf{x}_1, \mathbf{n}_1) - \det(\mathbf{x}_1, \mathbf{x}_3) \det(\mathbf{x}_2, \mathbf{n}_1) + \det(\mathbf{x}_1, \mathbf{x}_2) \det(\mathbf{x}_3, \mathbf{n}_1),
\]

and this is equivalent to

\[
0 = \det(a, b) \cdot \det(c, d) + \det(a, c) \cdot \det(d, b) + \det(a, d) \cdot \det(b, c).
\]

The consequences are

\[
0 = b(p_1, I_1, p_2, I_2) + b(p_2, I_1, p_1, I_2) =: \text{asym}_{p_1, p_2, I_1, I_2}(b),
\]
\[
0 = b((i, j), I) + b((j, i), I) =: \text{diffsym}_{i, j, I}(b),
\]
\[
0 = b((i, j), I) + b((j, k), I) + b((k, i), I) =: \text{linearity}_{i, j, k, I}(b),
\]
\[
0 = \sum_{j=1}^{d+1} (-1)^j b((I_i)_i\in\mathbb{N}_{d+1}\setminus\{j\}) b(I_j, J) =: \text{quadratic}_{f_I}(b).
\]

(4.25) \hspace{1cm} E_{\text{det}}^1(n, d) := \{ \text{asym}_{p_1, p_2, I_1, I_2} : p_1, p_2 \in \mathbb{N}_n, I_1 \in (\mathbb{N}_n^2)^k, I_2 \in (\mathbb{N}_n^2), k, l \geq 0, k + l + 2 = d \}
\]
\[\cup \{ \text{linearity}_{i, j, k, I} : i, j, k \in \mathbb{N}_n, I \in (\mathbb{N}_n^2) \}
\]
\[E_{\text{det}}^2(n, d) := \{ \text{quadratic}_{I, J} : I \in (\mathbb{N}_n^2)^{d+1}, J \in (\mathbb{N}_n^2) \}
\]
\[\text{SysDet}(n, d) := (E_{\text{det}}^1(n, d) \cup E_{\text{det}}^2(n, d)) \to \mathbb{R}^d).
\]

Lemma 4.15 The solution set of $\text{SysDet}(n, d)$ is exactly the set of all determinants

\[
L(\text{SysDet}(n, d)) = \{ b = b^f : f_e : \mathbb{N}_n \to \mathbb{R}^d \}.
\]

Remark 4.16 Note that (4.23) follows from (4.22), since $\text{diffsym}_{i, j, I}(b) = \text{linearity}_{i, j, k, I}(b) - \frac{1}{2} \text{linearity}_{i, j, j, I}(b)$. Furthermore, (4.22) is symmetric in the variables $i, j, k$ in view of (4.23): A cyclic shift $(i, j, k) \mapsto (j, k, i)$ leaves (4.22) invariant since $\text{linearity}_{i, j, k, I}(b) = \text{linearity}_{j, i, k, I}(b)$. Interchanging two of the variables $i, j, k$, such as $(i, j, k) \mapsto (j, i, k)$, yields (4.24) multiplied by $-1$ due to (4.23). Analogously we have that (4.25) is invariant under permutations of the pairs $I_1, I_2, \ldots, I_{d+1} \in \mathbb{N}_n^2$ forming the sequence $I$. Independently, the pairs $J_1, J_2, \ldots, J_{d-1}$ can be interchanged. Again, by using (4.22) we get the unpermutated (4.25), possibly after multiplication by $-1$. More precisely, for two permutations $\alpha : \mathbb{N}_{d+1} \to \mathbb{N}_{d+1}$, $\beta : \mathbb{N}_{d-1} \to \mathbb{N}_{d-1}$ and $I \in (\mathbb{N}_n^2)^{d+1}, J \in (\mathbb{N}_n^2)^{d-1}$ we have

\[
\text{quadratic}_{\{I_{\alpha(i)}\}_{i\in\mathbb{N}_{d+1}}, \{J_{\beta(i)}\}_{i\in\mathbb{N}_{d-1}}}(b) = (\pm 1) \cdot \text{quadratic}_{I, J}(b)
\]

Additionally, we can interchange the two components within each pair $(i, j) \mapsto (j, i)$ of $I$ and $J$ independently, which leaves (4.25) invariant (possibly up to multiplication by $-1$) in view of (4.23).

For the special case of $d = 2$, the pairs in $I$ and $J$ may even be interchanged between these sequences: $\text{quadratic}_{\{I_1, I_2, I_3\}, \{J_1\}}(b) = -\text{quadratic}_{\{J_1, I_2, I_3\}, \{I_1\}}(b)$. For $d \geq 3$ this is not true.

At this point we can state the transformation of Task 4.1 into several instances of Task 4.45 for $m = m(d, n)$-dimensional homogeneous linear systems with extra polynomial equations of degree 2.
Lemma 4.17 For $n > d$ there is an embedding $f_e$ from $(\mathbb{N}_n, \rho)$ into a suitable Minkowski space $\mathbb{M}^d$ if and only if at least one system 

$$\text{SysEmD}(\rho, d, s) := \left( (E_{\text{conv}}(d, s) \cup E^d_{\text{det}}(n, d)) \cup E^q_{\text{det}}(n, d), W_{\text{conv}}(d, s, \rho), S_{\text{conv}}(d, s) \right)$$

with nontrivial $s : (\mathbb{N}_n^2)^d \to \{0, \pm 1\}$ is admissible.

To state the analogous transformation from Task 4.3 into several instances of Task 4.17 we first need some notation and also discuss briefly the size of the system.

The number of unknown variables $m = n^{2d}$ is itself polynomial in $n$ of degree $2d$. E.g., for $d = 2$ we have $m = m(2, n) = n^4$.

Two functions $f, g : \mathbb{N}_n \to \mathbb{R}^d$ are called affinely equivalent if there is some affine linear bijection $a : \mathbb{R}^d \to \mathbb{R}^d$ with $f = a \circ g$, i.e., $f(i) = a(g(i))$ for all $i \in \mathbb{N}_n$. This describes an equivalence relation in the set $(\mathbb{R}^d)^{\mathbb{N}_n}$. We identify $(\mathbb{R}^d)^{\mathbb{N}_n}$ with $\mathbb{R}^{dn}$.

Two vectors $\mathbf{r}, \mathbf{g} \in \mathbb{R}^m$ are called positive equivalent if there is some $\lambda > 0$ with $\mathbf{r} = \lambda \mathbf{g}$. This describes an equivalence relation in the set $\mathbb{R}^m$.

Two vectors $\mathbf{r}, \mathbf{g} \in \mathbb{R}^m$ are called direction equivalent if there is some $\lambda \neq 0$ with $\mathbf{r} = \lambda \mathbf{g}$. This describes an equivalence relation in the set $\mathbb{R}^m$.

Theorem 4.18 The metric space $(\mathbb{N}_n, \rho)$ with $n \geq d+1$ can be embedded into a suitable Minkowski space $\mathbb{M}^d$ if and only if there is a nontrivial (i.e., not equal to zero) function $s : (\mathbb{N}_n^2)^d \to \{-1, 0, 1\}$ (called sign function) such that the homogeneous linear system $\text{SysEmD}(\rho, d, s)$ of equations and inequalities in $\mathbb{R}^{n^{2d}}$ with extra polynomial equations of degree 2 is admissible.

There is a one-to-one relation between

1. all affine equivalence classes (i.e., equivalence classes with respect to the relation that two embeddings are affinely equivalent) of embeddings $f_e : \mathbb{N}_n \to \mathbb{R}^d$ of $(\mathbb{N}_n, \rho)$ into a suitable Minkowski space $\mathbb{M}^d$ which are full dimensional (i.e., $\dim f_e(\mathbb{N}_n) = d$),

2. all direction equivalence classes (i.e., equivalence classes with respect to direction equivalence) in the union of $L(\text{SysEmD}(\rho, d, s))$ for all $s \in \{-1, 0, 1\}(\mathbb{N}_n^2)^d$, $s \neq 0$.

We note that a direction equivalence class $C$ in the union of $L(\text{SysEmD}(\rho, d, s))$, $s \neq 0$, has the form $C = [b]_\succ \cup [-b]_\prec$. Here $[b]_\succ$ denotes the equivalence class of $b \in \mathbb{R}^{n^{2d}}$ with respect to positive equivalence. So $C$ is connected with two systems $\text{SysEmD}(\rho, d, s)$ and $\text{SysEmD}(\rho, d, -s)$, where $s = \text{sign} \ b$ and $b$ is a solution of $\text{SysEmD}(\rho, d, s)$. Thus, the affine equivalence classes of full dimensional embeddings are in one-to-one relation to all positive equivalence classes of the union of $L(\text{SysEmD}(\rho, d, s))$, where $s$ traverses a subset of all non-trivial sign functions $s : (\mathbb{N}_n^2)^d \to \{0, \pm 1\}$ which contains of all pairs $s, -s$ exactly one representative.

We note that it is very difficult to solve the tasks 4.3 and 4.17 for non-linear systems. It can even be impossible in practice to solve such tasks. Nevertheless, this approach can be used and extended with further tools to solve some “easy” instances of the embedding tasks. The present chapter will discuss some possibilities to solve such “easy” instances.

4.2.2 Embedding into a given polytopal Minkowski space

Now we consider the special embedding problems where we are given a fixed Minkowski space and a fixed metric space. We will only consider Minkowski spaces whose unit ball $B$ is a polytope.

We assume that the unit ball $B$ of $\mathbb{M}^d$ is given by the real $k \times d$-Matrix $A$ as $B = \{ \mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq 1 \}$, where the column vector 1 consists of exactly $k$ components all equal to 1.

For such Minkowski spaces, we transform (both) Task 4.3 and Task 4.2 to Task 4.17 of solving one linear inhomogeneous system in $\mathbb{R}^{d(n-1)}$. Then we analyze the combinatorial structure of the solution polytope (solution set) to solve the corresponding embedding problem. Note that it is possible to solve Task 4.17 for linear systems.
CHAPTER 4. EMBEDDING METRIC SPACES INTO A MINKOWSKI SPACE

For each function \( f_e : (\mathbb{N}_n, \rho) \to \mathbb{R}^d \) we define its embedding vector \( e_{f_e} \), which is the vector formed by all the \( d(n-1) \) coordinates of \( g(2) = f_e(2) - f_e(1), \ldots, g(n) = f_e(n) - f_e(1) \). E.g., for \( d = 2 \) we have \( e_{f_e} = (x_2 - x_1, y_2 - y_1, x_3 - x_1, y_3 - y_1, \ldots, x_n - x_1, y_n - y_1)' \). Thus, defining additionally \( g(1) = 0 \), we have that \( g \) is a translation of \( f_e \) by \(-f_e(1)\). By \( g_e(i) \) we denote the corresponding part in the embedding vector \( e = (e_1, \ldots, e_{d(n-1)}) \) or the zero vector in the sequel. E.g., for \( d = 2 \) we have that \( g_e(2) = (e_3, e_4)' \).

**Theorem 4.19** For a given metric space \((\mathbb{N}_n, \rho)\) and a polytopal Minkowski space \( M^d = \mathbb{R}^d(\{ x \in \mathbb{R}^d : Ax \leq 1 \}) \) we have that the set \( L \) of all embedding vectors of embeddings \( f_e : (\mathbb{N}_n, \rho) \to \mathbb{R}^d \) is the union of some faces of the bounded and full-dimensional polytope \( P \subset \mathbb{R}^{d(n-1)} \) defined by \( n(n-1)k \) inequalities:

\[
P := \{ e \in \mathbb{R}^{d(n-1)} : A(g_e(i) - g_e(j)) \leq \rho(i,j)1 \quad \forall i, j \in \mathbb{N}_n, i \neq j \}.
\]

More precisely,

\[
L = \bigcap_{1 \leq i < j \leq n} \bigcup_{r \in \mathbb{N}_n} P \cap \{ e : A_r(g_e(i) - g_e(j)) = \rho(i,j) \} =: L'
\]

is the union of all faces of \( P \) which, for each pair \((i, j)\), are contained in a hyperplane \( A^*(g_e(i) - g_e(j)) = \rho(i,j) \), where \( A^* \) is an arbitrary row of \( A \) depending on \((i, j)\). If \( P \) does not have such faces, \((\mathbb{N}_n, \rho)\) cannot be embedded into the given space \( \mathbb{M}^d \).

Under certain circumstances we already know the only possible embedding of some metric space into a suitable Minkowski space up to affine transformations. For each solution of the (linear or quadratic) system in the variables \( b(\cdots) \), the proof of Theorem 4.18 constructively yields an embedding of a given metric space into a suitable Minkowski plane. If we have this information, then we can replace the polytope \( P \subset \mathbb{R}^{d(n-1)} \) which has to be computed if we apply Theorem 4.19 by another polytope \( P \subset \mathbb{R}^d \) in a space of smaller dimension (provided that \( n-1 > d \)).

**Theorem 4.20** The set of matrices \( M \in \mathbb{R}^{d \times d} \) with the property that the linear transformation \( Mf_e(1), Mf_e(2), \ldots, Mf_e(n) \) of the given function \( f_e : \mathbb{N}_n \to \mathbb{R}^d \) is an embedding of a given metric space \((\mathbb{N}_n, \rho)\) into the given polytopal Minkowski space \( M^d = \mathbb{R}^d(\{ x \in \mathbb{R}^d : Ax \leq 1 \}) \) is the union of some faces of the full-dimensional polyhedron \( P \subset \mathbb{R}^{d \times d} \) defined by \( n(n-1)k \) inequalities:

\[
P := \{ M \in \mathbb{R}^{d \times d} : AM(f_e(i) - f_e(j)) \leq \rho(i,j)1 \quad \forall i, j \in \mathbb{N}_n, i \neq j \}.
\]

More precisely,

\[
L := \bigcap_{1 \leq i < j \leq n} \bigcup_{r \in \mathbb{N}_n} P \cap \{ M : A_rM(f_e(i) - f_e(j)) = \rho(i,j) \}
\]

is the union of all faces of \( P \) which for each pair \((i, j)\) are contained in a hyperplane \( A^*M(f_e(i) - f_e(j)) = \rho(i,j) \), where \( A^* \) is an arbitrary row of \( A \) depending on \((i, j)\).

Note that the set of all real \( d \times d \)-matrices also forms a vector space of dimension \( d^2 \) with the usual operations of addition and scalar multiplications. So we can easily identify \( \mathbb{R}^{d \times d} \) with \( \mathbb{R}^{d^2} \).

### 4.3 Simplifying the analytical systems

The system \( \text{SysEmD}(\rho, d, s) \) of equations and inequalities in \( \mathbb{R}^{n^2d} \) considered in Theorem 4.18 has more variables \((n^2d)\) and restrictions \( (O(n^3d)) \) than are really necessary to describe the problem. Therefore the variables are not independent from each another. Remember that the corresponding points have only \( nd \) degrees of freedom!

In this section we will discuss some possible simplifications of the systems. First we describe the general principles of this kind of simplification. Second we present, as far as possible, a simplified system for the embedding problem for general data, so using the special structure. In Chapter 6 we deal with systems where these principles reduced the size of the problem very well, since the considered metric spaces were simple.
4.3.1 General simplification principles

But we start with some general simplifications that even can be detected and applied by some algorithm.

Still, they cannot tell in advance how much the system can be simplified.

Linear hull simplification

Assume that $S$ contains dependent variables and that $S_l$ is a subsystem of $S$ with only affinely linear restrictions. Then we have $L(S) \subset L(S_l)$, and every simplification in $S_l$ can be used to simplify $S$. 

Lemma 4.21 Assume that $S_l$ is a subsystem of $S$ in $\mathbb{R}^d$, and that $S_l$ is equivalent to $S_s$. If the function $\phi : \mathbb{R}^m \to \mathbb{R}^d$ is injective with $\phi(\mathbb{R}^m) \supset L(S_l)$, then also

$$S' := ((S \setminus S_l) \cup S_s) \circ \phi$$

is equivalent to $S$ with respect to $\phi$.

Since linear systems can be solved algorithmically, we are interested in (affinely) linear subsystems $S_l$. For $\phi$ we choose an (affinely) linear parametrization of the affine hull of $L(S_l)$. Using Lemma 4.21 we can compute a simplified system $S'$ for $S$.

Removing duplicates

After this step we will remove duplicates under the restrictions of the system $S = (E, W, S)$.

If we have one restriction $f \in E \cap S$, then we know that $S$ is not admissible. If some other restriction $f \in E \cup W \cup S$ is contained more than once in $S$, then we can reduce the number of restrictions by considering the equivalent system $(E, W \setminus \{f\}, S)$. The same simplification can be done if we have two “equivalent” restrictions $f_1, f_2 \in E \cup W \cup S$, i.e., if there is some $f : \mathbb{R}^d \to \mathbb{R}$ and for $i = 1, 2, x \in \mathbb{R}^d$ we have that

- $f(x) = 0$ if and only if $f_i(x) = 0$ in case $f_i \in E$,
- $f(x) \geq 0$ if and only if $f_i(x) \geq 0$ in case $f_i \in W$,
- $f(x) > 0$ if and only if $f_i(x) > 0$ in case $f_i \in S$.

Then we can leave out one restriction or know that the system is not admissible.

In the special case of affine linear restrictions, these conditions are equivalent to

$$f_i = \lambda f$$

for some $\lambda \in \mathbb{R}$ with

- $\lambda \neq 0$ in case $f_i \in E$,
- $\lambda > 0$ in case $f_i \in W \cup S$.

In general, the condition (4.31) is still sufficient for equivalence. So maybe we can reduce $S$ further after detecting such trivial equivalent restrictions. Note that for linear restrictions this step will not influence the complexity of the simplification achieved after the linear hull simplification stated above. But nevertheless, it may reduce the complexity of the system and thus the running time of this first step, if applied in advance.

To detect trivial equivalent restrictions without much additional effort we use the technique of normalization. After this step, trivial equivalent restrictions will show up as identical restrictions up to one exception: if $f_1 \in E'$ is equivalent to $f_2 \in W' \cup S'$, then besides $f_1 = f_2$ also $f_1 = -f_2$ is possible.
One possible normalization of the polynomial $f(x) = \sum_{i \in \mathbb{N}_d^2} a_i x^i$ with $f \neq 0$ (the zero function can be deleted from $E$ and $W$; if it belongs to $S$ then $S$ is not admissible) is $\frac{1}{2}f$ with $\lambda = \max\{|a_i| : i \in \mathbb{N}_d^2\}$ for $f \in W \cup S$. For $f \in E$ it is more complicated to determine the sign of $\lambda$ correctly: $\lambda = a_{i^*}$ where $i^*$ is the lexicographically smallest index with $|a_{i^*}| = \max\{|a_i| : i \in \mathbb{N}_d\}$.

Of course, other normalizations may be more convenient, for example number theoretic normalizations if we have rational coefficients (making all coefficients to integers).

This trivial simplification step does not always reduce the problem size. But using normalizations and ordered data structures, it can be implemented without much additional costs. This simplifies the task of generating all restrictions.

Linear dependence of nonlinear restrictions

Finally, we try to reduce the number of nonlinear restrictions. For example, if $f, g, f + g \in E$, then we can delete one of these three functions from $E$.

Lemma 4.22 As long as there are $f, f_1, \ldots, f_k \in E$, $\lambda_i \in \mathbb{R}$ with $f = \sum \lambda_i f_i$, we can replace $E$ by $E \setminus \{f\}$.

Note that this reduction requires more computational effort. We used this principle just for linear dependencies known in advance. Similar statements for restrictions not just in $E$ are possible, but after we achieved that every non-linear restriction is in $E$, Lemma 4.22 is the most interesting one.

4.3.2 Simplifying the general embedding system

Linear hull simplification

The system $\text{SysEmD}(\rho, d, s)$ of equations and inequalities in $\mathbb{R}^{n^2\!d}$ can be simplified to a system in $\mathbb{R}^m$ with $m = \binom{n^2\!d}{d}$.

Proposition 4.23 The system $\text{SysDet}_k(n, d) := (E_{\text{det}}^k(n, d), \emptyset, \emptyset)$ is a subsystem of $\text{SysDet}(n, d)$ and of $\text{SysEmD}(\rho, d, s)$. Its solution set $L := L(\text{SysDet}_k(n, d))$ is a linear subspace of dimension $m = \binom{n^2\!d}{d}$.

Let $M := M_{n,d} := \{I \in \mathbb{N}_n^d : 1 < I_1 < I_2 < \cdots < I_d \leq n\}$ denote the set of all strictly monotone increasing sequences of $d$ integers between 2 and $n$. The function $\phi : \mathbb{R}^{(\mathbb{N}_n^d)^d} \rightarrow \mathbb{R}^M$, $b \mapsto (I \mapsto b((1, I_i)_{i \in \mathbb{N}_d}))$ is a linear bijection from $L$ to $\mathbb{R}^M$.

For the inverse $\phi : \mathbb{R}^M \rightarrow L$ of $\phi|_L$ we have for all $\bar{b} : M \rightarrow \mathbb{R}$, that $\phi(\bar{b}) \in L$ is a function $(\mathbb{N}_n^d)^d \rightarrow \mathbb{R}$. We describe this function explicitly by its image of $J \in (\mathbb{N}_n^d)^d$:

\[
\phi(\bar{b})(J) = \sum_{p \in \{1, 2\}^d} (-1)^{\sum_{i \in \mathbb{N}_d} p_i} h(\bar{b})(I_{p, I_i})_{i \in \mathbb{N}_d},
\]

where $h(\bar{b})$ is the following function from $I \in \mathbb{N}_n^d$ to $\mathbb{R}$

\[
h(\bar{b})(I) = \begin{cases} 0 & \text{if } |\{I_i : i \in \mathbb{N}_d\}| < d \text{ or } \min(I) = 1, \\
\text{sign} \cdot \delta \cdot \bar{b}(I_{\delta(i)})_{i \in \mathbb{N}_d} & \text{otherwise, with } (I_{\delta(i)})_{i \in \mathbb{N}_d} \in M.\end{cases}
\]

Here $\delta : \mathbb{N}_d \rightarrow \mathbb{N}_d$ denotes the unique permutation which sorts $I$ strictly monotone increasing. Its sign (or signature) $\text{sign} \delta = +1 (-1)$ if there is an even (odd) number of transpositions (permutation interchanging exactly two elements) whose composition is $\delta$.

Note that $\mathbb{R}^{(\mathbb{N}_n^d)^d}$ is identified with $\mathbb{R}^{n^2\!d}$, that $|M| = m$, and that we identify (e.g., by lexicographical ordering) $\mathbb{R}^m$ and $\mathbb{R}^M$.

By our definition, for $b \in \mathbb{R}^{(\mathbb{N}_n^d)^d}$ with $\bar{b} := \psi(b) \in \mathbb{R}^M$ and any $I = (I_1, \ldots, I_d) \in M \subset \mathbb{N}_n^d$, we have

\[
\bar{b}(I_1, \ldots, I_d) = b((1, I_1), (1, I_2), \ldots, (1, I_d)).
\]
So \( \psi \) is really a projection onto some of the \( n^{2d} \) coordinates.

For \( d = 2 \) we can write \((4.32)\) using the abbreviation \( \hat{b} := h(b) \) as

\[
\phi(\hat{b})(i, j), (k, l)) = \hat{b}(i, k) - \hat{b}(i, l) - \hat{b}(j, k) + \hat{b}(j, l),
\]

(4.34)

and \((4.33)\) as

\[
\hat{b}(i, j) = \begin{cases} 
0 & \text{if } i = j \lor i = 1 \lor j = 1, \\
\hat{b}(i, j) & \text{if } 1 < i < j, \\
-\hat{b}(j, i) & \text{if } 1 < j < i.
\end{cases}
\]

(4.35)

Of course, beneath the projection \( \psi \) (and thus the parametrization \( \phi \)) given in Proposition \((4.23)\) there are other possibilities. None of them really retains all the symmetry in the points of the embedding problem. Our approach isolates the first point 1 as special. Another possibility uses the sequence from 1 to \( n \) and the corresponding differences, which retains to some extent the symmetry of a regular \( n \)-gon.

### Linear dependence of nonlinear restrictions

**Lemma 4.24** \( \text{SysDet}(n, d) \) is equivalent to \((E_{\text{det}}^{d}(n, d) \cup E_{\text{det, red}}^{d}(n, d), \emptyset, \emptyset)\) with

\[
E_{\text{det, red}}^{d}(n, d) := \{ \text{quadratic}_{I,J} : I \in \{(1)\} \times \{n\}^{d+1}, J \in \{(1) \times \{n\}^{d+1}, \}
\]

\[
1 < I_{1,2} < I_{2,2} < \cdots < I_{d+1,2}, 1 < J_{1,2} < J_{2,2} < \cdots < J_{d-1,2}, \quad \left[ (I_{k})_{k \in N_{d+1}} \cap (J_{k})_{k \in N_{d-1}} \leq d - 2 \right] \quad \text{if } d \geq 3,
\]

(4.36)

\[
E_{\text{det, red}}^{d}(n, 2) := \{ \text{quadratic}_{(1,a),(1,b),(1,c)_{(1,d)}} : 1 < a < b < c < d \leq n \}.
\]

(4.37)

**Corollary 4.25** The system \( \text{SysEmD}(\rho, d, s) \) is equivalent to an \((n^{1-d})\)-dimensional homogeneous linear systems with extra polygonal equations of degree 2, which has at most \( n^{2d} \) linear equations and strict linear inequalities (together), \((\frac{n}{d})^{d+1} \in O(n^{3d+2})\) weak linear inequalities and at most

\[
\sum_{k=0}^{d-2} \binom{n-1}{k} \binom{n-k-1}{d+1-k} \binom{n-d-2}{d-1-k} \leq \binom{n-1}{d+1} \binom{n-1}{d-1} \in O(n^{2d})
\]

quadratic equations. For \( d = 2 \) this bound of \((\frac{n-1}{3})(n-4)\) can be decreased by a factor 4 to at most \((\frac{n-1}{4})\) quadratic equations.

Especially for \( n < d + 3 \), the system \( \text{SysEmD}(\rho, d, s) \) is equivalent to a homogeneous linear system.

Note that by Wolfe [38] for at least one \( s \) for each \( \rho \) the system \( \text{SysEmD}(\rho, d, s) \) is admissible if \( n \leq d + 2 \), since all metric spaces with \( d + 2 \) points can be embedded into the \( \ell_{\infty} \)-space.

### 4.3.3 Using subsystems

Recall that \((E', W', S')\) is a subsystem of \((E, W, S)\) if \( E' \subset E, W' \subset W \) and \( S' \subset S \). Obviously, for the solution set \( L' \) and \( L \) the converse relation \( L \subset L' \) holds.

Now we can use the result \( L' \) and auxiliary information about \((E', W', S')\) possibly to simplify \((E, W, S)\). If \( L' = \emptyset \), then also \( L = \emptyset \). If some restriction of \( f \in E' \cup W' \cup S' \) turned out to be redundant, i.e., \( L' = L(E' \setminus \{f\}, W' \setminus \{f\}, S' \setminus \{f\}) \), then \( f \) is also redundant in \((E, W, S)\) and can be left out.

Even more interesting, if \( f \in W' \) turned out to be an implicit equation in \((E', W', S')\), i.e., if \( f(L') = \{0\} \), then \( f \) is an implicit equation in \((E, W, S)\), too. Thus \( f \) can be used to reduce the dimension of the system.

For our embedding systems there are some “natural” subsystems, namely the embedding systems of metric subspaces of \((\mathbb{N}, \rho)\). These subsystems are smaller in both the number of restrictions and also the dimension. More precisely, the solution set of the subsystem is an unbounded
cylinder. After some linear transformation all restrictions do not depend on some of the variables. Leaving out these variables we get a smaller system, whose solution also describes the solution of the subsystem.

4.4 Algorithmical solvability

From Theorem 4.0 and Lemma 4.17 we get

Proposition 4.26 The general decision problem on embedding \((\mathbb{N}_n, \rho)\) in \(\mathbb{M}^d\) (Task 4.1) is algorithmically decidable if \(\rho\) is explicitly given by real algebraic numbers.

Note that a decision procedure as meant in Proposition 4.26 needs tools from the theory of real closed fields. Thus, for real instances of this problem we can not expect to get an answer to this decision problem by computers from today, due to the enormous costs (time and memory) of this approach.

From Theorem 4.19 and well known facts from the theory of polyhedra we obtain

Corollary 4.27 Task 4.4 describing all \(\mathbb{M}^d\)-embeddings of a given metric space \((\mathbb{N}_n, \rho)\) (explicitly given by real algebraic numbers) into the given polytopal Minkowski space \(\mathbb{M}^d(B)\), given by a matrix \(A\) of real algebraic numbers in the representation \(B = \{ x \in \mathbb{R}^d : Ax \leq 1 \}\), can be solved algorithmically. In particular, the special decision problem on embedding (Task 4.3) for the same given data is algorithmically solvable.

And in the same way we can conclude from Theorem 4.18 and Corollary 4.25 our

Corollary 4.28 Task 4.3 describing all affine equivalence classes of embeddings of given metric spaces \((\mathbb{N}_n, \rho)\) (explicitly given by real algebraic numbers) into a suitable Minkowski space \(\mathbb{M}^d\), can be solved algorithmically if \(n \leq d + 2\).

As we will really do in Chapter 6, Proposition 4.28 also holds for special metrics \(\rho\), even if \(n > d + 2\), if the linear restrictions (many triangle inequalities which hold as equation) give degenerate, lower dimensional solution sets.

But here is one more statement in the spirit of (and included in) Theorem 4.26, which only uses the well developed technique of quadratic programming:

Proposition 4.29 The general decision problem on embedding \((\mathbb{N}_5, \rho)\) in \(\mathbb{M}^2\) (Task 4.7, for \(d = 2, n = 5\)) is algorithmically decidable if \(\rho\) is explicitly given with values in a class of numbers providing quadratic programming. For rational \(\rho\) we only need rational arithmetics.

4.5 Reducing the number of systems which need to be checked for admissibility

In Lemma 4.17 and Theorem 4.18 we used the fact that there are only finitely many sign functions \(s : (\mathbb{N}_n^d) \rightarrow \{-1, 0, 1\}\). But for practical situations, even for small values of \(n\) and \(d\), this number \(3^{2n^d}\) of all possible sign function does not allow to iterate and try all these functions.

There are three ways allowing to consider just some candidates of sign function instead of considering all.

First we can use the idea from Subsection 4.3.3. Omitting an arbitrary point of the metric space, we get a smaller instance of the same problem. By solving this subproblem recursively, we get a complete list of possible sign functions for fewer points. Extending only these sign functions, maybe combining the results for all \(n\) such subsystems, omitting one point each, we have restricted the number of cases. For the investigations in this paper this approach was sufficient to solve the problems.
Second, if we have a list of oriented matroids, we only need to take possible sign functions which are the chirotope of some rank 3 oriented matroid (the big oriented matroid associated to the points).

Last, we can consider some arrangement of hyperplanes in the space of determinant-functions \( b \), see (4.10). More precisely, we consider the partitioning of the linear subspace \( L(\text{SysDet}_{l}(n, d)) \) of \( \mathbb{R}^{n_d} \). Remember that \( L(\text{SysDet}_{l}(n, d)) \) has dimension \( \binom{n-1}{d} \). The hyperplanes are all coordinate hyperplanes \( b(I) = 0 \) for \( I \in \mathbb{N}^d \). Now each face of the partitioning of \( L(\text{SysDet}_{l}(n, d)) \) corresponds to a sign function \( s \) which may be realizable. For \( d = 2, n = 4 \) we get this way 98 possible sign functions. All of them are realizable. For \( d = 2 \) and \( n = 5 \) we got 174436 sign functions (up to symmetry: 839). In this case there is exactly one additional quadratic restriction \( q(b) = 0 \).

Considering the sign of \( q \) within the faces of the partitioning, it turns out that all points of 2350 faces (up to symmetry: 22) satisfy \( q(b) = 0 \), 142992 of these sign functions (678) are not realizable since \( q(b) \neq 0 \) within the corresponding faces, and the remaining 29094 faces (139) contain points with \( q(b) = 0 \) and also points with \( q(b) \neq 0 \). So we can restrict ourselves to check \( \frac{31444}{2} \) sign functions to decide embeddability and to describe all embeddings of a fixed metric with 5 points into the plane.

Note that similar questions, counting the number of different positions of \( n \) points, are also studied in the literature. Numbers of Euclidean order types of points in general position are given in [1], oriented matroids and abstract order types are considered in [16].

### 4.6 One application

With the technique of this chapter we can easily prove Proposition 9 from the survey [28]. Additionally we can describe the equality case of this geometric inequality.

**Theorem 4.30** For each convex quadrilateral \( abcd \) in a Minkowski plane, where

- \( \langle a, d \rangle \) and \( \langle b, c \rangle \) intersect on the side of \( \langle a, b \rangle \) opposite to \( c \) and \( d \) (or are parallel) and
- \( \langle a, b \rangle \) and \( \langle c, d \rangle \) intersect on the side of \( \langle b, c \rangle \) opposite to \( a \) and \( d \) (or are parallel),

the following relation holds:

\[
\|ab\| + \|bc\| \leq \|ad\| + \|dc\|. \tag{4.38}
\]

Figure 4.1 illustrates the prerequisites. The equation \( \|ab\| + \|bc\| = \|ad\| + \|dc\| \) implies that \( abcd \) is a parallelogram or, alternatively for \( x := c - d \) and \( y := b - a \), the segment \( \overline{xy} \subset \partial B \) is a part of the unit circle. Additionally, in that second case we have that \( \|ab\| + \|bc\| = \|ad\| + \|dc\| = \|ac\| \).

![Figure 4.1: Sketch to Theorem 4.30](image)

**Corollary 4.31** If \( abcd \) is a convex quadrilateral with \( \|ab\| = \|bc\| = \|cd\| = 1 \) and \( \|ad\| > 1 \), then \( \langle a, b \rangle \) and \( \langle c, d \rangle \) intersect on that side of \( \langle b, c \rangle \) which is opposite to \( a \) and \( d \).

### 4.7 Proofs

**Proof of Theorem 4.7** We have to show that \( f_x : X \to \mathbb{R}^d \) is an embedding of \( (X, \rho) \) into a suitable Minkowski space \( \mathbb{M}^d \) if and only if the set \( U \) defined by (4.11) is bounded and in weak convex position.
We can assume that $U$ linearly spans $\mathbb{R}^d$, otherwise we consider $f_c$ as function into an affine linear subspace of lower dimension. This does neither change the property of weak convex position, nor of the existence of a suitable Minkowski space. Note that we can easily embed a $(d-1)$-dimensional Minkowski space into a hyperplane of a suitable $d$-dimensional Minkowski space by choosing a bipyramid as unit ball.

By definition, $f_c$ is an embedding into some Minkowski space $\mathbb{M}_d$ with unit ball $B$ if and only if
\[
\|f_c(x) - f_c(y)\|_B = \rho(x, y) \quad \text{for all } x, y \in X.
\]
Now $\|f_c(x) - f_c(y)\|_B = \rho(x, y) = 0$ holds for all $x \in X$. For $x \neq y$ we get that $\rho(x, y) > 0$. The system (4.39) is equivalent to $\|\rho(x, y)^{-1}(f_c(x) - f_c(y))\|_B = 1$ for all $x, y \in X$ with $x \neq y$. This is equivalent to $U \subset \partial B$.

If $U$ is bounded and in weak convex position, i.e., $U \subset \partial \text{conv } U$, we consider the set $B := \text{cl convex } B$. $B$ is a centered (since $U$ is centrally symmetric), compact convex body with $\partial \text{conv } U = \partial B$. Thus $B$ is the unit ball of a Minkowski space $\mathbb{M}_d(B)$ so that $f_c$ is an embedding of $(X, \rho)$ into $\mathbb{M}_d(B)$.

If contrarily $f_c$ is an embedding into the Minkowski space $\mathbb{M}_d$ with unit ball $B$, we have $U \subset \partial B$. Thus $U$ is bounded. Every vector $x \in U$ belongs to the boundary of $B \supset \text{conv } U$ and to $\partial B$, too. Thus we also have $x \in \partial(\text{conv } U)$. Consequently, $U$ is in weak convex position. □

**Proof of Theorem 4.8** We can assume that $U$ linearly spans $\mathbb{R}^d$, i.e., that $k = d$.

First assume that $U$ is in weak convex position. We verify the inequalities (4.2). Take arbitrary $x_1, x_2, \ldots, x_k, x_{k+1} \in U$. If $V(x_1, \ldots, x_k) = 0$ then (4.2) holds trivially. Otherwise, the vectors $x_1, \ldots, x_k$ span $\mathbb{R}^d$, thus there are $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ with $x_{k+1} = \lambda_1 x_1 + \cdots + \lambda_k x_k$. Since $U$ is centered, we can achieve that all $\lambda_i \geq 0$ ($i = 1, \ldots, k$) by possibly interchanging $x_i$ with $-x_i \in U$. This does not modify (4.2) since the volumes $V(a_1, b, x, c, \ldots, 0) = V(a_1, b, -x, c, \ldots, 0)$ are invariant under inversion of a spanning vector. Consequently, $x_{k+1}$ belongs to the convex cone spanned by $x_1, \ldots, x_k$ with apex $0$. Since $U$ is in weak convex position, $x_{k+1}$ cannot belong to the interior of the simplex with vertices $0, x_1, \ldots, x_k$. Thus $\lambda_1 + \cdots + \lambda_k \geq 1$ (since the hyperplane through $x_1, \ldots, x_k$ is characterized for $x_{k+1}$ as $\lambda_1 + \cdots + \lambda_k = 1$). We get that
\[
\lambda_1 V(x_1, \ldots, x_k) + \cdots + \lambda_k V(x_1, \ldots, x_k) \geq V(x_1, \ldots, x_k),
\]
which turns out to be (4.2), since
\[
\lambda_1 V(x_1, \ldots, x_k) = V(\lambda_1 x_1, \lambda_2 x_2, \cdots, x_k) = V(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k, x_2, \cdots, x_k) = V(x_{k+1}, x_2, \cdots, x_k) = V(x_2, \cdots, x_k) + \lambda_1 V(x_1, x_3, \ldots, x_k), \quad \ldots, \quad \text{and } \lambda_k V(x_1, \ldots, x_k) = V(x_1, \ldots, x_{k-1}, x_{k+1}).
\]

For the other direction we assume that (4.2) holds for all $x_1, x_2, \ldots, x_k, x_{k+1} \in U$ and prove by contradiction that $U$ is in weak convex position. If $U$ were not in weak convex position, $U \not\subset \partial(\text{conv } U)$, there must be some $u \in U$ with $u \not\in \partial(\text{conv } U)$, thus $u \in \text{int}(\text{conv } U)$.

If $u = 0$, then we have the following contradiction to inequality (4.2): set $x_{k+1} := u = 0$ and take $x_1, \ldots, x_k$ as $k$ linearly independent vectors of $U$ (note that $\dim U = \dim \text{aff } U = \dim \text{lin } U = k = d$). Then $V(x_1, \ldots, x_k) > 0$ but $V(x_1, \ldots, x_k, x_{k-1}, x_{k+1}) + V(x_1, \ldots, x_k, x_{k-2}, x_{k+1}) + \cdots + V(x_2, \ldots, x_{k+1}) = 0$. Thus we have $u = 0$.

Consider the ray $[0, u]$ which intersects $\partial(\text{conv } U)$ in some point $u' = \mu u$ for $\mu > 1$. This intersection must exist if $U$ is bounded. If $U$ is not bounded, then we replace $U$ by some centered subset $U' \subset U$ such that still $u \in \text{int}(\text{conv } U')$ and $u \in U'$. Such a subset $U'$ exists: there must be some simplex contained in $\text{conv } U$ with $u$ in its interior. For each of its $d+1$ vertices $v$ we can find by Carathéodory’s Theorem (Theorem 1.43) $d+1$ points of $U$ such that $v$ belongs to their convex hull. Taking $U'$ as all these (at most) $(d+1)^2$ points, plus $u$ itself, and all corresponding symmetric points, we get a finite subset of $U$ as wanted. Thus we can additionally assume that $U$ is a finite set.

Again by Carathéodory’s Theorem there are affinely independent points $x_1, \ldots, x_k \in U$ with $u' = \Delta(x_1, \ldots, x_k)$, since $u'$ is contained in some face of $\partial(\text{conv } U)$ which is of dimension $k-1$. Thus there are real numbers $\lambda_1', \ldots, \lambda_k' \geq 0$ with $u' = \lambda_1' x_1 + \cdots + \lambda_k' x_k$ and $\lambda_1' + \cdots + \lambda_k' = 1$. So we have with $\lambda_i := \lambda_i'/\mu$ that $u = \lambda_1 x_1 + \cdots + \lambda_k x_k$ and $\lambda_1, \ldots, \lambda_k \geq 0$ and $\lambda_1 + \cdots + \lambda_k < 1$. Multiplying by $V(x_1, \ldots, x_k) > 0$ yields
\[
\lambda_1 V(x_1, \ldots, x_k) + \cdots + \lambda_k V(x_1, \ldots, x_k) < V(x_1, \ldots, x_k),
\]
which contradicts (since again $\lambda_1 V(\mathbf{f}_1, \ldots, \mathbf{f}_k) = V(\mathbf{f}_2, \ldots, \mathbf{f}_k, \mathbf{u})$, \ldots, and $\lambda_k V(\mathbf{f}_1, \ldots, \mathbf{f}_k) = V(\mathbf{f}_1, \ldots, \mathbf{f}_{k-1}, \mathbf{u})$, see above) the inequality (4.2) for $\mathbf{f}_{k+1} = \mathbf{u}$. This contradiction completes our proof.

Proof of Proposition 4.11

We show that if there is a lower dimensional embedding, then also a full dimensional embedding can be constructed. Assume that $f_\epsilon : (\mathbb{N}_n, \rho) \rightarrow \mathbb{M}_d^d$, $i \mapsto \mathbf{s}_i$ is an embedding where $L := \text{aff} \{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$ has dimension $\dim L < d$. Without loss of generality we can assume that $L$ is a linear subspace, i.e., $\mathbf{o} \in L$. Otherwise we can consider the translation $f_\epsilon' := f_\epsilon - f_\epsilon(1)$ instead of $f_\epsilon$, which is an embedding of $(\mathbb{N}_n, \rho)$ into $\mathbb{M}_d^d$, too. Next we consider a ( inclusion) maximal affinely independent set $S \subset \{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$. $S$ has exactly $\dim L + 1 < n$ elements. Thus there is some $k \in \mathbb{N}_n$ with $\mathbf{s}_k \notin S$. Note that $\mathbf{s}_i \neq \mathbf{s}_j$ for all $i \neq j$ since $\rho(i, j) > 0$.

We will extend the unit ball $\tilde{B} := B(\mathbb{M}_d^d) \cap L$ of the linear subspace $L$ of $\mathbb{M}_d^d$ to a unit ball in some linear subspace $L'$ of $\mathbb{R}^d$ with dimension $\dim L' = \dim L + 1$. For this we fix any direction $\mathbf{r} \in \mathbb{R}^d \setminus L$ and define $B := \tilde{B} + \mathbf{r}/\|\mathbf{r}\|$ as prism over $\tilde{B}$, and $L' := \text{lin B}$. Now we shift $\mathbf{s}_k$ a little bit in direction $\mathbf{r}$ to $\mathbf{s}_k' := \mathbf{s}_k + \epsilon \mathbf{r}$. If $\epsilon > 0$ is small enough, all length stays the same, $\|\mathbf{s}_k B\|_B = \|\mathbf{s}_k' B\|_B = \|\mathbf{s}_k B\|_B$ for all $j \in \mathbb{N}_n \setminus \{k\}$, but $\dim \{\mathbf{s}_1, \ldots, \mathbf{s}_k', \ldots, \mathbf{s}_n\} = \dim L + 1$.

If $\dim L' = \dim L + 1 < d$, we repeat this procedure $d - 1 - \dim L$ times. We obtain an embedding with full dimension $d$, and thus a solution of $\text{SysEm}(\rho, d)$.

Proof of Lemma 4.14

This follows from the fact that there are only finitely many restrictions in $S = (E, W, S)$. The relation $L(S_{\text{weak}}) \subset R^+ L(S_{\text{weak}}) \subset L(S)$ is obvious. If $\mathbf{r} \in L(S)$, then we get that $\lambda := \min \{ f(\mathbf{r}) : f \in S \}$ exists and $\lambda > 0$. Thus $\lambda \mathbf{r} \in L(S_{\text{weak}})$ and consequently $L(S) \subset R^+ L(S_{\text{weak}})$.

Proof of Lemma 4.15

First we show that each vector $b = b f_\epsilon \in \mathbb{R}^{(N_d)}$ for some function $f_\epsilon : \mathbb{N}_n \rightarrow \mathbb{R}^d$ is a solution of the system $\text{SysDet}(n, d)$.

The equations (4.18) and (4.19) are the well known antisymmetry and (multi-) linearity of the determinant function. We show the Grassmann identity (4.20). Consider the right hand side

$$r = \sum_{j=1}^{d+1} (-1)^j \det((\mathbf{f}_{j1})_{i\in \mathbb{N}_{d+1}\setminus\{j\}}) \det((\mathbf{r}_j, (\mathbf{n}_i)_{i\in \mathbb{N}_{d-1}})

of (4.20). $r$ is a real function on $2d$ vectors $\mathbf{r}_1, \ldots, \mathbf{r}_{d+1}, \mathbf{n}_1, \ldots, \mathbf{n}_{d-1} \in \mathbb{R}^d$, which is a linear function in each of the $2d$ vectors if we fix the others. Consequently, $r = 0$ follows for all arguments if this is true for all $\mathbf{r}_1, \ldots, \mathbf{r}_{d+1}$ being standard unit vectors $\mathbf{e}_1, \ldots, \mathbf{e}_d$. Thus two of the $d + 1$ vectors $\mathbf{r}_1, \ldots, \mathbf{r}_{d+1}$ must be equal, say $\mathbf{r}_{k} = \mathbf{r}_1$ or $k < l$. Then $d - 1$ summands of $r$ are zero ($j \notin \{k, l\}$) and the remaining summands add to zero,

$$(-1)^k \det((\mathbf{f}_{k1})_{i\in \mathbb{N}_{d+1}\setminus\{k\}}) \det((\mathbf{f}_{k1}, (\mathbf{n}_i)_{i\in \mathbb{N}_{d-1}}) + (-1)^l \det((\mathbf{f}_{l1})_{i\in \mathbb{N}_{d+1}\setminus\{l\}}) \det((\mathbf{f}_{l1}, (\mathbf{n}_i)_{i\in \mathbb{N}_{d-1}}) = ((-1)^k \det((A, B, \mathbf{r}, C) + (-1)^l \det((A, B, \mathbf{r}, C) \det((\mathbf{n}_i)_{i\in \mathbb{N}_{d-1}}) = 0.

Note that $B$ stands for a matrix with $k - l - 1$ columns $\mathbf{f}_{k+1}, \ldots, \mathbf{f}_{l-1}$. Thus $\det(A, B, C)$ if $k - l - 1$ is even, and $\det(A, B, C) = -\det(A, B, C)$ if $k - l - 1$ is odd.

Consequently, we get $r = 0$ as wanted.

Equation (4.18) yields for $a = f_\epsilon(p_{11}) - f_\epsilon(p_{12})$ and $b = f_\epsilon(p_{21}) - f_\epsilon(p_{22})$ that $\text{asymp}_{p_{12}, p_{21}, I_{12}, I_{21}}(b) = 0$, (4.19) yields for $\lambda = \mu = -1$, $a = f_\epsilon(i) - f_\epsilon(j)$, $b = f_\epsilon(j) - f_\epsilon(k)$, with $\lambda \mu + \mu b = f_\epsilon(k) - f_\epsilon(i)$ that $0 = \text{linearity}_{i, j, k, l}(b)$, and (4.20) applied to $\mathbf{y}_i = f_\epsilon(I_{i1}) - f_\epsilon(I_{i2})$ and $\mathbf{y}_j = f_\epsilon(J_{i1}) - f_\epsilon(J_{i2})$ yields that quadratic $f_{i,j}(b) = 0$. Thus we have $b f_\epsilon \in L(\text{SysDet}(n, d))$.

Now we will show the converse direction. Given any $b \in L(\text{SysDet}(n, d))$, we will construct a function $f_\epsilon : \mathbb{N}_n \rightarrow \mathbb{R}^d$ such that $b = b f_\epsilon$.

We show that without loss of generality we can assume that $e := b((i + 1, 1)_{i\in \mathbb{N}_d}) = b((2, 1), (3, 1), \ldots, (d + 1, 1)) \neq 0$. If we could not find a permutation $\sigma$ of the (abstract) points $1, \ldots, n$ with $b(\sigma(i + 1), \sigma(1))_{i\in \mathbb{N}_d} \neq 0$, then this implies that $b(I) = 0$ for all $I \in (\mathbb{N}_d^d)$. Using an induction argument we show the statement $H(k)$: that $b(I) = 0$ for all $I$ where the first $k$ pairs in $I$ have a common second value: $I_{1,2} = I_{2,2} = \cdots = I_{k,2}$. $H(d)$ follows from the hypothesis, $b((\sigma(i + 1), \sigma(1))_{i\in \mathbb{N}_d} = 0$ for all permutations $\sigma$, together with the trivial relations that $b(I) = 0$ if $I_{i,1} = I_{i,2} = a$ (using $3b(I) = \pm \text{linearity}_{\mathbf{a}, \mathbf{a}, \mathbf{a}, (I_{11}), (I_{12})(b)) or that $I_{i} = I_{j}$ for $i < j$. Now for $k = d - 1, \ldots, 1, 0$ we can show that from $H(k + 1)$ also $H(k)$ follows. So assume that $I_{1,2} = I_{2,2} = \cdots = I_{k,2} = 0$ and $I_{k,1} = I_{k,2} = a$. Then $b(I) = 0$ for all $I$ where the first $k$ pairs in $I$ have a common second value.
\( \cdots = I_{k,2} = i \) holds, and that \( I_{k+1} = (a, c) \). But now \( b(I) = b(I_1, \ldots, I_k, (a, c), I_{k+2}, \ldots, I_d) = \left( b(I_1, \ldots, I_k, (a, i), I_{k+2}, \ldots, I_d) + b(I_1, \ldots, I_k, (i, c), I_{k+2}, \ldots, I_d) \right) = -0 + 0 = 0. \) Finally, \( H(0) \) means that \( b \) is identically zero. In this case \( b = b^c \) for the trivial function \( f_c \equiv 0. \)

Thus we assume that for \( E := ((i+1,1)_{0} \leq (2,1), (3,1), \ldots, (d+1,1)) \) it holds that \( c := b(E) \neq 0. \) We directly construct \( f_c : \mathbb{N}_n \rightarrow \mathbb{R}^d \) as

\[
 f_c(a) := \left( b((a, 1), (3, 1), \ldots, (d+1,1)), b((2,1), (a, 1), (4,1), \ldots, (d+1,1)), \ldots, b((2,1), (d+1,1), (a,1)) \right).
\]

The \( c \)-th coordinate of \( f_c(a) \) \( (a \in \mathbb{N}_n, c \in \mathbb{N}_d) \) is exactly

\[
 (f_c(a))_c := \begin{cases} 
 b(J^{(a,1),c}) & \text{if } c = 1 \\
 \frac{1}{c} b(J^{(a,1),c}) & \text{if } c > 1,
\end{cases}
\]

where the sequence \( J^{(i,j),c} \in (\mathbb{N}_n^d \cap E) \) is almost \( E \), except for the \( c \)-th pair which is \( (i,j) \),

\[
 J^{(i,j),c}_k := \begin{cases} 
 (i, j) & \text{for } k = c \\
 (k+1, 1) & \text{for } k \in \mathbb{N}_d \setminus \{c\}.
\end{cases}
\]

Because of the linear relations \( 0 = \text{linearity}_{*,*,*}(b) \) and \( 0 = \text{asym}_{*,*,*}(b) \), the \( c \)-th coordinate of the difference vector \( f_c(i) - f_c(j) \),

\[
 (f_c(i) - f_c(j))_c := \begin{cases} 
 b(J^{(i,1),c}) - b(J^{(j,1),c}) & \text{if } c = 1 \\
 \frac{1}{c} \left( b(J^{(i,1),c}) - b(J^{(j,1),c}) \right) & \text{if } c > 1,
\end{cases}
\]
equals the value \( b(J^{(i,j),c}) \) for \( c = 1 \) or \( \frac{1}{c} b(J^{(i,j),c}) \) for \( c > 1 \).

Now let \( I \in (\mathbb{N}_n^d \cap E) \) be a fixed sequence. We will show that \( b^c(I) = b(I) \) which finishes this proof. Beneath the linear relations we also need the Grassmann identity quadricities_{*,*}(b) = 0. We use another induction argument and expand the determinant along the first row, to show the following statement \( S(k, I_k, \ldots, I_d), k \in \mathbb{N}_d \):

\[
 \det \begin{pmatrix} 
 b(I^{k,k}_1, \ldots, I^{k,k}_d) & \ldots & b(I^{k,k}_1, \ldots, I^{k,d}_d) \\
 \vdots & \ddots & \vdots \\
 b(I^{d,k}_1, \ldots, I^{k,k}_d) & \ldots & b(I^{d,k}_1, \ldots, I^{d,d}_d)
\end{pmatrix} = \det((b(J^{(i,j)}))_{i,j=k,\ldots,d} = e^{d-k} b((2,1), \ldots, (k,1), I_k, \ldots, I_d). \]

The statement \( S(d, I_d) \) is trivially satisfied: \( \det((b(J^{d,d}_1))) = b(J^{d,d}_1) = e^0 b((2,1), \ldots, (d,1), I_d) \). Now we assume that for \( 1 \leq k < d \) all the statements \( S(k+1,* \cap I_d) \) are true and show the same for \( S(k, I_k, \ldots, I_d) \).

\[
 \det((b(J^{i,j})(i,j=k,\ldots,d)) = b(J^{i,k}_1, \ldots, I^{i,k}_d) + b(J^{k,k}_1, \ldots, I^{k,k}_d - b(J^{k+1,k}_1, \ldots, I^{k+1,k}_d) \det((b(J^{i,j})(i,k+1,\ldots,d,j=k,k+2,\ldots,d)) + 
\end{pmatrix} = (d-1)^{d-k} b(J^{d,d}_1) \det((b(J^{i,j})(i,k+1,\ldots,d,j=k,k+2,\ldots,d)).
\]

Using the statements \( S(k+1, I_{k+1}, \ldots, I_d), S(k+1, I_k, I_{k+2}, \ldots, I_d), \ldots, S(k+1, I_k, \ldots, I_{d-1}) \), we get

\[
 \det((b(J^{i,j})(i,j=k,\ldots,d)) = b(J^{i,k}_1, \ldots, I^{i,k}_d) e^{d-k-1} b((2,1), \ldots, (k+1,1), I_{k+1}, \ldots, I_d) + 
- b(J^{k+1,k}_1, \ldots, I^{k+1,k}_d) e^{d-k-1} b((2,1), \ldots, (k+1,1), I_{k+1}, \ldots, I_d) + 
\end{pmatrix} = e^{d-k-1} b((2,1), \ldots, (k+1,1), I_{k+1}, \ldots, I_d) b(J^{k,k}_1) + 
- b((2,1), \ldots, (k+1,1), I_{k+1}, \ldots, I_d) b(J^{k+1,k}_1) + 
\end{pmatrix} = (d-1)^{d-k} b((2,1), \ldots, (k+1,1), I_{k+1}, \ldots, I_{d-1}) b(J^{i,k}_1).
\]

The last expression in parentheses now equals \( b((2,1), \ldots, (k,1), I_{k+1}, \ldots, I_d) e \) by some linear relations, and \( \det((2,1), \ldots, (k,1), I_{k+1}, \ldots, I_d, e) = 0 \) instead of the first argument: \( \sum_{j=1}^{d} (-1)^j b(J^{(i,j)}(I, k+1, \ldots, j)) b(J^{(i,j)}(I', J', J'')) \), where \( J' = ((2,1), \ldots, (k,1)) \) and \( J'' = ((k+2,1), \ldots, (d+1,1)) \), and \( I' = ((2,1), \ldots, (k+1,1), I_{k+1}, \ldots, I_d) \). The first \( k-1 \) summands are zero since the pair \( (h+1,1) = I'_h \) is also present in \( J' \), thus \( b(J', I'_h, J'') = 0 \) by the asymmetry. The \( k \)-th summand is \( (-1)^k b((2,1), \ldots, (k,1), I_k, \ldots, I_d) e \), and the remaining summands occur in the above formula up to the sign \( (-1)^h \).
independent, too. Thus there is a uniquely defined bijective affinely linear map $f_e$ of $S(I, I_1, \ldots, I_d)$ to the scalar $e$. This theorem is similar to Lemma 4.17. The additional relation between $f_e : (N_n, \rho) \to M^d$ into the polynomial Minkowski space $M^d = M^d(B)$, $B := \{ x \in R^d : Ax \leq 1 \}$, and $L'$ is the union of some special faces of a polytope $P$. It is not difficult to see that all inequalities in (1.29) are linear in the vector $e$, thus $P$ is a polyhedron. Note that $\phi$ belongs to the interior of $P$. $P$ is bounded since $B$ is bounded: consider $j = 1$.

Now assume that $e \in L$ is the vector of embedding function $f_e$, i.e., $\|f_e(i) - f_e(j)\|_B = \rho(i, j)$ for all $i, j \in N_n$. Then $f_e(i) - f_e(j) = g_e(i) - g_e(j)$ holds. Thus for all $i \neq j$ we have that $\|g_e(i) - g_e(j)\|_B = 1$ and that $u_{i,j} := \frac{1}{\rho(i, j)}(g_e(i) - g_e(j)) \in \partial B \subset B$. From $u_{i,j} \in B$ we can conclude that $A(g_e(i) - g_e(j)) \leq \rho(i, j) 1$, i.e., that $e \in P$. Since $u_{i,j} \in \partial B$, it must belong to a facet of $B$ which is described by the $r$-th row of $A$ via $A_r(g_e(i) - g_e(j)) = \rho(i, j)$. This gives us $e \in L'$ and $L \subset L'$.

This arguments can also be used for the other direction. If $e \in L'$, then the function $f_e : N_n \to M^d$, defined via $f_e(i) := g_e(i)$ for $i \in N_n$ turns out to be an embedding of $(N_n, \rho)$ into $M^d(B)$ with embedding vector $e$.

**Proof of Theorem 4.20** Analogous to the proof of Theorem 4.19 The linearity of the system follows from the representation $M = \sum_{i = 1, \ldots, d} m_{i,j} E_{i,j}$ where all $m_{i,j}$ is $\in \mathbb{R}$ and all the matrices $E_{i,j}$ have exactly one nonzero component $1$ at position $(i, j)$. The zero matrix (as vector in $\mathbb{R}^d$) is an interior point of $P$. The polyhedron $P$ is bounded (i.e., a polytope) if and only if the vectors $f_e(1), \ldots, f_e(n)$ affinely span the space $M^d$, otherwise it is not pointed.

**Proof of Lemma 4.21** For all $\eta \in \mathbb{R}^m$ we have that $\eta \in L(S')$ if and only if $\phi(\eta) \in L(S)$. Thus $\phi(L(S')) \subset L(S)$. Since additionally $\phi(\mathbb{R}^m) \supset L(S) \supset L(S)$, we get that $\phi(L(S')) = L(S)$. Since $\phi$ is injective, its restriction $\phi : L(S') \to L(S)$ is bijective.

**Proof of Proposition 4.23** The function $\psi : \mathbb{R}^{[0,1]^d} \to \mathbb{R}^M$, $b \mapsto (I \mapsto b(((1, I)), i \in N_n))$ is obviously a linear projection.
We consider the function $\phi : \mathbb{R}^M \to \mathbb{R}(\mathbb{N}_0^d)^d$ which is defined by (4.32) and (4.33). The function $h$, mapping $b \in \mathbb{R}^M$ to a function in $\mathbb{R}(\mathbb{N}_0^d)^d$, is a linear function. Thus also $\phi$ is linear.

It is not difficult to see that $b \in L$ implies that for all $I \in \mathbb{N}_0^d$

$$b(\psi(b))(I) = b((1, l_i)_{i \in \mathbb{N}_0^d}).$$

(4.41)

More precisely, for $I \in M$ we get (4.41) directly where $\sigma$ is the identity permutation in (4.33). Interchanging two elements within $I$ and $\sigma$ at the same time keeps (4.41) valid, both sides are multiplied by $-1$. The left hand side changes its sign together with $r$, the right hand side due to the antisymmetry of $b$. If $I$ contains a 1, i.e., $I_i = 1$ for some $i \in \mathbb{N}_0^d$, or if $I$ contains a number more than once, i.e., $I_i = I_j$ for $1 \leq i < j \leq d$, then both sides of (4.41) are zero. For the left hand side this follows from its construction, the right hand side is zero because of (4.22) in case of $I_i = I_j$ and of (4.24) in case of $I_i = 1$.

From (4.11), (4.32) and the relation

$$b(J) = \sum_{p \in \{1,2\}^d} (-1)^{\sum_{i \in \{1,2\}} p_i} b((1, J, l_p)_{i \in \mathbb{N}_0^d}),$$

where $J \in (\mathbb{N}_0^d)^d$ and $b \in L$, a consequence from the linearity (4.24), which can be generalized to every position using (4.22), we get that $\psi(b)(I) = b(J)$, i.e., that $\psi(b) = b$.

For every $b \in \mathbb{R}^M$ we show that $\phi(b) \in L$. We have to verify (4.22) and (4.24) for $b := \phi(b)$. Interchanging the $i$-th and $j$-th pair in the sequence $I \in (\mathbb{N}_0^d)^d$ yields interchanged sequences $(J, p_i)_{i \in \mathbb{N}_0^d}$, such that the terms $b((J, p_i)_{i \in \mathbb{N}_0^d})$ are multiplied by $-1$, and the same happens to $\phi(b)(J)$. So (4.22) holds. If we consider $\phi(b)((i, j), I) + \phi(b)((j, k), I) + \phi(b)((k, i), I) = \sum_{I \in \mathbb{N}_0^d}$ and $i, j, k \in \mathbb{N}_0$, this equals

$$\sum_{p \in \{1,2\}^d} (-1)^{1+\sum_{i \in \{1,2\}} p_i} (b(i, J, l_p)_{i \in \mathbb{N}_0^d} + b(j, J, l_p)_{i \in \mathbb{N}_0^d} + b(k, J, l_p)_{i \in \mathbb{N}_0^d}) +$$

$$+ \sum_{p \in \{1,2\}^d} (-1)^{2+\sum_{i \in \{1,2\}} p_i} (b(j, J, l_p)_{i \in \mathbb{N}_0^d} + b(k, J, l_p)_{i \in \mathbb{N}_0^d} + b(i, J, l_p)_{i \in \mathbb{N}_0^d})$$

$$= 0,$$

and (4.24) is verified.

Now for $J \in M$ the chain of equations $\psi(\phi(b))(J) = \phi(b)((1, J, l_p)_{i \in \mathbb{N}_0^d}) = (-1)^{\sum_{i \in \mathbb{N}_0^d} b(i, J, l_p)_{i \in \mathbb{N}_0^d}}$ follows easily.

Thus the restriction $\psi : L \to \mathbb{R}^M$ is bijective and $\phi$ is its inverse. Since $|M| = \binom{n}{d-1} = m$, we get that $L$ is an $m$-dimensional.

**Proof of Lemma 4.24.** The functions quadratic$_{I,J}$ are linear in $I$ and $J$ in the following sense: for $b \in \mathbb{L}(\text{SymDet}_d(n, d))$, $I' \in (\mathbb{N}_0^d)^d$ and $J \in (\mathbb{N}_0^d)^d-1$ we have that quadratic$_{(I,J), (I'), J'}(b) = \text{quadratic}_{((I,J), (I'), J')}(b) = \text{quadratic}_{(I,J), (I', J')}(b)$. The same holds true for any other pair of the sequences $I$ and $J$. Thus we know that quadratic$_{I,J}(b) = 0$ for all $I \in (\mathbb{N}_0^d)^d+1$, $J \in (\mathbb{N}_0^d)^d-1$, if we know that quadratic$_{I,J}(b) = 0$ for all $I, J$ with $I_{d+1} = J_{d+1} = 1$ and that $b \in \mathbb{L}(\text{SymDet}_d(n, d))$.

Using the symmetry of the functions quadratic$_{I,J}$, which we already discussed in Remark 4.16, we can additionally restrict ourselves to the case that $I_{d+1} = I_{d+2} = \cdots = I_{d+1,2}$ and $J_{d+1} = J_{d+2} = \cdots = J_{d+1,2}$. We can also restrict to nontrivial cases $I < I_1 < I_2 < \cdots < I_{d+1,2}$ and $J_1 < J_2 < J_3 < \cdots < J_{d+1,2}$, since otherwise $I_1 = (1, 1)$ or $J_1 = (1, 1)$ or one of the sequences $I, J$ contains two identical pairs. In each of these trivial cases we have that all summands $b((I, l_1)_{i \in \mathbb{N}_0^d+1}) b((J, l_1)_{i \in \mathbb{N}_0^d+1})$ are zero, $j \in \mathbb{N}_0^d+1$, provided that $b \in \mathbb{L}(\text{SymDet}_d(n, d))$.

Now assume that $|\{I_1 \in \mathbb{N}_0^d+1 \cap \{J_1 \in \mathbb{N}_0^d+1 \}| = d - 1$, i.e., each element of the sequence $J$ is also contained in $I$. Assume that $I_x$ and $I_y$ are the two elements of $I$ not contained in $J$, with $x < y$. Thus $b(I_x, J) = 0$ for all $j \in \mathbb{N}_0^d \setminus \{x, y\}$. Thus quadratic$_{I,J}(b) = (-1)^{x+y} b(I_x, I_y) b(I_x, J) + (-1)^{y+1} b(I_x, I_y) b(I_y, J) = (-1)^{x+y} b(I_x, I_y) b(I_x, J) + (-1)^{x+y} b(I_x, I_y) b(I_y, J) = (-1)^{x+y} b(I_x, I_y) b(I_x, J) - b(I_x, J) b(I_y, J) = 0.$
CHAPTER 4. EMBEDDING METRIC SPACES INTO A MINKOWSKI SPACE

In the two-dimensional case \( d = 2 \) there is even more symmetry within quadratic \( i, j \), which allows to interchange the pair in \( J \) with a pair in \( I \), see again Remark 4.16. This completes the proof. \( \square \)


For the number of quadratic equations we count all pairs \((I, J)\) with the same value \( k := \left| \{ I_i \in \mathbb{N}_{d+1} \cap \{ J_j \in \mathbb{N}_{d+1} \} \right. \). All the numbers \( I_{s,1} \) and \( J_{s,2} \) belong to \( \{2, \ldots, n\} \). So we can choose \( k \) of the \( n-1 \) numbers which belong to both \( I \) and \( J \), \( d+1-k \) of the remaining \( n-1-k \) numbers to belong to \( I \setminus J \), and \( d-1-k \) of the remaining \( n-d-2 \) numbers to belong to \( J \setminus I \). Then \( I \) and \( J \) are uniquely determined since they are sorted. For \( n \leq d+2 \) we have \( E_{\text{det}, d}(n,d) = \emptyset \).

The second upper bound \( \left( \begin{array}{c} n-1 \end{array} \right) \left( \begin{array}{c} n-1 \end{array} \right) \in O(n^{2d}) \) is the number of all pairs \((I, J)\) with \( I_{s,1} = \langle I, J \rangle, 1 < I_{1,2} < I_{2,2} < \cdots < I_{d+1,2} \) and \( 1 < J_{1,2} < J_{2,2} < \cdots < J_{d-1,2} \). \( \square \)

Proof of Proposition 4.26 If all values of \( \rho(i,j) \) are rational, then the Proposition follows directly from Theorem 4.20 and Lemma 4.17 since by scaling we can make the system \( \text{SysEmD}(\rho, d, s) \) polynomial over \( Z \). But this is also possible for non-rational real algebraic numbers \( \rho(i,j) \). For each such number we can introduce a new variable which is considered as unknown, but we can add one polynomial equation and two inequalities to ensure that this variable has exactly the value of \( \rho(i,j) \). The extended system now is feasible if and only if \( \text{SysEmD}(\rho, d, s) \) is feasible. This can be decided algorithmically. \( \square \)

Proof of Proposition 4.29 Leaving out all non-linear restrictions of the system \( \text{SysEmD}(\rho, 2, s) \), we get

\[
\text{SysEmDL}(\rho, 2, s) := \left( E_{\text{conv}}(2,s) \cup E_{\text{det}}^s(5,2), W_{\text{conv}}(2,s,\rho), S_{\text{conv}}(2,s) \right).
\]

For the one quadratic restriction \( q := \text{quadratic}_{(1,2), (1,3), (1,4), (1,5)} \) with \( q(b) = -b((1,3), (1,4))b((1,2), (1,5)) + b((1,2), (1,4))b((1,3), (1,5)) \) we have that \( L(\text{SysEmD}(\rho, 2, s)) = \{ b \in L(\text{SysEmD}(\rho, 2, s)) : q(b) = 0 \} \).

The main idea is to solve both quadratic optimization problems \( q(b) \rightarrow \min \) and \( q(b) \rightarrow \max \) within the “polyhedron” \( P := L(\text{SysEmD}(\rho, 2, s)) \). But \( P \) is not bounded and also not closed in general. We intersect \( P \) by a hyperplane and obtain \( P_\varepsilon := \{ b \in P : \varepsilon b = 1 \} \), where \( \varepsilon \) is relint \( P \), and have that \( P \cap \{ b : q(b) = 0 \} = \emptyset \) if and only if \( P_\varepsilon \cap \{ b : q(b) = 0 \} = \emptyset \). \( P_\varepsilon \) is bounded and does not contain \( 0 \) in its closure. Now consider \( P^* := \text{cl} P_\varepsilon \cap \{ b : q(b) = 0 \} \). If \( \text{SysEmD}(\rho, 2, s) \) is admissible, then \( P^* \neq \emptyset \) if and only if \( \min_{b \in \text{cl} P_\varepsilon} q(b) \leq 0 \leq \max_{b \in \text{cl} P_\varepsilon} q(b) \). This condition can be decided algorithmically, since \( \text{cl} P_\varepsilon \) is a polyhedron. If \( P^* \) is not empty, then there is a solution \( b \in P^* \). Although \( b \) is not necessarily a solution of \( \text{SysEmD}(\rho, 2, s) \), since \( b \) may violate some strict inequality \( 0 < s_k(b) = s(I)b(I) \), \( b \) is a solution of \( \text{SysEmD}(\rho, 2, s') \) for some other nontrivial sign function \( s' : (\mathbb{N}_n^2)^2 \rightarrow \{-1, 0, 1\} \) for \( b \).

Thus \( (\mathbb{N}_5, \rho) \) can be embedded into a suitable Minkowski plane \( M^2 \) if and only if for at least one nontrivial sign function \( s \) we have that \( \min_{b \in \text{cl} P_\varepsilon} q(b) \leq 0 \leq \max_{b \in \text{cl} P_\varepsilon} q(b) \), where \( P_\varepsilon := \{ b \in L(\text{SysEmD}(\rho, 2, s)) : \varepsilon b = 1 \} \).

Proof of Theorem 4.30 We define \( s_1 = a, s_2 = b, s_3 = c \) and \( s_4 = d \), and \( f_\varepsilon : N_4 \rightarrow \mathbb{R}^2, i \mapsto s_i \). Again we use the determinants encoded in \( b := b_{ij} \), and for \( i, j \in N_4 \) we define \( e_{ij} := b((i, i + 1), (j, j + 1)) (i, j \mod 4) \). Since \( a, b, c, d \) are the vertices of a convex quadrilateral, we can assume that \( e_{i, i+1} > 0 \) (i mod 4). From the hypotheses we conclude that \( e_{1,3} \geq 0 \), since \( \langle a, b \rangle \) and \( \langle c, d \rangle \) intersect on the side of \( \langle b, c \rangle \) opposite to \( a \) and \( d \) or parallel, and that \( e_{2,4} \leq 0 \), since \( \langle a, d \rangle \) and \( \langle b, c \rangle \) intersect on the side of \( \langle a, b \rangle \) opposite to \( \epsilon \) and \( d \) or parallel.

Now we use the inequality (4.11) two times, once for \( I = (2,3), (3,4), (1,2) \),

\[
\| \overline{ab} \| |b((2,3), (3,4))| \leq \| \overline{bc} \| |b((3,4), (1,2))| + \| \overline{cd} \| |b((2,3), (1,2))| \Rightarrow \| \overline{ab} \| e_{2,3} \leq \| \overline{bc} \| e_{1,3} + \| \overline{cd} \| e_{1,2},
\]

(4.42)

and once for \( I = (1,2), (1,4), (2,3) \),

\[
\| \overline{bc} \| |b((1,2), (1,4))| \leq \| \overline{ab} \| |b((1,4), (2,3))| + \| \overline{cd} \| |b((1,2), (2,3))| \Rightarrow \| \overline{bc} \| (e_{1,2} + e_{1,3}) \leq \| \overline{ab} \| (e_{2,1} + e_{2,3}) + \| \overline{cd} \| e_{1,2}.
\]

(4.43)
Now assume that \( \|\overrightarrow{ab}\| + \|\overrightarrow{bc}\| = \|\overrightarrow{ad}\| + \|\overrightarrow{cd}\| \). If \( e_{1,3} = e_{2,4} = 0 \) holds, then \( \overrightarrow{abcd} \) is a parallelogram and our claim is true. Otherwise we have \( e_{1,3} > 0 \) or \( e_{2,4} < 0 \), and consequently, \( \overrightarrow{c - b} \neq \overrightarrow{0 - a} \) or \( \overrightarrow{c - b} \neq \overrightarrow{d - a} \). Furthermore, the triangle inequalities (4.32) and (4.33) must be satisfied as equations, \( \|\overrightarrow{ad}\| e_{2,3} = \|\overrightarrow{cd}\| e_{1,2} + \|\overrightarrow{bc}\| e_{1,3} \) and \( \|\overrightarrow{bc}\| (e_{1,2} + e_{1,3}) = \|\overrightarrow{ad}\| e_{1,2} + \|\overrightarrow{ab}\| (e_{2,1} + e_{2,3}) \). Since the vectors \( \overrightarrow{f} = \overrightarrow{c - d}, \overrightarrow{g} = \overrightarrow{0 - a}, \overrightarrow{h} = \overrightarrow{c - a}, \overrightarrow{i} = \overrightarrow{d - b} \), we get that \( \overrightarrow{b - a} \in \overrightarrow{f - c} - \overrightarrow{b - c} \). At least one of these points belongs to the interior of the corresponding segment, thus this segment belongs to the unit circle, as well as \( \overrightarrow{i} \). This also implies that \( \overrightarrow{c - a} \in \overrightarrow{b - b} \), which gives \( \|\overrightarrow{f}\| = \|\overrightarrow{g}\| + \|\overrightarrow{h}\| \). \( \Box \)

**Proof of Corollary 4.31** Assume that the claim is not true, i.e., \( (a, b) \) and \( (c, d) \) do not intersect or they intersect on the side of \( (a, b) \) opposite to \( b \) and \( c \). We distinguish to cases. The first case occurs if \( (a, d) \) and \( (b, c) \) intersect on the side of \( (a, b) \) opposite to \( c \) and \( d \) or are parallel. Then we get by Theorem 4.30 that \( \|\overrightarrow{ab}\| + \|\overrightarrow{ad}\| \leq \|\overrightarrow{bc}\| + \|\overrightarrow{cd}\| \). Otherwise, \( (a, d) \) and \( (b, c) \) must intersect on the side of \( (c, d) \) opposite to \( a \) and \( b \), which gives \( \|\overrightarrow{cd}\| + \|\overrightarrow{ad}\| \leq \|\overrightarrow{ab}\| + \|\overrightarrow{bc}\| \). In both cases we get a contradiction \( 2 < 1 + \|\overrightarrow{ad}\| \leq 2 \). \( \Box \)
Chapter 5

Algorithmical solution of parametrized linear systems

The subject of this chapter is to discuss the algorithmical solution of families of linear systems. Each member is a linear system of equations and inequalities in $\mathbb{R}^d$ as introduced in Section 1.18, Definition 1.46, but its representation depends polynomially on some parameter.

5.1 Systems to solve

In this section we will precisely describe the systems we want to solve. Shortly we will explain how more general systems are slightly transformed into more suitable ones.

5.1.1 Homogeneous and inhomogeneous systems

We will focus on homogeneous linear systems $S = (E, W, S)$ in $\mathbb{R}^d$, all whose restrictions $f \in E \cup W \cup S$ are linear functions. Consequently, for each restriction $f$ there is a row-vector $a \in \mathbb{R}^d$ such that $f(x) = ax = \sum_{i=1}^{d} a_ix_i$.

For each row-vector $a \in \mathbb{R}^d$ we denote by $\text{lin}_a$ the linear function $\text{lin}_a(x) = ax$ from $\mathbb{R}^n$ to $\mathbb{R}$, by $H^0(a)$ the (homogeneous) hyperplane in $\mathbb{R}^d$ normal to $a$ (provided $a \neq 0$), $H^+(a) := \{ x \in \mathbb{R}^d : ax \geq 0 \}$ the corresponding positive open half-space, and by $H^0(a) := \{ x \in \mathbb{R}^d : ax \geq 0 \}$ the corresponding closed half-space.

For $i \in \mathbb{N}_m$ we denote by $A_i = A_{i,*}$ the $i$-th row of the matrix $A \in \mathbb{R}^{m \times d}$.

We can describe each homogeneous linear systems by some coefficient matrix $A$.

**Definition 5.1** The linear system $\text{Mat}(m, d, A, e, w, s)$ with $e \in \mathbb{N}^0$ equations, $w \in \mathbb{N}^0$ weak inequalities and $s \in \mathbb{N}^0$ strict inequalities described by the system matrix $A \in \mathbb{R}^{m \times d}$, with $m = e + w + s$, is the system

$$\text{Mat}(m, d, A, e, w, s) = \{ \{ \text{lin}_{A_i} : 1 \leq i \leq e \}, \{ \text{lin}_{A_i} : e + 1 \leq i \leq e + w \}, \{ \text{lin}_{A_i} : e + w + 1 \leq i \leq m \} \}.$$  \hspace{1cm} (5.1)

Note that $L(\text{Mat}(m, d, A, e, w, s)) = \bigcap_{i=1}^{m} H^0(A_i) \cap \bigcap_{i=e+1}^{e+w} H^+(A_i) \cap \bigcap_{i=e+1}^{m} H^0(A_i)$.

With the well known technique of homogenization (see, for example, [39]) we can transform each inhomogeneous linear system $S = (E, W, S)$ in $\mathbb{R}^d$ into an homogeneous linear systems $S_{\text{Hom}}$ in $\mathbb{R}^{d+1}$ with $|E|$ equations, $|W|$ weak inequalities and $|S| + 1$ strict inequalities by adding a homogeneous variable. Consider the function

$$\text{Homog} : \mathbb{R}^d \rightarrow \mathbb{R}^+ \times \mathbb{R}^d, \quad (x_1, \ldots, x_d) \mapsto (1, x_1, x_2, \ldots, x_d)$$  \hspace{1cm} (5.2)

with reverse transform

$$\text{ReHomog} : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad (x_0, x_1, \ldots, x_d) \mapsto \frac{1}{x_0} (x_1, x_2, \ldots, x_d).$$  \hspace{1cm} (5.3)
We transform an affine linear function $f : \mathbb{R}^d \to \mathbb{R}$, $f(x) = ax + b$, into

$$f_{\text{Hom}} : \mathbb{R}^{d+1} \to \mathbb{R}, (x_0, x_1, \ldots, x_d) \mapsto x_0b + a(x_1, \ldots, x_d).$$  \hspace{1cm} (5.4)

Note that for $x_0 \neq 0$ we have $f_{\text{Hom}}(x) = x_0 f(\text{ReHomog}(x))$. Transferring the notation to sets $F$ of functions, $F_{\text{Hom}} := \{ f_{\text{Hom}} : f \in F \}$, we can define

$$(E, W, S)_{\text{Hom}} := (E_{\text{Hom}}, W_{\text{Hom}}, S_{\text{Hom}} \cup \{(x_0, x_1, \ldots, x_d) \mapsto x_0\}).$$  \hspace{1cm} (5.5)

**Proposition 5.2** The inhomogeneous linear system $S$ in $\mathbb{R}^d$ is equivalent to the homogeneous linear system $S_{\text{Hom}}$ in $\mathbb{R}^{d+1}$ in the following sense:

- $\blacksquare$ $x$ is a solution vector of $S$ if and only if $\text{Homog}(x)$ is a solution vector of $S_{\text{Hom}}$.
- $\blacksquare$ $\mathfrak{z}$ is a solution vector of $S_{\text{Hom}}$ if and only if $\text{ReHomog}(\mathfrak{z})$ is a solution vector of $S$.
- $\blacksquare$ The function $\text{Homog}$ restricted as map from $L(S)$ into $L(S_{\text{Hom}}) \cap \{1\} \times \mathbb{R}^d$ is bijective.

Note that there is also a one-to-one relation between the combinatorial structure (face lattice) of the solution sets.

5.1.2 Removing strict inequalities

In the literature linear systems of the form $Ax \geq b$ are often considered, where no strict inequalities are allowed. In our notation this means to consider only linear systems $(\emptyset, W, \emptyset)$. Every system $(E, W, S)$ is equivalent to $(\emptyset, W \cup E \cup -E, S)$. So we can easily achieve $E = \emptyset$, but this transformation makes the system a little larger and discards some information about its solution.

Strict inequalities cannot be replaced in such a simple matter. But on the other hand they can be handled just like weak inequalities by storing and interpreting this extra information.

If we are only interested in admissibility of a homogeneous linear system, we can use Lemma 5.1 to transform the system into a second, inhomogeneous linear system of the same size (just some of the absolute values change from 0 to 1) preserving admissibility.

A more direct approach preserving much of the structure of the solution set is the following.

**Lemma 5.3** Each linear system $S = (E, W, S)$ is almost identical to $S' = (E, W \cup S, \emptyset)$: If $S$ is admissible, then $\text{rel int} L(S') \subset L(S) \subset L(S')$, and for each (relatively open) face $f$ of the polyhedron $L(S')$ all or none of its points belong to $L(S)$. A point $p$ of the polyhedron $L(S')$ also belongs to $L(S)$ if and only if $p$ does not belong to any set $\{x : s(x) = 0\}$ with $s \in S$.

5.1.3 Coefficients of the linear functions

We say that the linear system $\text{Mat}(m, d, A, e, w, s)$ is an integer, rational, or algebraic linear system if $A \in \mathbb{Z}^{m \times d}$, $A \in \mathbb{Q}^{m \times d}$, or $A \in \mathbb{A}^{m \times d}$, respectively. Obviously, every rational linear system can be easily transformed into an equivalent integer linear system.

Theoretically, even more real numbers can be used for coefficients, by investing both a huge combinatorial and also some manual effort: introduce a new parameter for the number, solve the system with this parameter instead of the number in mind (by computer, see below), and then check manually which part of the solution contains the required constant.

5.1.4 Polynomial coefficients

We are interested in solving not just one system but a whole family of linear systems $S_p$. All coefficients $A_{i,j,p}$ of the linear functions in $S_p$ should be (maybe poly-variate) polynomials $A_{i,j} \in \mathbb{Z}[X_1, \ldots, X_k]$, more precisely $A_{i,j,p} = A_{i,j}(p)$. We are interested in the solution of all these linear systems with $p \in P$ for a given semi-algebraic set $P \subset \mathbb{R}^k$. 
5.1.5 Assumptions

We assume that we are given a \( m \times d \)-matrix \( A \) of polynomials in \( \mathbb{Z}[X_1, \ldots, X_k] \), a semi-algebraic set \( P \) (as union of solution sets of systems in \( \mathbb{R}^k \) of finitely many polynomial restrictions in \( \mathbb{Z}[X_1, \ldots, X_k] \)) and numbers \( e, w, s \in \mathbb{N}_0 \) with \( e + w + s = m \).

For all \( p \in P \) we are interested in the solution of the linear system \( S_p := \text{Mat}(m, d, A(p), e, w, s) \).

In addition we assume that the solution set (of the weakened system) \( L(\text{Mat}(m, d, A(p), e, w + s, 0)) \) is pointed for every \( p \in P \), i.e., that it does not contain any complete line. If this property cannot be guaranteed in advance, then it is possible to achieve this by inserting artificial restrictions \( x_i \geq 0 \) (thus we have to consider a lot of such systems) or by suitable projections (if the directions of contained lines are known).

If we fix the parameter, then we consider the problem of solving one concrete linear system \( \text{Mat}(m, d, A, e, w, s) \) with \( A \subset \mathbb{R}^{m \times d} \).

5.2 Solution of a linear system

In this section we will discuss the answer that we expect when solving a (parametrized) linear system. So we describe all possible answers that our algorithms should supply.

5.2.1 Admissibility

We start our discussion with Task 1.48, i.e., of deciding admissibility. For a concrete linear system \( S \) in \( \mathbb{R}^d \) the answer can only be “yes” or “no” in the sense of a boolean value. In general, we will expect both answers for a family of systems if \( p \) ranges over all \( P \), “yes” for all systems \( S_p \) with \( p \in P_{\text{yes}} \) and “no” for all \( p \in P_{\text{no}} \) with \( P_{\text{yes}} \cup P_{\text{no}} = P \).

As we will show later, the sets \( P_{\text{yes}} \) and \( P_{\text{no}} \) are semi-algebraic sets in \( \mathbb{R}^k \).

So the answer of Task 1.48 for the family \( (S_p)_{p \in P} \) of linear systems the answer describes finitely many distinct cases. The semi-algebraic set \( P_{\text{no}} \) contains all \( p \in P \) for which \( S_p \) is not admissible. Each of the remaining \( n \in \mathbb{N}_0 \) cases is described by a semi-algebraic set \( P_c \) and the solution vector \( \mathbf{r} \in \mathbb{Z}^d \) (respectively, \( A^d \)).

For the whole family \( (S_p)_{p \in P} \) of linear systems the answer describes finitely many distinct cases. The semi-algebraic set \( P_{\text{no}} \) contains all \( p \in P \) for which \( S_p \) is not admissible. Each of the remaining \( n \in \mathbb{N}_0 \) cases is described by a semi-algebraic set \( P_c \) and the solution vector \( \mathbf{r} \in \mathbb{Z}^d \), where \( c \in \mathbb{N}_n \). So the following must hold:

\[
\mathbf{r}(p) \in L(S_p) \quad \forall p \in P_c \forall c \in \mathbb{N}_n, \tag{5.6}
\]

\[
P = P_{\text{no}} \cup \bigcup_{c \in \mathbb{N}_n} P_c. \tag{5.7}
\]

5.2.2 Solution vector

Now we consider Task 1.49, i.e., finding one solution vector of the system \( S \). For concrete systems \( S \) we expect the symbolic answer “the system is not admissible” or a solution vector \( \mathbf{r} \in L(S) \). For integer linear systems (algebraic linear systems) we also expect \( \mathbf{r} \) to belong to \( \mathbb{Z}^d \) (respectively, \( A^d \)).

For the whole family \( (S_p)_{p \in P} \) of linear systems the answer describes finitely many distinct cases. The semi-algebraic set \( P_{\text{no}} \) contains all \( p \in P \) for which \( S_p \) is not admissible. Each of the remaining \( n \in \mathbb{N}_0 \) cases is described by a semi-algebraic set \( P_c \) and the solution vector \( \mathbf{r} \in \mathbb{Z}^d \), where \( c \in \mathbb{N}_n \). So the following must hold:

\[
\mathbf{r}(p) \in L(S_p) \quad \forall p \in P_c \forall c \in \mathbb{N}_n, \tag{5.6}
\]

\[
P = P_{\text{no}} \cup \bigcup_{c \in \mathbb{N}_n} P_c. \tag{5.7}
\]

5.2.3 Solution set

Now we consider Task 1.37 asking for a description of the solution set.

Polyhedral cone (without strict inequalities)

For this subsection we only consider linear systems without strict inequalities. In other words, instead of \( S = \text{Mat}(m, d, A, e, w, s, \mathbf{r}) \) we will consider the weakened system \( S' = \text{Mat}(m, d, A, e, w + \mathbf{r}) \).
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With vertices $P, Q$ which can be described in the following way which is quite standard. Enhancing the inequality $R$ originates from an inhomogeneous linear system in the origin. Assume that the homogeneous linear system $S$.

Remark 5.4 Assume that the homogeneous linear system $S$ in $\mathbb{R}^{d+1}$ without strict inequalities originates from an inhomogeneous linear system in $\mathbb{R}^d$ as described in Subsection 5.1.1 and weakening the inequality $0 > 0$ to $0 \geq 0$. Then the solution set of the original system is a polyhedron $Q$ which can be described in the following way which is quite standard.

Assume that $L(S) = \text{cone}\{\eta_1, \ldots, \eta_t\}$. Then

$$Q = \text{conv}\{p_1, \ldots, p_k\} + \text{cone}\{h_1, \ldots, h_t\},$$

with vertices $p_j = \frac{1}{h}v$ for the generators $\eta_i = \left(\begin{array}{c} h \\ v \end{array}\right)$, $v \in \mathbb{R}^d$, with $h > 0$ (w. l. o. g. for all $i = 1, \ldots, k, i = j$) and directions $h_j = v$ from these generators $\eta_i$ with $h = 0$ (w. l. o. g. for all $i = k+1, \ldots, t$, with $i = k+j$ and $l = t-k$). Note that for $k = 0$ we get $Q = \emptyset$, regardless of how large $l$ is.

There is a similar correspondence for systems involving strict inequalities.

The additional incidence matrix

For some aspects it is important to relate the basic elements of the $\mathcal{H}$-representation (defining inequalities) and the basic elements of the $\mathcal{V}$-representation (generators) of the polyhedral cone to each other. This incidence relation describes whether a generator $\eta_i$ $(i \in \mathbb{N}_n)$ belongs to the hyperplane $\mathcal{H}^0(A_j)$ or $\mathcal{H}^0(\alpha_i(p))$ defined by the corresponding equation or inequality $(j \in \mathbb{N}_m)$ or not. So we define the incidence matrix $I$ of $A$ and $(\eta_1, \ldots, \eta_t)$ as the $t \times m$ boolean matrix $I = (I_{i,j})$ where $I_{i,j}$ is true if and only if $A_j \eta_i = 0$.

If $I_{i,j}$ is false, then $A_j \eta_i > 0$, and $j > e$ must hold if $\eta_i$ is a solution of $S' = \text{Mat}(m, d, A, e, w + s, 0)$.

For each facet $f$ of $L(S')$ there is at least one $j$ describing $f$ as $f = L(S') \cap \mathcal{H}^0(A_j)$. But in general $j$ need not be unique, and not each $j \in \mathbb{N}_m$ defines a facet in this way. Using only the incidence matrix $I$ we can extract the complete face lattice, i.e., the combinatorial structure of $L(S')$.

Although this incidence matrix can be easily calculated for concrete linear systems, the same procedure becomes costlier for parametrized solutions.

Typically the partitioning $P = \bigcup P_c$ will have to be refined, to get basic parameter sets $P_c$ such that the boolean $m \times d$-matrix $I^c$ is the incidence matrix of $(p, G_i^c(p), G_2^c(p), \ldots, G_e^c(p))$ for all $p \in P_c$.

A part of the face lattice

The information $G = (\eta_1, \ldots, \eta_t), I \in \{\text{true, false}\}^{t \times m}$ now already allows to answer Task 5.1, of the original system $S$ including strict inequalities. So we have that $C := \text{cone}\{\eta_1, \ldots, \eta_t\} = L(S')$ is the solution set of the weakened system. If we have the strict inequality $A_j \eta > 0, e + w + 1 \leq j \leq m$, within the system $S$ such that $C \subset \mathcal{H}^0(A_j)$, then obviously we have $L(S) = \emptyset$. This condition is equivalent to the existence of a column $I_{s,j} \in \{\text{true, false}\}^t$ with only true values within the last $s$
columns of $I$, $m - s < j < m$. If, conversely, for each strict inequality with number $j$ there is a generator $\eta_j$, $i \in \mathbb{N}_+$, not incident with inequality $j$, i.e. $I_{ij}$ is false, then $\text{rel int} \ C \subset L(S)$ and $S$ is admissible. In fact, $L(S)$ is the union of the relative interiors of all faces $f$ of $C$ which are not contained in any hyperplane defined by a strict inequality of $\mathcal{S}$. If we identify a face $f$ of $C$ with the index set $F = F_f$ of its generators, $f = \text{cone} \{ \eta_i : i \in F_f \}$, then we get
\[
L(S') = \bigcup_F \text{rel int} \{ \eta_i : i \in F \}
\]
and
\[
L(S_p) = \bigcup_{F \text{ with } \forall j \in \mathbb{N}_+ \setminus \mathbb{N}_+ \exists i \in F : I_{ij}} \text{rel int} \{ \eta_i : i \in F \}.
\]
For both unions $F$ ranges over all faces of $C$,
\[
F \in \left( \bigcap_j \{ i \in \mathbb{N}_+ : I_{ij} \} : J \subset \mathbb{N}_+ \right).
\]
Because of this representation, we can directly use $(G, I)$ as answer to Task 5.47 regarding $S$ if $\mathcal{S}$ is admissible, and otherwise we get the symbolic answer “not admissible”.

We summarize the discussion so far by the following

**Definition 5.5** Let $c, w, s, d \in \mathbb{N}_0$ be nonnegative integers and $m := c + w + s$. For $A \in \mathbb{R}^{m \times d}$ the **full solution** of $\mathcal{S} := \text{Mat}(m, d, A, c, w, s)$ is either the symbolic value “not admissible” or the triple $(t, G, I)$ with $t \in \mathbb{N}_0$, $G \in \{ \text{true} \}^{d \times m}$, and $I \in \{ \text{true, false} \}^{\mathbb{N}_+ \times m}$. The full solution $h$ is called **valid** if it correctly describes the solution set $L(S)$.

- “not admissible” is valid if $L(S) = \emptyset$.
- $(t, G, I)$ is valid if $L(S) \neq \emptyset$, $W := L(\text{Mat}(m, d, A, c, w, s, 0)) = \text{cone} \{ G_1, G_2, \ldots, G_t \}$, $W \neq \text{cone} \{ G_1, G_2, \ldots, G_t \} \setminus \{ G_i \}$ for all $i \in \mathbb{N}_+$, $G_i \neq G_j$ for all $1 \leq i < j \leq t$, and if further $I$ is the incidence matrix of $A$ and $G$.

A **full solution** of the family of linear systems $(\mathcal{S}_p)_{p \in P}$ with $\mathcal{S}_p = \text{Mat}(m, d, \Delta(p), c, w, s)$ and $\Delta \in \mathbb{Z}[X_1, \ldots, X_k]^{m \times d}$ is the triple $(P_{no}, n, C)$ of the semi-algebraic set $P_{no}$, the number of admissible cases $n$, and the sequence $C = (P_c, t_c, G^c, I^c)_{c \in \mathbb{N}_n}$, describing these cases via the parameter set $P_c$, the number of generators $t_c$, the generator sequence $G^c \in (\mathbb{Z}[X_1, \ldots, X_k]^{d \times c})^{e}$ and the incidence matrix $I^c \in \{ \text{true, false} \}^{t_c \times m}$. This full solution is valid if

- $P = P_{no} \cup \bigcup_{c \in \mathbb{N}_n} P_c$ is correctly partitioned into semi-algebraic sets $P_c$ and $P_{no}$, and
- the evaluated full solutions $h(p)$ are valid full solutions of $\mathcal{S}_p$: for $p \in P_{no}$ we define $h(p) := \text{“not admissible”}$, and for $p \in P_c$ with $c \in \mathbb{N}_n$, we define $h(p) := (t_c, G^c(p), I^c)$.

### 5.2.4 Special cases for the influence of parameters

At this point it is clear that we have to deal with semi-algebraic sets and their representation quite often, since we need them to describe exactly a full solution of a family of parametrized linear systems.

Semi-algebraic sets can be represented as finite union of basic semi-algebraic sets. These are themselves represented as solution set of a system of polynomial equations and inequalities. The key point for us, taken from the theory of real closed fields, is that we can decide whether or not such a system is admissible, and that we can calculate projections of semi-algebraic sets. This provides a way to implement the common set operations (union, intersection, complement) and to decide the common set relations containment and equality. Finally we can construct semi-algebraic sets \{ $x$ \} if $x \in A^d$ and find some $x \in P \cap A^d$ if $\emptyset \neq P \subset \mathbb{R}^d$ for the semi-algebraic set $P$.

But actually, the author did not use an implement of semi-algebraic sets in the way described so far. Whenever it is possible to use a more adapted implementation for a given problem, we should try it. This may be the important step from designing an algorithm, which will answer a special question in finite time only theoretically, to a computer program which will really return the answer in acceptable time without stopping because of exhausted memory.
Remark 5.6 Indeed, in Chapter 6 we need the special case of linear systems depending only on
$k = 1$ real parameter. Semi-algebraic sets in $\mathbb{R}^1 = \mathbb{R}$ are easily described in a direct way as (sorted,
minimal) disjoint unions of real intervals, including the sets $\mathbb{R} = (-\infty, \infty)$, $(a, b)$, \{a\} and so on.
All finite endpoints of these intervals are algebraic numbers.

The construction of these representations from their solution set representation requires the
comparison of real algebraic numbers in the above representation, the factorization of mono-variate
polynomials, calculating the number of distinct real roots of mono-variate polynomials and the sign
of the function values between the roots. All these ingredients are provided by special libraries for
computer implementations. The author used the CORE-library \cite{12}.

Remark 5.7 We mention another very special case of semi-algebraic sets: polyhedra. More
precisely, we consider semi-algebraic sets where all defining equations and inequalities are affine linear
functions. Such a set is the finite disjoint union of the relative interiors of convex polyhedra.

Within the context of this chapter on solving linear parametric systems such sets occur within the
full solution if

- all but one column $j^*$ in the system matrix $A$ do not depend on any parameter, $A_{i,j} \in \mathbb{Z}$
  \quad $\forall i \in \mathbb{N}_m, j \in \mathbb{N}_d \setminus \{j^*\}$, and
- the column $j^*$ of $A$ only contains homogeneous linear polynomials
  \quad $A_{i,j^*}(p) = \sum_{l=1}^{k} z_{i,l} p^l, i \in \mathbb{N}_m, z_{i,l} \in \mathbb{Z} \quad (l \in \mathbb{N}_k)$.

If $S_p$ is the homogenization of a linear system $Mx \geq b$ (the symbol $\geq$ is not meant to define this
system exactly, but tells that the system is defined by component-wise relations $=, >$ or $\geq$) with
constant matrix $M \in \mathbb{Z}^{m \times d}$, and if $b \in \mathbb{Z}[X_1, \ldots, X_k]^m$ depends linearly on $X_1, \ldots, X_k$, then all
parameter sets are linear. The linear system in (4.29), which provides an answer to the special
$M_d$-embedding problem (Task 4.4, where $M_d$ has polytopal unit ball) in Theorem 4.19, has this
structure if we regard the distances of the metric $\rho$ as parameters.

5.3 Certificates

In this section we explain the concept of certificates for the answers defined in the previous section.

A certificate is some data structure such that a verification algorithm can confirm the exactness
of the answer. Thus, this verification algorithm should be simple in some sense. First, for a
computer implementation its running time and memory requirements should be moderate. Second
the mathematical correctness of the verification algorithm, and thus also for the verified answer,
should be simple to prove, or even better it should be obvious.

Thus certificates are a way to separate the solution process into two independent parts:

- First we search the answer of some task, including certificates for it. This part may last
  longer and allows to use heuristic algorithms suited exactly to the given input.
- Second we can check again if our solution is correct using the certificate. Additionally, if the
  verification algorithm is simple enough, we can check the exactness without computer, even
  though we would never try to execute the first part by hand due to its complexity.

Remark 5.8 At this point we want to stress that an automatic verification of an answer with
certificate by computer can be trusted, if the verification algorithm is

- theoretically exact for any input data (regardless of size and special cases) on some theoretical
  machine (like a Turing-machine). This means that the algorithm stops with the correct
  answer that the certificate is correct or not.
- Additionally the implementation must react appropriately if it cannot simulate the theoretical
  machine because of restricted resources. This means that the failure of the process due to
  machine limitations (typically there is not enough memory) must be reported to the user.
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Additional tests of preconditions at the beginning of sub-algorithms and also tests afterwards for postconditions are useful to find errors of the implementation. Using such techniques, we can increase the probability that the implementation really does what it theoretically should do.

Note that we give no certificates that the assumption on $L(S_p)$ to be pointed (see Subsection 5.1.3) is satisfied. All verification algorithms will not require this assumption to be true. On the other side, the existence of answers, as introduced in Section 5.2 indeed depends on this assumption. So the solution-finding algorithms may fail or produce wrong results that will not pass the verification procedure described in this section.

5.3.1 Certificates for non-admissibility

We start with certificates for the fact that $S$ is not admissible.

**Definition 5.9** A certificate $C_{\text{impl}, i^*}$ for the fact that $i^*$ is an implicit equation of $\text{Mat}(m, d, A, e, w, s)$ is a vector $C_{\text{impl}, i^*} \in \mathbb{R}^m$. This certificate is called valid if

$$
\sum_{i=1}^{m} C_{i}^{\text{impl}, i^*} A_{i, *} = 0 \quad (5.8)
$$

and

$$
C_{i}^{\text{impl}, i^*} \geq 0 \quad \forall i : e < i \leq m \quad \text{and} \quad C_{i}^{\text{impl}, i^*} > 0. \quad (5.9)
$$

A certificate $\underline{C}_{\text{impl}, i^*, P}$ for the fact that $i^*$ is an implicit equation of $\text{Mat}(m, d, A(p), e, w, s)$ for all $p \in P$ is a vector $\underline{C}_{\text{impl}, i^*, P} \in \mathbb{Z}[X_1, \ldots, X_k]^m$. It is valid if $\underline{C}_{\text{impl}, i^*, P}(p)$ is a valid certificate for the fact that $i^*$ is an implicit equation of $\text{Mat}(m, d, A(p), e, w, s)$.

**Lemma 5.10** If $C_{\text{impl}, i^*}$ is a valid certificate for the fact that $i^*$ is an implicit equation of $S = \text{Mat}(m, d, A, e, w, s)$, then $A_{i^*, *} \mathfrak{r} = 0$ is an implicit equation of $S$, i.e., $L(S) \subset \mathcal{H}^0(A_{i^*})$.

**Proof** Assume $\mathfrak{r} \in L(S)$ and that the certificate $C_{\text{impl}, i^*}$ is valid. Then we get $C_{i}^{\text{impl}, i^*} A_{i, *} \mathfrak{r} \geq 0$ for all $i \in \mathbb{N}_m$ by (5.9). With (5.8) it follows that $\sum_{i=1}^{m} C_{i}^{\text{impl}, i^*} A_{i, *} \mathfrak{r} = 0$, thus $C_{i}^{\text{impl}, i^*} A_{i, *} \mathfrak{r} = 0$ for all $i \in \mathbb{N}_m$, especially $C_{i}^{\text{impl}, i^*} A_{i^*, *} \mathfrak{r} = 0$. Since $C_{i}^{\text{impl}, i^*} \neq 0$, we get $\mathfrak{r} \in \mathcal{H}^0(A_{i^*})$. $\square$

**Definition 5.11** A certificate $C_{\text{nonAdm}}$ for the fact that $S = \text{Mat}(m, d, A, e, w, s)$ is not admissible is a pair $(i^*, C_{\text{impl}, i^*}) \in \{e + w + 1, \ldots, m\} \times \mathbb{R}^m$. It is valid if $C_{\text{impl}, i^*}$ is a valid certificate for the fact that $i^*$ is an implicit equation of $S$. A basic certificate $\underline{C}_{\text{nonAdm}, P}$ for the fact that $S_p = \text{Mat}(m, d, A(p), e, w, s)$ is not admissible for all $p \in P$ is a pair $(i^*, \underline{C}_{\text{impl}, i^*, P}) \in \{e + w + 1, \ldots, m\} \times \mathbb{Z}[X_1, \ldots, X_k]^m$. It is valid if for all $p \in P$ the evaluated certificate $\underline{C}_{\text{nonAdm}, P}(p)$ is valid for the fact that $S_p = \text{Mat}(m, d, A(p), e, w, s)$ is not admissible.

**Lemma 5.12** If $C_{\text{nonAdm}}$ is a valid certificate for the fact that $S = \text{Mat}(m, d, A, e, w, s)$ is not admissible, then $S$ is not admissible, i.e., $L(S) = \emptyset$.

**Proof** Since $i^*$ corresponds to a strict inequality by Lemma 5.10 this is a contradiction for every solution vector if would exist. $\square$

Note that it may be necessary to give a finite number of valid basic certificates $\underline{C}_{\text{nonAdm}, no^c}$, $c = 1, \ldots, x$, together with semi-algebraic parameter sets $P_{no^1}, \ldots, P_{no^x}$ and with $\bigcup_{c \in [x]} P_{no^c} = P_{no}$.

A certificate for the fact that $S$ is admissible is in fact simply a solution vector $\mathfrak{r}$. This answer itself needs no further certificate, since it can directly be verified that $\mathfrak{r} \in L(S)$.

5.3.2 Certificates for upper bounds on the dimension of the solution set

**Definition 5.13** A certificate for the statement that the solution set of $S = \text{Mat}(m, d, A, e, w, s)$ is at most $x$-dimensional is a set $C_{\text{dim} \leq x}$ of cardinality $d - x$ whose elements are numbers $i \in \mathbb{N}_e$ or pairs $(i, C_{\text{impl}}) \in \{e + 1, \ldots, e + w\} \times \mathbb{R}^m$. Furthermore, $A_{\text{dim} \leq x}$ is the submatrix of $A$ formed by the $d - x$ rows whose indices $i$ occur in $C_{\text{dim} \leq x}$. This certificate is valid if
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• \( A_{\dim \leq x} \) has full column rank, \( \text{rank}(A_{\dim \leq x}) = d - x \), and

• for each \((i^*, C_{\text{impl}, i^*}) \in C_{\dim \leq x}\) the certificate \( C_{\text{impl}, i^*} \) is valid for the fact that \( i^* \) is an implicit equation of \( S \).

Again, substituting all real numbers by polynomials yields the definition of a basic certificate of the fact that the solution set of \( S_p \) is at most \( x \)-dimensional for all \( p \in P \) as a set \( C_{\dim \leq x, p} \) of \( d - x \) numbers in \( \mathbb{N} \), and pairs \((i, C_{\text{impl}, i}) \in \{e + 1, \ldots, e + w\} \times \mathbb{Z}[X_1, \ldots, X_k]^m \). \( C_{\dim \leq x, p} \) is valid if \( C_{\dim \leq x, p}(p) \) is valid for the fact \( \dim L(S_p) \leq x \) for all \( p \in P \).

Note that the numbers \( i \) are indices of explicit and implicit equations of the system \( S_p \) whose solution set is a linear subspace of dimension \( x \) in \( \mathbb{R}^d \).

Again, standard arguments from linear algebra yield

Lemma 5.14 If \( C_{\dim \leq x} \) is a valid certificate for the statement that the solution set of \( S \) is at most \( x \)-dimensional, then \( \dim L(S) \leq x \) is true.

In general, this certificate is not unique.

5.3.3 Certificates for lower bounds on the dimension of the solution set

A certificate for the statement that \( \dim L(S) \geq x \geq 0 \) is a set \( C_{\dim \leq x} \subset \mathbb{R}^m \) of \( x \) linearly independent solution vectors.

It is valid if for all \( \mathbf{r} \in C_{\dim \leq x} \) we have that \( \mathbf{r}(p) \in L(S) \) and the matrix \( G \in \mathbb{R}^{d \times x} \), whose columns are formed by all \( \mathbf{r} \in C_{\dim \leq x} \), has full column rank, i.e., \( \text{rank} G = x \).

Remark 5.15 The full solution \((t, G, I)\) of \( S \) contains enough information such that it can be used as certificate to verify \( \dim L(S) \geq x \) (if \( x = \dim L(S) \) is the correct dimension) without a need to compute the rank of a matrix formed by some generators. Instead of this, it is sufficient to check that the generators are really solution vectors, that \( I \) is the incidence matrix of \( A \) and \( G \), and that \( I \) has some combinatorial properties (see below). The analogous statement for an upper bound derived from a full solution with verified incidence relations is not true.

5.3.4 Certificate for full solution

A certificate for the fact that \((t, G, I)\) is a valid full solution of \( S \) is simply a certificate for an (sharp) upper bound on the dimension of the solution set. This is sufficient since the incidence matrix encodes a lot of information. Of course, an incidence matrix has some special properties. Thus the verification algorithm has to check these properties, including the dimension of corresponding solution set.

Definition 5.16 We say that a boolean matrix \( I \) is the incidence matrix of an \( x \)-dimensional polyhedral cone if there is a linear system \( S \) with valid full solution \((t, G, I)\) according to Definition 5.5 (for appropriate \( t, G \)) and with \( \dim L(S) = x \), and where \( L(S) \) is pointed.

Now we give another definition for the same concept of the structure of the incidence matrix, but based on some combinatorial, recursive property.

Definition 5.17 We say that a boolean matrix \( I \in \{\text{true, false}\}^{t \times m} \) has the recursive \( x \)-incidence property, if

• \( I \) has no rows, \( t = 0 \), in case of \( x = 0 \), and

• \( I \) has exactly one row, \( t = 1 \), and there is at least one false value, in case of \( x = 1 \),

• in case \( x > 1 \) there is a non-empty set \( R \subset \mathbb{N}_m \) of columns such that the corresponding submatrix \( I' := I_{*,R} \) of \( I \) satisfies the following.
• Each row of I' contains at least one true value and each column of I' contains at least one false value.
• For every column c ∈ N|R| of I', the submatrix K of I' containing exactly all rows i incident with c (i.e., where I',c is true) has the recursive (x − 1)-incidence property.

Lemma 5.18 If I ∈ \{true, false\}^{t×m} is the incidence matrix of an x-dimensional polyhedral cone, then it has the recursive x-incidence property. A suitable subset R ⊂ Nm of columns of I (and in the same way for all matrices which are accessed recursively) can be constructed as follows. We identify each column of I with the set of incident row indices. Then we consider the partial order ⊆ in the set of these columns of I which have at least one value false. Taking now the indices of all maximal such columns as R will do.

Proof Consider inductively all facets of the x-dimensional solution set L(S).

Note that we can reduce R in such a way that it contains exactly one index for every set of identical (non-maximal non-trivial) rows of I.

The converse statement of Lemma 5.18 is not true in general. But together with some prerequisites relating I to generators and the upper bound on the dimension we get the statement in the desired direction, see Theorem 5.20 below.

Definition 5.19 A certificate for the statement that h is a valid full solution of \( S = \text{Mat}(m, d, A, e, w, s) \) is either a certificate \( \text{FullSol}^h = \text{C}_{\text{nonAdm}} \) for the statement that \( S \) is not admissible, or it is a pair \( \text{FullSol}^h = (x, C_{\dim\leq x}) \) of the actual dimension \( x \in \mathbb{N}^0 \) and a certificate for the statement that the solution set of \( S \) is at most x-dimensional. In case \( \text{FullSol}^h = \text{C}_{\text{nonAdm}} \) this certificate is valid if it is valid for non-admissibility and if \( h \) = “not admissible”. The certificate \( \text{FullSol}^h = (x, C_{\dim\leq x}) \) is valid if

- \( h = (t, G, I) \) with \( t \in \mathbb{N}^0, G \in (\mathbb{R}^d)^l, I \in \{\text{true, false}\}^{l×m} \),
- the certificate \( C_{\dim\leq x} \) is valid for the statement \( \dim L(S) \leq x \),
- all generators \( G_i \) (\( i \in \mathbb{N}_t \)) solve the weakened system: \( A_j G_i \geq 0 \) for all \( j \in \mathbb{N}_m \), and \( A_j G_i = 0 \) if \( j \in \mathbb{N}_e \),
- \( I \) is the incidence matrix of \( A \) and \( G \), i.e., \( A_j G_i = 0 \) if and only if \( I_{i,j} \) is true (for all \( i \in \mathbb{N}_t, j \in \mathbb{N}_m \)),
- no strict inequality is an implicit equation, i.e., the \( (e + w + 1) \)-rd to m-th column of \( I \) has at least one value false, and
- \( I \) has the recursive x-incidence property.

A certificate for the statement that \( h \) is a valid full solution of the family of systems \( \text{Mat}(m, d, A, e, w, s, P) \) is a sequence \( \{\text{FullSol}^h, p\} \) where \( p \) is in \( \mathbb{N}^0 \) and \( \text{FullSol}^h, p \) is valid if \( \text{FullSol}^h \) is a valid certificate for the statement that \( h \) is a valid full solution of \( S \)

\[
\text{Theorem 5.20} \quad \text{If } \text{FullSol}^h \text{ is a valid certificate for the statement that } h \text{ is a valid full solution of } S, \text{ then } h \text{ is a valid full solution of } S.
\]

If \( \text{FullSol}^h, p \) is a valid certificate for the statement that \( h \) is a valid full solution of the family of systems \( (S_p)_{p \in P} \), then \( h \) is a valid full solution of the family \( (S_p)_{p \in P} \).

Proof We assume that \( \text{FullSol}^h = (x, C_{\dim\leq x}) \) is a valid certificate for the full solution \( h = (t, G, I) \). It is sufficient to show that \( h \) is a valid full solution of \( S = \text{Mat}(m, d, A, e, w, s) \). This follows if we can show that \( W := L(\text{Mat}(m, d, A, e, w + s, 0)) = \text{cone}\{G_1, G_2, \ldots, G_t\} \) and that all generators \( G_1, \ldots, G_t \) represent distinct extremal rays of \( W \).
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We already know that \( C := \text{cone}\{G_i : i \in \mathbb{N}_1\} \subset W \), that \( \dim W \leq x \) (by Lemma 5.14) and that \( I \) is the incidence matrix of \( A \) and \( G \). For \( x \leq 1 \) we get \( t = x \) by Definition 5.17 and the claim follows easily.

Let \( R \) be the set of columns of \( I \) as in Definition 5.17. We define the cone \( W' \) by the inequalities of \( S \) corresponding to \( R \), and all implicit and explicit equations of \( S \) in the following way: \( W' := \text{lin} W \cap \bigcap_{i \in R} H^{0+}(A_i) \). So we still have \( C \subset W \subset W' \), \( \dim W' \leq x \). Additionally, we know that for every facet \( f \) of \( W' \) there is some \( i \in R \) with \( f = W' \cap H^0(A_i) \).

A simple induction argument, based on the fact that \( I' = I_{-,R} \) contains at least one false value in every column, yields that \( W \) is pointed. Consequently, \( W' \) and is the conical hull of all its facets. We can now show inductively that \( W' \subset C \) and that all generators represent distinct extremal rays. Finally we get \( C = W \) by the chain \( C \subset W' \subset C \).

The theorem follows now from Lemma 5.12.

\[ \square \]

**Theorem 5.21** Each polynomial family of linear systems \( (S_p)_{p \in P} \) with \( S_p = \text{Mat}(m,d,A(p),e,w,s) \) has a valid full solution and a corresponding valid certificate \( C^\text{FullSol,}A,P \) for it provided that the solution set \( L(S_p) \) is pointed for every \( p \in P \).

**Proof** Follows from the theory of polyhedra for every concrete linear system \( S_p \). In general, only the certificate for an implicit equation can occur in infinitely many possibilities. These certificates itself belong to a polyhedral cone for some dual system. If we only consider extremal such certificates, then there are only finitely many combinatorial possibilities. See also Algorithm 5.27.

\[ \square \]

5.4 Algorithm

Within this section we will shortly explain how parametrized linear systems can be solved, i.e., how to compute a valid full solution and valid certificates to proof its correctness.

We will not discuss how concrete integer or even algebraic systems can be solved efficiently. In fact, we will use such algorithms to solve sub-tasks. Such algorithms are quite standard now, especially for rational systems. For integer linear systems the author relied on polymake [18]. For algebraic systems an adaption of cddlib [17] for real algebraic numbers provided by CORE [12] was used.

Polynomials with exact integer coefficients can be represented by a finite number of coefficients. We used the CORE library support for mono-variate polynomials.

5.4.1 Verification algorithms

All certificates introduced in the previous section can be verified in a straight forward manner, based on simple combinatorial tests and more involved tests on sign conditions \( q(p) > 0 \) or \( q(p) = 0 \) of polyvariate polynomials \( q \in \mathbb{Z}[X_1,\ldots,X_k] \) for all arguments \( p \) within some given semi-algebraic set \( P \).

We note that the verification that a matrix \( A \) has full rank also can be reduced to conditions \( q(p) = 0 \) for some polynomials \( q \) which are determinants of submatrices of \( A \).

Finally, the verification procedure has to perform set operations and check set relations for semi-algebraic sets.

**Remark 5.22** We note that these verification algorithms can easily be modified to calculate maximal parameter sets where the evaluated certificate is valid. More precisely, given the basic certificate \( D = C^\text{monAdm,Q} \) for the fact that \( S_p \) is not admissible for all \( p \in Q \), the validity algorithm determines the semi-algebraic set \( P = P(D) \) of all those parameter values \( p \in \mathbb{R}^k \) for which \( D(p) \) is a valid certificate for the fact that “not admissible” is a valid full solution of \( S_p \). Then \( D \) is valid if and only if \( Q \subset P \). \( D \) is valid as basic certificate for the statement that \( S_p \) is not admissible for all \( p \in P \).
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We could also say that the sequence of one pair, (\((P, D)\)), is a valid certificate for the statement that \((P, 0, (\cdot))\) is the full solution of the family \((S_p)_{p \in P}\).

For the second type, the pair \(D = (x, C_{\dim < x} Q)\) of \(x \in \mathbb{N}^0\) and a basic certificate \(C_{\dim < x} Q\) for the fact that \(\dim L(S_p) \leq x\) for all \(p \in P\), together with \(t \in \mathbb{N}^0\), \(G \in (\mathbb{Z}[X_1, \ldots, X_k]^d)^t\), \(I \in \{\text{true, false}\}^{t \times m}\), we can determine in the same way the semi-algebraic set \(P = P(D)\) of all \(p \in \mathbb{R}^k\) for which \(D(p)\) is a valid certificate for the fact that \((t, G(p), I)\) is a valid full solution of \(S_p\).

We could also say that the sequence of one pair, \((\langle P, D \rangle)\), is a valid certificate for the statement that \((\emptyset, 1, (P, t, G, I))\) is a full solution of the family \((S_p)_{p \in P}\).

5.4.2 Full search

We start with some simple algorithm which is not meant to be practical. But it shows that the full solution can be found in finite time.

Before we do that, we will discuss the solution of a homogeneous system of \((d - 1)\) linear equations in \(\mathbb{R}^d\).

**Definition 5.23** Let \(R\) be a commutative ring, \(R = \mathbb{R}\) or \(R = \mathbb{Z}[X_1, \ldots, X_k]\). The generalized cross product \(\wedge (A_1, \ldots, A_{d-1})\) of \(d - 1\) row vectors \(A_i \in \mathbb{R}^d\) \((i \in \mathbb{N}_{d-1})\) is another vector \(\sigma \in \mathbb{R}^d\) defined as the formal determinant

\[
\wedge (A_1, \ldots, A_{d-1}) := \det \begin{pmatrix} A_{1,1} & \cdots & A_{1,d} \\ \vdots & \ddots & \vdots \\ A_{d-1,1} & \cdots & A_{d-1,d} \\ \sigma_1 & \cdots & \sigma_d \end{pmatrix},
\]

where \(\sigma_1, \ldots, \sigma_d\) are the standard unit vectors in \(\mathbb{R}^d\) and the determinant must be expanded using Laplace’s formula along the last row. If \(A\) denotes the \((d - 1) \times d\) matrix with rows \(A_1, \ldots, A_{d-1}\), we also write \(\wedge A\) for \(\wedge (A_1, \ldots, A_{d-1})\).

From standard linear algebra we know the following.

**Lemma 5.24** Consider the set \(S := \{ \sigma \in \mathbb{R}^d : A\sigma = 0 \}\), for a matrix \(A \in \mathbb{R}^{(d-1) \times d}\). Then we always have \(\dim S \geq 1\) and \(\dim S = 1\) if and only if \(\wedge A \neq 0\) if and only if \(\operatorname{rank} A = d - 1\). For \(\dim S = 1\) we additionally have that \(S = \mathbb{R} : \wedge A\).

Note that Lemma 5.24 can be seen as Cramer’s rule for homogeneous systems.

**Algorithm 5.25** Input: A polynomial family of linear systems \((S_p)_{p \in P}\) with \(S_p = \operatorname{Mat}(m, d, A(p), e, w, s)\), via its matrix \(A \in \mathbb{Z}[X_1, \ldots, X_k]^{m \times d}\) and numbers \(e, w, s, d \in \mathbb{N}^0\) with \(m := e + w + s\) and the semi-algebraic set \(P \subset \mathbb{R}^d\).

Output: A valid full solution \((P_{\text{iso}}, n, C)\) of the family \((S_p)_{p \in P}\).

Iterate through each set \(E \subset \mathbb{N}_m\) of \((d - 1)\) indices of rows of \(A\), and do the following

- Calculate the generalized cross product \(\sigma_E = \wedge A_E \in \mathbb{Z}[X_1, \ldots, X_k]^d\), where \(A_E\) is the \((d - 1) \times d\)-matrix of all rows of \(A\) whose index belongs to \(E\).
- Check where \(\sigma_E(p)\) and \(-\sigma_E(p)\) are solutions of \(S'_p = \operatorname{Mat}(m, d, A(p), e, w + s, 0)\) by calculating for \(\sigma = \pm 1\)
  \[
  S(\sigma_E) := P \cap \bigcap_{i=1}^e \{ p : A_i(p)\sigma_E(p) = 0 \} \cap \bigcap_{i=e+1}^m \{ p : \sigma A_i(p)\sigma_E(p) \geq 0 \}
  \]
- Calculate for all \(i \in \mathbb{N}_m\) the incidence relation set \(I(E, i) = \{ p : A_i(p)\sigma_E(p) = 0 \}\).
Now we can calculate
\[ G := \{ \sigma \tau_E : \sigma \in \{ \pm 1 \}, S(\sigma \tau_E) \neq \emptyset, E \subset \mathbb{N}_m, |E| = d - 1 \}, \]
as well as the finite set \( \mathcal{F} \) of semi-algebraic sets
\[ \mathcal{F} := \{ S(\tau) : \tau \in G \} \cup \{ I(E, i) : E \subset \mathbb{N}_m, |E| = d - 1, i \in \mathbb{N}_m \}. \]

Next we partition the set \( P \) such that all (non-empty) sets in \( \mathcal{F} \) are unions of basic sets of the partition. We calculate the set of all intersections \( \mathcal{D} := \{ \bigcap_{\tau \in \mathcal{F}} F : \mathcal{F} \subset \mathcal{F} \} \) and the set of its minimal elements \( \mathcal{B} := \{ F \in \mathcal{D} : F \neq \emptyset, \forall F_2 \in \mathcal{D} \setminus \{ F \} : F_2 \not\subset F \}. \)

We initialize the variables \( P_{\text{no}} \) and \( C \) as empty sets. We iterate through all sets \( P' \in \mathcal{B} \) and do the following:

- Form the candidate set of generators and assign numbers to it \( G' := \{ \tau \in G : P' \subset S(\tau) \} =: \{ \tau_1, \ldots, \tau_h \}. \)
- Form the candidate incidence matrix \( I := (I_{i,j})_{i \in \mathbb{N}_h, j \in \mathbb{N}_m} = (A_j(\tau)_{i,j} = 0\forall p \in P')_{i,j} \), i.e., for \( \tau = \pm \tau_E \) we set \( I_{i,j} = I(E,j) \).
- Check if there is some strict inequality which is an implicit equation, i.e., if one of the \((e + w + 1)\)-st to \(m\)-th column of \( I' \) has only values true. If this is the case, \( P_{\text{no}} \) is replaced by \( P_{\text{no}} \cup P' \) and the remaining steps for the current \( P' \) will be skipped.
- Assign a new integer \( c \) (starting from 1) to this case via \( P_c := P' \).
- Find an index set \( G'' \) of generators which are distinct and not zero with respect to the incidence relation, \( G'' := \{ i \in \mathbb{N}_h : I_i \neq \{ \text{true} \}^m, \forall 1 \leq j < i : I_i \neq I_j, \} =: \{ i_1, \ldots, i_{t_c} \}. \) Thus \( t_c \) denotes \( |G''| \), and \( I_{i_c}, I_{j_c} \) denote the \( i \)-th and \( j \)-th row of \( I \), respectively.
- Set \( G' := (\tau_{i_1}, \ldots, \tau_{i_{t_c}}) \) and \( I' := (I_{i,j})_{i \in \mathbb{N}_{t_c}, j \in \mathbb{N}_m} \).
- Insert \( (P, t_c, G', I') \) into \( C \).

Now \( (P_{\text{no}}, |C|, C) \) is the output of the algorithm.

**Lemma 5.26** Algorithm 5.25 yields a valid full solution for all parameter values where the matrix \( A \) has full column rank (i.e., where all rows of \( A(p) \) linearly span \( \mathbb{R}^d \)).

For parameter values where the system has a solution, the rank condition is equivalent to the condition that the solution set is pointed, i.e., there is no (affine) line all whose points are solutions of the given system.

**Proof** Follows from the theory of polyhedra: each pointed polyhedral cone is the conical hull of its generators. Each generator \( \eta \) itself belongs to a one-dimensional intersection of facet hyperplanes. By Lemma 5.24 we get that \( \eta = \lambda \bigwedge \mathcal{A}_E(p) \) for some \( E \subset \mathbb{R}^m, |E| = d - 1 \), and \( \lambda \in \mathbb{R} \). If \( \eta \) belongs to every defining hyperplane, then we get \( \eta = 0 \) provided that \( \mathcal{L}(S) \) is pointed. Each nonzero vector belonging to the intersection of \( d - 1 \) hyperplanes and satisfying all weakened equations and inequalities is a generator. Two generators belong to the same ray if and only if they have identical incidences to all defining hyperplanes. \( \square \)

The same strategy can be used to find certificates, which are essentially certificates for implicit equations.

**Algorithm 5.27** Input: A polynomial family of linear systems \( (S_p)_{p \in P} \) as in Algorithm 5.25 and the output \( \mathcal{h} = (p_{\text{no}}, n, C) \) of Algorithm 5.25.

Output: Certificate \( C_{\text{FullSol}}(\mathcal{h}, P) \) for the fact that \( \mathcal{h} \) is a full solution of \( (S_p)_{p \in P} \).

Iterate through each inequality index \( i^* \) with \( e < i^* \leq m \) and each set \( E = \{ E_1, \ldots, E_d \} \subset \mathbb{N}_m \setminus \{ i^* \} \) of \( d \) additional row indices of \( A \).
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• Calculate a solution \( z \in \mathbb{Z}[X_1, \ldots, X_k]^{d+1} \) of the (dual) linear system \( zA_{e,E} = 0 \) via

\[
\begin{align*}
\mathcal{I}_E, E := \bigwedge A_{e,E}.
\end{align*}
\]

Here \( A_{e,E} \) denotes the \((d+1) \times d\) matrix of rows wit numbers \( i^*, E_1, \ldots, E_d \) of \( A_e \), of which we consider the transposed matrix.

• Check where for \( z := \pm \mathcal{I}_E, E \) the certificate for an implicit equation \( i^* \)

\[
\mathcal{C}_z^{\text{impl}, i^*} \in \mathbb{Z}[X_1, \ldots, X_k]^{m}.
\]

\[
\mathcal{C}_z^{\text{impl}, i^*} := \begin{cases}
\mathcal{I}_1 & \text{if } j = i^* \\
\mathcal{I}_{j+1} & \text{if } j = E_i \\
0 & \text{otherwise},
\end{cases}
\]

is valid:

\[
S(z) := \{ p \in P : \mathcal{I}_1(p) > 0 \land \forall i \in \mathbb{N}_d : E_i > e \Rightarrow \mathcal{I}_{j+1}(p) \geq 0 \}.
\]

Initialize \( \mathcal{C}^{\text{FullSol}, h, P} \) to an empty sequence. Next refine the partitioning \( P = P_{\text{no}} \bigcup \bigcup_{p \in P_c} P_c \) with respect to all sets \( S(z) \). For every basic parameter set \( Q \subset P_{\text{no}} \), i.e., where \( S_p \) is not admissible for \( p \in Q \), we can choose a valid \( \mathcal{C}_z^{\text{impl}, i^*} \), with \( i^* \in \mathbb{N}_m \), \( i^* > e + w \), to form a basic certificate \( D := \mathcal{C}^{\text{nonAdm}, Q} := (i^*, \mathcal{C}_z^{\text{impl}, i^*}) \) for this fact, and append \( (Q, D) \) to the list \( \mathcal{C}^{\text{FullSol}, h, P} \).

After possibly further refinements of the partitioning of \( P \) with respect to the rank of all submatrices of up to \( d \) rows of \( A(p) \), we can find for each basic parameter set \( Q \not\subset P_{\text{no}} \) with corresponding solution \((P_c, i^*, C^*, \bar{P}^*)\), i.e., \( Q \subset P_c \), a maximal set of \( y \) linearly independent explicit and implicit equations in \( A \). Defining \( x := d - y \) we can form \( D := (x, \mathcal{C}^{\text{lim}, x, Q}) \), where \( \mathcal{C}^{\text{lim}, x, Q} \) is the set representing these equations, containing the index \( i \in \mathbb{N}_e \) for chosen explicit equations, and containing the pair \((i, \mathcal{C}_z^{\text{impl}, i^*})\) for chosen implicit equations determined in the first phase. Again we append \( (Q, D) \) to the list \( \mathcal{C}^{\text{FullSol}, h, P} \).

**Lemma 5.28** Algorithm \([5.27]\) yields a valid certificate \( \mathcal{C}^{\text{FullSol}, h, P} \) for the fact that \( h \) is a full solution of the system \( (S_p)_{p \in P} \), provided that the matrix \( A(p) \) has full column rank for all \( p \in P \).

**Proof** If \( i^* > e \) is an implicit equation of \( S_p \), then the linear system \( x \in L(S_p), A_e(x) \geq 1 \) has no solution. By a variant of Farkas' Lemma, the linear system \( A_i^3 = 0, A_i \geq 1, A_i \geq 0 \) for all \( i \in \mathbb{N}_m \setminus \mathbb{N}_e \) (for \( A := A(p) \)) has a solution \( z \in \mathbb{R}^m \). Replacing each equation \( A_i x \geq 0 \) by two inequalities, \( A_i x \geq 0 \) and \( -A_i x \geq 0 \), we get a solution \( x' \in \mathbb{R}^{m+1} \) of \( A'' x' = 0, x' \geq 0, x' \geq 1, \)

\[
\begin{bmatrix}
A'' \\
-A''
\end{bmatrix} \mathcal{J} .
\]

With \( B := \begin{bmatrix} A'' & -A'' \end{bmatrix} \in \mathbb{R}^{(2d+m+e) \times (m+e)} \) this becomes \( B^T x' \geq 0, x' > 0 \)

(\( I_{m+e} \) denotes the identity-matrix of size \( m + e \), \( e \) is the \( i \)-th unit vector in \( \mathbb{R}^{m+e} \). \( B \) has full column rank. Thus the cone \( C := \{ x' \in \mathbb{R}^{m+e} : B^T x' \geq 0 \} \) is the conical hull of its generators \( g \).

All generators can be chosen as \( g = \pm \bigwedge B_F \), where \( B_F \) is a matrix consisting of \( m + e - 1 \) linearly independent rows of \( B,F \subset \mathbb{N}_{2d+m+e} \). At least one generator \( g \) satisfies \( g \geq 0 \), since otherwise \( z_{i^*} = 0 \) for all \( i \in C \). We fix one such generator \( g = \pm \bigwedge B_F \in \mathbb{R}^{m+e} \).

Consider the vector \( \eta \in \mathbb{R}^m \) with \( \eta := g_i \) if \( i > e \) and \( \eta := g_{i+m} \) for \( i \leq e \). By the definition of \( A' \), we have \( \sum_{i \in \mathbb{N}_m} \eta_i A_i = A'' \eta = A'' g = 0 \). For all \( i \in \mathbb{N}_m \setminus \mathbb{N}_e \) it is \( \eta_i \geq 0 \) and also \( \eta_i > 0 \). Thus \( \eta \) is a valid certificate for the fact that \( i^* \) is an implicit equation of \( S_p \).

The set \( I := \{ i \in \mathbb{N}_m : 2d + i \notin F \lor (i \leq e \land 2d + i + m \notin F) \} \) contains at least the indices of all non-zero elements from \( \eta \). Furthermore, \( A_j \) has rank at least \( |I| - 1 \). So by Lemma \([5.29]\) we can extend the set \( I \) to a set \( E \cup \{i^*\} \) of \( d + 1 \) row indices of \( A \) such that \( \eta = \mathcal{C}_A^{\text{impl}, i^*} \), where \( \lambda > 0 \) and \( \mathcal{C}_z^{\text{impl}, i^*} \in \mathbb{Z}[X_1, \ldots, X_k]^m \) is defined by \([5.10]\) with \( z := \pm \mathcal{I}_E, i^* \) (\( \pm \) chosen appropriately).

Thus we get a valid certificate \( \pm \mathcal{I}_E, i^*(p) \) for every implicit inequality \( i^* \). The claim follows. \( \square \)

**Lemma 5.29** Assume that \( A' \eta = 0, \eta \in \mathbb{R}^m, A \in \mathbb{R}^{m \times d} \), rank \( A = d \), and that \( \eta_i = 0 \) for all \( i \in \mathbb{N}_m \setminus I \), and that \( A_{j, i} \) has almost full row rank, rank \( A_{j, i} \geq |I| - 1 \). Then there is some \( E \supset I \), \( E \subset \mathbb{N}_m \), with \( |E| = d + 1 \) and rank \( A_{E, i} = d \). For each such \( E \) there is some real \( \lambda \) with \( \eta = \lambda \eta' \), where \( \eta' \) is \( z := \bigwedge A_{E, i} \in \mathbb{R}^m \) mapped into \( \mathbb{R}^m \) by \( \eta' = z \) and \( \eta'_{|\mathbb{N}_m \setminus E} = 0 \).

**Proof** The existence of \( E \) follows from our assumption that rank \( A = d \). The linear subspace \( L := \{ \eta \in \mathbb{R}^m : A' \eta = 0, \eta_i = 0 \forall i \in \mathbb{N}_m \setminus I \} \) of \( \mathbb{R}^m \) has dimension 1 and obviously contains \( \eta' \neq 0 \). \( \square \)
5.4.3 Instantiation and generalizing

For practical situations, Algorithms 5.25 and 5.27 are much too slow. On the other hand they can be quite usable if we have an oracle telling us the “right” sets of indices defining generators or implicit equations. We can stop to search further such sets if the verification process of our certificates so far is successful.

We obtain such an oracle, telling us the complete structure of the full solution of one basic parameter set, by computing the full solution of a linear system without parameters, obtained by evaluating the parametric system at special (if possible: at rational) values.

Algorithm 5.30 Input: A polynomial family of linear systems \((S_p)_{p \in P}\) with 
\[
S_p = \text{Mat}(m, d, A(p), e, w, s).
\]
Output: Full solution \(\overline{h}\) of the system and a certificate \(\underline{h}^{\text{FullSol}} \in P\) for the fact that \(\overline{h}\) is a full solution of \((S_p)_{p \in P}\).

We initialize the set of open parameters \(P_0 := P\). As long as \(P_0 \neq \emptyset\) we choose some \(p^* \in P\). Here we prefer values \(p^* \in \text{int} P \cap \mathbb{Q}^k\). We can find such rational points if \(P_0\) has full dimension \(k\). Otherwise we can find at least algebraic parameters, \(p^* \in P_0 \cap \mathbb{A}^k\). Now we evaluate all the polynomials of the system at \(p^*\) and get a system \(S := \text{Mat}(m, d, A(p^*), e, w, s)\) with rational or with algebraic coefficients, without parameters.

For this system we use classical methods to find a full solution \(h\) of \(S\) and a corresponding certificate \(C^{\text{FullSol}, h}\). If \(S\) is admissible, we need to compute a complete description of the solution set (generators and incidence matrix) and maybe some certificates for implicit equations. These certificates, one of which is also necessary if \(S\) is not admissible, correspond to one (extremal) solution of a corresponding dual system (see the proof of Lemma 5.28 for details).

Then we generalize this concrete solution \(h\) of \(S_p\) and its certificate \(C^{\text{FullSol}, h}\) to a basic polynomial solutions and certificates. After this we determine the parameter set where this combination is valid and update our answer.

First consider how to generalize a full solution \(h = (t, G, I)\) with \(t \in \mathbb{N}^d\), \(G \in (\mathbb{R}^d)^I\) and 
\[
I \in \{\text{true}, \text{false}\}^{1 \times m}, \ t \text{ and } I \text{ stays the same, but for } G = (\eta_1, \ldots, \eta_n) \text{ we have to find polynomial generalizations. For each generator } \eta_i, \ i \in \mathbb{N}_n, \ \text{there are at least } d - 1 \ \text{linearly independent rows of } A \ \text{incident with it. We choose one subset } E \ \text{of } d - 1 \ \text{incident row indices which are linearly independent in } A(p^*). \ \text{Choosing the correct sign, we arrive at an algebraic polynomial}\]
\[
g := \pm A_E \in \mathbb{Z}[X_1, \ldots, X_k]^d, \ \text{with } \eta_i = \lambda g_i(p) \text{ for some real } \lambda > 0 \text{ (nice point to check our implementation). Together with the (at this moment unknown) parameter set } P \text{ the generalized full solution is } (Q, t, (g_i)_{i \in \mathbb{N}_n, I}).
\]

The corresponding certificate \(C^{\text{FullSol}, h}\) will be generalized to 
\[
D := (x, C_{\text{dim} < x, Q}),
\]
where for \(i \in \mathbb{N}_n\) we have 
\[
i \in C_{\text{dim} < x, Q} \text{ if and only if } i \in C_{\text{dim} < x}. \text{ But for } (i, C^{\text{impl}, i}) \in C_{\text{dim} < x} \text{ with } i \in \{e + 1, \ldots, e + w\} \text{ and } C^{\text{impl}, i} \in \mathbb{R}^m \text{ we have to generalize the certificate } C^{\text{impl}, i} \text{ that } i \text{ is an implicit equation of } S. \text{ As noted earlier, we rely on the fact that } C^{\text{impl}, i} \text{ is an extremal such certificate. Let } E := \{j \in \mathbb{N}_n : C^{\text{impl}, j} \neq 0\} \text{ denote the indices of non-zero entries in } C^{\text{impl}, i}. \text{ Then we can assume that } A(p^*) \text{ has rank } |E| - 1. \text{ If } |E| < d + 1, \text{ then we add further rows indices to } E \text{ such that the corresponding submatrix } A_E \text{ still has almost full row-rank. As in Algorithm 5.27, we can now calculate an polynomial certificate } C^{\text{impl}, i} : \ \text{embedding } \sigma \bigwedge A_i = \mathbb{Z}[X_1, \ldots, X_k]^{d+1} \text{ correctly into } \mathbb{Z}[X_1, \ldots, X_k]^{m}, \text{ but choose the sign } \sigma = \pm 1 \text{ so that } C^{\text{impl}, i} = \lambda C^{\text{impl}, i}(p^*) \text{ for some } \lambda > 0. \text{ Then we put } (i, C^{\text{impl}, i}) \text{ into } C_{\text{dim} < x, Q}.
\]

The same has be done if \(S\) was not admissible, the full solution \(h = \text{“not admissible”}\) tells us to put \(Q\), which comes from the generalized certificate, into a union defining \(P_{\text{no}}\). Its certificate \(C^{\text{FullSol}, h} = C^{\text{nonAdm}} = (i^* : C^{\text{impl}, i^*}) \in \{e + w + 1, \ldots, m\} \times \mathbb{R}^m\) is generalized as described above into 
\[
D := (x, C^{\text{nonAdm}, Q}) \text{ with } C^{\text{nonAdm}, Q} = (i^* : C^{\text{impl}, i^*}), \text{ where } C^{\text{impl}, i^*} \text{ generalizes } C^{\text{impl}, i^*}.
\]

The last step is to compute the set \(Q\) of parameters for which our certificate are valid, see Remark 5.22. Here we must get \(p^* \in Q\). Note that \(\{p^*\} = Q\) is possible.

Finally we update \(P_o\) with \(P_o \setminus P_n\) and start from the beginning, unless \(P_o\) is empty now.

\footnote{We can give one dual system for each possible implicit equality and a fixed set \(E\) of dimension \(d + 1\) or one dual system to find just one contradiction of dimension \(m\), with \(y_0 \geq 0\) for \(e + 1 \leq i \leq m\) and \(\sum_{i=e+1}^{m} y_i = 1\).}
Note that a variant of Algorithm 5.30 can decide admissibility with the same idea in a faster way than determining the full solution.

Remark 5.31 In practice it turned out to be quite awful to calculate the systems with algebraic numbers, because there was not enough memory. On the other side, there are cases which only consist of real algebraic numbers. So it seems that it cannot be avoided by this approach. But there is a chance to avoid some of these cases, by generalizing neighborly solutions to its boundary. This makes sense as long as there are just changes to the incidence matrix with respect to non-facets.

5.5 Further notes

First we note that the question of admissibility for every polynomial system with polynomial parameters can be answered by giving two semi-algebraic sets \( T, N \), such that \( S(p) \) is admissible if \( p \in T \) and not admissible if \( p \in N \). This follows directly from the theory of semi-algebraic sets, since \( T \) is just a projection of the semi-algebraic set of all solution pairs \( (p, \tau) \) and \( N \) its complement (in \( P \)).

More discussion on semi-algebraic sets can be found in [8] and [3].

Second the algorithms discussed here need not always produce a result on computers of today, due to its high consumption of memory and time. Thus it is necessary to use much information about the problems in advance!

Remark 5.32 During research and documentation, the verification procedure was applied various times, using at least three different computers having different hardware and using (slightly) different operating systems, without rejecting the exactness of the result\(^2\). Thus we can neglect spontaneous influences as well as hardware problems. Using established libraries for number support and standard data types (C++ - Standard-Template-Library) which are used by many people decrease the possibility of errors in the used libraries. Thus the most remarkable source of errors is the fact that no human is error free. Having this in mind, the result checked automatically by a computer program which was carefully designed and tested using the mentioned guidelines, is as sure as any other mathematical proof written by a human.

\(^2\) Up to now I left out the certificate for some implicit equations for five concrete algebraic systems, where \( p = 1 + \sqrt{2}/2 \), or \( p = \tau = \text{RootOf}(1, x^3 - 2x^2 + x - 1) \approx 1.754877 \), see the next chapter. Assuming that we solved the concrete linear system \( S_p \) correctly, we can still trust our results.
Chapter 6

2-distance sets in Minkowski planes

In this chapter we will study and classify sets of points in Minkowski planes possessing at most two distinct non-zero distance values.

6.1 Definitions and classification

**Definition 6.1** For a given Minkowski plane \( M^2 \) we call a set \( S \subset M^2 \) and, more precisely, the pair \((M^2, S)\), a 2-distance set if \( S \) contains at least two elements and the set
\[
\text{dist}(M^2, S) := \{ \|s_i - s_j\|_{M^2} : s_i, s_j \in S, s_i \neq s_j \}
\]
contains at most two elements. We denote the set of all 2-distance sets \((M^2, S)\), where \( M^2 \) is an arbitrary Minkowski plane, by \( \mathcal{C}_2 \). If the cardinality of \( \text{dist}(M^2, S) \) is only 1, then \((M^2, S)\), and therefore \( S \), are usually called equilateral sets.

**Example 6.2** In the Euclidean plane \( E^2 \) there are the following 2-distance sets:

\[
\begin{align*}
\text{dist}(E^2, S) &= \{1, \sqrt{2}\}, \{1, \sqrt{3}\}, \{1, \sqrt{2} + \sqrt{3}\}, \{1, \sqrt{3}\}, \{1, \sqrt{2} + \sqrt{3}\}, \text{ and } \{1, \frac{1}{2}(1 + \sqrt{5})\}, \text{ respectively.}
\end{align*}
\]

There are many 2-distance sets which are almost the same concerning geometric intuition. Furthermore, since we are studying all Minkowski planes at the same time, we need to stress the connection between the 2-distance set and its corresponding plane.

**Definition 6.3** We call two 2-distance sets \((M^2, S)\) and \((\tilde{M}^2, \tilde{S})\) strongly equivalent, \((M^2, S) \equiv_s (\tilde{M}^2, \tilde{S})\), if and only if there is an isometry \( \phi : M^2 \to \tilde{M}^2 \) (see Definition 1.3) and a positive real number \( \lambda \) such that \( \phi(S) = \lambda \tilde{S} \).

Note that \( \phi \) can only be an affine linear function, see Theorem 1.17.

It is not difficult to see that the part ‘equivalent’ in the notation of \( \equiv_s \) is correct.

**Proposition 6.4** The relation \( \equiv_s \) of strong equivalence is an equivalence relation in the set \( \mathcal{C}_2 \), i.e.,

1. \((M^2, S) \equiv_s (\tilde{M}^2, \tilde{S}) \) for all \((M^2, S) \in \mathcal{C}_2 \),
2. \((\mathbb{M}^2, S) \equiv_s (\tilde{\mathbb{M}}^2, \tilde{S})\) holds if and only if \((\tilde{\mathbb{M}}^2, \tilde{S}) \equiv_s (\mathbb{M}^2, S)\), and

3. from \((\mathbb{M}^2, S) \equiv_s (\tilde{\mathbb{M}}^2, \tilde{S})\) and \((\tilde{\mathbb{M}}^2, \tilde{S}) \equiv_s (\mathbb{M}^2, \tilde{S})\) it follows that \((\mathbb{M}^2, S) \equiv_s (\tilde{\mathbb{M}}^2, \tilde{S})\).

As the name ‘strongly equivalent’ suggests, this relation \(\equiv_s\) partitions the domain \(\mathcal{C}_2\) into quite a lot of equivalence classes \([C]_s = \{ C' \in \mathcal{C}_2 : C' \equiv_s C \}\), where \(C \in \mathcal{C}_2\). We now look at some weaker equivalence relations which allow a complete description of its equivalence classes.

**Definition 6.5** We call two 2-distance sets \((\mathbb{M}^2, S)\) and \((\tilde{\mathbb{M}}^2, \tilde{S})\) affinely equivalent, \((\mathbb{M}^2, S) \equiv_a (\tilde{\mathbb{M}}^2, \tilde{S})\), if and only if there is an affine linear function \(A : \mathbb{M}^2 \to \tilde{\mathbb{M}}^2\) and a positive real number \(\lambda\) such that \(A(S) = \lambda S\) and \(\|A\mathbf{r} - A\mathbf{n}\|_{\tilde{\mathbb{M}}^2} = \|\mathbf{r} - \mathbf{n}\|_{\mathbb{M}^2}\) for all \(\mathbf{r}, \mathbf{n} \in S\).

If \(g\) denotes the metric of \(\mathbb{M}^2\), \(g(\mathbf{r}, \mathbf{n}) = \|\mathbf{r} - \mathbf{n}\|_{\mathbb{M}^2}\), then \(\mathbb{M}^2|_S := (S, g|_{S \times S})\) is a subspace of the metric space \(\mathbb{M}^2\) in a straightforward way. The function \(A\) in Definition 6.5 is therefore an isometry between the metric spaces \(\mathbb{M}^2|_S\) and \(\tilde{\mathbb{M}}^2|_{A(S)}\).

The relation \(\equiv_a\) of affine equivalence is an equivalence relation in \(\mathcal{C}_2\), too.

It is known that every 2-distance set \(S\) contains at most 9 points, see \([34, \text{Theorem 3}]\) for more general \(k\)-distance sets in Minkowski spaces. Thus each 2-distance set \(S\) is finite, and the set \(\text{dist}(\mathbb{M}^2, S)\) contains a positive minimum.

**Definition 6.6** The normalized induced metric of a 2-distance set \(\mathbb{M}^2, S\) is the function \(S \times S \to \mathbb{R}, (\mathbf{r}, \mathbf{n}) \mapsto \frac{\inf \text{dist}(\mathbb{M}^2, S)}{\|\mathbf{r} - \mathbf{n}\|_{\mathbb{M}^2}}\).

For the purpose of classification we will consider the relative position of the points and the normalized induced metric.

There are two basic concepts for the relative position of points in real planes or higher dimensional real vector spaces.

The first one is the (acyclic) oriented matroid associated by the point configuration \(s_1, \ldots, s_n\). Its equivalence classes are called order types. From the theory of oriented matroids we use the following notion, see for example \([6]\).

**Definition 6.7** The chirotope (or basis orientation) of the point configuration \((s_1, \ldots, s_n)\) of points in \(\mathbb{R}^2\) is the antisymmetric function

\[
\chi(i_1, i_2, i_3) := \text{sign} \det \begin{pmatrix} 1 & 1 & 1 \\ s_{i_1} & s_{i_2} & s_{i_3} \end{pmatrix} \in \{+,-,0\}, \quad i_1, i_2, i_3 \in \mathbb{N}_n.
\]

Obviously, the points \(s_i\) are identified with the column vector of its coordinates.

The second basic concept for the relative position of points in the plane contains even more information about the points. Roughly speaking, this concept adds the information whether or not two straight lines – each defined by containing two points of the configuration – intersect. If they intersect, this concept also describes the ordering of the intersection point and the two defining points along each of the two lines.

Following \([21]\), this concept can be described by a circular sequence of permutations, also called allowable sequence. Another way is to use the big oriented matroid, as described in \([6]\). But in view of Chapter 4 we will use a function determining the sign of some determinants \(s : (\mathbb{N}^2_+)^3 \to \{-1,0,1\}\). The introduction of this function was motivated by transforming the expression including the absolute value function into a simpler expression without \(\mid \cdot \mid\).

**Definition 6.8** A relative full position function of a subset \(S\) of a Minkowski plane \(\mathbb{M}^2\) is a function \(f_p : S \times S \times S \times S \to \{+,-,0\}\), \(f_p(\mathbf{r}, \mathbf{n}, \mathbf{u}, \mathbf{w}) := \text{sign} \det(\mathbf{n} - \mathbf{r}, \mathbf{w} - \mathbf{u})\). The abstract full position function of the sequence \(s_1, \ldots, s_n\) of points in \(\mathbb{R}^2\) is defined as \(F : \mathbb{N}^4 \to \{+,-,0\}\), \((i, j, k, l) \mapsto f_p(s_i, s_j, s_k, s_l)\).\

\(^1\)This definition can be extended for arbitrary sets \(S\) with \(\inf \text{dist}(\mathbb{M}^2, S) > 0\), which includes all 2-distance sets.
Remark 6.9 The chirotope $\chi$ can be obtained from the abstract full relative position function by $\chi(a,b,c) = f^F(a,b,c)$.

We have that $f_p(x,y,z,w) = +1$ if and only if $x - z$ and $y - w$ are not collinear and $y - w$ is between $x - z$ and $-x - z$ with respect to positive orientation of $\partial B$.

We further note that the (abstract) full position function will not distinguish “different” collinear configurations. Assume that $s_1, s_2$ and $s_3$ belong to a line and that there are no other points in $S$. Intuitively, there are three different relative positions, since all $s_1$ should be different: exactly one point is between the two others. But in all cases the full position function is identical to zero. Adding another point not on the same line, we can distinguish all three cases. Nevertheless, since we also know the induced metric by these points, it is not necessary to treat this – not so complicated case of collinear points – in a special way.

**Definition 6.10** We call two 2-distance sets $(M^2, S)$ and $(\tilde{M}^2, \tilde{S})$ fully equivalent, $(M^2, S) \equiv_f (\tilde{M}^2, \tilde{S})$, if and only if there is a bijection $\varphi : S \rightarrow \tilde{S}$ such that the relative full position functions $f_p$ of $S$ and $\tilde{f}_p$ of $\tilde{S}$ as well as the corresponding normalized induced metrics are equal up to $\varphi$ and, possibly, a reorientation:

$$f_p(x,y,z,w) = \tilde{f}_p(\varphi(x),\varphi(y),\varphi(z),\varphi(w)) \quad \forall x,y,z,w \in S$$

or

$$f_p(x,y,z,w) = -\tilde{f}_p(\varphi(x),\varphi(y),\varphi(z),\varphi(w)) \quad \forall x,y,z,w \in S$$

and

$$\frac{1}{\inf\text{dist}(M^2,S)} \|x - y\|_{M^2}^2 = \frac{1}{\inf\text{dist}(\tilde{M}^2,\tilde{S})} \|\varphi(x) - \varphi(y)\|_{\tilde{M}^2}^2 \quad \forall x,y \in S.$$

We can describe every equivalence class of full equivalence in $C_z$ with representative $(M^2, S = \{s_1, \ldots, s_n\})$ by the following quantities:

1. the number $n = |S|$ of points,
2. the ratio $r := r(M^2, S) := \frac{\max\text{dist}(M^2,S)}{\min\text{dist}(M^2,S)}$,
3. the set $L \subset \binom{\mathbb{N}_n}{2}$ describing all “large” distances among points in $S$: $\{i,j\} \in L \iff \|s_i - s_j\|_{M^2}^2 = \max\text{dist}(M^2,S)$, and
4. an abstract full position function $F : \mathbb{N}_n^4 \rightarrow \{+,-,0\}$.

This description is unique up to permutation of the points and up to reorientation of the plane, which is the same as replacing $F$ by $-F$. Having this in mind, we get another equivalence relation if we ignore the value of $r$.

**Definition 6.11** We call two 2-distance sets $(M^2, S)$ and $(\tilde{M}^2, \tilde{S})$ similar, $(M^2, S) \sim (\tilde{M}^2, \tilde{S})$, if and only if $|S| = n = \tilde{n} = |\tilde{S}|$ and it is possible to number the elements of $S$ and of $\tilde{S}$ such that the sets of large distances satisfy $L = \tilde{L}$, and the relative full position functions are equal up to reorientation, i.e., $\{F,-F\} = \{\tilde{F},-\tilde{F}\}$.

**Definition 6.12** We call two 2-distance sets $(M^2, S)$ and $(\tilde{M}^2, \tilde{S})$ weakly equivalent, $(M^2, S) \equiv_w (\tilde{M}^2, \tilde{S})$, if and only if with a suitable numbering of the elements of $S$ and $\tilde{S}$ the numbers of points satisfy $|S| = n = \tilde{n} = |\tilde{S}|$, the sets of large distances satisfy $L = \tilde{L}$, and the chirotopes fulfill $P = \pm \tilde{P}$.

By now we have seen five different equivalence relations in $C_z$: strong equivalence $\equiv$, affine equivalence $\equiv_a$, full equivalence $\equiv_f$, similarity $\sim$, and weak equivalence $\equiv_w$.

**Definition 6.13** We call each equivalence class $T = [C] = [C] := \{C' \in C_z : C' \sim C\}$ of similarity of a 2-distance set $C \in C_z$ a 2-distance configuration. In the same way we call the equivalence class $[C]_s$ of strong equivalence of $C$ strong 2-distance configuration, the equivalence
class \([C]_a\) of affine equivalence of \(C\) affine 2-distance configuration, the equivalence class \([C]_f\) of full equivalence of \(C\) full 2-distance configuration, and the equivalence class \([C]_w\) of weak equivalence of \(C\) weak 2-distance configuration.

**Proposition 6.14** In the sequence \(\equiv_s, \equiv_a, \equiv_f, \sim, \equiv_w\) of the introduced relations, each following relation is more general than the preceding one: for all \(C \in \mathcal{E}_s\) we have

\[
[C]_s \subset [C]_a \subset [C]_f \subset [C]_w = [C]_{\sim} \subset [C]_w.
\]

Note that there are 2-distance sets \(C\) with \([C]_s \neq [C]_a \neq [C]_f \neq [C] \neq [C]_w\), for example we can take any \(C \in D(T_{30}^5, 1.73)\), see the following section. But also \([C]_s = [C]_a = [C]_f = [C] = [C]_w\) is possible for some 2-distance sets \(C\).

**Definition 6.15** We call a 2-distance set \((M^2, S)\) a maximal 2-distance set, if \(S\) is not contained in some larger 2-distance set \((M^2, S')\) \(\in \mathcal{E}_s\), i.e., \(S \subset S'\) is not possible. We call a (full, strong) 2-distance configuration maximal, if all its members are maximal 2-distance sets.

### 6.2 Results

Since \(n\) is bounded by 9, there are only finitely many 2-distance configurations, and thus there are only finitely many possibilities for the characteristics \(n, L\) and \(F\). We will show a complete list thereof. Each 2-distance configuration \(T\), which could by described by \(n, L\) and \(F\) (we will use figures instead), is the union of some full 2-distance configurations \(T_r\), described by \(n\), the real number \(r, L\), and \(F\). Thus the set of these full 2-distance configuration is characterized by the corresponding set \(R := R(T) \subset [1, \infty)\) of suitable values \(r\). Attaching this set \(R(T)\) to each 2-distance configuration gives a complete classification of full 2-distance configurations. At a further step, we can add another parameter \(p\) – besides \(r\) – belonging to a simple, at most two dimensional geometric set \(P = P(T, r)\). We will describe (or at least illustrate) these parameter sets as well as the construction of corresponding 2-distance sets \(S = S(T, r, p)\). One suitable Minkowski plane \(M^2(T, r, p)\) can be constructed from \(S\) and the metric \((L\) and \(r)\), see Chapter 4 proof of Theorem 4.7. Then each affine 2-distance configuration \(A\) is represented exactly once, by some \(T\), a corresponding \(r \in R(T)\) and some \(p \in P(T, r)\) as \(A = [(M^2(T, r, p), S(T, r, p))]_a\).

For strong 2-distance configurations there is no such classification using just finite dimensional parameters. But for each affine 2-distance configuration \(A\) we can give precise conditions for the unit ball of \(M^2\) such that \((M^2, S(T, r, p)) \in A\): there must be some set \(F := F(T, r, p)\) contained in the boundary of the unit ball.

#### 6.2.1 Visualization

We will visualize 2-distance configurations \(T = [(M^2, S)]\) by drawing all points of the set \(S = \{s_1, \ldots, s_n\}\) as small “double”-balls \(\bullet\). The 2-distance set was chosen so that \(\text{dist}(M^2, S) = \{1, r\}\), including the case \(\text{dist}(M^2, S) = \{1\}\) for equilateral sets \(S\). We connect \(s_i, s_j\) by a blue straight line if \(\|s_i - s_j\| = 1\). Otherwise, i.e., if they have a large distance \(\|s_i - s_j\| = r > 1\), we connect them by a red dashed line. We do not visualize the unit circle of the corresponding Minkowski plane \(M^2\), because one suitable \(M^2\) can always be constructed from this picture together with the value of \(r\).

From these pictures we can extract \(n\). Assigning labels \(s_1, \ldots, s_n\) to the points in an arbitrary way, we can also extract \(L\), and the abstract relative full position function \(F\). In most cases it holds that two lines \(C\) \((s_i, s_j)\) and \((s_k, s_l)\), which seem to be parallel in the picture, are really parallel. But there are two exception \(T_8^4\) and \(T_{10}^4\), where two lines are close to be parallel, but really they intersect right \((T_8^4)\) and left \((T_{10}^4\) respectively to the picture.

#### 6.2.2 Full 2-distance configurations

Now we show a list of all 94 different 2-distance configurations \(T = T_k^n\), where \(k\) is some number to yield unique symbols. All different full 2-distance configurations are obtained as \(D = D(T, r) =\)
\{ (M^2, S) \in T : r(M^2, S) = r^* \}, where T is a 2-distance configuration and r^* \in R(T). This set R(T) is specified in this list. We denote by \tau the real root of the polynomial \( x^3 - 2x^2 + x - 1, \)

\( \tau = \frac{1}{6} \sqrt[3]{100 + 12 \sqrt{69}} + \frac{2}{3} \sqrt[3]{100 + 12 \sqrt{69}} + \frac{1}{3} \approx 1.754877. \)
CHAPTER 6. 2-DISTANCE SETS IN MINKOWSKI PLANES

<table>
<thead>
<tr>
<th>$T_{19}^5$</th>
<th>$T_{20}^5$</th>
<th>$T_{21}^5$</th>
<th>$T_{22}^5$</th>
<th>$T_{23}^5$</th>
<th>$T_{24}^5$</th>
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</thead>
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<td>$\mathcal{R}(T_{20}) = {2}$</td>
<td>$\mathcal{R}(T_{21}) = {2}$</td>
<td>$\mathcal{R}(T_{22}) = {2}$</td>
<td>$\mathcal{R}(T_{23}) = {2}$</td>
<td>$\mathcal{R}(T_{24}) = {1 + \frac{\sqrt{2}}{2}, 2}$</td>
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</tbody>
</table>

<table>
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<tr>
<th>$T_{25}^5$</th>
<th>$T_{26}^5$</th>
<th>$T_{27}^5$</th>
<th>$T_{28}^5$</th>
<th>$T_{29}^5$</th>
<th>$T_{30}^5$</th>
</tr>
</thead>
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<tr>
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<td>$\mathcal{R}(T_{26}) = {\sqrt{3}, 2}$</td>
<td>$\mathcal{R}(T_{27}) = {2}$</td>
<td>$\mathcal{R}(T_{28}) = {1 + \frac{\sqrt{2}}{2}, 2}$</td>
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<td>$\mathcal{R}(T_{30}) = {2}$</td>
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<th>$T_{35}^5$</th>
<th>$T_{36}^5$</th>
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<td>$\mathcal{R}(T_{32}) = {2}$</td>
<td>$\mathcal{R}(T_{33}) = {2}$</td>
<td>$\mathcal{R}(T_{34}) = {2}$</td>
<td>$\mathcal{R}(T_{35}) = {2}$</td>
<td>$\mathcal{R}(T_{36}) = {2}$</td>
</tr>
</tbody>
</table>
6.2.3 Maximal 2-distance configurations

We can compress the classification of 2-distance configurations by listing the 11 maximal 2-distance configurations. Then each of the 83 remaining 2-distance configurations is contained as subconfiguration in at least one of the following.

We will skip similar lists of maximal full or affine 2-distance configurations because they do not represent the informations much easier.

6.2.4 Affine 2-distance configurations

Each affine 2-distance configuration \( A \) is represented exactly once in the following way. Take a full 2-distance configuration \( D = D(T, r^*) \), where \( r^* \in R(T) \) and \( T = T_{\emptyset}^0 \). Sometimes, \( D \) is itself an affine 2-distance configuration; in this case \( A = D \). To simplify notations, we introduce for these cases a purely formal parameter \( p = p \in \mathbb{R}^0 =: P(T, r^*) \) in the 0-dimensional real vector space. The coordinates of the points of \( S(T, r^*, p) \) are piecewise polynomials in \( \mathbb{Z}[r^*] \). More precisely, the set \( R(T) \) will occasionally be split into several (finitely many) subsets, \( R(T) = \bigcup_{i=1}^k R(T, i) \).

For some \( i \) the case \( A = D \) may occur. Then there are \( 2n \) mono-variate polynomials, \( S_{T,i} \in (\mathbb{Z}[X]^{2})^n \), which define \( S(T, r^*, p) := \{s_1, \ldots, s_n\} \) by evaluating the polynomials at \( X = r^* \in R(T, i) \): \( (s_1, \ldots, s_n) := S_{T,i}(r^*) \).

For some other cases one or more points of \( S \) may be allowed to move within some bounds, so that \( [(M^2, S)]_f \) stays the same but \( [(M^2, S)]_a \) changes (note that \( M^2 \) depends on \( S, L \) and \( r^* \)). It turned out that there are at most two degrees of freedom, and that the influence of the parameter can be chosen to be affinely linear. Thus, there is some degree of freedom \( d \in \{0, 1, 2\} \) (we already discussed \( d = 0 \)), and \( 2n \) polynomials in \( d + 1 \) variables \( r \) and \( p_1, \ldots, p_d \), summarized as \( S_{T,i} \in (\mathbb{Z}[X, p_1, \ldots, p_d]^{2})^n \). Fixing the first parameter \( X = r^* \) in \( S_{T,i} \), we get an affinely linear function \( S_{T,i}(r^*) : \mathbb{R}^d \to (\mathbb{R}^2)^n \). In addition, the parameters \( p \in \mathbb{R}^d \) are restricted to belong to a polyhedron, or, more precisely, to the union \( P(T, i, r^*) \subset \mathbb{R}^d \) of interiors of some well defined faces of a polyhedron \( P'(T, i, r^*) \). Note that we have chosen each parameter \( p_j \) so that at least...
one coordinate of one of the points of $S$ coincides with $p_j$, possibly after multiplication with a real factor in $\mathbb{Z}[r]$. Note that in almost all cases with $d = 2$ we could take both coordinates of just one point $s$ from $S$ as parameters. In all these cases the parameter range $P(T, i, r^*)$ coincides, up to scaling (in both coordinate directions), with the range where $s$ can move. The vertices of $P'(T, i, *r)$ are again defined by mono-variate polynomial coordinates. The combinatorial structure of $P'(T, i, r)$ is identical for all $r \in R(T, i)$, and $P$ always consists of the same faces of $P'$ with respect to the labeled vertices.

Summarizing, we have for each 2-distance configuration $T$ a refinement of $R(T)$, $R(T) = \bigcup_{i=1, \ldots, k(T)} R(T, i)$. For each subcase $i = 1, \ldots, k(T)$ there is a construction $S = S(T, i, r, p)$ of $n$ points in the plane which is polynomial in $r \in R(T, i)$ and linear in some additional parameter $p \in R^d$, with $d \in \{0, 1, 2\}$. This parameter $p$ can be chosen from a well defined parameter range $P(T, i, r, p) \subset R^d$. Now each affine 2-distance configuration $A$ is uniquely determined by one $T$, $i \in \mathbb{N}_{k(T)}$, $r \in R(T, i)$ and $p \in P(T, i, r)$ as $A = [(\mathbb{M}^2(T, i, r, p))_a]$, where $\mathbb{M}^2$ is constructed appropriately from $S$ and the metric.

Even though this classification can be precisely described by algebraic and combinatorial expressions, we will just give pictures. Of course, these pictures cannot cover all informations. For each $T$ and $i \in \mathbb{N}_{k(T)}$, where the full 2-distance configuration is affinely unique, i.e., $D = A$ or $P(T, i) = \mathbb{R}^d$, we will show nothing. Otherwise, we will state $R(T, i)$ precisely, fix some $r^* \in R(T, i)$ and show a floating point approximation of $r^*$, take $p^* \in P(T, i, r^*)$ as the mean of all vertices of $P'(T, i, r^*)$, show a picture of $S(T, i, r^*, p^*)$ and also illustrate for each $s_i \in S(T, i, r^*, p^*)$ its domain if $p \in P(T, i, r^*)$ varies. In case $d = 2$, i.e., if the parameter space is more than a point or a segment, then also $P(T, i, r^*)$ is shown in a separate picture right to $S(T, i, r^*, p^*)$. Each face (vertex, edge or the interior) of $P'$ is drawn in a way indicating whether or not it belongs to $P$. Vertices belong to $P$ if they are shown as small black filled balls, otherwise a little larger gray circle line is drawn around the point. Edges belong to $P$ if they are drawn as black line and not as dotted brown line. Polygons belong to $P$ if they are dark and crosswise shaded.

In fact, the pictures are a little bit more complicated. Instead of just drawing $P$ and $P'$ we have drawn sometimes a larger area $P_a$, but in a style showing that it does not belong to $P$. This area is the set of all parameters $p$ such that $S(T, i, r^*, p)$ is a 2-distance set belonging to the same full 2-distance configuration. But due to some symmetry within the combinatorial structure of $T$, for all $p \in P_a \setminus P$ there is some $p' \in P$ such that $S(T, i, r^*, p)$ is an affine image of $S(T, i, r^*, p')$, or more precisely, $(\mathbb{M}^2(T, i, r^*, p), S(T, i, r^*, p')) \equiv (\mathbb{M}^2(T, i, r^*, p'), S(T, i, r^*, p'))$. One can guess the exact description of $P_a$ with respect to its boundary from $P$.

The list of all full 2-distance configurations which contain more than one affine 2-distance configuration follows.
$T_{10}^d, d = 2$
\(P'\) is 4-gon
\[r^* \approx 1.854\]
\[R(T_{10}^d, 3) = (1 + \frac{\sqrt{2}}{2}, 2)\]

$T_{13}^d, d = 2$
\(P'\) is 3-gon
\[r^* \approx 2\]
\[R(T_{13}^d, 3) = \{2\}\]

$T_{15}^d, d = 2$
\(P'\) is 3-gon
\[r^* \approx 2\]
\[R(T_{15}^d, 1) = \{1, 2\}\]

$T_{15}^d, d = 2$
\(P'\) is 3-gon
\[r^* \approx 2\]
\[R(T_{15}^d, 1) = \{2\}\]

$T_{16}^d, d = 1$
\(P'\) is segment
\[r^* \approx 2\]
\[R(T_{16}^d, 1) = \{2\}\]

$T_{16}^d, d = 1$
\(P'\) is segment
\[r^* \approx 2\]
\[R(T_{16}^d, 1) = \{2\}\]

$T_{24}^d, d = 1$
\(P'\) is segment
\[r^* \approx 2\]
\[R(T_{24}^d, 3) = \{2\}\]

$T_{24}^d, d = 1$
\(P'\) is segment
\[r^* \approx 2\]
\[R(T_{24}^d, 3) = \{2\}\]

$T_{30}^d, d = 2$
\(P'\) is 3-gon
\[r^* \approx 1.755\]
\[R(T_{30}^d, 3) = \{\tau\}\]

$T_{30}^d, d = 2$
\(P'\) is 4-gon
\[r^* \approx 1.877\]
\[R(T_{30}^d, 4) = (\tau, 2)\]

$T_{30}^d, d = 2$
\(P'\) is 4-gon
\[r^* \approx 2\]
\[R(T_{30}^d, 5) = \{2\}\]

$T_{30}^d, d = 2$
\(P'\) is 3-gon
\[r^* \approx 2\]
\[R(T_{30}^d, 2) = (1 + \frac{\sqrt{2}}{2}, \tau)\]
6.2.5 Strong 2-distance configurations

We will not discover a complete classification of strong 2-distance configurations. It is not possible to parametrize all strong 2-distance configurations by parameters \( p \in \mathbb{R}^d \) of a finite dimensional real vector space, i.e., with \( d < \infty \). On the other hand, some affine 2-distance configurations \( A = ([M^2(T,i,r^*,p),S(T,i,r^*,p)])_\alpha \) are in fact also strong 2-distance configurations! In these cases, the unit ball \( B \) of \( M^2(T,i,r^*,p) \) is a polygon which is uniquely determined by \( S(T,i,r^*,p) \). In general, each 2-distance set \( S = S(T,i,r^*,p) \) (together with the metric determined by \( L \) and \( r \)) determines a set \( F = F(T,r^*,p) \) which must be part of the unit circle \( F(T,r^*,p) \subset \partial B \) of \( M^2 \). This condition \( F(T,r^*,p) \subset \partial B \) is necessary and also sufficient for \( M^2 \) such that \( S \) induces the correct metric in \( M^2 \).

If \( F \) is in strong convex position, i.e., each point of \( F \) is a vertex of \( \text{conv} F \), then there is a smooth and strictly convex unit ball \( B \) with \( (M^2(B),S) \in A \). In some sense there are quite a lot of unit balls containing \( F \) in its boundary. Otherwise, i.e., if there are three collinear points \( a, b, c \in F \), say with \( b \in \text{rel int} \overline{ac} \), then by convexity we get \( \overline{ac} \subset \partial B \). Consequently, there is no strictly convex unit ball \( B \) with \( (M^2(B),S) \in A \). If there is another such segment \( \overline{ac} \) in \( \partial B \), because \( \overline{ac} \in F \) and \( \partial \in \text{rel int} \overline{ac} \), which is not collinear with \( \overline{ac} \), then \( c \) must be a vertex of \( B \). Thus \( \partial B \) cannot be smooth. Finally, if every point of \( F \) is a vertex of \( B \) or belongs to the interior of a segment in \( \partial B \) (due to the stated reasons), then \( B \) is uniquely determined by \( F \).

Luckily, the behavior of \( S(T,i,r^*,p) \) does not depend “too much” on \( r^* \) and \( p \). In fact, as long as \( T \) and \( i \) are fixed, all points of one face (relatively open) of the parameter region \( P' = P'(T,i,r^*) \) will show the same behavior. More precisely, we know the combinatorial structure of the set \( F \) as well as the remaining geometric symmetry (the isometry group) of \( S \) for each face of each set \( P'(T,i,r) \), regardless of \( r \in R(T,i) \).

Instead of presenting all this information, we restrict ourselves just to distinguish three types of affine 2-distance configurations \( A \).

1. There is some strictly convex Minkowski plane \( M^2 \) with \( (M^2,S) \in A \).
2. There is no strictly convex Minkowski plane \( M^2 \) with \( (M^2,S) \in A \), but there are planes \( M^2 \) whose unit ball is not a polygon with \( (M^2,S) \in A \), i.e., \( M^2 \) can be “partially” strictly convex.
3. There is a uniquely defined plane \( M^2 \) with \( (M^2,S) \in A \). This plane \( M^2 \) has a unit ball which is a \( 2k \)-gon \( (k \in \{2,3,4\}) \).

The following pictures are similar to the visualization of affine 2-distance configurations. For each subcase, where \( T \) is a 2-distance configuration and \( i \in \mathbb{N}_{k,r} \), there is at least one picture, representing \( A = ([M^2(T,i,r,p),S(T,i,r,p)])_\alpha \) for all \( r \in R(T,i) \) and \( p \in \hat{P}' \subset P \). Here \( \hat{P}' \) contains the new information provided with this classification. \( \hat{P}' \) is the union of some faces of \( P \), drawn with the same conventions as previously used to define \( P \).
### Strictly convex planes possible

- $T^2_2, d = 0$
  - $r^* \approx 1$
  - $R(T^2_2, 1) = (1)$

- $T^2_3, d = 0$
  - $r^* \approx 1.5$
  - $R(T^2_3, 1) = (1, 2)$

- $T^2_4, d = 0$
  - $r^* \approx 2$
  - $R(T^2_4, 1) = (1, \infty)$

- $T^4_2, d = 0$
  - $r^* \approx 1.854$
  - $R(T^4_2, 2) = (1 + \sqrt{2}, 2)$

- $T^4_3, d = 0$
  - $r^* \approx 1.354$
  - $R(T^4_3, 2) = (1 + \sqrt{2}, 2)$

- $T^4_4, d = 0$
  - $r^* \approx 1.707$
  - $R(T^4_4, 2) = (1 + \sqrt{2}, 2)$

- $T^3_10, d = 2$
  - $r^* \approx 1.854$
  - $R(T^3_{10}, 2) = (1 + \sqrt{2}, 2)$

- $T^4_{10}, d = 2$
  - $r^* \approx 1.75$
  - $R(T^4_{10}, 2) = (\frac{3}{2}, 2)$

- $T^4_{11}, d = 2$
  - $r^* \approx 1.5$
  - $R(T^4_{11}, 1) = (1, 2)$

- $T^5_{26}, d = 0$
  - $r^* \approx 1.618$
  - $R(T^5_{26}, 1) = (\frac{1 + \sqrt{5}}{2})$

### “Partially” strictly convex planes possible

- $T^2_6, d = 0$
  - $r^* \approx 2$
  - $R(T^2_6, 2) = (2)$

- $T^3_7, d = 1$
  - $r^* \approx 2$
  - $R(T^3_7, 1) = \{2\}$

- $T^2_8, d = 0$
  - $r^* \approx 2$
  - $R(T^2_8, 2) = \{2\}$

- $T^3_9, d = 0$
  - $r^* \approx 1.854$
  - $R(T^3_9, 2) = (1 + \sqrt{2}, 2)$

- $T^4_6, d = 0$
  - $r^* \approx 2$
  - $R(T^4_6, 1) = \{2\}$

- $T^2_7, d = 0$
  - $r^* \approx 1.5$
  - $R(T^2_7, 1) = (1, 2)$

- $T^3_8, d = 1$
  - $r^* \approx 1.5$
  - $R(T^3_8, 1) = (1, 2)$

- $T^2_9, d = 1$
  - $r^* \approx 2$
  - $R(T^2_9, 2) = \{2\}$

- $T^3_{10}, d = 2$
  - $r^* \approx 1.707$
  - $R(T^3_{10}, 2) = (1 + \sqrt{2}, 2)$

- $T^4_{10}, d = 2$
  - $r^* \approx 1.854$
  - $R(T^4_{10}, 3) = (1 + \sqrt{2}, 2)$

- $T^5_{10}, d = 2$
  - $r^* \approx 2$
  - $R(T^5_{10}, 4) = \{2\}$

- $T^4_{11}, d = 0$
  - $r^* \approx 2$
  - $R(T^4_{11}, 2) = \{2\}$
Unique polygonal unit ball
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<thead>
<tr>
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<th>$T_{6}^{5}$, $d = 1$</th>
<th>$T_{7}^{5}$, $d = 0$</th>
<th>$T_{8}^{5}$, $d = 0$</th>
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<th>$T_{12}^{5}$, $d = 0$</th>
<th>$T_{13}^{5}$, $d = 1$</th>
<th>$T_{14}^{5}$, $d = 0$</th>
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<td>$r^* \approx 2$</td>
<td>$R(T_{11}^{5}, 3) = {2}$</td>
<td>$r^* \approx 2$</td>
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<th>$T_{26}^{5}$, $d = 0$</th>
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<th>$T_{30}^{5}$, $d = 0$</th>
<th>$T_{31}^{5}$, $d = 0$</th>
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6.2.6 Surprising results

Really a surprise was the behavior of the Euclidean 2-distance set $T_{26}^5$ formed by the vertices of a regular pentagon. The corresponding 2-distance configuration consists only of affinely regular pentagons! The reason is that each diagonal is parallel to a corresponding side, which is trivially deduced from the definition of similarity of 2-distance sets. But there are no other 2-distance sets with the same set of large distances, not depending on the position!

There are Minkowski planes having no such 2-distance set! But there are also planes, far from the Euclidean one, which have such a 2-distance set “in every position”, i.e., that one can fix two of the points arbitrarily and still can extend them to such a 2-distance set. On the other hand, such planes are rare; they have a “kind” of symmetry. For example, all planes whose isometry group is homeomorphic to $D_5$ (the symmetry group of the regular pentagon) have this property. But there are other examples.

6.3 Verification

6.3.1 Overview

The correctness of all the results stated in the previous section follows from the way of obtaining these results. The last two chapters describe the main tools and techniques even in a more general context. Theorem 4.18 is the key result for getting affine 2-distance configurations.

For this we have to solve systems of equations and inequalities. Each system is polynomial (at most quadratic) and the coefficients themselves depend polynomially on the real parameter $r$.

Instead of solving these systems, we used simplified auxiliary systems. Leaving out all non-linear restrictions of the system $\text{SysEmD}(\rho, d, s)$, we get

$$\text{SysEmDL}(\rho, d, s) := (E_{\text{conv}}(d, s) \cup E_{\text{det}}^1(n, d), W_{\text{conv}}(d, s, \rho), S_{\text{conv}}(d, s)).$$
Thus the condition \( r \in L(\text{SysEmDL}(\rho, d, s)) \) is necessary for the condition \( r \in L(\text{SysEmD}(\rho, d, s)) \).

Chapter 5 shows how to solve such parametrized linear systems. Due to the very special metrics

\[
\rho = \rho(L, r) : (i, j) \mapsto \begin{cases} 
0 & \text{if } i = j \\
 r & \text{if } \{i, j\} \in L \\
1 & \text{otherwise},
\end{cases}
\]

the solutions of \( \text{SysEmDL}(\rho, d, s) \) are really nice. First the dimension of the solution set is at most 3, including the homogeneous variable. Second with only one exception this solution set was also the solution set of the original system: \( L(\text{SysEmDL}(\rho, d, s)) = L(\text{SysEmD}(\rho, d, s)) \). This was a big surprise! This result was obtained by checking whether or not all quadratic restrictions are satisfied by all solution vectors of \( \text{SysEmDL}(\rho, d, s) \).

Finally, to obtain a classification of affine 2-distance configurations, we had to find out which solution vectors belong to the same equivalence class.

The results about strong 2-distance configurations, i.e., on the structure of the fixed part in the unit circle \( \partial B \), can be obtained purely combinatorially, namely from the incidence matrix contained in the solution of \( \text{SysEmDL}(\rho, d, s) \).

### 6.3.2 Completeness of the classification

First we want to discuss how we get a complete list of 2-distance configurations and corresponding full 2-distance configurations.

For \( n = 2 \) there is just one possibility.

For \( n \geq 3 \) we study “candidates” of full 2-distance configurations as quadruples of \((n, r, L, F)\) (number of points, ratio of the two distances, set of large distances, and abstract full position function) which are the characteristics (up to the trivial modifications: permutations and inversion of orientation) of each full 2-distance configuration. Besides the infinitely many candidates of full 2-distance configurations we also define candidates \((n, L, F)\) of 2-distance configurations. A candidate is called realizable if there is some 2-distance set \((M^2, S)\) possessing exactly these three or four parameters.

Remember the following obvious statement: Omitting an arbitrary point \( s \) of a 2-distance-set \( S \) in \( M^2 \) with at least 3 points yields again a 2-distance-set \( S \setminus \{s\} \) in \( M^2 \). Each “sub-candidate” of a realizable candidate \((n, r, L, F)\) with \( n \geq 3 \) has to be realizable. A sub-candidate is obtained by deleting one abstract point \( i \in [n] \) from the given data and by renumbering the others in a straightforward way. We have to take care if the sub-candidate corresponds to an equilateral set.

In this case \( L = \emptyset \) has to be considered as the same as \( L = \begin{pmatrix} [n] \\ 2 \end{pmatrix} \), and \( r \) has to be 1 in the sub-candidate, not depending on the original value.

We construct a list \( X \) of all (full) 2-distance configurations by alternately solving two sub-problems for each \( 3 \leq n \leq 9 \). For \( n = 10 \) we get the known answer that there are no realizable candidates at all. This implies \( n < 9 \).

The first subproblem is to obtain combinatorially a complete – up to trivial modification – list \( L_n \) of all these candidates with \( n \) points all whose “sub-candidates” are realizable. At this time we know a complete list \( X_n \) of realizable candidates \((n', L, F, R)\) with \( n' < n \). There are only finitely many combinations of \((L, F)\). To each pair we maintain a semi-algebraic set \( R \subseteq \mathbb{R} \) of possible values for \( r \). Since for each class of trivially modified candidates we need only one representative, it is not necessary to iterate through all possible pairs \((L, F)\). Note that the number of all pairs

\[
L \subseteq \begin{pmatrix} [n] \\ 2 \end{pmatrix}, N \to \{+, -, 0\},
\]

grows very fast with \( n \). The main idea for a reduction of cases is to extend – one after another – each candidate \((L, F)\) of \( X \) with \( n - 1 \) points by addition of another point in all possible ways to a new candidate for \( L_n \). For \( n - 1 \) new direction vectors \((s_1 - s_n, \ldots, s_{n - 1} - s_n)\) we have to choose an arbitrary position within the cyclic order of all directions. We used a recursive approach to choose these positions one after another. After each decision we check all sub-candidates which can already be extracted from this incomplete candidate. The sub-candidates may already be known as \((n', L', F', R')\) in \( X \). In this case we update \( R \) with
CHAPTER 6. 2-DISTANCE SETS IN MINKOWSKI PLANES

If one sub-candidate is not found, or if \( R = \emptyset \), then we can omit the incomplete candidate. Otherwise we accept it and try recursively to insert the next direction in all possible ways. If all directions were chosen and all sub-candidates were checked and \( R \neq \emptyset \), the candidate \((L, F, R)\) will be added to \( L_n \) unless a trivial modification of \((L, F, R)\) is already contained in \( L_n \).

The second subproblem is to determine analytically for each candidate \((L, F, R)\) from \( L_n \) the set \( R' \) of all \( r \in R \) such that \((n, r, L, F)\) is realizable. If \( R' \neq \emptyset \), then \((n, L, F, R')\) is added to \( X \).

This procedure yielded 132 candidates \((L, F)\) in \( L_n \), 97 of which were not realizable. For \( n \geq 6 \) all candidates in \( L_n \) were realizable.

6.3.3 Realizability of candidates

Assume that \((n, r, L, F)\) is a candidate for a full 2-distance configuration for all \( r \in R \). Is this candidate \((n, r, L, F)\) realizable?

If the full position function \( F \) is everywhere 0, then the corresponding 2-distance set \( S \) is at most 1-dimensional. Then there is just one (up to trivial modification) realizable candidate with \( n = 2 \), only one with \( n = 3 \), and no other. So this case \( F \equiv 0 \) can be checked easily, and in the following we can assume that \( \dim S = 2 \).

Using the first part of Theorem 4.18, it is sufficient to check whether or not the system \( \text{SysEmD}(\rho(L, r), 2, s) \) is admissible. Note that the sign function \( s : (\mathbb{N}_2^2) \rightarrow \{-1, 0, 1\} \) can be directly constructed from \( F : \mathbb{N}_2^2 \rightarrow \{+,-,0\} \) as \( s(i, j, k, l) = F(i+j, k+l) \).

As noted earlier, we first check for which \( r \in R' \) the linear system \( \text{SysEmDL}(\rho(L, r), 2, s) \) is admissible. Note that this system contains for each variable \( x_i \) a condition \( x_i > 0 \), \( x_i < 0 \) or \( x_i = 0 \), thus the inequality matrix has full column rank for all \( r \in R \). So all the tools of Chapter 5 are available for the parametrized family \((\text{SysEmDL}(\rho(L, r), 2, s))_{r \in R} \) of systems of linear equations and inequalities. In particular, we used Algorithm 5.30 to determine a verified full solution of this family. Of course, we solved a simplification of the system as described in Section 4.3. But the discussion is a little bit easier if we formulate the performed steps using the original system in \( R^n \).

Note that for equilateral sets \((L = 0)\) we only consider \( r = 1 \) \((R_d := \{1\})\), and otherwise \( r > 1 \) by definition \((R_d := (1, \infty))\).

An application of Algorithm 5.30 gives us the set \( R' := \{r \in R_d : L(\text{SysEmDL}(\rho(L, r), 2, s)) \neq \emptyset \} \).

All candidates \((n, L, F)\) with \( \mathbb{R}' = \emptyset \) are not realizable as 2-distance configuration. Due to some luck, the necessity of the system \( \text{SysEmDL}(\rho(L, r), d, s) \) is a very strong criterion. All remaining candidates \((n, r, L, F)\) for full 2-distance configurations, i.e., where \( r \in R' \), are realizable. This was shown by examples. Indeed, it was easy to find some solution vector of \( \text{SysEmDL}(\rho(L, r), 2, s) \) as the sum of some (up to one exception: of all) generators of the solution set of \( \text{SysEmDL}(\rho(L, r), 2, s) \).

Note that we used two heuristics to find necessary and also sufficient conditions for \((n, r, L, F)\) to be realizable: \( L(\text{SysEmDL}(\rho(L, r), 2, s)) \neq \emptyset \) is a necessary condition (which was checked in a deterministic way). We also constructed special embeddings \( f : (\mathbb{N}_n, \rho(L, r)) \rightarrow \mathbb{R}^2 \) (using polynomials) with full abstract position function \( F \), which is sufficient for \((n, r, L, F)\) to be realizable. The calculations showed that these two heuristics were good enough, i.e., that the sets of described candidates are the same for the necessary and also the sufficient conditions. Note that the sufficiency of \( L(\text{SysEmDL}(\rho(L, r), 2, s)) \neq \emptyset \) for \((n, r, L, F)\) to be realizable also follows from the classification of affine 2-distance configurations.

All candidates \((n, L, F)\) from \( L_n \) with \( R' \neq \emptyset \) yield our list of 2-distance configurations \( T^n_\nu \). With \( R(T^n_\nu) := R' \) also the classification of all full distance-configurations \( D = D(T^n_\nu, r) \), where \( r \in T^n_\nu \), is correct.

6.3.4 Affine 2-distance configurations

With the tools described so far, the given classification of full 2-distance configurations was obtained and also shown to be correct. We checked a large number of candidates, operated on semi-algebraic sets in \( \mathbb{R}^3 \), and checked the parametrized linear systems \( \text{SysEmDL}(\rho(L, r), 2, s) \) for admissibility.
At this point, the complete solutions of these systems were only needed to "guess" examples, showing that the remaining candidates were really realizable.

But we also obtained a complete classification of affine 2-distance configurations by analyzing the full solution of the linear systems. There are three main issues for this analysis. First, we will investigate the solution set \( X := L(\text{SysEmD}(\rho(L, r), 2, s)) \) from \( H := L(\text{SysEmDL}(\rho(L, r), 2, s)) \).

Second we have to reduce symmetrical copies of the same affine 2-distance configuration from \( X \). And last, we will transform the reduced solution set \( Y \) – or more precisely its positive equivalence classes – into a set consisting of exactly one (polynomial) representative of every affine 2-distance configuration.

Note that we identify the \( n^4 \)-dimensional space \( \mathbb{R}^{n^4} \) with the set \( \mathbb{R}^{(n_2)^2} \) of functions \( b : (\mathbb{N}_n^2)^2 \to \mathbb{R} \).

6.3.5 Nonlinear restrictions

We consider a fixed full 2-distance configuration \( D = D(T, r) \), with \( r \in R \), which is represented by \((n, r, L, F)\). We are interested in a description of the set \( L(\text{SysEmD}(\rho(L, r), 2, s)) := X \). We already know a certificated solution of the linear parametrized system \( \text{SysEmDL}(\rho(L, r), 2, s) \) with solution set \( H \supset X \). \( H \) is the union of some faces of the polyhedral cone \( C := \text{cone}(G_1, \ldots, G_t) \subset \mathbb{R}^m \), where the generators \( G_i \) depend polynomially on \( r \).

Let \( q : (\mathbf{r}, \mathbf{y}) \mapsto \sum_{i,j \in \mathbb{N}_m} q_{i,j} \mathbf{r}_i \mathbf{y}_j \) be a symmetric bilinear form on \( \mathbb{R}^m \), i.e., \( q_{i,j} = q_{j,i} \). For each face \( F \) of \( C \) we can decide whether or not a quadratic equation \( q(\mathbf{r}, \mathbf{r}) = 0 \) holds for all \( \mathbf{r} \in F \).

**Lemma 6.16** For a symmetric bilinear form \( q \) we have that \( q(\mathbf{r}, \mathbf{r}) = 0 \) for all \( \mathbf{r} \in \text{cone}(\mathbf{g}_1, \ldots, \mathbf{g}_k) := L \) if and only if \( q(\mathbf{r}, \mathbf{y}) = 0 \) for all \( \mathbf{r}, \mathbf{y} \in \{ \mathbf{g}_1, \ldots, \mathbf{g}_k \} \).

**Proof** The sufficiency is clear by \( q(\sum \lambda_i \mathbf{g}_i, \sum \mu_j \mathbf{g}_j) = \sum \lambda_i \mu_j q(\mathbf{g}_i, \mathbf{g}_j) \), the necessity follows from \( q(\mathbf{g}_i, \mathbf{g}_i) = q(\mathbf{g}_i + \mathbf{g}_j, \mathbf{g}_i + \mathbf{g}_j) = 0 \) using the formula
\[
q(\mathbf{g}_i, \mathbf{g}_j) = \frac{1}{2} \left( q(\mathbf{g}_i + \mathbf{g}_j, \mathbf{g}_i + \mathbf{g}_j) - q(\mathbf{g}_i, \mathbf{g}_i) - q(\mathbf{g}_j, \mathbf{g}_j) \right) \quad (6.1)
\]

Using (6.1) we can calculate symmetric bilinear forms \( q_1, \ldots, q_e \), each representing one of the \( e := \binom{m+1}{4} \) quadratic restrictions quadratic \( c_{i,j} \in E_{\text{det,vol}}(n, 2) \) in the form \( q_i(\mathbf{r}, \mathbf{r}) = 0 \). Remember that these restrictions are the extra conditions for \( X \) in \( H \):
\[
X = \{ \mathbf{r} \in H : q_i(\mathbf{r}, \mathbf{r}) = 0 \quad \forall 1 \leq i \leq e \}.
\]

With Lemma 6.16 we can test whether or not all quadratic restrictions are satisfied by all vectors in \( C \). Since \( \text{relint} C \subset H \subset C \) and all quadratic forms are continuous, this test also yields the answer whether or not all solution vectors in \( H \) are also solution vectors of \( \text{SysEmD}(\rho(L, r), 2, s) \), i.e., if \( H = X \) holds true.

**Investigation 6.17** For all full 2-distance configurations \( D(T, r) \), \( r \in R(T) \), with \( T \neq T_{10} \) we have that
\[
L(\text{SysEmD}(\rho(L, r), 2, s)) = L(\text{SysEmDL}(\rho(L, r), 2, s))
\]

Thus, up to one exception we have that \( X = H \).

6.3.6 The exception which is not polyhedral

**Investigation 6.18** The exception is the (full) 2-distance configuration \( D(T_{10}^5, 2) = T_{10}^5 \). So we have \( R(T_{10}^5) \subset \{ 2 \} \). The solution set \( H \) of the corresponding linear system is 4-dimensional, the cone \( C = \text{cone}(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4) \) over a usual 3-dimensional tetrahedron \( \text{conv} \{ \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4 \} \). \( C \) has 4 one-dimensional faces (\( \text{cone} \{ \mathbf{g}_1 \} \)), 6 two-dimensional faces (cones over the edges of the tetrahedron) and 4 three-dimensional faces. Since \( n = 5 \), exactly one quadratic equation has to be considered. The solution set \( X = L(\text{SysEmD}(\rho(L, r), 2, s)) \) is instead not convex. But the closure \( \text{cl} X \) contains
all generators $g_1, \ldots, g_4$ of $C$ and 4 of the 6 faces generated by the edges $\overline{g_1g_2}$, by $\overline{g_2g_3}$, by $\overline{g_3g_4}$ and by $\overline{g_4g_1}$. We can easily parametrize $S$ as $S := \{(0,0), (1,0), (0,1), (-1,\lambda), (\mu,2)\}$ with $\lambda, \mu \in (0,1)$. The generators correspond to $\lambda, \mu \in \{0,1\}$, $g_1$ to $\lambda = \mu = 0$, $g_2$ to $(\lambda, \mu) = (0,1)$, $g_3$ to $\lambda = \mu = 1$, and $g_4$ to $(\lambda, \mu) = (1,0)$. The two-dimensional faces of $C$, for which one of the parameters $\lambda, \mu$ is constant, are part of $\partial X \setminus X$, and all their points are extremal in $C$. But for $\overline{g_1g_2}$ and $\overline{g_3g_4}$ both parameters change from 0 to 1, and no relatively interior point belongs to $X$. It is clear that points within $\overline{g_1g_2}$ cannot be extremal in $X$. So $X$ is in fact a 3-dimensional (2 coordinates describing the affine 2-distance configuration, and the homogeneous variable) surface, which affinely spans a 4-dimensional subspace in some higher-dimensional space.

Thus we can set $k(T^{6}_{10}) := 1$, $R(T^{6}_{10}, 1) = \{2\}$, $d := 2$, $S(T^{6}_{10}, 1) = \{(0,0), (1,0), (0,1), (-1,1), (1,0), (0,1)\}$, and $(p_2, 2) \in \mathbb{Z}[X, p_1, p_2]^2$ and $P(T^{6}_{10}, 1, 2) = \{(0,1) = \text{int conv}\{0,0, (0,1), (1,0), (1,1)\}$ to get the required classification of corresponding affine 2-distance configurations.

6.3.7 Reducing the symmetry

Theorem 6.19 Assume that $(n, r, L, F)$ represents a full 2-distance configuration $D = D(T, r)$ and that we have two solutions $\overline{f_1, f_2} \in L(\text{SysEmD}(\rho(L, r), 2, s)) = X$. Then there are two embeddings $f_1^{\lambda_1}, f_2^{\lambda_2} : N_n \rightarrow \mathbb{R}^2$ of $(N_n, \rho(L, r))$ into suitable Minkowski planes with $\overline{f_1} = \overline{b^{\lambda_1}}$ and $\overline{f_2} = \overline{b^{\lambda_2}}$, using the notation introduced via (4.10). These embeddings are unique up to affine transformations, i.e., for all $\overline{f_1} : N_n \rightarrow \mathbb{R}^2$ with $\overline{f_1} = \overline{b^{\lambda_1}}$ there is some affine transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $f_1(i) = T(f_1(i))$ for all $i \in N_n$. The corresponding affine 2-distance configurations $A_1$ and $A_2$, containing the 2-distance sets $\{f_1(1), \ldots, f_1(n)\}$ and $\{f_2(1), \ldots, f_2(n)\}$ (in suitable planes each), respectively, are well defined by $\overline{f_1}$ and $\overline{f_2}$.

We have that $A_1 = A_2$ if and only if there is a permutation $\sigma : N_n \rightarrow N_n$ and a scalar $\lambda \neq 0$ with $\overline{f_1} = \lambda \sigma(\overline{f_2})$ in the following sense: $f_1(i, j, k, l) = \lambda f_2((\sigma(i), \sigma(j), (\sigma(k), \sigma(l)))$ for all $i, j, k, l \in N_n$. In this case we have necessarily $L = \sigma(L)$, $\text{sign}(\lambda) \cdot F = \sigma(F) := (i, j, k, l) \mapsto F(\sigma(i), \sigma(j), (\sigma(k), \sigma(l)))$.

Also the converse is true: if $\overline{f} \in L(\text{SysEmD}(\rho(L, r), 2, s))$, if the permutation $\sigma : N_n \rightarrow N_n$ and a scalar $\lambda \neq 0$ satisfy $L = \sigma(L)$, $\text{sign}(\lambda) \cdot F = \sigma(F)$, then $\overline{f} = \lambda \sigma(\overline{f})$ belongs to $X$. Thus $\overline{f}$ determines the same affine 2-distance configuration as $\overline{f}$

Corollary 6.20 For each realizable candidate $(n, L, F)$ there is a finite group $G = \{a_1, \ldots, a_g\}$ of linear transformations $a_i \in \mathbb{R}^{[\mathbb{N}_g]^2}$ such that (using the notation of Theorem 6.19) $A_1 = A_2$ holds if and only if there is some $i \in \mathbb{N}_g$ and $\lambda > 0$ with $\overline{f_1} = \lambda a_i(\overline{f_2})$. All $a_i$ are orthogonal transformations for the usual scalar product in $\mathbb{R}^m$: $\overline{f} \cdot \overline{f} = (a_i(\overline{f}))^t a_i(\overline{f})$ for all $\overline{f} \in \mathbb{R}^m$.

Proof There are only finitely many permutations $\sigma : N_n \rightarrow N_n$. Restricting to all $\sigma$ with $L = \sigma(L)$ and $\pm F = \sigma(\pm F)$ we define the linear functions $a_\sigma(\overline{f}) := \sigma(\overline{f})$ if $F = \sigma(F)$ and $a_\sigma(\overline{f}) := -\sigma(\overline{f})$ if $-F = \sigma(F)$.

If for some candidate $(n, L, F)$ the corresponding group $G$ is trivial, i.e., if $G$ only contains the identity, then the set of corresponding affine 2-distance configurations coincides with all positive equivalence classes of $X = L(\text{SysEmD}(\rho, d, s))$. Especially $G$ is trivial for $T^{6}_{10}$, the exceptional 2-distance configuration discussed in 6.3.6.

Otherwise, if $G$ contains more than the trivial symmetry, we want to find a subset $Y$ of $X = L(\text{SysEmD}(\rho(L, r), 2, s))$ such that each positive equivalence class of $Y$ corresponds to exactly one equivalence class in $X$ with respect to the relation $\overline{f} \equiv \overline{f}' \Leftrightarrow \overline{f} = \lambda a_i(\overline{f})$ for some $\lambda > 0$ and $i \in \mathbb{N}_g$. We will sketch a general method for this task for polyhedral sets

$$X = \bigcup_{i \in \mathbb{N}_g} \text{rel int cone } Q_i,$$

where all $Q_i$ are finite sets in $\mathbb{R}^m$ (note again that $m = n^2$). Furthermore, we know that $X$ is closed under the transformations, i.e., for all $\overline{f} \in X$ and $g \in G$ we have $g(\overline{f}) \in X$. We assume that the representation of $X$ is closed under $G$ in the same way: for all $g \in G$ and $i \in \mathbb{N}_g$ there is some $j \in \mathbb{N}_k$ which describes the image $g(\text{cone } Q_i) = \text{cone } Q_j$ of cone $Q_i$ under $g$. 


Consequently, this induces an equivalence relation in $\mathbb{N}_k$: $i \equiv j$ if there is some $g \in G$ with $g(\text{cone } Q_i) = \text{cone } Q_j$. For given $X$ and $G$ we can compute these equivalence classes. But it still can happen that there are linearly independent vectors $x, y$ within the same basic cone rel int cone $Q_i$ with $x = g(y)$. For this it is necessary that $g(\text{cone } Q_i) = \text{cone } Q_i$ and that $g$ is not identical (up to scalar multiplication) within cone $Q_i$. It is necessary and also sufficient for this case that $g(\text{cone } Q_i) = \text{cone } Q_i$ and that for some $q \in Q_i$ its image $g(q)$ is linearly independent from $q$, see for example the affine 2-distance configurations for $T^3$ and $T^4$. Then we have to split rel int cone $Q_i$ into more polyhedral pieces. Thus we have to divide rel int cone $Q_i$ by at least one hyperplane $h = \{x : n \cdot x = 0\}$, which should have $q$ and $g(q)$ on opposite sides. As normal vector we choose the difference $n := q - g(q) \neq 0$. Since $g$ is an orthogonal transformation, we get that $n'(q + g(q)) = q' - (g(q))'g(q) = 0$, thus we really have $n'q = -n'g(q)$. Now we can replace every cone $C = \text{rel int cone } Q_j$ (at least all cones with $j \equiv i$) which is intersected by $h$ but not contained in $h$ by three relatively open cones $C \cap \{x : n_1 \cdot x < 0\}$, $C \cap \{x : n_1 \cdot x = 0\}$, and $C \cap \{x : n_1 \cdot x > 0\}$. The same procedure has to be repeated for all transformations of $h$ by symmetries in $q' \in G$, i.e., by the hyperplanes $\{x : n'q'(x) = 0\}$ with normal vector $g'^t(n)$. After this refinement process we have again a representation of $X = \bigcup_{i \in \mathbb{N}_k} \text{rel int cone } Q_i'$ by cones which are closed under $G$. Again we can determine which basic cones belong together via transformations in $G$. It is possible that further refinement steps are necessary to reduce the symmetry of basic cones, but only finitely many times.

Finally, all transformations with $g(\text{cone } Q_i) = \text{cone } Q_i$ are really invariant on cone $Q_i$. Choosing exactly one cone from every equivalence class yields the desired subset $Y$ of $X$. Note that for aesthetic reasons we did not choose the representatives of the equivalence classes independent from each other at random. Instead, we sorted the classes by decreasing dimension. For each class we have chosen randomly one representative with a maximal number of generators which coincide with generators of formerly chosen representatives.

### 6.3.8 Construction of polynomial representatives of affine 2-distance Configurations

Up to now we determined a polyhedral cone $Y \subset \mathbb{R}^m$ all whose positive equivalence classes represent exactly one affine 2-distance configuration contained in the full 2-distance configuration $D = D(T, r)$. $T$ itself is known via $(m, L, F)$, and $r$ is some real number in the semi-algebraic set $R$, i.e., $R$ is either an interval or simply $R = \{r^*\}$. $Y$ is known in the form

$$Y = \bigcup_{i \in \mathbb{N}_k} \text{rel int cone } Q_i$$

with $k \in \mathbb{N}$, where for all $i \in \mathbb{N}_k$

$$Q_i = \{g_{i, 1}(r), g_{i, 2}(r), \ldots, g_{i, h(i)}(r)\}$$

for $l(i) \in \mathbb{N}$ generators $g_{i, j} \in \mathbb{Z}[X]^m$ ($j \in \mathbb{N}_{l(i)}$). Note that for all $i \in \text{rel int cone } Q_i$ the incidence relation to the restrictions in SysEmDL$(\rho(L, r), 2, s)$ is known, especially for all $I \in (\mathbb{P}_n)^3$ we know whether $w_{\rho(L, r), 2, s}^I(1) = 0$ or $w_{\rho(L, r), 2, s}^I(1) > 0$ holds. The answer only depends on $i$ and $l$, and it is constant within rel int cone $Q_i$. This information is needed afterwards to derive the informations on strong 2-distance configurations.

Furthermore, let $x$ be the dimension of (the linear hull of) $Y$. Note that this is the same as the dimension of $L(\text{SysEmDL}(\rho(L, r), 2, s))$. If $T \neq T^0_0$, then $x$ is contained in the valid certificate for a valid full solution of $\text{SysEmDL}(\rho(L, r), 2, s)$. For $T = T^0_0$ we define $x = 3$ as an exception, but $\dim Y = \dim L(\text{SysEmDL}(\rho(L, r), 2, s)) = 4$. We define $d := x - 1$ as the dimension of the parameter space.

There must be three “abstract” points $a, b, c \in \mathbb{N}$ which are “not collinear” regarding $F$, i.e., that $F(a, b, a, c) \neq 0$ is satisfied. Without loss of generality (we renumber the points otherwise) we assume that $F(1, 2, 1, 3) = +$. We use almost the same construction like the one in the proof of Lemma 4.13 but omitting to divide all $y$-coordinates by some constant. For each $r \in L(\text{SysEmDL}(\rho(L, r), 2, s)) = H$, which is as function $r : (\mathbb{N}^2_4)^2 \to \mathbb{R}$, we define

$$S(r) := ((\phi(1, j, (1, 3)), \phi((1, 2), (1, j))))_{j \in \mathbb{N}} \in (\mathbb{R}^2)^n.$$
In fact, the map $S$ is a linear projection, leaving out some coordinates of $x$, and arranging the other coordinates in the desired structure. Thus it is clear that the image of $Y$ and of $H$ under $S(\cdot)$ is again polyhedral, with dimension at most $x = d + 1$, and vertices which are images of the generators $g_{i,j}$ ($i \in \mathbb{N}_k, j \in \mathbb{N}(n_i)$). Additionally, $S(\cdot)$ is injective unless $T = T_0^k$.

For $x \in H$ we consider $S(x)$, which is an embedding of $\mathbb{N}_n$ into $\mathbb{R}^2$. We consider the determinants defined by this embedding $S(x)$ using \textbf{4.10}. For all $I := ((x_1, y_1), (x_2, y_2)) \in (\mathbb{N}_n^2)^2$ we get that

\[
b^{S(x)}(I) = \det(S(x)(x_1) - S(x)(x_2), S(x)(y_1) - S(x)(y_2)) \\
= \det(\begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}) = x_1 - x_2, y_1 - y_2.
\]

i.e., $b^{S(x)} = x_1 - x_2, y_1 - y_2$. Since $F, 1, 2, 3 = +, we get that $b^{S(x)}$ and $x$ are positively equivalent, as wanted. We can parametrize all positive equivalence classes of the image $S(Y)$ using a parameter range $P(r) \in \mathbb{R}^d$ which is linear in this parameter $p$.

Obviously, for fixed $r$ the parameter range $P(r)$ is again a polyhedral set with the same combinatorial structure as $Y$. In general, the coordinates of the vertices describing this polyhedral set are rational functions in $r$. It is not difficult to modify the parametrization such that all coordinates are polynomials in $r$.

Remark 6.21 If we choose carefully from $S$ the points $s_1, s_2, s_3$ which fix the origin $(s_1)$ and the coordinate axes, then in most cases only $d$ of the $2n$ coordinates of the points in $S$ were not constant, while $p$ moves within $P(r)$. Only for $T = T_0^k$ with $d = 1$ there are two points which can "move" together along a horizontal line segment, always admitting a fixed difference of $x$-coordinates.

Using these techniques, we obtain a parametrization $S(T, i, r, p)$, the parameter range $P \in \mathbb{R}^d$ representing each affine 2-distance configuration exactly once (so $P$ represents $Y$), and also the extended parameter range $P_a \in \mathbb{R}^d$, which corresponds to $X = L(SystemD(\rho(L, r), 2, s))$, by a polynomial description.

6.3.9 Strong 2-distance configurations

Remember that the idea behind the system $SystemD(\rho(L, r), 2, s)$ and behind the system $SystemD(\rho(L, r), 2)$ was the weak convex position of all the points $\frac{s_i - s_j}{\rho(x, y)}$ in $U$ derived from $S = \{s_1, \ldots, s_n\}$ and the metric $\rho$. This weak convex position is necessary and also sufficient for the existence of a unit ball $B$ (i.e., a convex centered body in $\mathbb{R}^2$) which contains $U$ in its boundary. Exactly if the unit ball $B$ of the Minkowski plane $\mathbb{M}^2$ satisfies $U \subset \partial B$, then the metric induced by $S$ in $\mathbb{M}^2$ is $\rho$.

Up to now we answered the question whether or not such a suitable Minkowski plane exists. In Section \textbf{6.2.5} we have seen how the structure of $U$ determines the conditions which $B$ must satisfy: in which cases $\partial B$ must contain some line segment, in which cases $\partial B$ must contain some vertices or even in which cases $B$ is uniquely determined by $U$. These questions can be answered by knowing for each $a, b, c \in U$ whether or not $b \in rel \ \text{int} \ \mathfrak{a}$. This condition can be checked combinatorially from the abstract full position function and the incidence relation of the corresponding solution vector $x \in Y$ with the corresponding triangle inequality.

By using the full position function we know in advance the cyclic ordering of the vectors from $U$ along the unit circle.

Finally, for $b \in rel \ \text{int} \ \mathfrak{a}$ we formulate this condition precisely.

Proposition 6.22 Let $\rho = \rho(L, r)$, $x \in L(SystemD(\rho, 2, s))$ and $S = \{s_1, \ldots, s_n\} = S(x)$. Furthermore, assume that for $I = ((i_1, j_1), (i_2, j_2), (i_3, j_3)) \in (\mathbb{N}_n)^3$ the vectors

\[
a = \frac{1}{\rho(i_1, j_1)}(s_{i_1} - s_{j_1}), b = \frac{1}{\rho(i_2, j_2)}(s_{i_2} - s_{j_2}), c = \frac{1}{\rho(i_3, j_3)}(s_{i_3} - s_{j_3})
\]
and −a of U are in this cyclical order and pairwise distinct. Then \( b \in \text{rel int } \mathcal{U} \) holds if and only if \( \mathbf{x} \) is incident to \( w_{\rho,2,s}^{1} \), i.e., if \( w_{\rho,2,s}^{1}(\mathbf{x}) = 0 \).
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Symbols

\((X, \rho)\)  
metric space, [6]

\((\mathcal{M}^2, S)\)  
2-distance set, [71]

\((\mathcal{V}, \cdot, +)\)  
real linear vector space, [7]

\((a, b)\)  
open interval \(\{ x \in \mathbb{R} : a < x < b \}\), [6]

\((a, b]\)  
half open interval \(\{ a \leq x < b \}\), [6]

\((i, j)\)  
pair of \(i\) and \(j\), [6]

\(A\)  
System matrix defining real polytopal unit ball, [42]

\(A + B\)  
Minkowski sum, [9]

\(A_B = A_{B, \mathcal{M}^d}\)  
system of Busemann angular bisectors, [20]

\(A_G\)  
system of Glogovskij angular bisectors, [20]

\(A_\mu\)  
system of \(\mu\)-bisectors, [20]

\(B\)  
unit ball, [11]

\(F\)  
abstract full position function, [72]

\(H(a, b) := \{ x \in \mathbb{R}^d : a^t x \leq b \}\)  
(closed) (affine) half-space, [67]

\(I\)  
isoperimetrix, [12]

\(U_x(\varepsilon)\)  
\(\varepsilon\)-neighborhood, [7]

\([a, b]\)  
bilinear skew-symmetric form, [12]

\([a, b]\)  
half open interval \(\{ x \in \mathbb{R} : a \leq x < b \}\), [6]

\([a, b]\)  
closed interval \(\{ x \in \mathbb{R} : a \leq x \leq b \}\), [6]

\(A\)  
real algebraic numbers, [10]

\(\angle \text{bac}\)  
angle with apex \(a\) and sides containing \(b\) and \(c\), [20]

\(\angle (r_1, r_2)\)  
angle with sides \(r_1, r_2\), [20]

\(\mathcal{C}_z\)  
set of all 2-distance sets, [71]

\(\mathbb{E}^d\)  
Euclidean \(d\)-dimensional space, [9]

\(\mathbb{M}^2\)  

\(\mathbb{M}_d\)  
Minkowski spaces \((\mathbb{R}^d, \rho)\), [11]

\(\mathbb{N}\)  
set of positive integers, [6]

\(\mathbb{N}_n\)  
\(\{1, 2, \ldots, n\}\), [6]

\(\mathbb{P}_n\)  
\(\mathbb{N}^n \setminus \{(i, i) : i \in \mathbb{N}\}\), [6]

\(\mathcal{X} = \{ Y \in \mathcal{P}(X) : |Y| = n \}\)  
\(n\)-power set of \(X\), [6]

\(\mathbb{N}^0\)  
set of non-negative integers, [6]

\(\mathcal{P}(X) = \{ Y : Y \subset X \}\)  
power set of \(X\), [6]

\(\mathbb{Q}\)  
field of rational numbers, [6]

\(\mathbb{R}\)  
field of real numbers, [6]

\(\mathbb{R}^d\)  
real linear vector space of dimension \(d\), set of \(d\)-tuples, [8]

RootOf\((n, f)\)  
\(n\)-smallest real root of non-zero monovariate polynomial \(f \in \mathbb{Z}\)[X], [10]

\(\mathbb{Z}\)  
ring of integers, [6]

\(|\lambda|\)  
absolute value of \(\lambda\), [6]

\(\text{aff}\{ r_i : i \in I \}\)  
affine hull, [8]

\(\partial A\)  
boundary, [7]

\(\beta(a)\)  
maximal determinant among unit vectors, [13]

\(\wedge(A_1, \ldots, A_{d-1})\)  
generalized cross product, [66]

\(|X|\)  
cardinality of \(X\), [6]

\(\text{charcone } P\)  
characteristic cone of \(P\), [17]
SYMBOLS

\( \chi \)
chirotepe, [72]

\( \text{cl } A \)
closure, [7]

\( \text{cone } A \)
convex conical hull, [10]

\( \text{conv } A \)
convex hull, [9]

\( \dim \mathcal{V} \)
dimension of \( \mathcal{V} \), [8]

\( \rho(\epsilon, M) = \inf_{m \in M} \rho(\epsilon, m) \)
distance of \( \epsilon \in M^d \) to a set \( M \subset M^d \), [11]

\( \rho_c \)
Euclidean metric, [9]

\( \| \cdot \|_2 \)
Euclidean norm, [9]

\( \equiv_a \)
affinely equivalent, [72]

\( \equiv_f \)
fully equivalent, [73]

\( \equiv_s \)
strongly equivalent, [71]

\( \equiv_w \)
weakly equivalent, [73]

\( f_p \)
relative full position function, [72]

\( \det [a, b] \)
determinant, [13]

\( \text{int } A \)
interior of \( A \), [7]

\( \| \overrightarrow{ab} \| \)
length of the segment \( \overrightarrow{ab} \), [11]

\( \lambda_1 (X) \)
arc length of a curve, [20]

\( \lambda_2 \)
Lebesgue-measure, [20]

\( (x_1, \ldots, x_n)_C \)
\( x_1, \ldots, x_n \) are ordered along \( C \), [14]

\( \text{lin} \{ x_i : i \in I \} \)
linear subspace spanned by \( x_i \in I \), [5]

\( \langle x, \eta \rangle \)
straight line through \( x \) and \( \eta \), [9]

\( \mu_a \)
area angular measures, [20]

\( \mu_l \)
arc length angular measure, [20]

\( \| : : \| \)

\( \hat{A} \)
normalized representation \( \hat{A} \) of the system of angular bisectors \( A \), [20]

\( \hat{x} \)
normalization of \( x \), [11]

\( o \in \mathcal{V} \)
origin, [7]

\( [x, \eta] \)
ray with starting point \( x \) passing through \( \eta \), [9]

\( \text{rel bd } A \)
relative boundary, [7]
Theses for the dissertation
“Selected Problems from Minkowski Geometry”,
submitted by Dipl.-Math. Nico Düvelmeyer,
Technische Universität Chemnitz, Fakultät für Mathematik

1. Generalizing various metric properties of the Euclidean angular bisector, there are different possibilities to define angular measures and angular bisectors in normed linear planes. We consider angular bisectors of angles which are characterized by an arbitrarily chosen angular measure, and by two further approaches described in the literature. We characterize all normed linear (=Minkowski) spaces in which two of these approaches yield the same bisector for each angle of the space, in any dimension.

2. For two bisectors defined by angular measures $A_{\mu_1}$ and $A_{\mu_2}$, we have $A_{\mu_1} \equiv A_{\mu_2}$ if and only if the two measures are the same, i.e., $\mu_1 = \mu_2$.

3. We consider two special angular measures in a Minkowski space, corresponding to the area of the unit ball sector and to the length of the unit circle arc, respectively. These two angular measures are equal for all angles of a Minkowski plane if and only if the unit ball is equiframed.

4. A Minkowski space $M^d$ of dimension $d \geq 3$ is Euclidean if each two-dimensional subspace has an equiframed unit ball.

5. The bisectors due to Busemann and due to Glogovskij are equal in $M^d$ for each angle if and only if for $d = 2$ we have a Radon plane, or if $M^d$ is Euclidean in case $d \geq 3$.

6. If all angular bisectors in $M^d$, as defined by Busemann or Glogovskij, coincide with the angular bisectors $A_{\mu}$ for some fixed angular measure $\mu$, then $M^d$ is Euclidean with its standard angular measure $\mu$.

7. We transform the task of embedding a metric space with $n$ points into a suitable Minkowski space $M^d$ of dimension $d \geq 3$ into an analytic representation. The transformed problem consists of a (possibly large) number of systems of linear inequalities with additional quadratic constraints. Using this transformation and algorithmic tools we can answer some questions regarding embeddings of finite metric spaces into suitable Minkowski spaces. This includes a complete classification of possible embeddings up to affine transformations if $n \leq d + 2$. For all $n$ the existence of such an embedding can be algorithmically decided.

8. For the special case of metric spaces with just two different values for nonzero distances (2-distance sets) we determine all such metrics which are embeddable into a suitable Minkowski plane. Additionally, we give a complete classification of all corresponding embeddings up to affine transformations.

9. The classification mentioned above was algorithmically obtained and also algorithmically verified.

10. For the purpose of the classification of 2-distance sets an algorithm for solving a family of parametrized linear systems of inequalities was developed. More precisely, for linear systems of inequalities all whose coefficients are polynomials in $\mathbb{Z}[X]$, an algorithm solves all these systems for real values $X = x$ at the same time. Of course, the implementation uses exact arithmetics and exact comparison of real algebraic numbers.
Lebenslauf

von Dipl.-Math. Nico Düvelmeyer

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Erklärung

Ich erkläre an Eides Statt, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Chemnitz, den 8. Juni 2006

Nico Düvelmeyer