Well-posedness and causality for a class of evolutionary inclusions

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Nomenclature

\( \mathbb{N} \) set of integers greater than or equal to 1
\( \mathbb{R} \) set of reals
\( \mathbb{C} \) set of complex numbers
\( (a, b) \) open interval \( \{ x \in \mathbb{R} \mid a < x < b \} \)
\( [a, b) \) half-closed interval \( \{ x \in \mathbb{R} \mid a < x \leq b \} \)
\( [a, b] \) closed interval \( \{ x \in \mathbb{R} \mid a \leq x \leq b \} \)
\( \mathbb{R}_{<a} \) \( (-\infty, a) \)
\( \mathbb{R}_{>a} \) \( (a, \infty) \)
\( \mathbb{R}_{\geq a} \) \( [a, \infty) \)
\( \mathbb{R}_{\leq a} \) \( (-\infty, a] \)
\( \oplus \) direct sum
\( \otimes \) tensor product
\( \partial \Omega \) boundary of a set \( \Omega \)
\( A^B \) set of mappings \( f : B \to A \)
\( \mathcal{P}(X) \) power set of a set \( X \)
\( B(x, r) \) open ball around \( x \) with radius \( r \)
\( B[x, r] \) closed ball around \( x \) with radius \( r \)
\( \hookrightarrow \) weak convergence
\( \chi_A \) characteristic function of a set \( A \), i.e.
\[
\chi_A(x) = \begin{cases} 
1 & \text{if } x \in A, \\
0 & \text{otherwise}. 
\end{cases}
\]
\( \mathcal{H} \) Hausdorff-metric
\( \mathcal{F} \) Fourier transform on \( L_2(\mathbb{R}) \)
\( \mathcal{L}_\nu \) Fourier-Laplace transform on the weighted \( L_2 \)-space \( H_\nu \)
\( \rho(T) \) resolvent set of an operator \( T \)
<table>
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<th>Nomenclature</th>
<th>Description</th>
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<tbody>
<tr>
<td>( \sigma(T) )</td>
<td>spectrum of an operator ( T )</td>
</tr>
<tr>
<td>([M]A)</td>
<td>pre-set of a set ( M ) under the relation ( A )</td>
</tr>
<tr>
<td>( A[M] )</td>
<td>post-set of a set ( M ) under the relation ( A )</td>
</tr>
<tr>
<td>( B(\mathbb{R}) )</td>
<td>Borel-( \sigma )-algebra over ( \mathbb{R} )</td>
</tr>
<tr>
<td>( L(X) )</td>
<td>space of all bounded, linear operators ( T : X \to X )</td>
</tr>
<tr>
<td>( L_2(\mu; H) )</td>
<td>space of ( H )-valued square integrable functions, defined on some measure space ( (\Omega, \Sigma, \mu) )</td>
</tr>
<tr>
<td>( L_2(\Omega) )</td>
<td>space of ( \mathbb{C} )-valued, square integrable functions, defined on ( \Omega \subseteq \mathbb{R}^n )</td>
</tr>
<tr>
<td>( C(\Omega) )</td>
<td>space of ( \mathbb{C} )-valued, continuous functions, defined on ( \Omega \subseteq \mathbb{R} )</td>
</tr>
<tr>
<td>( C^k(\Omega) )</td>
<td>space of ( \mathbb{C} )-valued, ( k )-times continuously differentiable functions, defined on ( \Omega \subseteq \mathbb{R} )</td>
</tr>
<tr>
<td>( C^\infty(\Omega) )</td>
<td>( \bigcap_{k \in \mathbb{N}} C^k(\Omega) )</td>
</tr>
<tr>
<td>( C_c(\Omega) )</td>
<td>space of ( \mathbb{C} )-valued, continuous functions, defined on ( \Omega \subseteq \mathbb{R} ) with compact support</td>
</tr>
<tr>
<td>( C_c^\infty(\Omega) )</td>
<td>( C_c(\Omega) \cap C^\infty(\Omega) )</td>
</tr>
<tr>
<td>( \text{supp} f )</td>
<td>support of the function ( f )</td>
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Introduction

This thesis gives a unified Hilbert space approach to well-posedness of differential inclusions of the form

\[(u, f) \in \partial_0 M(\partial_0^{-1}) + A.\]  

(0.1)

Here \(\partial_0\) denotes the time derivative, \(A\) is a so-called maximal monotone set-valued operator and \(M(\partial_0^{-1})\) is an operator, which describes the behaviour of the underlying material. The concept of maximal monotonicity, even in Banach-spaces is studied in several books and articles and the author refers to the book of S. Hu and N. Papageorgiou [25] as well as to the books of G. Morusau [39] and H. Brezis [8] for a very detailed introduction to this topic. A fundamental characterization of maximal monotonicity in the Hilbert-space case was given by G. Minty in 1962 [37], who stated, that a monotone operator \(A\) is maximal monotone if and only if \(1 + A\) is surjective (see also Theorem 1.6 of this thesis). In the case of linear, single-valued operators this condition is well-known for generators of contraction semigroups (cf. [30, 20]). As an important class of maximal monotone operators, we would like to mention the subgradients of convex, lower semicontinuous, proper functions. It was proved by R. T. Rockafellar in 1970 [46], that these subgradients are indeed maximal monotone operators. For a definition and properties of subgradients we refer to I. Cioranescu, V. Barbu and T. Precupanu [3, 10]. Later on, this concept was generalized by F. Clarke to gradients of locally Lipschitz-continuous functions in [11].

Problems, which can be described by (0.1) arise in many fields, such as ordinary and partial differential equations including hysteresis phenomena ([52, 5, 24]), variational inequalities as well as switched dynamical systems ([34]). Beside the deterministic case, the concept of stochastic differential inclusions introduced by P. Krée [32] is studied in several works. We refer to [41], where, similar to the strategy we want to follow, an approximation argument is used to prove the well-posedness of a class of inclusions. For a numerical treatment of differential inclusions we refer to [16] and the references therein.

The form given in (0.1) is based on R. Picard’s article [43], where it was shown, that most of the equations arising in classical mathematical physics posses this form, where \(A\) is a skew-selfadjoint operator. The well-posedness and causality of such problems were proved and a lot of examples were studied. We wish to extend this theory to differential inclusions, where the skew-selfadjoint \(A\) is replaced by a maximal monotone operator. The classical way to study well-posedness of differential inclusions is by a semigroup approach (cf. [25, 26]). Indeed, in [31] Y. Komura has developed the concept of contractive, nonlinear semigroups generated by maximal monotone operators in Hilbert spaces (see also [8, p. 113 ff.]). Later on this concept was extended to the Banach-space case by M.G. Crandall and A. Pazy in [12, 13] and by T. Kato in [29, 28] for the nonautonomous single-valued operator case. Another way of guaranteeing well-posedness is to ensure that the stationary problem is well-posed. This was done by R. Showalter in his book [48, IV, Theorem 6.1.]. However, the idea presented in this thesis, is to look at the operator in
as an operator in time and space, following the strategy of [43], and to establish the time derivative in a suitable Hilbert space, such that it becomes a normal and continuously invertible operator (this idea was also worked out by R. Picard, cf. [45] and is introduced in section 2.1 of this thesis). In fact, there are examples of problems of the form (0.1), which are well-posed, but whose stationary version does not have a solution for every right-hand side $f$.

Beside the question of well-posedness, we shall address the problem of causality. Causality plays a crucial role in mathematical physics and we will show, that our solution operator is indeed causal. For the theory of causal mappings we refer to the book of V. Lakshmikantham, S. Leela and F.A. Merae [33]. In our setting, causality can be characterized by applying a Theorem of Paley-Wiener type (cf. [47, 56]).

The thesis is structured in three chapters. The first chapter gives an introduction to maximal monotone operators, where the concept of set-valued mappings is replaced by interpreting the operator as a relation. Most of the results in the Sections 1.1 up to 1.3 can be found in standard books dealing with maximal monotone operators. We refer to [39, 8, 20, 15]. In Section 1.4, we introduce a new class of relations, so-called uniformly bounded relations. This class allows us to generalize the framework of operator-valued material laws, introduced in [43], to those, which are relation-valued (see Section 2.6).

In the second chapter, a detailed study of well-posedness of (0.1) for a large class of material operators $M(\partial_0^{-1})$ is presented. This is done by establishing the time derivative in a suitably weighted $L^2$-space (Section 2.1) and proving well-posedness for special material operators of the form $M_0 + \partial_0^{-1}M_1$ in Section 2.2 (this covers the so called (P)-degenerated case in [43], which usually arises in the study of parabolic equations). After that, we study the causality of our solution mapping and consider perturbation problems, where we use the results of Section 1.3. In the last sections of Chapter 2 we extend the developed solution theory to initial value problems and differential inclusions on $\mathbb{R}$ as the time line, where we make use of the concept of uniformly bounded relations, introduced in Section 1.4.

In the third chapter we apply the solution theory, developed in the previous chapters, to a coupled system of diffusion and deforming equations including hysteresis phenomena. This model was studied by R. Showalter and U. Stefanelli in [50], who proved the existence and uniqueness of a solution, by reducing the equations to the stationary problem and apply a general theorem presented in [48, IV, Theorem 6.1]. They deal with complicated boundary conditions on different parts of the boundary and study different types of hysteresis models, covered by maximal monotone operators. However, for simplicity we reduce ourselves to easier boundary conditions do not study the structure of the monotone relation in detail. As a benefit of our solution theory we can omit regularity assumptions on the given data as well as relax coercitivity conditions on the operators.

In the appendix we give an introduction to the theory of tensor calculus in Hilbert spaces and provide the required theorems of tensor products of Hilbert spaces and linear operators. The basic concepts can also be found in the book of J. Weidmann [54] as well as in the book of R. Picard and D. McGhee [41]. Moreover, we recall the definition and basic properties of the Hausdorff metric. There are several references for this topic and we like to mention [22, 18] for instance.

In general, we assume that the reader is familiar with the main results of functional analysis,
Hilbert space theory and the theory of unbounded operators and we like to refer to some classical books \cite{56, 55, 30, 11, 54, 23}.

First of all I like to express my gratitude to my supervisor Prof. Dr. Rainer Picard for suggesting this topic and for uncountable moments of helpful and enlightening discussions. Furthermore I like to thank Marcus Waurick, Daniel Karrasch, Henrik Freymond, Jan Mankau, Samuel Littig and Marcus Köhler for several valuable scientific and non-scientific discussions. Moreover, I am very grateful to the whole institute of analysis at the technical university of Dresden. And last but not least I like to thank my parents and my whole family for supporting me in every period of my life.
1. Maximal monotone relations

We begin our thesis by introducing the concept of maximal monotone relations. In most books, these relations are also called maximal monotone operators, and the operators are assumed to be set-valued. Since this coincides with the well-known mathematical structure of a relation, we prefer to use this notion. There are a lot of studies in the field of maximal monotone relations. We shall only refer to a few, e.g. [15], where the focus is laid on the study of differential inclusions, [8, 3, 39] and [25] where the topic of monotone operators is studied in detail. We restrict ourselves to relations on a complex Hilbert space $H$. Throughout this chapter let $H$ be a Hilbert space with inner product $\langle \cdot | \cdot \rangle$, induced norm $| \cdot |$ and $A \subseteq H \oplus H$, i.e. a relation.

1.1. The Theorem of Minty

In article [37] George J. Minty gives a celebrated characterization for maximal monotonicity of monotone relations. To get familiar with the notion of relations instead of mappings, we like to paraphrase the proof. At first we fix some notation in order to handle relations appropriately.

**Definition 1.1.** For $M \subseteq H$ we define the post-set of $M$ under $A$ by

$$A[M] := \{ y \in H | \exists x \in M : (x, y) \in A \},$$

and the pre-set of $M$ under $A$ by

$$[M]A := \{ x \in H | \exists y \in M : (x, y) \in A \}.$$

We define the inverse relation $A^{-1}$ by

$$A^{-1} := \{ (x, y) \in H \oplus H | (y, x) \in A \}.$$

Moreover, we introduce an algebraic structure on the set of relations by

$$\lambda A + B := \{ (x, y) \in H \oplus H | \exists z_0 \in A[\{x\}], z_1 \in B[\{x\}] : y = \lambda z_0 + z_1 \}.$$  

where $\lambda \in \mathbb{C}$ and $A, B \subseteq H \oplus H$.

**Definition 1.2.** $A$ is called monotone, if for all $(u, v), (x, y) \in A$ the following property is satisfied:

$$\text{Re}\langle y - v | x - u \rangle \geq 0.$$
A is called \textit{strictly monotone}, if there exists a constant $c \in \mathbb{R}_{>0}$ such that:

$$\text{Re}(y - v|x - u) \geq c|x - u|^2$$

for all $(u,v), (x,y) \in A$.

The relation $A$ is \textit{maximal monotone}, if $A$ is monotone and for every monotone relation $B \subseteq H \oplus H$ with $A \subseteq B$ it follows $A = B$.

Finally, for $c \in \mathbb{R}$ the relation $A$ is called \textit{$c$-monotone}, if $A + c$ is monotone and \textit{$c$-maximal monotone}, if $A + c$ is maximal monotone.

To model monotonicity in Banach-spaces one deals with the so called duality map and arrives at accretive relations ([9, 26]).

\textit{Remark 1.3.} If $A$ is monotone, then the property of being maximal monotone can be characterized in the following way: If $(x,y) \in H \oplus H$ satisfies the condition

$$\forall (u,v) \in A: \text{Re}(y - v|x - u) \geq 0,$$

then $(x,y) \in A$.

We follow the ideas presented in [8], to prove the Theorem of Minty. For doing so, we state a Theorem of Min-Max-type.

\textbf{Proposition 1.4} (Min-Max-Theorem ([8 Theorème 1.1])). \textit{Let $m, n \in \mathbb{N}$ and $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m$ be compact, convex subsets. Moreover, let $K : X \times Y \to \mathbb{R}$ be a mapping which satisfies the following two conditions:

1. $\forall x \in X : y \mapsto K(x,y)$ is concave and upper semicontinuous on $Y$,
2. $\forall y \in Y : x \mapsto K(x,y)$ is convex and lower semicontinuous on $X$.

Then:

$$\min_{x \in X} \max_{y \in Y} K(x,y) = \max_{y \in Y} \min_{x \in X} K(x,y).$$

\textbf{Proof.} Obviously for all $y \in Y$

$$\min_{x \in X} K(x,y) \leq \min_{x \in X} \max_{y \in Y} K(x,y)$$

holds and so

$$\max_{y \in Y} \min_{x \in X} K(x,y) \leq \min_{x \in X} \max_{y \in Y} K(x,y).$$

\footnote{A function $f : M \to \mathbb{R}$ is called \textit{upper semicontinuous}, if for each $x \in M$

$$f(x) \geq \limsup_{y \to x} f(y)$$

and \textit{lower semicontinuous}, if for each $x \in M$

$$f(x) \leq \liminf_{y \to x} f(y).$$}
Let \( \varepsilon > 0 \) and define the mapping \( K_\varepsilon : X \times Y \to \mathbb{R} \) by
\[
K_\varepsilon(x, y) := K(x, y) + \varepsilon|x|^2 \quad (x \in X, y \in Y).
\]
Clearly, \( K_\varepsilon \) is convex and lower semicontinuous in the first and concave and upper semicontinuous in the second variable. Moreover, we define the upper semicontinuous function
\[
f_\varepsilon : Y \to \mathbb{R}, \quad y \mapsto \min_{x \in X} K_\varepsilon(x, y),
\]
where the minimum exists due to the compactness of \( X \). We find an element \( E_y \in X \) with
\[
f_\varepsilon(y) = K_\varepsilon(E_y, y) \quad (y \in Y).
\]
By the upper semicontinuity of \( f_\varepsilon \) and the compactness of \( Y \) there exists \( y^* \in Y \) such that:
\[
\min_{x \in X} K_\varepsilon(x, y^*) = f_\varepsilon(y^*) = \max_{y \in Y} f_\varepsilon(y) = \max_{y \in Y} \min_{x \in X} K_\varepsilon(x, y).
\]
Let now \( x \in X, y \in Y \) and \( t \in (0, 1) \). Then we estimate:
\[
K_\varepsilon(x, ty + (1 - t)y^*) \geq tK_\varepsilon(x, y) + (1 - t)K_\varepsilon(x, y^*) \\
\geq tK_\varepsilon(x, y) + (1 - t)f_\varepsilon(y^*).
\]
For \( x = E_{ty+(1-t)y^*} \) we conclude
\[
f_\varepsilon(ty + (1 - t)y^*) \geq tK_\varepsilon(E_{ty+(1-t)y^*}, y) + (1 - t)f_\varepsilon(y^*)
\]
and since \( f_\varepsilon(y^*) \geq f_\varepsilon(ty + (1 - t)y^*) \) we get
\[
f_\varepsilon(y^*) \geq K_\varepsilon(\xi_t, y) \quad (t \in (0, 1), y \in Y), \tag{1.1}
\]
where \( \xi_t := E_{ty+(1-t)y^*} \). Then for all \( x \in X \) we obtain:
\[
K_\varepsilon(x, ty + (1 - t)y^*) \geq K_\varepsilon(\xi_t, ty + (1 - t)y^*) \\
\geq tK_\varepsilon(\xi_t, y) + (1 - t)K_\varepsilon(\xi_t, y^*). \tag{1.2}
\]
Let \( (t_n)_{n \in \mathbb{N}} \in (0, 1)^\mathbb{N} \) be a sequence with \( t_n \to 0 \) as \( n \to \infty \). Since \( X \) is compact, there exists a subsequence \( (t_{n_j})_{j \in \mathbb{N}} \) such that \( (\xi_{t_{n_j}})_{j \in \mathbb{N}} \) converges. We denote the limit by \( \xi_0 \in X \). From (1.2) it follows that
\[
K_\varepsilon(\xi_0, y^*) \leq \liminf_{j \to \infty} K_\varepsilon(\xi_{t_{n_j}}, y^*) \\
= \liminf_{j \to \infty} (t_{n_j} K_\varepsilon(\xi_{t_{n_j}}, y^*) + (1 - t_{n_j})K_\varepsilon(\xi_{t_{n_j}}, y^*)) \\
\leq \limsup_{j \to \infty} K_\varepsilon(x, t_{n_j}y^* + (1 - t_{n_j})y^*) \\
\leq K_\varepsilon(x, y^*)
\]
for all \(x \in X\). Therefore \(\xi_0 = E_y^*\) and with (1.1) we get
\[
f_\varepsilon(y^*) \geq \liminf_{j \to \infty} K_\varepsilon(\xi_{t_n_j}, y) \geq K_\varepsilon(E_y^*, y) \quad (y \in Y).
\]
So it follows that
\[
\max_{y \in Y} K_\varepsilon(E_y^*, y) \leq f_\varepsilon(y^*) = \min_{x \in X} K_\varepsilon(x, y^*)
\]
and thus
\[
\min x \max y K_\varepsilon(x, y) \leq \max y \min x K_\varepsilon(x, y).$$
Since \(X\) is compact, there exists a constant \(C > 0\) with
\[
K_\varepsilon(x, y) = K(x, y) + \varepsilon|x|^2 \leq K(x, y) + \varepsilon C \quad (x \in X, y \in Y)
\]
and so we get
\[
\min x \max y K_\varepsilon(x, y) \leq \max y \min x K_\varepsilon(x, y) \leq \max y \min x K(x, y) + \varepsilon C.
\]
Since this holds for all \(\varepsilon > 0\) the missing inequality follows.

**Proposition 1.5** (\[8, Theoreme 2.1\]). Let \(C \subseteq H\) be closed and convex and \(A\) monotone. Then for each \(y \in H\) there exists an element \(x \in C\) such that
\[
\Re\langle \eta + x | \xi - x \rangle \geq \Re\langle y | \xi - x \rangle
\]
for all \(\xi \in C\) with \((\xi, \eta) \in A\).

**Proof.** Without loss of generality let \(y = 0\) (otherwise consider the relation \([A] - (0, y) := \{(u, v) | (u, v + y) \in A\}\), which is also monotone). Moreover, let \((\xi, \eta) \in A\) where \(\xi \in C\) and define:
\[
S[\xi, \eta] := \{x \in C | \Re\langle \eta + x | \xi - x \rangle \geq 0\}.
\]
We have to show, that
\[
\bigcap_{(\xi, \eta) \in A, \xi \in C} S[\xi, \eta] \neq \emptyset. \tag{1.3}
\]
First we show, that \(S[\xi, \eta]\) is convex, bounded and closed.

(a) Let \(u, v \in S[\xi, \eta]\) and \(t \in (0, 1)\). Then
\[
\Re\langle \eta + tu + (1 - t)v | \xi - tu - (1 - t)v \rangle = \Re\langle \eta + tu + (1 - t)v | \xi \rangle
\]
1.1. Theorem of Minty

\[= t \text{Re} \langle \eta + u | \xi - u \rangle + (1 - t) \text{Re} \langle \eta + v | \xi - v \rangle + t |u|^2 + (1 - t) |v|^2 - |tu + (1 - t)v|^2 \geq 0.\]

Hence, \(tu + (1 - t)v \in S[\xi, \eta].\)

(b) Let \(x \in S[\xi, \eta]\) and compute

\[0 \leq \text{Re} \langle \eta + x | \xi - x \rangle = \text{Re} \langle \eta + x | \xi \rangle - \text{Re} \langle \eta | x \rangle - |x|^2 = \text{Re} \langle \eta | \xi \rangle + \text{Re} \langle \xi - \eta | x \rangle - |x|^2.\]

The Cauchy-Schwarz-inequality yields

\[|x|^2 \leq \text{Re} \langle \eta | \xi \rangle + \text{Re} \langle \xi - \eta | x \rangle \leq |\eta| |\xi| + |\xi - \eta| |x|.\]

From this inequality we conclude

\[|x| \leq \frac{1}{2} |\xi - \eta| + \sqrt{\frac{1}{4} |\xi - \eta|^2 + |\eta| |\xi|}\]

which shows the boundedness of \(S[\xi, \eta].\)

(c) The closedness of \(S[\xi, \eta]\) follows immediately from the continuity of the inner product.

The convexity and closedness implies the weak-closedness of \(S[\xi, \eta]\) (see [53, p. 120, Theorem 2]) and the boundedness of \(S[\xi, \eta]\) yields the weak-compactness. Therefore, for showing (1.3), it suffices to prove that \(\{S[\xi, \eta]: \xi \in C, (\xi, \eta) \in A\}\) has the finite intersection property. Let \(n \in \mathbb{N}\) be fixed and choose a finite sequence \((\xi_i, \eta_i)\) with \(\xi_i \in C\) for all \(i \in \{1, \ldots, n\}\).

We consider the \(n-\)simplex

\[K := \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n | \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0 \ (i \in \{1, \ldots, n\})\}\]

and define the following mapping

\[f : K \times K \rightarrow \mathbb{R} \quad (\lambda, \mu) \mapsto \sum_{i=1}^{n} \mu_i \text{Re} \left( \sum_{j=1}^{n} \lambda_j \xi_j + \eta_i \sum_{j=1}^{n} \lambda_j \xi_j - \xi_i \right).\]

It is not hard to see, that \(f\) satisfies the conditions of Proposition 1.4 and hence

\[\min_{\lambda \in K} \max_{\mu \in K} f(\lambda, \mu) = \max_{\mu \in K} \min_{\lambda \in K} f(\lambda, \mu).\]

Since the mapping

\[g : K \rightarrow \mathbb{R} \quad \lambda \mapsto \max_{\mu \in K} f(\lambda, \mu)\]
is lower semicontinuous, we find a $\lambda^0 \in K$ such that
\[
g(\lambda^0) = \min_{\lambda \in K} \max_{\mu \in K} f(\lambda, \mu) = \max_{\mu \in K} \min_{\lambda \in K} f(\lambda, \mu) \leq \max_{\mu \in K} f(\mu, \mu).
\]
So for all $\nu \in K$ it follows that
\[
f(\lambda^0, \nu) \leq \max_{\mu \in K} f(\lambda^0, \mu) = g(\lambda^0) \leq \max_{\mu \in K} f(\mu, \mu).
\]
Using the monotonicity of $A$, we compute for each $\mu \in K$
\[
f(\mu, \mu) = \sum_{i=1}^{n} \mu_i \text{Re} \left( \sum_{j=1}^{n} \mu_j \xi_j + \eta_i \sum_{j=1}^{n} \mu_j \xi_j - \xi_i \right)
= \sum_{i=1}^{n} \mu_i \text{Re} \left( \sum_{j=1}^{n} \mu_j \xi_j \sum_{j=1}^{n} \mu_j \xi_j - \xi_i \right) + \sum_{i=1}^{n} \mu_i \text{Re} \langle \eta_i \sum_{j=1}^{n} \mu_j \xi_j - \xi_i \rangle
= \sum_{i=1}^{n} \mu_i \text{Re} \langle \eta_i \sum_{j=1}^{n} \mu_j \xi_j - \xi_i \rangle
= \sum_{i,j=1}^{n} \mu_i \mu_j \text{Re} \langle \eta_i \xi_j - \xi_i \rangle
\leq 0.
\]
So for each $\nu \in K$ we get
\[
f(\lambda^0, \nu) = \sum_{i=1}^{n} \nu_i \text{Re} \left( \sum_{j=1}^{n} \lambda^0_j \xi_j + \eta_i \sum_{j=1}^{n} \lambda^0_j \xi_j - \xi_i \right) \leq 0.
\]
Since $C$ is convex we conclude that $x(\lambda^0) := \sum_{j=1}^{n} \lambda^0_j \xi_j \in C$ and with $(\nu^k_i)_{i \in \{1, \ldots, n\}} = (\delta_{ik})_{i \in \{1, \ldots, n\}}$ for $k \in \{1, \ldots, n\}$ we obtain
\[
\text{Re} \langle x(\lambda^0) + \eta_k | x(\lambda^0) - \xi_k \rangle \leq 0 \quad (k \in \{1, \ldots, n\}).
\]
This implies
\[
x(\lambda^0) \in \bigcap_{i=1}^{n} S[\xi_i, \eta_i]
\]
which shows the assertion.

With this proposition we are now able to prove Minty’s Theorem.
1.1. The Theorem of Minty

**Theorem 1.6** (G. Minty,[37]). Let $A$ be monotone. Then the following statements are equivalent:

(i) $A$ is maximal monotone,

(ii) $\forall \lambda > 0 : (1 + \lambda A)[H] = H$,

(iii) $\exists \lambda > 0 : (1 + \lambda A)[H] = H$.

Proof. (i) $\Rightarrow$ (ii): Let $\lambda > 0$ and $y \in H$. Then we apply Proposition 1.5 with $C := H$ and the monotone relation $\lambda A$ and get the existence of an element $x \in H$ such that

$$\text{Re} \langle \eta - (y - x) | \xi - x \rangle \geq 0 \quad ((\xi, \eta) \in \lambda A).$$

Since $\lambda A$ is maximal monotone, we conclude

$$(x, y - x) \in \lambda A$$

or equivalently

$$(x, y) \in 1 + \lambda A.$$

(ii) $\Rightarrow$ (iii): This implication holds trivially.

(iii) $\Rightarrow$ (i): Let $(x_0, y_0) \in H \oplus H$ with

$$\text{Re} \langle y - y_0 | x - x_0 \rangle \geq 0 \quad ((x, y) \in A). \quad (1.4)$$

Let $\lambda > 0$ be such that (iii) is satisfied and take $x_1 \in H$ such that

$$(x_1, x_0 + \lambda y_0) \in 1 + \lambda A.$$ 

Thus, there is $y_1 \in H$ with $(x_1, y_1) \in A$ and

$$x_0 + \lambda y_0 = x_1 + \lambda y_1.$$

From (1.4) we get

$$-\lambda |y_1 - y_0|^2 = \text{Re} \langle y_1 - y_0 | \lambda (y_0 - y_1) \rangle$$

$$\geq 0$$

and since $\lambda > 0$ it follows that $y_1 = y_0$. Analogously we estimate

$$-\frac{1}{\lambda} |x_1 - x_0|^2 = \text{Re} \langle x_1 - x_0 | \frac{1}{\lambda} (x_0 - x_1) \rangle$$

$$\geq 0$$

and so $x_1 = x_0$. Hence, $(x_0, y_0) \in A$ which shows the maximal monotonicity by Remark 1.3.
Remark 1.7. In the operator case, the assumption on $A$ being maximal monotone is well-known for generators (here $-A$) of contraction semigroups (see [19] p. 82 ff.).

We like to mention some examples for maximal monotone relations, which play a crucial rule in the topic of partial differential equations and inclusions.

Example 1.8. (a) Let $T : D(T) \subseteq H \to H$ be a skew-seldefjoint operator. Then $T$ is maximal monotone, since $\text{Re}(Tu|u) = 0$ for each $u \in D(T)$ and $\sigma(T) \subseteq i[\mathbb{R}]$.

(b) Let $T : D(T) \subseteq H \to H$ be a positive, selfadjoint operator. Then $T$ is maximal monotone, since $\sigma(T) \subseteq \mathbb{R}_{\leq 0}$.

(c) Let $\varphi : H \to (-\infty, \infty]$ be a lower semicontinuous, proper and convex function. Then its subgradient (for definition see [3, 39]) $\partial \varphi$ is maximal monotone. For a proof we refer to [46, 39].

1.2. Properties of maximal monotone relations

Now we want to collect some basic properties of maximal monotone relations, which are needed for dealing with differential inclusions. The properties and proofs can also be found in [39]. For further properties we refer to [8, 39, 25].

Remark 1.9. Let $A$ be maximal monotone and $x \in [H]A$. Then $A[\{x\}]$ is closed and convex. Indeed, let $(y_n)_{n \in \mathbb{N}} \in A[\{x\}]^\mathbb{N}$ with $y_n \to y \in H$ as $n \to \infty$. Then we obtain for all $(\xi, \eta) \in A$

$$0 \leq \text{Re}(y_n - \eta|x - \xi|) \to \text{Re}(y - \eta|x - \xi|)$$

and hence $\text{Re}(y - \eta|x - \xi|) \geq 0$. By the maximal monotonicity of $A$, we conclude $(x, y) \in A$. For $y, \tilde{y} \in A[\{x\}]$ and $t \in (0, 1)$ we observe for all $(\xi, \eta) \in A$

$$\text{Re}(ty + (1 - t)\tilde{y} - \eta|x - \xi|) = t\text{Re}(y - \eta|x - \xi|) + (1 - t)\text{Re}(\tilde{y} - \eta|x - \xi|) \geq 0.$$ 

Hence, $(x, ty + (1 - t)\tilde{y}) \in A$ as $A$ is maximal monotone.

We can apply the projection theorem on the set $A[\{x\}]$ for each $x \in [H]A$ and find a projector $P_x : H \to A[\{x\}]$. With this projector we define the following mapping:

$$A^0 : [H]A \to A[H]$$

$$x \mapsto P_x(0),$$

the so-called minimal section of $A$.

Proposition 1.10. Let $A$ be maximal monotone. Then $A$ is demiclosed, i.e. for each sequence $((x_n, y_n))_{n \in \mathbb{N}} \in A^\mathbb{N}$ with $x_n \to x \in H$ and $y_n \to y \in H$ as $n \to \infty$ it follows that $(x, y) \in A$.

Proof. Let $((x_n, y_n))_{n \in \mathbb{N}} \in A^\mathbb{N}$ be a sequence like above. Then for every $(\xi, \eta) \in A$ we estimate:

$$|\text{Re}(y_n - \eta|x_n - \xi|) - \text{Re}(y - \eta|x - \xi|)| \leq |\text{Re}(y_n - \eta|x_n - x|)| + |\text{Re}(y_n - y|x - \xi|)|$$
Therefore, we get
\[
\text{Re}(y - \eta|x - \xi|) \geq 0
\]
and since \( A \) is maximal monotone this implies that \((x, y) \in A\). \(\square\)

**Definition 1.11.** Let \( A \) be maximal monotone and \( \lambda > 0 \). Then we define by
\[
J_\lambda(A) := (1 + \lambda A)^{-1}
\]
the *resolvent of \( A \) to \( \lambda \).* Moreover, we define the *Yosida-Approximation of \( A \) to \( \lambda \) by*
\[
A_\lambda := \lambda^{-1}(1 - J_\lambda(A)).
\]

By the monotonicity of \( A \) it follows, that \( J_\lambda(A) \) and \( A_\lambda \) are mappings and by Theorem 1.6 they are defined on the whole space \( H \).

We will prove some properties of \( J_\lambda(A) \) and \( A_\lambda \).

**Proposition 1.12** ([39, Theorem 1.3]). Let \( A \) be maximal monotone and \( \lambda > 0 \). Then the following holds:

(a) \( J_\lambda(A) \) is Lipschitz-continuous\(^2\) with \(|J_\lambda(A)|_{\text{Lip}} \leq 1\).

(b) \( \forall x \in H : A_\lambda(x) \in A[\{J_\lambda(A)(x)\}] \).

(c) \( A_\lambda \) is monotone and Lipschitz-continuous with \(|A_\lambda|_{\text{Lip}} \leq \lambda^{-1}\).

(d) \( \forall x \in [H]A : |A_\lambda(x)| \leq |A^0(x)| \).

(e) \( \forall x \in [H]A : A_\lambda(x) \rightarrow A^0(x) \) for \( \lambda \rightarrow 0^+ \).

(f) The set \([H]A\) is convex and for all \( x \in H \) it holds that \( J_\lambda(A)(x) \rightarrow P_{[H]A}(x) \) as \( \lambda \rightarrow 0^+ \), where \( P_{[H]A} \) denotes the projector on the set \([H]A\).

**Proof.** (a) Let \( x, y \in H \). By the definition of \( J_\lambda(A) \) it follows, that
\[
(J_\lambda(A)(x), x) \in 1 + \lambda A
\]
which is equivalent to
\[
(J_\lambda(A)(x), \lambda^{-1}(x - J_\lambda(A)(x)) \in A.
\]

\(^2\)For a Lipschitz-continuous function \( F : D(F) \subset X \rightarrow Y \), where \( X, Y \) are metric spaces, we denote the best Lipschitz-constant of \( F \) by
\[
|F|_{\text{Lip}} := \inf \{ c \geq 0 : \forall x_0, x_1 \in D(F) : d_Y(F(x_0), F(x_1)) \leq cd_X(x_0, x_1) \}.
\]
Therefore we estimate
\[
\Re(\lambda^{-1}(x - J_\lambda(A)(x)) - \lambda^{-1}(y - J_\lambda(A)(y))|J_\lambda(A)(x) - J_\lambda(A)(y)| \geq 0,
\]
which in turn is equivalent to
\[
\Re(x - y|J_\lambda(A)(x) - J_\lambda(A)(y)) \geq |J_\lambda(A)(x) - J_\lambda(A)(y)|^2.
\]
By the Cauchy-Schwarz-Inequality we conclude
\[
|J_\lambda(A)(x) - J_\lambda(A)(y)| \leq |x - y|.
\]

(b) Let \(x, y \in H\). Then
\[
(x, y) \in A_\lambda \iff (x, \lambda y) \in 1 - J_\lambda(A) \\
\iff (x, x - \lambda y) \in J_\lambda(A) \\
\iff (x - \lambda y, x) \in 1 + \lambda A \\
\iff (x - \lambda y, \lambda y) \in \lambda A \\
\iff (x - \lambda y, y) \in A \\
\iff (J_\lambda(A)(x), y) \in A.
\]
So we see \(y = A_\lambda(x) \in A\{J_\lambda(A)(x)\}\).

(c) Let \(x, y \in H\). We calculate
\[
\Re(A_\lambda(x) - A_\lambda(y)|x - y) = \lambda^{-1}\Re(x - J_\lambda(A)(x) - (y - J_\lambda(A)(y))|x - y) \\
= \lambda^{-1}(|x - y|^2 - \Re(J_\lambda(A)(x) - J_\lambda(A)(y)|x - y)).
\]
From (a) we know that
\[
\Re(J_\lambda(A)(x) - J_\lambda(A)(y)|x - y) \leq |x - y|^2
\]
and thus
\[
\Re(A_\lambda(x) - A_\lambda(y)|x - y) \geq 0.
\]
For the Lipschitz-continuity we first observe
\[
x - y = J_\lambda(A)(x) - J_\lambda(A)(y) + \lambda(A_\lambda(x) - A_\lambda(y)).
\]
Using the monotonicity of \(A\) and (b) we get
\[
\Re(x - y|A_\lambda(x) - A_\lambda(y)) = \Re(J_\lambda(A)(x) - J_\lambda(A)(y)|A_\lambda(x) - A_\lambda(y)) \\
+ \lambda|A_\lambda(x) - A_\lambda(y)|^2 \\
\geq \lambda|A_\lambda(x) - A_\lambda(y)|^2.
\]
By the Cauchy-Schwarz-Inequality this yields
\[
|A_\lambda(x) - A_\lambda(y)| \leq \frac{1}{\lambda}|x - y|.
1.2. Properties of maximal monotone relations

(d) Let \( x \in [H]A \). Then \( x = J_\lambda(A)((1 + \lambda A^0)(x)) \) and so by (a)
\[
|A_\lambda(x)| = \lambda^{-1}|J_\lambda(A)((1 + \lambda A^0)(x)) - J_\lambda(A)(x)| \\
\leq \lambda^{-1}|x + \lambda A^0(x) - x| \\
= |A^0(x)|.
\]

(e) Let \( x \in [H]A \) and \( (\lambda_k)_{k \in \mathbb{N}} \in \mathbb{R}_{>0}^\mathbb{N} \) with \( \lambda_k \to 0 \) as \( k \to \infty \). By (d) the sequence \( (A_{\lambda_k}(x))_{k \in \mathbb{N}} \) is bounded and so there exists an element \( p \in H \) and a subsequence \( (\lambda_{k_j})_{j \in \mathbb{N}} \) such that
\[
A_{\lambda_{k_j}}(x) \rightharpoonup p \quad (j \to \infty).
\]
On the other hand we know that
\[
J_{\lambda_{k_j}}(A)(x) = x - \lambda_{k_j} A_{\lambda_{k_j}}(x) \to x \quad (j \to \infty).
\]
By (b) for all \( j \in \mathbb{N} \) we have \( (J_{\lambda_{k_j}}(A)(x), A_{\lambda_{k_j}}(x)) \in A \). So it follows by the demiclosedness of \( A \) (Proposition 1.10) that
\[
(x, p) \in A.
\]
Moreover, we get
\[
|\langle p, A_{\lambda_{k_j}}(x) \rangle| \leq |p||A^0(x)|
\]
and letting \( j \to \infty \) it follows that
\[
|p| \leq |A^0(x)|.
\]
By the definition of \( A^0 \) the element \( A^0(x) \) is the unique element in \( A\{x\} \) with the smallest norm and thus
\[
p = A^0(x).
\]
So we have shown, that
\[
A_{\lambda_{k_j}}(x) \to A^0(x) \quad (j \to \infty) \quad (1.5)
\]
and by (d) we estimate
\[
\limsup_{j \to \infty} |A_{\lambda_{k_j}}(x)| \leq |A^0(x)|. \quad (1.6)
\]
From (1.5) and (1.6) we get
\[
A_{\lambda_{k_j}}(x) \to A^0(x) \quad (j \to \infty)
\]
which completes the proof of part (e).

(f) We define \( C := \text{conv}([H]A) \). Let \( x \in H \) be fixed. Then we get for every \( (u, v) \in A \)
\[
\text{Re}\langle A_\lambda(x) - v|J_\lambda(A)(x) - u \rangle \geq 0.
\]
Since \( A_\lambda(x) = \lambda^{-1}(x - J_\lambda(A)(x)) \) we estimate
\[
\text{Re}\langle x - J_\lambda(A)(x) - \lambda v|J_\lambda(A)(x) - u \rangle \geq 0
\]
and therefore
\[
|J_\lambda(A)(x)|^2 \leq \Re \langle x - \lambda v | J_\lambda(A)(x) - u \rangle + \Re \langle J_\lambda(A)(x)|u \rangle \tag{1.7}
\]
\[
\leq |x - \lambda v|(|J_\lambda(A)(x)| + |u|) + |J_\lambda(A)(x)||u|
\]
\[
= |J_\lambda(A)(x)||x - \lambda v| + |x - \lambda v||u|.
\]
Thus, we have
\[
|J_\lambda(A)(x)| \leq \frac{1}{2}(|x - \lambda v| + |u|) + \sqrt{\frac{1}{4}(|x - \lambda v|^2 + |x - \lambda v||u|}. \tag{1.8}
\]

Let \((\lambda_k)_{k \in \mathbb{N}} \in \mathbb{R}_0^\mathbb{N}\) be a sequence with \(\lambda_k \to 0\) as \(k \to \infty\). According to (1.8) there exist a subsequence \((\lambda_{k_j})_{j \in \mathbb{N}}\) and \(q \in H\) such that
\[
J_{\lambda_{k_j}}(A)(x) \to q \quad (j \to \infty).
\]
By using (1.7) we estimate
\[
|q|^2 \leq \limsup_{j \to \infty} |J_{\lambda_{k_j}}(A)(x)|^2 \leq \Re \langle x|q - u \rangle + \Re \langle q|u \rangle \quad (u \in [H]A),
\]
which is equivalent to
\[
|q|^2 \leq \limsup_{j \to \infty} |J_{\lambda_{k_j}}(A)(x)|^2 \leq \Re \langle x|q - u \rangle + \Re \langle q|u \rangle \quad (u \in C). \tag{1.9}
\]
Hence,
\[
\Re \langle x - x|q - u \rangle \leq 0 \quad (u \in C). \tag{1.10}
\]
Since \(C\) is convex and closed, it is weakly-closed and since \((J_{\lambda_{k_j}}(A)(x))_{j \in \mathbb{N}} \in C^\mathbb{N}\) it follows that \(q \in C\). From (1.10) we derive
\[
|q - x|^2 \leq |q - x|^2 - 2\Re \langle q - x|q - u \rangle + |q - u|^2
\]
\[
= |u - x|^2
\]
for all \(u \in C\). Therefore \(q = P_C(x)\) and so (1.9) turns into
\[
\limsup_{j \to \infty} |J_{\lambda_{k_j}}(A)(x)|^2 \leq \Re \langle x|P_C(x)|u \rangle + \Re \langle P_C(x)|u \rangle \quad (u \in C).
\]
In particular for \(u = P_C(x)\) we get
\[
\limsup_{j \to \infty} |J_{\lambda_{k_j}}(A)(x)|^2 \leq |P_C(x)|^2
\]
and this inequality together with the weak convergence implies
\[
J_{\lambda_{k_j}}(A)(x) \to P_C(x) \quad (j \to \infty). \tag{1.11}
\]
1.2. Properties of maximal monotone relations

It remains to show, that $C = \overline{[H]A}$. To this end let $z \in C$. Then by (1.11) we know that

$$J_{\lambda_{k_j}}(A)(z) \to z \quad (j \to \infty)$$

for a nullsequence $(\lambda_{k_j})_{j \in \mathbb{N}} \in \mathbb{R}_{\geq 0}^\mathbb{N}$ and since $J_{\lambda_{k_j}}(A)(z) \in [H]A$ for all $j \in \mathbb{N}$ it follows that

$$z \in \overline{[H]A}$$. Since the other inclusion is trivial, this completes the proof.

For our study of differential inclusions in Chapter 2, we need to extend a maximal monotone relation on $H$ to $H$-valued $L_2$-spaces.

Definition 1.13. Let $(\Omega, \Sigma, \mu)$ be a measure space and $A \subseteq H \oplus H$. We define

$$A_{L_2(\mu; H)} := \{(u, v) \in L_2(\mu; H) \oplus L_2(\mu; H) \mid (u(\xi), v(\xi)) \in A \ (\xi \in \Omega \ \mu\text{-a.e.})\}$$

the extension of $A$ on $L_2(\mu; H)$.

We want to find sufficient conditions, which guarantee the maximal monotonicity of the above extension (see [39, p.31]).

Proposition 1.14. Let $(\Omega, \Sigma, \mu)$ be a measure space and $A \subseteq H \oplus H$ monotone. Then the following statements hold:

(a) $A_{L_2(\mu; H)}$ is monotone.

(b) If $\mu(\Omega) < \infty$ and $A$ is maximal monotone, then $A_{L_2(\mu; H)}$ is maximal monotone.

(c) If $(0, 0) \in A$ and $A$ is maximal monotone, then $A_{L_2(\mu; H)}$ is maximal monotone.

Proof. (a) Let $(u, v), (x, y) \in A_{L_2(\mu; H)}$. Then we estimate

$$\text{Re}(u - x|v - y)_{L_2(\mu; H)} = \int_{\Omega} \text{Re}(u(\xi) - x(\xi)|v(\xi) - y(\xi)) \ d\mu(\xi) \geq 0.$$  

Thus, $A_{L_2(\mu; H)}$ is monotone.

(b) We will use Theorem 1.6 to prove the maximal monotonicity of $A_{L_2(\mu; H)}$. Let $g \in L_2(\mu; H)$ and define

$$u := (1 + A)^{-1} \circ g \in H^\Omega.$$ 

Since $g$ is measurable and $(1 + A)^{-1}$ is continuous we conclude that $u$ is also measurable. Let $(x, y) \in A$ ($A \neq \emptyset$, since $A$ is maximal monotone) and define the following measurable mappings

$$\hat{x} : \Omega \to H$$

$$\omega \mapsto x$$
and

$$\hat{y} : \Omega \rightarrow H$$

$$\omega \mapsto y.$$ 

We know that $\hat{x}, \hat{y} \in L_2(\mu; H)$ since $\mu(\Omega) < \infty$. Moreover, $(\hat{x}, \hat{y}) \in A_{L_2(\mu; H)}$. We compute

$$\int_{\Omega} |u(\xi)|^2 d\mu(\xi) \leq 2 \int_{\Omega} |u(\xi) - \hat{x}(\xi)|^2 + |\hat{x}(\xi)|^2 d\mu(\xi)$$

$$= 2 \int_{\Omega} |(1 + A)^{-1}(g(\xi)) - (1 + A)^{-1}(\hat{x}(\xi) + \hat{y}(\xi))|^2 d\mu(\xi) + 2 \int_{\Omega} |\hat{x}(\xi)|^2 d\mu(\xi)$$

$$= 2 \int_{\Omega} |J_1(A)(g(\xi)) - J_1(A)(\hat{x}(\xi) + \hat{y}(\xi))|^2 d\mu(\xi) + 2 \int_{\Omega} |\hat{x}(\xi)|^2 d\mu(\xi)$$

$$\leq 2 \int_{\Omega} |g(\xi) - (\hat{x}(\xi) + \hat{y}(\xi))|^2 d\mu(\xi) + 2 \int_{\Omega} |\hat{x}(\xi)|^2 d\mu(\xi) < \infty.$$

So $u \in L_2(\mu; H)$ and thus $(1 + A_{L_2(\mu; H)})[L_2(\mu; H)] = L_2(\mu; H)$. Therefore, $A_{L_2(\mu; H)}$ is maximal monotone by Theorem 1.6.

(c) Let $g$ and $u$ be chosen as before. Since $(0, 0) \in A$ it follows, that $(0, 0) \in 1 + A$ and we conclude

$$\int_{\Omega} |u(\xi)|^2 d\mu(\xi) = \int_{\Omega} |u(\xi) - 0|^2 d\mu(\xi)$$

$$= \int_{\Omega} |J_1(A)(g(\xi)) - J_1(A)(0)|^2 d\mu(\xi)$$

$$\leq \int_{\Omega} |g(\xi) - 0|^2 d\mu(\xi) < \infty.$$

Like in (b) we get that $A_{L_2(\mu; H)}$ is maximal monotone.

\[\square\]

1.3. Sums of maximal monotone relations

An important issue for studying differential inclusions, is the question, whether the sum of two maximal monotone relations is again maximal monotone. Several answers on this question can be found, for instance in [39] or [26, Chapter 3, Section 3]. It turns out, that the boundedness or relative boundedness of a relation plays a crucial role for answering this question.

Lemma 1.15. Let $A \subseteq H \oplus H$ be maximal monotone and $C : H \rightarrow H$ be Lipschitz-continuous and monotone. Then $A + C$ is maximal monotone.
Proof. The monotonicity is clear. For showing the maximality, we use Minty’s Theorem. If $C = 0$ the statement is trivial. In the case $C \neq 0$ let $0 < \lambda < \frac{1}{\|C\|_{\text{Lip}}}$ and $f \in H$. Then

$$|u \mapsto J_\lambda(A)(f - \lambda C(u))|_{\text{Lip}} < 1.$$ 

According to the Contraction-Mapping-Theorem we know that there exists $u^* \in H$ such that

$$u^* = J_\lambda(A)(f - \lambda C(u^*)).$$ 

This, in turn, is

$$f \in (1 + \lambda(A + C))[H]$$ 

and thus

$$(1 + \lambda(A + C))[H] = H.$$

\[ \square \]

Corollary 1.16. Let $A, B \subseteq H \oplus H$ be maximal monotone. Then for all $\lambda \in \mathbb{R}_{>0}$ the relation $A + B_\lambda$ is maximal monotone.

The next Lemma can also be found in [25, Lemma 7.41], where the Banach space case is studied and maximal monotonicity is replaced by $m$-accretivity.

Lemma 1.17. Let $A, B \subseteq H \oplus H$ be maximal monotone and $y \in H$. Denote

$$x_\lambda := (1 + A + B_\lambda)^{-1}(y)$$

for $\lambda \in \mathbb{R}_{>0}$. If

$$\sup_{\lambda \in \mathbb{R}_{>0}} |B_\lambda(x_\lambda)| < \infty$$

then there exists $x \in H$ such that

$$(x, y) \in 1 + A + B.$$

Proof. Let $(\lambda_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}_{>0}$ be a sequence with $\lambda_n \to 0$ as $n \to \infty$. We want to prove that $(x_{\lambda_n})_{n \in \mathbb{N}}$ is a Cauchy-sequence. We estimate by using the monotonicity of $A$ and $B$

$$|x_{\lambda_n} - x_{\lambda_m}|^2 = \text{Re}\langle x_{\lambda_n} - x_{\lambda_m} | x_{\lambda_n} - x_{\lambda_m} \rangle$$

$$= -\text{Re}\langle x_{\lambda_n} - x_{\lambda_m} | y - x_{\lambda_n} - B_{\lambda_n}(x_{\lambda_n}) - (y - x_{\lambda_m} - B_{\lambda_m}(x_{\lambda_m})) \rangle$$

$$+ \text{Re}\langle x_{\lambda_n} - x_{\lambda_m} | B_{\lambda_m}(x_{\lambda_m}) - B_{\lambda_n}(x_{\lambda_n}) \rangle$$

$$\leq \text{Re}\langle x_{\lambda_n} - x_{\lambda_m} | B_{\lambda_m}(x_{\lambda_m}) - B_{\lambda_n}(x_{\lambda_n}) \rangle$$

$$\leq \text{Re}\langle \lambda_n B_{\lambda_n}(x_{\lambda_n}) + J_{\lambda_n}(B)(x_{\lambda_n}) - \lambda_m B_{\lambda_m}(x_{\lambda_m}) - J_{\lambda_m}(B)(x_{\lambda_m}) | B_{\lambda_m}(x_{\lambda_m}) - B_{\lambda_n}(x_{\lambda_n}) \rangle$$

$$= -\text{Re}\langle J_{\lambda_n}(B)(x_{\lambda_n}) - J_{\lambda_m}(B)(x_{\lambda_m}) | B_{\lambda_m}(x_{\lambda_m}) - B_{\lambda_n}(x_{\lambda_n}) \rangle$$

$$+ \text{Re}\langle \lambda_n B_{\lambda_n}(x_{\lambda_n}) - \lambda_m B_{\lambda_m}(x_{\lambda_m}) | B_{\lambda_m}(x_{\lambda_m}) - B_{\lambda_n}(x_{\lambda_n}) \rangle$$

$$\leq \text{Re}\langle \lambda_n B_{\lambda_n}(x_{\lambda_n}) - \lambda_m B_{\lambda_m}(x_{\lambda_m}) | B_{\lambda_m}(x_{\lambda_m}) - B_{\lambda_n}(x_{\lambda_n}) \rangle$$

$$\leq (\lambda_n + \lambda_m)C.$$
where \( C := 2 \left( \sup_{\lambda \in \mathbb{R}_{>0}} |B_\lambda(x_\lambda)| \right)^2 \). Thus, \((x_\lambda)_{n \in \mathbb{N}}\) is a Cauchy-sequence and therefore convergent. We denote its limit by \( x \in H \). Moreover, since \((B_\lambda_n(x_\lambda_n))_{n \in \mathbb{N}}\) is bounded, we find a weakly convergent subsequence (again labeled with the index \( n \)) with
\[
B_\lambda_n(x_{\lambda_n}) \to v \quad (n \to \infty)
\]
for some \( v \in H \). Also
\[
|J_{A_n}(B)(x_{\lambda_n}) - x|_H \leq \lambda_n |B_\lambda_n(x_{\lambda_n})|_H + |x_{\lambda_n} - x|_H \to 0 \quad (n \to \infty)
\]
and since \( B \) is demiclosed (Proposition 1.10), we get
\[
(x, v) \in B.
\]
Since \((x_{\lambda_n}, y - x_{\lambda_n} - B_\lambda_n(x_{\lambda_n})) \in A\) for all \( n \in \mathbb{N} \) we obtain by applying the demiclosedness of \( A \) that
\[
(x, y - x - v) \in A
\]
which is equivalent to
\[
(x, y) \in 1 + A + B.
\]

\(\square\)

**Lemma 1.18.** Let \( A, B \subseteq H \oplus H \) be maximal monotone with \([H]A \cap [H]B \neq \emptyset\) and \( y \in H \). We set
\[
x_\lambda := (1 + A + B_\lambda)^{-1}(y)
\]
for \( \lambda \in \mathbb{R}_{>0} \). Then for every \( x^* \in [H]A \cap [H]B \)
\[
|x_\lambda| \leq |y| + 2|x^*| + |A^0(x^*)| + |B^0(x^*)|
\]
holds for every \( \lambda \in \mathbb{R}_{>0} \).

**Proof.** We fix \( x^* \in [H]A \cap [H]B \) and calculate, by using the monotonicity of \( A \) and \( B_\lambda \)
\[
|x_\lambda - x^*|^2 = \Re\langle x_\lambda - x^* | x_\lambda - x^* \rangle
\]
\[
= -\Re\langle x_\lambda - x^* | y - x_\lambda - B_\lambda(x_\lambda) - A^0(x^*) \rangle + \Re\langle x_\lambda - x^* | y - x^* - B_\lambda(x_\lambda) - A^0(x^*) \rangle
\]
\[
\leq \Re\langle x_\lambda - x^* | y - x^* - B_\lambda(x_\lambda) - A^0(x^*) \rangle
\]
\[
= -\Re\langle x_\lambda - x^* | B_\lambda(x_\lambda) - B_\lambda(x^*) \rangle + \Re\langle x_\lambda - x^* | y - x^* - A^0(x^*) - B_\lambda(x^*) \rangle
\]
\[
\leq \Re\langle x_\lambda - x^* | y - x^* - A^0(x^*) - B_\lambda(x^*) \rangle
\]
\[
\leq |x_\lambda - x^*||y - x^* - A^0(x^*) - B_\lambda(x^*)|
\]
and thus
\[
|x_\lambda| - |x^*| \leq |x_\lambda - x^*| \leq |y - x^* - A^0(x^*) - B_\lambda(x^*)| \leq |y| + |x^*| + |A^0(x^*)| + |B^0(x^*)|.
\]
So we obtain
\[
|x_\lambda| \leq |y| + 2|x^*| + |A^0(x^*)| + |B^0(x^*)|
\]
for all \( \lambda \in \mathbb{R}_{>0} \). \(\square\)
Proposition 1.19 ([39] Theorem 1.7). Let \( A, B \subseteq H \oplus H \) be maximal monotone with \([H]A \cap [H]B \neq \emptyset\) and 
\[
\forall (x, y) \in A, \lambda \in \mathbb{R}_{>0} : \text{Re}(y|B_\lambda(x)) \geq 0.
\]
Then \( A + B \) is maximal monotone.

Proof. Let \( y \in H \) and \( x_\lambda := (1 + A + B_\lambda)^{-1}(y) \) for \( \lambda \in \mathbb{R}_{>0} \). We want to show that \( \sup_{\lambda \in \mathbb{R}_{>0}} |B_\lambda(x_\lambda)| < \infty \). Since \( (x_\lambda, y - x_\lambda - B_\lambda(x_\lambda)) \in A \) we conclude, that
\[
\text{Re}(y - x_\lambda - B_\lambda(x_\lambda)|B_\lambda(x_\lambda)) \geq 0
\]
and thus
\[
|B_\lambda(x_\lambda)|^2 \leq \text{Re}(y - x_\lambda|B_\lambda(x_\lambda)) \leq (|y| + |x_\lambda|)|B_\lambda(x_\lambda)|.
\]
Since \( (x_\lambda)_{\lambda \in \mathbb{R}_{>0}} \) is bounded according to Lemma 1.18, we get that \( (B_\lambda(x_\lambda))_{\lambda \in \mathbb{R}_{>0}} \) is bounded. Therefore Lemma 1.17 is applicable and we find \( x \in H \) such that
\[
(x, y) \in 1 + A + B.
\]
By Minty’s Theorem this means that \( A + B \) is maximal monotone.

\[
\square
\]

Definition 1.20. A relation \( D \subseteq H \oplus H \) is called bounded, if for all bounded sets \( M \subseteq H \) the post-set \( D[M] \) is also bounded.

Lemma 1.21. Let \( D \subseteq H \oplus H \) be bounded. Then \( \overline{D} \) is bounded.

Proof. Let \( M \subseteq H \) be bounded. For each \( y \in \overline{D}[M] \) there is an \( x_y \in M \) with \( (x_y, y) \in \overline{D} \) and we find an element \( (u_y, v_y) \in D \) with \(|u_y - x_y| < 1\) and \(|v_y - y| < 1\). Hence,
\[
\sup\{|y| \mid y \in \overline{D}[M]\} \leq \sup\{|y - v_y| + |v_y| \mid y \in \overline{D}[M]\} \leq 1 + \sup\{|v| \mid v \in D[M]\} + |B(0,1)| < \infty
\]
which shows the assertion.

\[
\square
\]

Proposition 1.22. Let \( A, B \subseteq H \oplus H \) be maximal monotone. Moreover, let \( A \) be bounded and \([H]A \cap [H]B \neq \emptyset\). Then \( A + B \) is maximal monotone.

Proof. Let \( y \in H \). For \( \lambda \in \mathbb{R}_{>0} \) we define
\[
x_\lambda = (1 + A + B_\lambda)^{-1}(y),
\]
which yields in particular \( (x_\lambda, y - x_\lambda - B_\lambda(x_\lambda)) \in A \). According to Lemma 1.18, the set \( \{x_\lambda \mid \lambda \in \mathbb{R}_{>0}\} \) is bounded and since \( A \) is a bounded relation we get \( \sup_{\lambda \in \mathbb{R}_{>0}} |y - x_\lambda - B_\lambda(x_\lambda)| < \infty \). Thus, for all \( \lambda \in \mathbb{R}_{>0} \):
\[
|B_\lambda(x_\lambda)| \leq |y - x_\lambda - B_\lambda(x_\lambda)| + |y - x_\lambda| \leq \sup_{\lambda \in \mathbb{R}_{>0}} |y - B_\lambda(x_\lambda) - x_\lambda| + |y| + \sup_{\lambda \in \mathbb{R}_{>0}} |x_\lambda| < \infty
\]
which shows that \( \sup_{\lambda \in \mathbb{R}_{>0}} |B_\lambda(x_\lambda)| < \infty \). Thus, by Lemma 1.17 there exists \( x \in H \) such that
\[
(x, y) \in 1 + A + B.
\]
Since \( y \in H \) was chosen arbitrarily, this implies the maximal monotonicity of \( A + B \).

\[
\square
\]
At last, we want to present a result, which generalizes the concept of $A$-bounded perturbations for an operator $A$ (cf. \[30\]). We begin with the definition of relative bounded relations.

**Definition 1.23.** Let $A, B \subseteq H \oplus H$ be two maximal monotone relations. $B$ is called $A$-bounded, if $[H]A \subseteq [H]B$ and there exist two non-decreasing function $b, c: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\forall x \in [H]A : |B^0(x)| \leq b(|x|)|A^0(x)| + c(|x|).$$

If $b|_{[H]A}$ is bounded, we call $\sup \{b(|x|) | x \in [H]A\}$ the $A$-bound of $B$.

**Proposition 1.24** ([26, Theorem 3.7, p. 336]). Let $A, B \subseteq H \oplus H$ be two maximal monotone relations. If $B$ is $A$-bounded with an $A$-bound strictly less than 1, then $A + B$ is maximal monotone.

**Proof.** We apply Lemma 1.17 to show the maximal monotonicity of $A + B$. For $y \in H$ we define

$$x_\lambda := (1 + A + B_\lambda)^{-1}(y) \quad (\lambda \in \mathbb{R}_{>0}).$$

According to Lemma 1.18, the set $\{x_\lambda | \lambda \in \mathbb{R}_{>0}\}$ is bounded and we set

$$R := \sup_{\lambda \in \mathbb{R}_{>0}} |x_\lambda|$$

For $\lambda \in \mathbb{R}_{>0}$ we estimate as $(x_\lambda, y - x_\lambda - B_\lambda(x_\lambda)) \in A$

$$|A^0(x_\lambda)| \leq |y - x_\lambda - B_\lambda(x_\lambda)|$$

$$\leq |y| + \sup_{\lambda \in \mathbb{R}_{>0}} |x_\lambda| + |B_\lambda(x_\lambda)|$$

$$\leq |y| + \sup_{\lambda \in \mathbb{R}_{>0}} |x_\lambda| + |B^0(x_\lambda)|$$

$$\leq |y| + \sup_{\lambda \in \mathbb{R}_{>0}} |x_\lambda| + b(|x_\lambda|)|A^0(x_\lambda)| + c(|x_\lambda|)$$

$$\leq |y| + R + |A^0(x_\lambda)| \sup_{\lambda \in \mathbb{R}_{>0}} b(|x_\lambda|) + c(R).$$

Since $\sup_{\lambda \in \mathbb{R}_{>0}} b(|x_\lambda|) < 1$, we conclude, that $\sup_{\lambda \in \mathbb{R}_{>0}} |A^0(x_\lambda)| < \infty$ and thus

$$|B_\lambda(x_\lambda)| \leq |B^0(x_\lambda)|$$

$$\leq b(|x_\lambda|)|A^0(x_\lambda)| + c(|x_\lambda|)$$

$$\leq \sup_{\lambda \in \mathbb{R}_{>0}} |A^0(x_\lambda)| + c(R)$$

for all $\lambda \in \mathbb{R}_{>0}$. This implies that we find an element $x \in H$ such that $(x, y) \in 1 + A + B$ and since $y$ was chosen arbitrarily we obtain the maximal monotonicity of $A + B$ by Minty’s Theorem. \qed
1.4. Uniformly bounded relations

In this section we introduce a new class of relations. For further examples of special classes and continuity concepts of maximal monotone relations we refer to [25]. The set of so-called uniformly bounded relations introduced here, will allow us to generalize the notion of material laws, which was introduced in [43, 53].

**Definition 1.25.** A relation \( A \subseteq H \oplus H \) is called **uniformly bounded**, if there exists a constant \( c \in \mathbb{R} > 0 \) such that
\[
\forall r > 0 : A[B[0, r]] \subseteq B[0, cr].
\]
For a uniformly bounded relation \( A \), we set
\[
c(A) := \inf \{ c \in \mathbb{R} > 0 | \forall r > 0 : A[B[0, r]] \subseteq B[0, cr] \}.
\]

We define a certain set of uniformly bounded relations by
\[
B_u(H) := \{ A \subseteq H \oplus H | [H]A = H, \forall x \in H : A[\{x\}] \text{ is closed, } A \text{ is uniformly bounded} \}.
\]

In the definition one could choose a subset of \( H \) as the common pre-set of the relations. However, for simplicity, we assume, that all relations are defined on the whole \( H \).

**Proposition 1.26.** For every \( A \in B_u(H) \) we have \( A[\{0\}] = \{0\} \). The functional \[3\]
\[
d : B_u(H) \times B_u(H) \rightarrow [0, \infty), \quad (A, B) \mapsto \sup_{x \in H \setminus \{0\}} \frac{1}{|x|} \mathcal{H}(A[\{x\}], B[\{x\}])
\]
is well-defined and \((B_u(H), d)\) is a complete metric space.

**Proof.** Let \( A \in B_u(H) \) and \( y \in A[\{0\}] \). By definition there exists \( c > 0 \) such that \( A[B[0, r]] \subseteq B[0, cr] \) for all \( r > 0 \). Since \( 0 \in \bigcap_{r>0} B[0, r] \) we conclude that \( y \in \bigcap_{r>0} B[0, cr] \) and hence \( y = 0 \). Let now \( B \in B_u(H) \). We shall show that \( d(A, B) < \infty \). For this we fix \( x \in H \) and choose \( y \in A[\{x\}] \) and \( z \in B[\{x\}] \). Then
\[
|y - z| \leq |y| + |z| \\
\leq c(A)|x| + c(B)|x| \\
= (c(A) + c(B))|x|.
\]
This implies
\[
\text{dist}(y, B[\{x\}]) \leq (c(A) + c(B))|x| \quad \text{and} \quad \text{dist}(z, A[\{x\}]) \leq (c(A) + c(B))|x|
\]
and thus, passing to the supremum over \( y \in A[\{x\}] \) and \( z \in B[\{x\}] \) respectively, yields
\[
\mathcal{H}(A[\{x\}], B[\{x\}]) \leq (c(A) + c(B))|x|.
\]

\( ^3 \mathcal{H} \) is the Hausdorff-metric (cf. Appendix part B).
1.4. Uniformly bounded relations

This shows
\[ d(A, B) \leq c(A) + c(B) < \infty. \]

The metric properties of \( d \) can be shown easily. It is left to prove the completeness of \((B_u(H), d)\).

Let \((A_n)_{n \in \mathbb{N}} \subseteq B_u(H)\) be a Cauchy-sequence. We first show, that \( \sup_{n \in \mathbb{N}} c(A_n) < \infty \). There exists \( n_0 \in \mathbb{N} \) such that
\[ d(A_n, A_{n_0}) \leq 1 \quad (n \geq n_0). \]

This implies that for all \( x \in H \) we have
\[ \mathcal{H}(A_n[\{x\}], A_{n_0}[\{x\}]) \leq |x| \quad (n \geq n_0). \]

By the triangle inequality for the metric \( \mathcal{H} \) we conclude
\[ \mathcal{H}(A_n[\{x\}], \{0\}) \leq |x| + \mathcal{H}(A_{n_0}[\{x\}], \{0\}) \quad (n \geq n_0). \]

By observing that for a bounded, closed, nonempty set \( B \subseteq H \)
\[ \mathcal{H}(B, \{0\}) = \max\{\sup_{x \in B} \text{dist}(x, \{0\}), \sup_{x \in \{0\}} \text{dist}(x, B)\} \]
\[ = \max\{\sup_{x \in B} |x|, \inf_{x \in \{0\}} |x|\} \]
\[ = \sup_{x \in B} |x|, \quad (1.12) \]

we conclude
\[ \mathcal{H}(A_n[\{x\}], \{0\}) \leq \sup_{y \in A_{n_0}[\{x\}]} |y| \quad (n \geq n_0). \]

Since \( x \in B[0, |x|] \) we estimate for all \( y \in A_{n_0}[\{x\}] \)
\[ |y| \leq c(A_{n_0})|x| \]

and thus
\[ \mathcal{H}(A_n[\{x\}], \{0\}) \leq (1 + c(A_{n_0}))*|x| \quad (n \geq n_0). \quad (1.13) \]

Now let \( r > 0 \) and \( x \in B[0, r], y \in A_n[\{x\}] \) for \( n \geq n_0 \). According to \((1.12)\) and \((1.13)\) we get
\[ |y| \leq (1 + c(A_{n_0}))|x| \leq (1 + c(A_{n_0}))r \]

and hence
\[ A_n[B[0, r]] \subseteq B[0, (1 + c(A_{n_0}))r] \quad (r > 0, n \geq n_0). \]

Thus, by setting
\[ c := \max\{c(A_1), \ldots, c(A_{n_0-1}), 1 + c(A_{n_0})\} \]

we obtain
\[ \sup_{n \in \mathbb{N}} c(A_n) \leq c. \]

From the Cauchy-property of \((A_n)_{n \in \mathbb{N}} \) we read off
\[ \mathcal{H}(A_n[\{x\}], A_m[\{x\}]) \leq |x| d(A_n, A_m) \to 0 \quad (n, m \to \infty) \]
for all $x \in H \setminus \{0\}$ and hence $(A_n([x]))_{n \in \mathbb{N}}$ is a Cauchy-sequence for all $x \in H$ (for $x = 0$ the sequence is the constant sequence $(\{0\})_{n \in \mathbb{N}}$). Since $(BC(H), \mathcal{H})$ is complete by Proposition \[B.4\] it follows that $(A_n([x]))_{n \in \mathbb{N}}$ is convergent. We define a relation $A \subseteq H \oplus H$ by

$$A([x]) = \lim_{n \to \infty} A_n([x]) \quad (x \in H).$$

We will show that $A \in B_u(H)$ and $A_n \to A$ with respect to $d$ as $n \to \infty$. By the definition of $A$ we obtain that $[H]A = H$ and $A([x])$ is closed for every $x \in H$ by the properties of the Hausdorff-metric. Let now $r > 0$ and fix $y \in A[B[0, r]]$. Hence, there exists $x \in B[0, r]$ with $(x, y) \in A$. By Lemma \[B.2\] there exists $y_n \in A_n([x])$ for $n \in \mathbb{N}$ such that

$$|y - y_n| \leq 2\mathcal{H}(A([x]), A_n([x])) \to 0 \quad (n \to \infty).$$

Thus, we find $n \in \mathbb{N}$ such that

$$|y| \leq |y - y_n| + |y_n| \leq |x| + c(A_n)|x| \leq (1 + c)|x| \leq (1 + c)r.$$

This shows

$$A[B[0, r]] \subseteq B[0, (1 + c)r] \quad (r > 0)$$

and hence $A \in B_u(H)$. It is left to prove, that $A_n \to A$ as $n \to \infty$. Let $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that

$$d(A_n, A_m) \leq \varepsilon \quad (n, m \geq n_0).$$

This implies that for all $n, m \geq n_0, x \in H$ one has

$$\mathcal{H}(A_n([x]), A_m([x])) \leq |x|d(A_n, A_m) \leq \varepsilon|x|.$$

Since $A_m([x]) \to A([x])$ as $m \to \infty$ for all $x \in H$ we conclude

$$\mathcal{H}(A_n([x]), A([x])) \leq \varepsilon|x| \quad (n \geq n_0)$$

and hence

$$d(A_n, A) \leq \varepsilon \quad (n \geq n_0).$$

This completes the proof.

**Proposition 1.27.** Let $(A_n)_{n \in \mathbb{N}} \in B_u(H)^{\mathbb{N}}$ be convergent and denote its limit by $A \in B_u(H)$. Then $\sup_{n \in \mathbb{N}} c(A_n) < \infty$ and

$$c(A) \leq \sup_{n \in \mathbb{N}} c(A_n).$$

**Proof.** In the proof of Proposition \[1.26\] it was already shown that $\sup_{n \in \mathbb{N}} c(A_n) < \infty$. Let $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that $d(A_n, A_m) < \frac{\varepsilon}{2}$ for $n \geq n_0$. Moreover, let $r > 0$ and fix $y \in A[B[0, r]], x \in A([x])$. Since

$$\mathcal{H}(A_n([x]), A([x])) \leq \frac{\varepsilon}{2}|x| \quad (n \geq n_0)$$

we find an element $y_n \in A_n([x])$ with

$$|y - y_n| \leq \varepsilon|x| \quad (n \geq n_0).$$
according to Lemma 3.2. Thus,

$$|y| \leq |y - y_n| + |y_n| \leq (\varepsilon + c(A_n))|x| \leq (\varepsilon + \sup_{n \in \mathbb{N}} c(A_n))r \quad (n \geq n_0)$$

and since $\varepsilon > 0$ was arbitrary, we conclude

$$A[0, r] \subseteq B[0, \sup_{n \in \mathbb{N}} c(A_n)r]$$

for all $r > 0$ and hence

$$c(A) \leq \sup_{n \in \mathbb{N}} c(A_n).$$

\[\square\]

**Proposition 1.28.** Let $A, B \in B_u(H)$ and $\lambda \in \mathbb{C}$. Then $\lambda A + B \in B_u(H)$ and $c(\lambda A + B) \leq |\lambda|c(A) + c(B)$.

**Proof.** By the definition of the sum of two relations and the scalar multiplication, we get

$$[H](\lambda A + B) = [H]A \cap [H]B = H.$$ Since $(\lambda A + B)[\{x\}] = \lambda[A(\{x\}) + B(\{x\})] \quad \text{for all } x \in H$$

we obtain the closedness and boundedness of $(\lambda A + B)[\{x\}]$. Let now $r > 0$ and $x \in \mathbb{B}[0, r]$, $y \in (\lambda A + B)[\{x\}]$. Then there exists $y_1 \in A[\{x\}], y_2 \in B[\{x\}]$ with $y = \lambda y_1 + y_2$. Hence, we estimate

$$|y| \leq |\lambda||y_1| + |y_2| \leq (|\lambda|c(A) + c(B))|x| \leq (|\lambda|c(A) + c(B))r$$

and thus

$$(\lambda A + B)[\mathbb{B}[0, r]] \subseteq \mathbb{B}[0, (|\lambda|c(A) + c(B))r]$$

for all $r > 0$. This shows $\lambda A + B \in B_u(H)$ and $c(\lambda A + B) \leq |\lambda|c(A) + c(B)$. \[\square\]

**Proposition 1.29.** Let $(A_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}} \in B_u(H)^{\mathbb{N}}, (\lambda_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ such that $A_n \to A \in B_u(H)$, $B_n \to B \in B_u(H)$ and $\lambda_n \to \lambda \in \mathbb{C}$ as $n \to \infty$. Then $\lambda_n A_n + B_n \to \lambda A + B$ as $n \to \infty$.

**Proof.** Using the triangle inequality, we estimate

$$d(\lambda_n A_n + B_n, \lambda A + B) \leq d(\lambda_n A_n + B_n, \lambda_n A + B) + d(\lambda_n A + B, \lambda A + B). \quad (1.14)$$

Let $x \in H$ and $n \in \mathbb{N}$. For $y \in (\lambda_n A_n + B_n)[\{x\}]$ and $z \in (\lambda_n A + B)[\{x\}]$ we find $y_1 \in A_n[\{x\}], y_2 \in B_n[\{x\}], z_1 \in A[\{x\}]$ and $z_2 \in B[\{x\}]$ such that $y = \lambda_n y_1 + y_2$ and $z = \lambda_n z_1 + z_2$. Thus,

$$\text{dist}(y, (\lambda_n A + B)[\{x\}]) = \inf \{|y - z| : z \in (\lambda_n A + B)[\{x\}]\}$$

$$\leq \inf \{|\lambda_n y_1 - \lambda_n z_1| + |y_2 - z_2| : z_1 \in A[\{x\}], z_2 \in B[\{x\}]\}$$

$$= |\lambda_n| \sup_{y_1 \in A_n[\{x\}] \atop y_2 \in B_n[\{x\}]} \text{dist}(y_1, A[\{x\}]) + \text{dist}(y_2, B[\{x\}])$$

and therefore

$$\sup_{y \in (\lambda_n A_n + B_n)[\{x\}]} \text{dist}(y, (\lambda_n A + B)[\{x\}]) \leq |\lambda_n| \mathcal{H}(A_n[\{x\}], A[\{x\}]) + \mathcal{H}(B_n[\{x\}], B[\{x\}]).$$
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Analogously, we obtain
\[
\sup_{y \in (\mathbf{A} + \mathbf{B}) \{\{x\}\}} \text{dist}(y, (\mathbf{A} + \mathbf{B}) \{\{x\}\}) \leq |\mathbf{A}| + |\mathbf{B}|
\]
and hence
\[
\mathcal{H}((\mathbf{A} + \mathbf{B}) \{\{x\}\}) \leq |\mathbf{A}| + |\mathbf{B}|
\]

This implies that
\[
d(\mathbf{A} + \mathbf{B}, \mathbf{A} + \mathbf{B}) \leq |\mathbf{A}| + |\mathbf{B}|
\]
for each \( n \in \mathbb{N} \) and hence \( d(\mathbf{A} + \mathbf{B}, \mathbf{A} + \mathbf{B}) \to 0 \) as \( n \to \infty \), since \( (\mathbf{A}) \) is bounded.

To obtain the convergence of the second term in \( (1.14) \), let \( n \in \mathbb{N} \) and \( x \in H \). Then we find for \( y \in (\mathbf{A} + \mathbf{B}) \{\{x\}\} \) two elements \( y_1 \in \mathbf{A} \{\{x\}\} \), \( y_2 \in \mathbf{B} \{\{x\}\} \) with \( y + \mathbf{A} \). Thus, we obtain
\[
\text{dist}(y, (\mathbf{A} + \mathbf{B}) \{\{x\}\}) \leq |\mathbf{A}| + |\mathbf{B}|
\]
and hence
\[
\sup_{y \in (\mathbf{A} + \mathbf{B}) \{\{x\}\}} \text{dist}(y, (\mathbf{A} + \mathbf{B}) \{\{x\}\}) \leq |\mathbf{A}| + |\mathbf{B}|
\]

Analogously, we find
\[
\sup_{y \in (\mathbf{A} + \mathbf{B}) \{\{x\}\}} \text{dist}(y, (\mathbf{A} + \mathbf{B}) \{\{x\}\}) \leq |\mathbf{A}| + |\mathbf{B}|
\]
and summarizing, we estimate
\[
d(\mathbf{A} + \mathbf{B}, \mathbf{A} + \mathbf{B}) \leq |\mathbf{A}| + |\mathbf{B}|
\]
which yields the asserted convergence by \( (1.14) \). \( \square \)

**Proposition 1.30.** Let \( (\mathbf{A}) \) be \( B_\ast(H) \) such that \( \sum_{k=1}^{\infty} c(A_k) < \infty \). Then \( A := \sum_{k=1}^{\infty} A_k \in B_\ast(H) \) with \( c(A) \leq \sum_{k=1}^{\infty} c(A_k) \).

**Proof.** By Proposition 1.28 we know that \( \sum_{k=1}^{n} A_k \in B_\ast(H) \) for all \( n \in \mathbb{N} \). Since \( B_\ast(H) \) is complete, it suffices to show that \( \sum_{k=1}^{n} A_k \{\{x\}\} \) is a Cauchy-sequence. Let \( m, n \in \mathbb{N} \) with \( n > m \). We have to study the term
\[
\mathcal{H}(\sum_{k=1}^{n} A_k \{\{x\}\}, \sum_{k=1}^{n} A_k \{\{x\}\})
\]
for \( x \in H \). Let \( x \in H \) and \( y \in \sum_{k=1}^{m} A_k \{\{x\}\} \). This means that we find elements \( y_k \in A_k \{\{x\}\} \) for \( k \in \{1, \ldots, m\} \) with \( y = \sum_{k=1}^{m} y_k \). For \( z_k \in A_k \{\{x\}\} \) with \( k \in \{m+1, \ldots, n\} \) we set
\[
z := y + \sum_{k=m+1}^{n} z_k \in \sum_{k=1}^{n} A_k \{\{x\}\}.
\]
Therefore

\[ \text{dist}(y, \sum_{k=1}^{n} A_k[\{x\}]) \leq |y - z| \]

\[ \leq \sum_{k=m+1}^{n} |z_k| \]

\[ \leq \sum_{k=m+1}^{n} c(A_k)|x| \]

and hence

\[ \sup_{y \in \sum_{k=1}^{m} A_k[\{x\}]} \text{dist}(y, \sum_{k=1}^{n} A_k[\{x\}]) \leq |x| \sum_{k=m+1}^{n} c(A_k). \]

In the same way, we find for arbitrary \( z \in \sum_{k=1}^{n} A_k[\{x\}] \) elements \( z_k \in A_k[\{x\}] \) for \( k \in \{1, \ldots, n\} \) with \( z = \sum_{k=1}^{n} z_k \). By setting \( y := \sum_{k=1}^{m} z_k \) we define an element of \( \sum_{k=1}^{m} A_k[\{x\}] \) and estimate

\[ \text{dist}(z, \sum_{k=1}^{m} A_k[\{x\}]) \leq |z - y| \]

\[ \leq \sum_{k=m+1}^{n} |z_k| \]

\[ \leq \sum_{k=m+1}^{n} c(A_k)|x| \]

and therefore

\[ \sup_{z \in \sum_{k=1}^{n} A_k[\{x\}]} \text{dist}(z, \sum_{k=1}^{m} A_k[\{x\}]) \leq |x| \sum_{k=m+1}^{n} c(A_k). \]

This shows

\[ \mathcal{H}(\sum_{k=1}^{n} A_k[\{x\}], \sum_{k=1}^{m} A_k[\{x\}]) \leq |x| \sum_{k=m+1}^{n} c(A_k) \]

for all \( x \in H \), which yields

\[ d(\sum_{k=1}^{n} A_k, \sum_{k=1}^{m} A_k) \leq \sum_{k=m+1}^{n} c(A_k) \to 0 \quad (n, m \to \infty). \]

This shows that \( A := \sum_{k=1}^{\infty} A_k \) exists in \( B_u(H) \). Furthermore we get by Proposition 1.28 that \( c(\sum_{k=1}^{n} A_k) \leq \sum_{k=1}^{n} c(A_k) \) holds for all \( n \in \mathbb{N} \) and by Proposition 1.27 it follows that

\[ c(A) \leq \sup_{n \in \mathbb{N}} c(\sum_{k=1}^{n} A_k) \leq \sum_{k=1}^{\infty} c(A_k). \]

\[ \square \]
1.4. Uniformly bounded relations

**Definition 1.31.** Let \( A, B \subseteq H \oplus H \). Then the relation \( AB \) is defined by

\[
AB = \{ (x, y) \in H \oplus H \mid B[\{x\}] \cap \{y\} \neq \emptyset \}.
\]

**Lemma 1.32.** Let \( A, B \in B_u(H) \). Moreover, let \( A \) be demiclosed in the first argument, i.e. 
\( \forall ((x_n, y_n))_{n \in \mathbb{N}} \in A^\mathbb{N} : (x_n \to x \land y_n \to y) \Rightarrow (x, y) \in A, \) and \( B[\{x\}] \) be weakly closed for all \( x \in H \). Then \( AB \in B_u(H) \) and \( c(AB) \leq c(A)c(B) \).

**Proof.** \( [H](AB) = H \) is trivial. Let \( x \in H \) and \( (y_n)_{n \in \mathbb{N}} \in ((AB)[\{x\}])^\mathbb{N} \) with \( y_n \to y \) as \( n \to \infty \). By the definition of \( AB \) there exists a \( z_n \in H \) with \( (x, z_n) \in B, (z_n, y_n) \in A \) for every \( n \in \mathbb{N} \).
Hence, \( |z_n| \leq c(B)|x| \) for all \( n \in \mathbb{N} \) and thus we find a weak-convergent subsequence \( (z_{n_k})_{k \in \mathbb{N}} \), whose limit will be denoted by \( z \). Since \( B[\{x\}] \) is weak-closed, we conclude \( (x, z) \in B \). Moreover, by the demiclosedness of \( A \) we obtain \( (z, y) \in A \) and thus \( (x, y) \in AB \), which shows the closedness of \( (AB)[\{x\}] \). Let now \( r > 0 \) and \( x \in B[0, r], y \in (AB)[\{x\}] \). Then there exists a \( z \in H \) with \( (x, z) \in B, (z, y) \in A \). Thus, we estimate

\[
|y| \leq c(A)|z| \leq c(A)c(B)|x| \leq c(A)c(B)r.
\]

This shows the boundedness of \( (AB)[\{x\}] \) and moreover

\[
(AB)[B[0, r]] \subseteq B[0, c(A)c(B)r].
\]

Hence, \( AB \in B_u(H) \) with \( c(AB) \leq c(A)c(B) \).

We like to discuss the extensions of uniformly bounded relations on an \( H \)-valued \( L_2 \)-space like in Proposition 1.14.

**Proposition 1.33.** Let \( A \in B_u(H) \) and \( (\Omega, \Sigma, \mu) \) a measure space. We define the extension of \( A \) on the space \( L_2(\mu; H) \) like in Definition 1.13 by

\[
A_{L_2(\mu; H)} := \{ (u, v) \in L_2(\mu; H) \oplus L_2(\mu; H) \mid (u(t), v(t)) \in A \text{ for } \mu - \text{a.e. } t \in \Omega \}.
\]

Then \( A_{L_2(\mu; H)} \) is bounded.

**Proof.** Let \( D \subseteq L_2(\mu; H) \) be bounded and \( y \in A_{L_2(\mu; H)}[D] \), i.e. there is \( x \in D \) with \( (x, y) \in A_{L_2(\mu; H)} \). We estimate

\[
|y|_{L_2(\mu; H)} = \left( \int_{\Omega} |y(t)|^2 \, dt \right)^{\frac{1}{2}}
\]

\[
\leq c(A) \left( \int_{\Omega} |x(t)|^2 \, dt \right)^{\frac{1}{2}}
\]

\[
\leq c(A) \sup_{z \in D} |z|_{L_2(\mu; H)}
\]

and hence, \( A[D] \) is bounded.
To justify the topology, we have chosen for the set of uniformly bounded relations, we like to discuss, whether the set of maximal monotone, uniformly bounded relations is closed under this topology.

**Lemma 1.34.** Let \((A_n)_{n \in \mathbb{N}} \in B_u(H)\) be a convergent sequence with limit \(A \in B_u(H)\). Assume that there is \(c \in \mathbb{R}\) such that \(A_n\) is \(c\)-monotone for all \(n \in \mathbb{N}\). Then \(A\) is \(c\)-monotone as well.

**Proof.** Let \((u, v), (x, y) \in A\). Then by Lemma 3.2 there exist two sequences \((v_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \in H^\mathbb{N}\) with \(v_n \in A_n\{u\}, y_n \in A_n\{x\}\) for all \(n \in \mathbb{N}\) and \(v_n \to v\) as well as \(y_n \to y\) as \(n \to \infty\). Thus,

\[
\Re\langle u - x|v - y \rangle = \lim_{n \to \infty} \Re\langle u - x|v_n - y_n \rangle \geq c|u - x|^2.
\]

**Lemma 1.35.** Let \((A_n)_{n \in \mathbb{N}} \in B_u(H)\) with \(A_n \to A \in B_u(H)\) as \(n \to \infty\). Moreover, let \(A_n[H] = H\) and \(A_n^{-1}\) be a Lipschitz-continuous mapping for all \(n \in \mathbb{N}\) with

\[
\sup_{n \in \mathbb{N}} |A_n^{-1}|_{\text{Lip}} < \infty.
\]

Then \(A[H]\) is dense in \(H\).

**Proof.** Let \(y \in (A[H])^\perp\) and \(\varepsilon > 0\). Moreover, we set \(c := \sup_{n \in \mathbb{N}} |A_n^{-1}|_{\text{Lip}}\) and \(x_n := A_n^{-1}(y)\) for \(n \in \mathbb{N}\). Then we estimate

\[
|x_n| = |A_n^{-1}(y) - A_n^{-1}(0)| \leq c|y|
\]

for all \(n \in \mathbb{N}\). Furthermore we find an \(N \in \mathbb{N}\) such that for all \(n \geq N\) and \(x \in H\)

\[
\mathcal{H}(A_n\{x\}, A\{x\}) \leq \frac{\varepsilon}{2}|x|.
\]

Thus, we can choose elements \(z_n \in A\{x_n\}\) with

\[
|y - z_n| \leq \varepsilon|x_n| \leq c\varepsilon|y|
\]

for all \(n \geq N\). We compute

\[
|z_n|^2 = \langle y|y \rangle = \langle y|y - z_n \rangle \leq |y||y - z_n| \leq c\varepsilon|y|^2
\]

and since \(\varepsilon\) was chosen arbitrarily, it follows that \(y = 0\). 

**Proposition 1.36.** Let \((A_n)_{n \in \mathbb{N}} \in B_u(H)\) be a convergent sequence of maximal monotone relations with limit \(A \in B_u(H)\). Then \(\overline{A}\) is maximal monotone and bounded.

**Proof.** By Lemma 1.34 \(A\) is monotone and an easy approximation argument shows, that \(\overline{A}\) is also monotone. We use Theorem 1.6 to show the maximal monotonicity of \(\overline{A}\) and prove that \((1 + A) [H] = H\). By assumption \((1 + A_n)[H] = H\) and \((1 + A_n)^{-1}\) is a Lipschitz-continuous mapping for all \(n \in \mathbb{N}\) with

\[
\sup_{n \in \mathbb{N}} |(1 + A_n)^{-1}|_{\text{Lip}} \leq 1.
\]
1.4. Uniformly bounded relations

By Proposition 1.28 \((1 + A_n)_{n \in \mathbb{N}} \in B_u(H)^\mathbb{N}\) and from Proposition 1.29 it follows that \(1 + A_n \to 1 + A\) as \(n \to \infty\). Using Lemma 1.35 we conclude, that \((1 + A)H\) is dense in \(H\). According to Lemma 1.34 \((1 + A)^{-1}\) is a Lipschitz-continuous mapping with Lipschitz-constant less than or equal to 1. Let \(y \in H\). Then there exists a sequence \((y_n)_{n \in \mathbb{N}} \in (1 + A)[H]^\mathbb{N}\) with \(y_n \to y\) as \(n \to \infty\). This implies, that \(((1 + A)^{-1}(y_n))_{n \in \mathbb{N}}\) is a Cauchy-sequence and hence convergent. Thus, \(y \in (1 + A)H\) which shows \(H = (1 + A)H = (1 + A)[H]\). Therefore \(A\) is maximal monotone. The boundedness of \(A\) follows from Lemma 1.21.

Corollary 1.37. Let \((A_n)_{n \in \mathbb{N}} \in B_u(H)^\mathbb{N}\) be a sequence of maximal monotone relations with \(\sum_{n=1}^{\infty} c(A_n) < \infty\). Then \(\sum_{n=1}^{\infty} A_n\) is maximal monotone and bounded.

Proof. By Proposition 1.30 the relation \(\sum_{n=1}^{\infty} A_n \in B_u(H)\). Moreover, \(\sum_{n=1}^{k} A_n\) is maximal monotone for all \(k \in \mathbb{N}\) according to Proposition 1.22. Thus, \(\sum_{n=1}^{\infty} A_n\) is maximal monotone and bounded by Proposition 1.36.

Proposition 1.38. Let \((c_n)_{n \in \mathbb{N}} \in \mathbb{R}_{\geq 0}^\mathbb{N}\) and \((A_n)_{n \in \mathbb{N}} \in B_u(H)^\mathbb{N}\) such that \(\sum_{n=1}^{\infty} c_n < \infty\) and \(\sum_{n=1}^{\infty} c(A_n) < \infty\). Moreover, let \(A_n\) be \(c_n\)-maximal monotone for each \(n \in \mathbb{N}\). Then \(\sum_{n=1}^{\infty} A_n\) is \(\sum_{n=1}^{\infty} c_n\)-maximal monotone.

Proof. For all \(k \in \mathbb{N}\) we obviously have

\[
\sum_{n=1}^{k} (A_n + c_n) = \sum_{n=1}^{k} A_n + \sum_{n=1}^{k} c_n
\]

and since the right side converges in \(B_u(H)\) as \(k \to \infty\), we conclude by using Proposition 1.29

\[
\sum_{n=1}^{\infty} (A_n + c_n) = \sum_{n=1}^{\infty} A_n + \sum_{n=1}^{\infty} c_n.
\]

By Corollary 1.37 \(\sum_{n \in \mathbb{N}} (A_n + c_n)\) is maximal monotone and therefore so is \(\sum_{n \in \mathbb{N}} A_n + \sum_{n \in \mathbb{N}} c_n = \sum_{n \in \mathbb{N}} A_n + \sum_{n \in \mathbb{N}} c_n\), which shows the assertion. \(\square\)
2. Evolutionary inclusions

In this chapter we want to discuss the well-posedness of evolutionary inclusions, that means, differential inclusions of the form

\[(u, f) \in \partial_0 M(\partial_0^{-1}) + A,\]  

where \(\partial_0\) denotes the time derivative, the operator \(M(\partial_0^{-1})\) can be interpreted as a material law and \(A\) is a maximal monotone relation. Many results are known for this problem under suitable assumptions on \(M(\partial_0^{-1}), A\) and \(f\) (cf. [48, Chapter IV], [25], [39]). In [48] and [25] the problem is studied in a Banach space setting. However, we want to restrict ourselves to the Hilbert space case, avoiding any regularity assumptions on our given function \(f\). Furthermore we do not want to assume any continuity condition on our relations as well as any kind of coercitivity. For those concepts we again refer to [39], [25], [48]. The main idea of proving the well-posedness, is to identify the right side of (2.1) as the sum of two maximal monotone relations plus a (maybe) small real number, and then apply the results of Chapter 1.3. For doing this, we introduce the time derivative as a continuously invertible operator in a suitable Hilbert space. This strategy can be found for instance in [43]. Also the problem of causality should be addressed, which means, that our solution \(u\) of (2.1) given on a time interval \([0, a]\) for some \(a \in \mathbb{R} \geq 0\) does only depend on the behaviour of \(f\) on the same interval \([0, a]\), or in other words, the solution of \(u\) until a certain time does not depend on the future behaviour of \(f\) (for an exact definition of causality, we refer to [33], however a definition in our framework is given in Chapter 2.3).

2.1. The time derivative

We want to establish a derivative on a weighted \(L_2\)-space, and follow the idea presented in [45, Chapter 1.4]. We restrict ourselves to \(L_2(\mathbb{R}_{\geq 0})\) instead of taking the whole real line as domain. The case of \(L_2(\mathbb{R})\) is handled in Chapter 2.6. We define the derivative operator \(\partial\) on \(L_2(\mathbb{R}_{\geq 0})\) by taking the closure of

\[\partial|_{C_c^\infty(\mathbb{R}_{\geq 0})} : C_c^\infty(\mathbb{R}_{\geq 0}) \subseteq L_2(\mathbb{R}_{\geq 0}) \rightarrow L_2(\mathbb{R}_{\geq 0})\]

\[\phi \mapsto \phi'.\]

Here \(C_c^\infty(\mathbb{R}_{\geq 0})\) denotes the space of the infinite times differentiable functions with compact support on \(\mathbb{R}_{\geq 0}\). This operator is known to be skew-symmetric. Moreover, we define for \(\nu \in \mathbb{R}_{\geq 0}\) the Hilbert space \(H_{\nu,0}\) as the space of square-integrable functions with respect to the measure

\[1\text{With the usual identification of functions, which are equal almost everywhere.}\]
\[ \mu_\nu : B(\mathbb{R}_{\geq 0}) \to \mathbb{R}_{\geq 0} \]
\[ A \mapsto \int_A e^{-2\nu x} \, dx. \]

One easily sees that the operator
\[ e^{\nu m} : L^2(\mathbb{R}_{\geq 0}) \to H_{\nu, 0} \]
\[ f \mapsto (x \mapsto e^{\nu x} f(x)) \]
is unitary and we set \((e^{\nu m})^{-1} = e^{-\nu m}\). So the operator \(\hat{\partial}_\nu := e^{\nu m} \hat{\partial} e^{-\nu m}\) is skew-symmetric on \(H_{\nu, 0}\). For \(\phi \in C_c^\infty(\mathbb{R}_{>0})\) we compute:
\[ \hat{\partial}_\nu \phi = e^{\nu m} \hat{\partial} e^{-\nu m} \phi = e^{\nu m}(-\nu e^{-\nu m} \phi + e^{-\nu m} \phi') = \phi' - \nu \phi \]
and thus \(\phi' = (\hat{\partial}_\nu + \nu) \phi\). This leads to the definition for the derivative \(\partial_{\nu, 0} = \hat{\partial}_\nu + \nu\) on \(H_{\nu, 0}\).
Since \(e^{\nu m}\) and \(e^{-\nu m}\) are bijections on \(C_c^\infty(\mathbb{R}_{>0})\), it follows that
\[ \partial_{\nu, 0} = \partial|_{C_c^\infty(\mathbb{R}_{>0})} H_{\nu, 0} \otimes H_{\nu, 0}. \]

Using the skew-symmetry of \(\hat{\partial}_\nu\) we obtain for all \(u \in D(\partial_{\nu, 0})\):
\[ \text{Re} \langle \hat{\partial}_\nu u|u \rangle_{H_{\nu, 0}} = -\text{Re} \langle u| \hat{\partial}_\nu u \rangle_{H_{\nu, 0}} = -\text{Re} \langle \hat{\partial}_\nu u|u \rangle_{H_{\nu, 0}} \]
and thus
\[ \text{Re} \langle \partial_{\nu, 0} u|u \rangle_{H_{\nu, 0}} = \nu |u|_{H_{\nu, 0}}^2. \]
The application of the Cauchy-Schwarz-Inequality on the left hand side leads to
\[ |u|_{H_{\nu, 0}} \leq \frac{1}{\nu} |\partial_{\nu, 0} u|_{H_{\nu, 0}} \]
and thus \(\partial_{\nu, 0}^{-1}\) exists as a linear, bounded operator on the range of \(\partial_{\nu, 0}\) with \(\|\partial_{\nu, 0}^{-1}\| \leq \frac{1}{\nu}\). Moreover, we are able to show that \(\partial_{\nu, 0}\) is surjective. Since \(\partial_{\nu, 0}\) is closed and continuously invertible, it suffices to prove the density of \(\partial_{\nu, 0}[H_{\nu, 0}]\) in \(H_{\nu, 0}\). Let \(\phi \in C_c^\infty(\mathbb{R}_{>0})\) and define the function
\[ u(x) := \int_0^x \phi(t) \, dt \quad (x \in \mathbb{R}_{\geq 0}). \]
2.1. The time derivative

If we prove that \( u \in D(\partial_{v,0}) \) and \( \partial_{v,0} u = \phi \) we have shown the density of the range of \( \partial_{v,0} \), since \( C_c^\infty(\mathbb{R}_>0) \) is dense in \( H_{v,0} \). At first, we prove that \( u \in H_{v,0} \) by calculating with Tonelli’s Theorem

\[
\int_0^\infty |u(x)|^2 e^{-2v x} \, dx = \int_0^\infty \left( \int_0^x \phi(t) \, dt \right)^2 e^{-2v x} \, dx
\]

\[
= \int_0^\infty \left( \int_0^x \phi(t) e^{-\frac{2}{\nu} t} e^{\frac{2}{\nu} t} \, dt \right)^2 e^{-2v x} \, dx
\]

\[
\leq \int_0^\infty \int_0^1 |\phi(t)|^2 e^{-\nu t} \, dt \frac{1}{\nu} (e^{\nu x} - 1) e^{-2v x} \, dx
\]

\[
= \int_0^\infty \frac{1}{\nu} \int_0^\infty e^{-\nu x} - e^{-2v x} \, dx |\phi(t)|^2 e^{-\nu t} \, dt
\]

\[
= \int_0^\infty \frac{1}{\nu^2} e^{-\nu t} \left( 1 - \frac{1}{2} e^{-\nu t} \right) |\phi(t)|^2 e^{-\nu t} \, dt
\]

\[
\leq \frac{1}{\nu^2} |\phi|_{H_{v,0}}^2.
\]

Now we show that \( u \in D(\partial_{v,0}) = D(\partial_v) \) which is equivalent to \( e^{-\nu m} u \in D(\partial_v) \). We set \( a := \inf \text{supp} \phi > 0 \) and choose a sequence \((\psi_n)_{n \in \mathbb{N}} \in C_c^\infty(\mathbb{R}_>0)^N\) satisfying the following properties:

- \( \sup_{n \in \mathbb{N}} |\psi_n|_{\infty} < \infty \),
- \( \psi_n = 1 \) on \([a,n]\) for all \( n \in \mathbb{N} \),
- \( \sup_{n \in \mathbb{N}} |\psi_n'|_{\infty} < \infty \).

For a function \( v \in L_2(\mathbb{R}_>0) \) with \( \inf \text{supp} v \geq a \) we obtain

\[
\int_0^\infty |\psi_n(t)v(t) - v(t)|^2 \, dt = \int_a^\infty |\psi_n(t)v(t) - v(t)|^2 \, dt
\]

\[
= \int_n^\infty |(\psi_n(t) - 1)v(t)|^2 \, dt
\]

\[
\leq \left( \sup_{n \in \mathbb{N}} |\psi_n|_{\infty} + 1 \right)^2 \int_n^\infty |v(t)|^2 \, dt
\]

\[
\rightarrow 0 \quad (n \to \infty).
\]
By definition of $u$ we see that $u \in C^\infty(\mathbb{R}_0)$ with $\text{supp} u \geq a$ and the same holds for $e^{-\nu m}u$. Furthermore $e^{-\nu m}u \in L_2(\mathbb{R}_0)$ and thus $(\psi_n e^{-\nu m}u)_{n \in \mathbb{N}} \in C_c^\infty(\mathbb{R}_0)^N$ converges to $e^{-\nu m}u$ in $L_2(\mathbb{R}_0)$. By the product rule we obtain

$$(\psi_n e^{-\nu m}u)'(t) = \psi_n'(t)(e^{-\nu m}u)(t) + \psi_n(t)(-\nu(e^{-\nu m}u)(t) + (e^{-\nu m}\phi)(t)) \quad (t \in \mathbb{R}_0)$$

for each $n \in \mathbb{N}$ and by the same argumentation as above, we conclude

$$\psi_n(-\nu e^{-\nu m}u + e^{-\nu m}\phi) \to -\nu e^{-\nu m}u + e^{-\nu m}\phi \text{ in } L_2(\mathbb{R}_0) \quad (n \to \infty).$$

Moreover, we estimate

$$\int_0^\infty |\psi_n'(t)e^{-\nu t}u(t)|^2 \, dt = \int_0^\infty |\psi_n'(t)e^{-\nu t}u(t)|^2 \, dt \leq \left( \sup_{n \in \mathbb{N}} |\psi_n'|_{\infty} \right)^2 \int_0^\infty |e^{-\nu t}u(t)|^2 \, dt \to 0 \quad (n \to \infty)$$

and hence

$$(\psi_n e^{-\nu m}u)' \to -\nu e^{-\nu m}u + e^{-\nu m}\phi \text{ in } L_2(\mathbb{R}_0) \quad (n \to \infty).$$

This implies that $e^{-\nu m}u \in D(\bar{\partial})$ and $\bar{\partial} (e^{-\nu m}u) = e^{-\nu m}(-\nu u + \phi)$. By the definition of the derivative operator $\partial_{\nu,0}$ this implies $\partial_{\nu,0}u = \phi$. So we have proved that $0 \in \rho(\partial_{\nu,0})$.

For later applications it is useful to compute $\|\partial_{\nu,0}^{-1}\|$ for $k \in \mathbb{N}$ precisely.

**Lemma 2.1.** Let $\phi \in H_{\nu,0}$. Then

$$(\partial_{\nu,0}^{-1}\phi)(x) = \int_0^x \phi(t) \, dt \quad (x \in \mathbb{R}_0 \text{ a.e.}).$$

**Proof.** Since $C_c^\infty(\mathbb{R}_0)$ is dense in $H_{\nu,0}$ we find a sequence $(\psi_n)_{n \in \mathbb{N}} \in C_c^\infty(\mathbb{R}_0)^N$ such that $\psi_n \to \phi$ in $H_{\nu,0}$ as $n \to \infty$. By the calculations above we know, that

$$(\partial_{\nu,0}^{-1}\psi_n)(x) = \int_0^x \psi_n(t) \, dt \quad (x \in \mathbb{R}_0).$$

Since $\partial_{\nu,0}^{-1}$ is continuous, we obtain $\partial_{\nu,0}^{-1}\psi_n \to \partial_{\nu,0}^{-1}\phi$ in $H_{\nu,0}$ and by choosing a suitable subsequence (again labeled by index $n$) we get $(\partial_{\nu,0}^{-1}\psi_n)(x) \to (\partial_{\nu,0}^{-1}\phi)(x)$ a.e. as $n \to \infty$. We estimate, by using the Cauchy-Schwarz-Inequality for each $x \in \mathbb{R}_0$:

$$\left| \int_0^x \psi_n(t) - \phi(t) \, dt \right| \leq \int_0^x |\psi_n(t) - \phi(t)|e^{-\nu t}e^{\nu t} \, dt$$
2.1. The time derivative

\[
\begin{align*}
\leq & \left( \int_0^x |\psi_n(t) - \phi(t)|^2 e^{-2\nu t} \, dt \right)^{1/2} \left( \int_0^x e^{2\nu t} \, dt \right)^{1/2} \\
\leq & \left| \psi_n - \phi \right|_{H_{r,0}} \frac{1}{\sqrt{2\nu}} (e^{2\nu x} - 1)^{1/2} \\
\to & \quad 0 \quad (n \to \infty).
\end{align*}
\]

Since on the other hand

\[
\int_0^x \psi_n(t) \, dt = (\partial_{\nu,0}^{-1} \psi_n)(x) \to (\partial_{\nu,0}^{-1} \phi)(x) \quad (x \in \mathbb{R}_{>0} \text{ a.e.})
\]

as \( n \to \infty \) we get

\[
(\partial_{\nu,0}^{-1} \phi)(x) = \int_0^x \phi(t) \, dt \quad (x \in \mathbb{R}_{>0} \text{ a.e.}).
\]

\[\square\]

**Proposition 2.2.** Let \( \nu \in \mathbb{R}_{>0} \). Then \( \|\partial_{\nu,0}^{-k}\| = \nu^{-k} \) for all \( k \in \mathbb{N} \).

**Proof.** We first consider the case \( k = 1 \).

We define a sequence \((\phi_n)_{n \in \mathbb{N}} \in H^1_{\nu,0}\) by

\[
\phi_n(x) := \frac{1}{\sqrt{n}} e^{\nu x} \chi_{[0,n]}(x) \quad (x \in \mathbb{R}_{>0}, n \in \mathbb{N}).
\]

Moreover, we define

\[
u_n(x) := \int_0^x \phi_n(t) \, dt \quad (x \in \mathbb{R}_{>0}, n \in \mathbb{N})
\]

and obtain

\[
\begin{align*}
\nu_n(x) &= \int_0^x \phi_n(t) \, dt \\
&= \frac{1}{\sqrt{n}} \int_0^x e^{\nu t} \chi_{[0,n]}(t) \, dt \\
&= \frac{1}{\sqrt{n}} \int_0^{\min\{x,n\}} e^{\nu t} \, dt \\
&= \frac{1}{\sqrt{n}} (e^{\nu \min\{x,n\}} - 1)
\end{align*}
\]
for all \( x \in \mathbb{R}_{>0} \). Moreover, we obtain that \( |\phi_n|_{H^{\nu}_0} = 1 \) for all \( n \in \mathbb{N} \) by calculating

\[
|\phi_n|_{H^{\nu}_0}^2 = \frac{1}{n} \int_0^\infty e^{2\nu x} \chi_{[0,n]}(x) e^{-2\nu x} \, dx = 1.
\]

By Lemma 2.1 we get \( u_n = \partial^{-1}_{\nu,0} \phi_n \) for all \( n \in \mathbb{N} \). Furthermore, the following equality holds:

\[
\phi_n(x) - \nu u_n(x) = \frac{1}{\sqrt{n}} (1 - e^{|\nu|}(x)) \quad (x \in \mathbb{R}_{>0}).
\]

Thus, we obtain

\[
\int_0^\infty |\phi_n(x) - \nu u_n(x)|^2 e^{-2\nu x} \, dx = \frac{1}{n} \int_0^\infty |e^{-\nu x} (1 - e^{\nu|n|}(x))|^2 \, dx \to 0 \quad (n \to \infty).
\]

Therefore,

\[
||\nu u_n|_{H^{\nu}_0} - 1| = ||\nu u_n|_{H^{\nu}_0} - |\phi_n|_{H^{\nu}_0}| \leq ||\nu u_n - \phi_n|_{H^{\nu}_0} \to 0 \quad (n \to \infty)
\]

and thus \( |u_n|_{H^{\nu}_0} \to \frac{1}{\nu} \) as \( n \to \infty \). Since \( u_n = \partial^{-1}_{\nu,0} \phi_n \) and \( |\phi_n|_{H^{\nu}_0} = 1 \) this implies

\[
\frac{1}{\nu} \leq ||\partial^{-1}_{\nu,0}||.
\]

Since we already know that \( ||\partial^{-1}_{\nu}\|| \leq \frac{1}{\nu} \), this completes the proof for \( k = 1 \).

Applying the continuity of \( \partial^{-k}_{\nu,0} \) for all \( k \in \mathbb{N} \) we derive from (2.3) that

\[
\partial^{-k}_{\nu,0} \phi_n - \nu \partial^{-k}_{\nu,0} u_n \to 0 \quad (n \to \infty) \text{ in } H^{\nu}_0.
\]

We will show by induction that \( |\partial^{-k}_{\nu,0} u_n|_{H^{\nu}_0} \to \nu^{-k+1} \) for all \( k \in \mathbb{N}_0 \). For \( k = 0 \) we already have shown the assertion. Let us assume it is true for some \( k \in \mathbb{N}_0 \). Then we compute by using \( u_n = \partial^{-1}_{\nu,0} \phi_n \):

\[
\left| |\partial^{-k}_{\nu,0} u_n|_{H^{\nu}_0} - \frac{1}{\nu^{k+2}} \right| \leq \left| |\partial^{-k}_{\nu,0} u_n|_{H^{\nu}_0} - \frac{1}{\nu} |\partial^{-k}_{\nu,0} \phi_n|_{H^{\nu}_0} \right| + \left| \frac{1}{\nu} - \frac{1}{\nu^{k+2}} \right|
\]

\[
\leq \left| \frac{1}{\nu} |\partial^{-k}_{\nu,0} \phi_n - \nu |\partial^{-k}_{\nu,0} u_n|_{H^{\nu}_0} \right| + \left| \frac{1}{\nu} \right| \left| \partial^{-k}_{\nu,0} \phi_n - \frac{1}{\nu^{k+1}} \right|
\]

\[
= \frac{1}{\nu} \left| |\partial^{-k}_{\nu,0} \phi_n|_{H^{\nu}_0} - \nu |\partial^{-k}_{\nu,0} u_n|_{H^{\nu}_0} + \frac{1}{\nu} \right| \left| \partial^{-k}_{\nu,0} \phi_n - \frac{1}{\nu^{k+1}} \right|
\]

\[
\to 0 \quad (n \to \infty).
\]
2.2. Well posedness of evolutionary inclusions for positive times

So we conclude for all \( k \in \mathbb{N} \)
\[
|\partial_{\nu,0}^{-k} \phi_n|_{H_{\nu,0}} = |\partial_{\nu,0}^{-(k-1)} u_n|_{H_{\nu,0}} \to \frac{1}{\nu^k} \quad (n \to \infty)
\]
and since \( |\phi_n|_{H_{\nu,0}} = 1 \) for all \( n \in \mathbb{N} \) we get
\[
\|\partial_{\nu,0}^{-k}\| \geq \frac{1}{\nu^k}.
\]
Since on the other hand one clearly has
\[
\|\partial_{\nu,0}^{-k}\| \leq \|\partial_{\nu,0}^{-1}\|^k = \frac{1}{\nu^k},
\]
we get the assertion. \( \square \)

2.2. Well posedness of evolutionary inclusions for positive times

Let \( A \subseteq H \oplus H \) be a maximal monotone relation. We set for \( \nu \in \mathbb{R}_{>0} \):
\[
A_{\nu} := A_{L_2(\mathbb{R}_{\geq 0}, \mu_{\nu}; H)}.
\]
Then \( A_{\nu} \) is also maximal monotone by Proposition 1.14 since \( \mu_{\nu}(\mathbb{R}_{\geq 0}) = \frac{1}{2\nu} < \infty \). Moreover, we have two bounded, linear operators \( M_0, M_1 : H \to H \) such that \( M_0 \) is selfadjoint, \( M_0|_{M_0[H]} \geq c_0 > 0 \) and \( \text{Re} M_1|_{(0)} M_0 \geq c_1 > 0 \). We will denote the orthogonal projector on \( M_0[H] \) by \( P \) and set \( Q := 1 - P \).

We identify the operators \( \partial_{\nu,0}, M_0, M_1 \) and \( P \) with their extensions on \( H_{\nu,0} \otimes H \) by taking the tensorproducts with the identities \( 1_{H_{\nu,0}} \) and \( 1_H \) respectively.\(^2\) We will study partial differential inclusions of the type
\[
(u, f) \in \partial_{\nu,0}M_0 + M_1 + A_{\nu}
\]
for a given right-hand side \( f \in H_{\nu,0} \otimes H \). We will show that this kind of problems are well-posed, i.e. we will prove the uniqueness, the existence\(^3\) and the continuous dependence on the given data of a solution. This problem was analyzed (with a more general material law) in \([13]\) in the case, when \( A \) is a skew-selfadjoint operator. By extending this theory to maximal monotone relations, we are in particular able to deal with positive, selfadjoint operators, like the Dirichlet- or Neumann-Laplace (cf. \([20]\)) as well as subgradients of lower semicontinuous, convex, proper mappings (\([3],[39],[10]\)). We start to discuss the uniqueness and continuous dependence of a solution on the given function \( f \).

Lemma 2.3. For every \( \eta \in \mathbb{R}_{>0}, 0 < c < c_1 \) we find \( \nu_0 \in \mathbb{R}_{>0} \) such that
\[
\forall u \in D(\partial_{\nu,0}M_0) : \text{Re}(\partial_{\nu,0}M_0 u + M_1 u|u|)_{H_{\nu,0} \otimes H} \geq \eta |P u|_{H_{\nu,0} \otimes H}^2 + c |Q u|_{H_{\nu,0} \otimes H}^2
\]
for all \( \nu \geq \nu_0 \). Especially for \( \eta = c \) we get
\[
\forall u \in D(\partial_{\nu,0}M_0) : \text{Re}(\partial_{\nu,0}M_0 u + M_1 u|u|)_{H_{\nu,0} \otimes H} \geq c |u|_{H_{\nu,0} \otimes H}^2.
\]
\(^2\)See the appendix part A for an introduction to tensorproducts of Hilbert spaces and linear operators.
\(^3\)Indeed we will show that \( (\partial_{\nu,0}M_0 + M_1 + A_{\nu})|_{H_{\nu,0} \otimes H} \) is dense in \( H_{\nu,0} \otimes H \).
Proof. Let \( \eta \in \mathbb{R}_{>0}, c \in (0, c_1) \) and define \( \nu_0 := \eta \left( \frac{1}{c_0} \right) \). Then we estimate for every \( u \in D(\partial_{\nu,0}M_0) \) and \( \nu \geq \nu_0 \):

\[
\text{Re}((\partial_{\nu,0}M_0 + M_1)u|u)_{H_{\nu,0} \otimes H} = \text{Re}((\partial_{\nu,0}M_0Pu|Pu)_{H_{\nu,0} \otimes H} + \text{Re}(M_1Pu|Pu)_{H_{\nu,0} \otimes H} + \text{Re}(M_1Qu|Pu)_{H_{\nu,0} \otimes H} + \text{Re}(M_1Qu|Qu)_{H_{\nu,0} \otimes H}

\geq \nu \eta |Pu|^2_{H_{\nu,0} \otimes H} - 2|\text{Re}(\nu c_0 - \|M_1\|)|Pu|^2_{H_{\nu,0} \otimes H} + c_1 |Qu|^2_{H_{\nu,0} \otimes H}

\geq \eta |Pu|^2_{H_{\nu,0} \otimes H} + c_1 |Qu|^2_{H_{\nu,0} \otimes H}.
\]

The following result yields the uniqueness and the continuous dependence of a solution on the given data.

**Proposition 2.4.** Let \( 0 < c < c_1 \) and \( \nu \in \mathbb{R}_{>0} \) such that \( \text{Re} \partial_{\nu,0}M_0 + M_1 \geq c \) and let \( B \subseteq (H_{\nu,0} \otimes H)^2 \) be monotone. If \( u, v, f, g \in H_{\nu,0} \otimes H \) are such that

\[
(u, f), (v, g) \in \partial_{\nu,0}M_0 + M_1 + B,
\]

then

\[
|u - v|_{H_{\nu,0} \otimes H} \leq c^{-1} |f - g|_{H_{\nu,0} \otimes H}.
\]

**Proof.** Let \( x, y \in H_{\nu,0} \otimes H \) such that \( (u, x), (v, y) \in B \) and

\[
f = \partial_{\nu,0}M_0u + M_1u + x,
\]

\[
g = \partial_{\nu,0}M_0v + M_1v + y.
\]

Since \( B \) is monotone we estimate:

\[
\text{Re}(f - g|u - v)_{H_{\nu,0} \otimes H} = \text{Re}((\partial_{\nu,0}M_0 + M_1)(u - v) + x - y|u - v)_{H_{\nu,0} \otimes H}

\geq \text{Re}((\partial_{\nu,0}M_0 + M_1)(u - v)|u - v)_{H_{\nu,0} \otimes H}

\geq c|u - v|^2_{H_{\nu,0} \otimes H}.
\]

By applying the Cauchy-Schwarz-Inequality on the left hand side we conclude:

\[
|u - v|_{H_{\nu,0} \otimes H} \leq c^{-1} |f - g|_{H_{\nu,0} \otimes H}.
\]

It remains to show the existence of a solution for “sufficiently many” right-hand sides. First we need two Lemmas.
Lemma 2.5. For $h \in \mathbb{R}$ we define the translation operator

$$
\tau_h : H_{\nu,0} \otimes H \to H_{\nu,0} \otimes H
$$

$$
f \mapsto \begin{cases} 
    f(t+h) & \text{if } t \geq -h, \\
    0 & \text{otherwise}
\end{cases}
$$

Then $\tau_h$ is linear, continuous and $\tau_h^* = e^{2\nu h} \tau_{-h}$.

Proof. The linearity of $\tau_h$ is obvious. For $f \in H_{\nu,0} \otimes H$ we calculate

$$
\int_0^\infty |(\tau_h f)(t)|^2 e^{-2\nu t} \, dt = \int_{\max(0,-h)}^\infty |f(t+h)|^2 e^{-2\nu t} \, dt
$$

$$
= \int_{\max(0,h)}^\infty |f(s)|^2 e^{-2\nu(s-h)} \, ds
$$

and thus

$$
\|\tau_h\| \leq e^{\nu h}.
$$

Let $g \in H_{\nu,0} \otimes H$. Then

$$
\langle \tau_h f, g \rangle_{H_{\nu,0} \otimes H} = \int_{\max(0,-h)}^\infty \langle f(t+h), g(t) \rangle e^{-2\nu t} \, dt
$$

$$
= \int_{\max(0,h)}^\infty \langle f(s), g(s-h) \rangle e^{-2\nu(s-h)} \, ds
$$

$$
= \langle f, e^{2\nu h} \tau_{-h} g \rangle_{H_{\nu,0} \otimes H},
$$

which shows

$$
\tau_h^* = e^{2\nu h} \tau_{-h}.
$$

Lemma 2.6. Let $u \in D(\partial_{\nu,0})$ for $\nu \in \mathbb{R}_{>0}$ and $h \in \mathbb{R}_{\leq 0}$. Then $\tau_h u \in D(\partial_{\nu,0})$ and $\partial_{\nu,0} \tau_h u = \tau_h \partial_{\nu,0} u$.

Proof. Since $u \in D(\partial_{\nu,0})$ there exists a sequence $(\phi_n)_{n \in \mathbb{N}} \subset C^\infty_c(\mathbb{R}_{>0}; H)^{\mathbb{N}}$ such that

$$
\phi_n \to u \text{ and } \phi'_n \to \partial_{\nu,0} u \text{ in } H_{\nu,0} \otimes H \text{ as } n \to \infty.
$$

According to Lemma 2.5, $\tau_h$ is continuous and since $\tau_h \phi'_n = (\tau_h \phi_n)'$ for all $n \in \mathbb{N}$, we obtain

$$
\tau_h \phi_n \to \tau_h u \text{ and } (\tau_h \phi_n)' \to \tau_h \partial_{\nu,0} u \text{ in } H_{\nu,0} \otimes H \text{ as } n \to \infty.
$$

Clearly $\tau_h \phi_n \in C^\infty_c(\mathbb{R}_{>0}; H)$ (note that $h \leq 0$) for all $n \in \mathbb{N}$ and thus $\tau_h u \in D(\partial_{\nu,0})$ with $\partial_{\nu,0} \tau_h u = \tau_h \partial_{\nu,0} u$. \qed
The main argument for proving the existence of solutions for our evolutionary inclusion, is to show the \( c \)-maximal monotonicity of \( \partial_{\nu,0}M_0 + M_1 \) for a certain constant \( c > 0 \), which is done in the next proposition. With this knowledge, we are able to apply the results presented in Section 1.3 to show the existence of a solution.

**Proposition 2.7.** Let \( 0 < c < c_1 \) and \( \nu_0 \in \mathbb{R}_{>0} \) such that \( \Re(\partial_{\nu,0}M_0 + M_1) \geq c \) for all \( \nu \geq \nu_0 \). Then there exists \( \nu_1 \geq \nu_0 \) such that \( \partial_{\nu,0}M_0 + M_1 - c \) is maximal monotone for every \( \nu > \nu_1 \).

**Proof.** The monotonicity is clear, since \( \Re(\partial_{\nu,0}M_0 + M_1) \geq c \) for \( \nu \geq \nu_0 \). For showing the maximal monotonicity we apply Minty’s Theorem. In fact we show that \( \partial_{\nu,0}M_0 + M_1 \) is surjective. Let \( f \in H_{\nu,0} \otimes H \) and consider the following fixed point problem on the Hilbert space \( H_{\nu,0} \otimes M_0[H] \):

\[
\begin{align*}
\nu &\in (PM_0P)^{-1}(P\partial_{\nu,0}^{-1}(f - M_1(QM_1Q)^{-1}Qf) + P\partial_{\nu,0}^{-1}(M_1(QM_1Q)^{-1}QM_1 - M_1)u^*). \\
&= (PM_0P)^{-1}(P\partial_{\nu,0}^{-1}(f - M_1(QM_1Q)^{-1}Qf) + P\partial_{\nu,0}^{-1}(M_1(QM_1Q)^{-1}QM_1 - M_1)u^*).
\end{align*}
\]

(2.4)

Here \( P : H_{\nu,0} \otimes H \to H_{\nu,0} \otimes H \) denotes the orthogonal projector on \( H_{\nu,0} \otimes M_0[H] \) and \( Q = 1 - P \). Keep in mind that we have already proved the invertibility of \( \partial_{\nu,0} \). We calculate the Lipschitz-seminorm of the operator on the right hand side and estimate

\[
|u| = \big| (PM_0P)^{-1}(P\partial_{\nu,0}^{-1}(f - M_1(QM_1Q)^{-1}Qf) + P\partial_{\nu,0}^{-1}(M_1(QM_1Q)^{-1}QM_1 - M_1)u^*) \big|_{\text{Lip}} \leq \frac{1}{c_0^\nu} \left( \frac{\|M_1\|}{c_0} + \|M_1\| \right).
\]

We define \( \nu_1 := \max \left\{ \nu_0, \frac{\|M_1\|}{c_0} \left( \frac{\|M_1\|}{c_1} + 1 \right) \right\} \). Then we get for each \( \nu > \nu_1 \)

\[
|u| = \big| (PM_0P)^{-1}(P\partial_{\nu,0}^{-1}(f - M_1(QM_1Q)^{-1}Qf) + P\partial_{\nu,0}^{-1}(M_1(QM_1Q)^{-1}QM_1 - M_1)u^*) \big|_{\text{Lip}} < 1
\]

and so the Contraction Mapping Theorem yields the existence of \( u^* \in H_{\nu,0} \otimes M_0[H] \), which satisfies (2.4). Additionally we find out that \( M_0u^* \in D(\partial_{\nu,0}) \). We define

\[
v^* := (QM_1Q)^{-1}(Qf - QM_1u^*) \in H_{\nu,0} \otimes \{0\}M_0
\]

and set \( u := u^* + v^* \). We calculate

\[
\begin{align*}
\partial_{\nu,0}M_0u + M_1 u &\quad= \partial_{\nu,0}M_0u^* + M_1 u^* + M_1 v^* \\
&\quad= \partial_{\nu,0}M_0u^* + M_1 u^* + M_1 (QM_1Q)^{-1}(Qf - QM_1u^*) \\
&\quad= \partial_{\nu,0}M_0u^* + M_1 u^* + PM_1(QM_1Q)^{-1}(Qf - QM_1u^*) + Qf - QM_1u^* \\
&\quad= \partial_{\nu,0}M_0u^* + PM_1u^* + PM_1(QM_1Q)^{-1}(Qf - QM_1u^*) + Qf.
\end{align*}
\]

Since \( u^* \) satisfies the fixed point equation (2.4), it follows that

\[
PM_0u^* = P\partial_{\nu,0}^{-1}(f - M_1(QM_1Q)^{-1}Qf) + P\partial_{\nu,0}^{-1}(M_1(QM_1Q)^{-1}QM_1 - M_1)u^*
\]

and thus

\[
\partial_{\nu,0}M_0u^* = Pf - PM_1(QM_1Q)^{-1}Qf + PM_1(QM_1Q)^{-1}QM_1u^* - PM_1u^*.
\]
2.2. Well posedness of evolutionary inclusions for positive times

Summarizing we get

$$\partial_{\nu,0}M_0u + M_1u = Pf + Qf = f.$$  

Hence, $\partial_{\nu,0}M_0 + M_1$ is surjective. Let $f \in H_{\nu,0} \otimes H$. Then we find $u \in D(\partial_{\nu,0})$ such that

$$(\partial_{\nu,0}M_0 + M_1 - c + c)u = cf,$$

which is equivalent to

$$(1 + \frac{1}{c}(\partial_{\nu,0}M_0 + M_1 - c))u = f.$$  

This shows the maximal monotonicity of $\partial_{\nu,0}M_0 + M_1 - c$ according to Theorem 1.6

Now we are able to prove the existence of a solution for a right side $f \in C_\infty_c(\mathbb{R}_{>0}; H)$, by using the results about sums of maximal monotone relations, discussed in Section 1.3.

**Proposition 2.8.** Let $f \in C_\infty_c(\mathbb{R}_{>0}; H)$ and assume that $(0, 0) \in A$. Then there exists $\nu_0 \in \mathbb{R}_{>0}$ such that for every $\nu \geq \nu_0$ we find $u \in H_{\nu,0} \otimes H$ such that

$$(u, f) \in \partial_{\nu,0}M_0 + M_1 + A_\nu.$$  

Proof. For $\lambda \in \mathbb{R}_{>0}$ we denote the Yosida-Approximation of $A_\nu$ by $A_{\nu,\lambda}$. Let $0 < c < c_1$ and $\nu_0 \in \mathbb{R}_{>0}$ such that $\partial_{\nu,0}M_0 + M_1 - c$ is maximal monotone for all $\nu \geq \nu_0$ (cf. Lemma 2.3). We want to apply Lemma 1.17 for showing the existence of $u \in H_{\nu,0} \otimes H$ such that

$$(u, f) \in \partial_{\nu,0}M_0 + M_1 + A_\nu$$  

which is equivalent to

$$\left( u, \frac{1}{c}f \right) \in \left( 1 + \frac{1}{c}(\partial_{\nu,0}M_0 + M_1 - c) + \frac{1}{c}A_\nu \right).$$

An easy computation shows that $(\frac{1}{c}A_\nu)_\lambda = \frac{1}{c}A_{\nu,\lambda}$. We use Corollary 1.16 to define $u_\lambda \in H_{\nu,0} \otimes H$ by

$$u_\lambda := \left( 1 + \frac{1}{c}(\partial_{\nu,0}M_0 + M_1 - c) + \frac{1}{c}A_{\nu,\lambda} \right)^{-1} \left( \frac{1}{c}f \right),$$

which in particular implies that $Pu_\lambda \in D(\partial_{\nu,0})$ for all $\lambda \in \mathbb{R}_{>0}$. We want to show that

$$\sup_{\lambda \in \mathbb{R}_{>0}} \left| \frac{1}{c}A_{\nu,\lambda} \right|_{H_{\nu,0} \otimes H} < \infty. \quad (2.5)$$

Since $(0, 0) \in A$, it follows that $(0, 0) \in A_\nu$ and thus $(0, 0) \in \left( \frac{1}{c}(\partial_{\nu,0}M_0 + M_1 - c) \right) \cap \left( \frac{1}{c}A_\nu \right)$. Lemma 1.18 yields

$$|u_\lambda|_{H_{\nu,0} \otimes H} \leq \frac{1}{c} |f|_{H_{\nu,0} \otimes H}$$

for all $\lambda \in \mathbb{R}_{>0}$. Next we want to find an upper bound for $|\partial_{\nu,0}Pu_\lambda|_{H_{\nu,0} \otimes H}$ for $\lambda \in \mathbb{R}_{>0}$. By the definition of $u_\lambda$ we obtain

$$\partial_{\nu,0}M_0u_\lambda + M_1u_\lambda + A_{\nu,\lambda}(u_\lambda) = f$$
for $\lambda \in \mathbb{R}_{>0}$. Let $h < 0$. One easily sees that $\tau_h A_{v, -\frac{h}{c}}(u_\lambda) = A_{v, -\frac{h}{c}}(\tau_h u_\lambda)$, since $(0, 0) \in A$ and by using Lemma 2.6 we find that

$$\partial_{v,0} M_0 \tau_h u_\lambda + M_1 \tau_h u_\lambda + A_{v, -\frac{h}{c}}(\tau_h u_\lambda) = \tau_h f.$$  

Since $A_{v, -\frac{h}{c}}$ is monotone, Proposition 2.4 yields that

$$|\tau_h P u_\lambda - P u_\lambda|_{H_{v,0} \otimes H} \leq |\tau_h u_\lambda - u_\lambda|_{H_{v,0} \otimes H} \leq \frac{1}{c} |\tau_h f - f|_{H_{v,0} \otimes H}$$

and thus

$$|h^{-1}(\tau_h P u_\lambda - P u_\lambda)|_{H_{v,0} \otimes H} \leq \frac{1}{c} |h^{-1}(\tau_h f - f)|_{H_{v,0} \otimes H} \leq \frac{1}{\sqrt{2\nu c}} |f'|_{\infty}.$$  

For $\phi \in C_c^\infty(\mathbb{R}_{>0};H)$ we observe with Lemma 2.5

$$\langle h^{-1}(\tau_h P u_\lambda - P u_\lambda) \phi \rangle_{H_{v,0} \otimes H} = \langle P u_\lambda | h^{-1}(e^{2\nu h - 1}) \phi \rangle_{H_{v,0} \otimes H} = \langle P u_\lambda | e^{2\nu h - 1} (\tau_h \phi - \phi) \rangle_{H_{v,0} \otimes H} + \langle P u_\lambda | h^{-1}(e^{2\nu h - 1}) \phi \rangle_{H_{v,0} \otimes H}.$$  

We compute

$$|e^{2\nu h - 1} (\tau - h \phi - \phi')^2|_{H_{v,0} \otimes H} = \int_0^\infty e^{2\nu h (t-h) - 2\nu h - 1} (\phi(t-h) - \phi(t)) \phi(t) e^{-2\nu t} \, dt$$

$$\to 0 \quad (h \to 0)$$

and

$$|h^{-1}(e^{2\nu h - 1}) \phi - 2\nu \phi|^2_{H_{v,0} \otimes H} = \int_0^\infty \frac{h^{-1}(e^{2\nu h - 1} - 2\nu h \phi(t) e^{-2\nu t}}{\leq 2(h^4 + 1)|\phi|^2_{L^2} e^{-2\nu t}} \, dt$$

$$\to 0 \quad (h \to 0),$$

by Lebesgue’s dominated convergence theorem. Therefore

$$\langle h^{-1}(\tau_h P u_\lambda - P u_\lambda) \phi \rangle_{H_{v,0} \otimes H} \to \langle P u_\lambda \phi \rangle_{H_{v,0} \otimes H} + \langle P u_\lambda | 2\nu \phi \rangle_{H_{v,0} \otimes H}$$

$$= \langle P u_\lambda | - \partial_{v,0} \phi + 2\nu \phi \rangle_{H_{v,0} \otimes H}$$

$$= \langle \partial_{v,0} P u_\lambda | \phi \rangle_{H_{v,0} \otimes H} \quad (h \to 0-).$$

Since $(h^{-1}(\tau_h P u_\lambda - P u_\lambda))_{h<0}$ is bounded and $C_c^\infty(\mathbb{R}_{>0};H)$ is dense in $H_{v,0} \otimes H$ it follows that

$$h^{-1}(\tau_h P u_\lambda - P u_\lambda) \to \partial_{v,0} P u_\lambda \quad (h \to 0-),$$

and thus

$$|\partial_{v,0} P u_\lambda|_{H_{v,0} \otimes H} \leq \frac{1}{\sqrt{2\nu c}} |f'|_{\infty}$$

for all $\lambda \in \mathbb{R}_{>0}$.  

2.2. Well posedness of evolutionary inclusions for positive times
Summarizing we get
\[
|A_{\nu,0}(u_\lambda)|_{H_{\cdot,0} \otimes H} = |f - M_1 u_\lambda - \partial_{\nu,0} M_0 P u_\lambda|_{H_{\cdot,0} \otimes H}
\leq |f|_{H_{\cdot,0} \otimes H} + \|M_1\| \frac{1}{c} |f|_{H_{\cdot,0} \otimes H} + \|M_0\| \frac{1}{\sqrt{2\nu c}} |f'|_{\infty},
\]
which implies the desired inequality (2.5). Hence, Lemma 1.17 yields the existence of \( u \in H_{\nu,0} \otimes H \) such that
\[
(u, \frac{1}{c} f) \in (1 + \frac{1}{c}(\partial_{\nu,0} M_0 + M_1 - c) + \frac{1}{c} A_\nu),
\]
which turns over into
\[
(u, f) \in \partial_{\nu,0} M_0 + M_1 + A_\nu.
\]

We now want to relax the condition \( (0, 0) \in A \) in Proposition 2.8 by just assuming \( 0 \in [H] A \).

**Proposition 2.9.** If \( 0 \in [H] A \) then there exists \( \nu_0 \in \mathbb{R}_{>0} \) such that for every \( \nu \geq \nu_0 \) the post set \((\partial_{\nu,0} M_0 + M_1 + A_\nu)[H_{\nu,0} \otimes H]\) is dense in \( H_{\nu,0} \otimes H \).

**Proof.** Since \( 0 \in [H] A \) we find an element \( z \in H \) with \( (0, z) \in A \). We define the maximal monotone relation \( B \) by
\[
B := \{[A] - (0, z) = \{(x, y - z) \in H \otimes H \mid (x, y) \in A\}
\]
and obtain \( (0, 0) \in B \). According to Proposition 2.8 we find \( \nu_0 \in \mathbb{R}_{>0} \), such that for all \( \nu \geq \nu_0 \)
\[
C_c^\infty(\mathbb{R}_{>0}; H) \subseteq (\partial_{\nu,0} M_0 + M_1 + B_\nu)[H_{\nu,0} \otimes H].
\]
An easy argumentation yields that \( B_\nu = [A_\nu] - (0, \hat{z}) \) where \( \hat{z} \in H_{\nu,0} \otimes H \) denotes the constant function with value \( z \in H \). Thus we find out that
\[
[C_c^\infty(\mathbb{R}_{>0}; H)] + \hat{z} \subseteq (\partial_{\nu,0} M_0 + M_1 + A_\nu)[H_{\nu,0} \otimes H]
\]
and obviously \([C_c^\infty(\mathbb{R}_{>0}; H)] + \hat{z} \) lies dense in \( H_{\nu,0} \otimes H \). \( \Box \)

We summarize our results of this section in the following Corollary.

**Corollary 2.10.** Let \( M_0, M_1 : H \rightarrow H \) be two bounded, linear operators. Assume that \( M_0 \) is selfadjoint, \( M_0|M_0[H] \geq c_0 > 0 \) and \( \text{Re} M_1||0||M_0 \geq c_1 > 0 \). Furthermore let \( A \subseteq H \oplus H \) be maximal monotone with \( 0 \in [H] A \). Then there exists \( \nu_1 \in \mathbb{R}_{>0} \), such that for every \( f \in H_{\nu,0} \otimes H \) we find an unique element \( u \in H_{\nu,0} \otimes H \), such that
\[
(u, f) \in \partial_{\nu,0} M_0 + M_1 + A_\nu
\]
for \( \nu \geq \nu_1 \). Moreover, the solution \( u \) depends continuously on the function \( f \).
2.3. Causality

We want to address the concept of causality in our framework, which plays a crucial role, when modelling physical phenomenas mathematically. For a general definition we refer to [33].

**Definition 2.11.** Let $H$ be a Hilbert space, $\nu > 0$ and $F : D(F) \subseteq H_{\nu,0} \otimes H \rightarrow H_{\nu,0} \otimes H$. Then $F$ is called *(forward)* causal, if for all $f, g \in D(F)$ with $\chi_{[0,a]}(m_0)(f - g) = 0$ for a certain $a \in \mathbb{R}_{\geq 0}$, it follows that also

$$\chi_{[0,a]}(m_0)(F(f) - F(g)) = 0.$$  

Here $(\chi_{[0,a]}(m_0)f)(t) := \chi_{[0,a]}(t)f(t) \in H$ for $t \in \mathbb{R}_{\geq 0}$.

Causality can be interpreted as the independence of the images of $F$ on the future behavior of their pre-images. We now want to show, that our solution operator $(\partial_{\nu,0}M_0 + M_1 + A_{\nu})^{-1}$ is causal in $H_{\nu,0} \otimes H$, but first we show an equivalent definition of causality and some general properties of causal mappings.

**Lemma 2.12.** Let $H$ be a Hilbert space and $\nu > 0$. A mapping $F : H_{\nu,0} \otimes H \rightarrow H_{\nu,0} \otimes H$ is causal if and only if

$$\forall a \in \mathbb{R}_{\geq 0} : \chi_{[0,a]}(m_0)F \chi_{[0,a]}(m_0) = \chi_{[0,a]}(m_0)F.$$  

**(2.6)**

**Proof.** Let $F$ be causal and $a \in \mathbb{R}_{\geq 0}$. For $f \in H_{\nu,0} \otimes H$ we obtain $\chi_{[0,a]}(m_0)(f) = \chi_{[0,a]}(m_0)f = 0$ and thus, by using the assumed causality of $F$

$$\chi_{[0,a]}(m_0)(F(f) - F(\chi_{[0,a]}(m_0)f)) = 0,$$

which yields the assertion. Let now (2.6) hold and fix $f, g \in H_{\nu,0} \otimes H$ with $\chi_{[0,a]}(m_0)(f - g) = 0$ for some $a \in \mathbb{R}_{\geq 0}$. Then we compute

$$\chi_{[0,a]}(m_0)(F(f) - F(g)) = \chi_{[0,a]}(m_0)(F(\chi_{[0,a]}(m_0)f) - F(\chi_{[0,a]}(m_0)g)) = \chi_{[0,a]}(m_0)(F(\chi_{[0,a]}(m_0)f) - F(\chi_{[0,a]}(m_0)f)) = 0,$$

which shows the desired causality of $F$. \qed

**Lemma 2.13.** Let $H$ be a Hilbert space and $\nu > 0$. Then the following statements hold:

a) Let $F : D(F) \subseteq H_{\nu,0} \otimes H \rightarrow H_{\nu,0} \otimes H$ and $G : D(G) \subseteq H_{\nu,0} \otimes H \rightarrow H_{\nu,0} \otimes H$ be two causal mappings and $\lambda \in \mathbb{C}$. Then $\lambda F + G$ is also causal.

b) Let $F : D(F) \subseteq H_{\nu,0} \otimes H \rightarrow H_{\nu,0} \otimes H$ and $G : D(G) \subseteq H_{\nu,0} \otimes H \rightarrow H_{\nu,0} \otimes H$ be two causal mappings. Then $F \circ G$ is causal.

c) Let $F_n : D(F_n) \subseteq H_{\nu,0} \otimes H \rightarrow H_{\nu,0} \otimes H$ be a causal mapping for every $n \in \mathbb{N}$. Define

$$F : D(F) \subseteq H_{\nu,0} \otimes H \rightarrow H_{\nu,0} \otimes H \quad u \mapsto \lim_{n \to \infty} F_n(u),$$

where $D(F) := \{u \in \bigcap_{n \in \mathbb{N}} D(F_n) \mid \lim_{n \to \infty} F_n(u) \text{ exists}\}$. Then $F$ is causal.
Proof. a) Let \( f, g \in D(\lambda F + G) = D(F) \cap D(G) \) with \( \chi_{[0,a]}(m_0)(f - g) = 0 \) for an \( a \in \mathbb{R}_\geq 0 \).

Then
\[
\chi_{[0,a]}(m_0)((\lambda F + G)(f) - (\lambda F + G)(g)) = \chi_{[0,a]}(m_0)(F(f) - F(g)) + \chi_{[0,a]}(m_0)(G(f) - G(g)) = \lambda \chi_{[0,a]}(m_0)(F(f) - F(g)) = 0.
\]

b) Let \( f, g \in D(F \circ G) \) with \( \chi_{[0,a]}(m_0)(f - g) = 0 \) for a certain \( a \in \mathbb{R}_\geq 0 \). We conclude, since \( \chi_{[0,a]}(m_0)(G(f) - G(g)) = 0 \):
\[
\chi_{[0,a]}(m_0)((F \circ G)(f) - (F \circ G)(g)) = \chi_{[0,a]}(m_0)(F(G(f)) - F(G(g))) = 0.
\]

c) Let \( f, g \in D(F) \) with \( \chi_{[0,a]}(m_0)(f - g) = 0 \) for an \( a \in \mathbb{R}_\geq 0 \). Then, according to the definition of \( F \), we find out that
\[
\lim_{n \to \infty} F_n(f) - F_n(g) = F(f) - F(g).
\]

Since the cut-off operator \( \chi_{[0,a]}(m_0) \) is continuous in \( H_{\nu,0} \otimes H \), we conclude
\[
\chi_{[0,a]}(m_0)(F(f) - F(g)) = \lim_{n \to \infty} \chi_{[0,a]}(m_0)(F_n(f) - F_n(g)) = 0.
\]

\[\square\]

Remark 2.14. The concept of causality is also meaningful for functions, defined on the whole real axis. In this case the definition should by modified, by taking the cut-off on the interval \((-\infty, a]\) for each \( a \in \mathbb{R} \). It turns out that the preceding results remain true in this situation.

Now we will consider our solution operator and prove the following Lemma.

Lemma 2.15. Let \( M_0, M_1 \) be as in Corollary 2.10. Let \( 0 < c < c_1 \) and \( a \in \mathbb{R}_\geq 0 \). Then there exists \( \nu_0 \in \mathbb{R}_\geq 0 \) such that for all \( \nu \geq \nu_0 \) it follows that
\[
\int_0^a |u(t)|^2 e^{-2\nu t} \, dt \leq c^{-1} \Re \int_0^a ((\partial_{\nu,0} M_0 + M_1)u(t)|u(t)|^2 e^{-2\nu t} \, dt
\]

for all \( u \in D(\partial_{\nu,0} M_0) \).

Proof. Let \( \phi \in C_\infty^c(\mathbb{R}_\geq 0; H) \) and \( \nu_0 \in \mathbb{R}_\geq 0 \) be chosen as in Lemma 2.3. Then we compute for \( \nu \geq \nu_0 \) with integration by parts
\[
\Re \int_0^a (M_0 \phi'(t)|\phi(t)|e^{-2\nu t} \, dt = \frac{1}{2} \Re \int_0^a (s \to \langle M_0 \phi(s)|\phi(s) \rangle)(t)e^{-2\nu t} \, dt
\]
\[ \begin{align*}
&= \nu \text{Re} \int_0^a \langle M_0 \phi(t) | \phi(t) \rangle e^{-2\nu t} \, dt + \frac{1}{2} \text{Re}(M_0 \phi(a) | \phi(a) \rangle e^{-2\nu a} \\
&\geq \nu c_0 \int_0^a |P \phi(t)|^2 e^{-2\nu t} \, dt.
\end{align*} \]

Thus, we compute with the same technique as in Lemma 2.3

\[ \begin{align*}
\text{Re} \int_0^a (M_0 \phi'(t) + M_1 \phi(t)) | \phi(t) \rangle e^{-2\nu t} \, dt &= \text{Re} \int_0^a \langle M_0 \phi'(t) | \phi(t) \rangle e^{-2\nu t} \, dt \\
&\quad + \text{Re} \int_0^a \langle M_1 Q \phi(t) | Q \phi(t) \rangle e^{-2\nu t} \, dt \\
&\quad + \text{Re} \int_0^a \langle M_1 P \phi(t) | Q \phi(t) \rangle e^{-2\nu t} \, dt \\
&\quad + \text{Re} \int_0^a \langle M_1 Q \phi(t) | P \phi(t) \rangle e^{-2\nu t} \, dt \\
&\quad + \text{Re} \int_0^a \langle M_1 P \phi(t) | P \phi(t) \rangle e^{-2\nu t} \, dt \\
&\geq (\nu c_0 - \| M_1 \|) \int_0^a |P \phi(t)|^2 e^{-2\nu t} \, dt + c_1 \int_0^a |Q \phi(t)|^2 e^{-2\nu t} \, dt \\
&\quad - 2\| M_1 \| \int_0^a |P \phi(t)| |Q \phi(t)| \, dt \\
&\geq (\nu c_0 - \| M_1 \|) \int_0^a |P \phi(t)|^2 e^{-2\nu t} \, dt + c_1 \int_0^a |Q \phi(t)|^2 e^{-2\nu t} \, dt \\
&\quad - \int_0^a \| M_1 \|^2 (c_1 - c)^{-1} |P u(t)|^2 + (c_1 - c) |Q u(t)|^2 \, dt \\
&\quad \geq c \left( \int_0^a |P \phi(t)|^2 e^{-2\nu t} \, dt + \int_0^a |Q \phi(t)|^2 e^{-2\nu t} \, dt \right) \\
&= c \int_0^a |\phi(t)|^2 e^{-2\nu t} \, dt.
\end{align*} \]
Since $C^\infty_c(\mathbb{R}_0^+;H)$ is a core of $\partial_{\nu,0}M_0 + M_1$ and the cut-off operator $\chi_{[0,a]}(m_0)$ is linear and continuous in $L^1(\mathbb{R}_0^+)$ and $L^2(\mathbb{R}_0^+)$, we conclude the assertion by a standard approximation argument.

**Proposition 2.16.** Let $M_0, M_1, A$ be as in Corollary 2.10. Then there exists $\nu_0 \in \mathbb{R}_0^+$, such that the solution operator $(\partial_{\nu,0}M_0 + M_1 + A_\nu)^{-1}$ is causal in $H_{\nu,0} \otimes H$ for each $\nu \geq \nu_0$.

**Proof.** Let $f, g \in (\partial_{\nu,0}M_0 + M_1 + A_\nu)[H_{\nu,0} \otimes H]$ and $a \in \mathbb{R}^+$. We find two pairs $(u, v), (x, y) \in A_\nu$ with $u, x \in D(\partial_{\nu,0}M_0)$ such that

$$(\partial_{\nu,0}M_0 + M_1)u + v = f$$

and

$$(\partial_{\nu,0}M_0 + M_1)x + y = g.$$ 

By using Lemma 2.15 and the monotonicity of $A$ we estimate

$$\int_0^a |u(t) - x(t)|^2 e^{-2\nu t} dt \leq c^{-1} \text{Re} \int_0^a ((\partial_{\nu,0}M_0 + M_1)(u(t) - x(t))) |u(t) - x(t)| e^{-2\nu t} dt$$

$$\leq c^{-1} \text{Re} \int_0^a ((\partial_{\nu,0}M_0 + M_1)(u(t) - x(t)) + v(t) - y(t)) |u(t) - x(t)| e^{-2\nu t} dt$$

$$= c^{-1} \text{Re} \int_0^a (f(t) - g(t)) |u(t) - x(t)| e^{-2\nu t} dt$$

and thus

$$\left(\int_0^a |u(t) - v(t)|^2 e^{-2\nu t} dt\right)^{\frac{1}{2}} \leq c^{-1} \left(\int_0^a |f(t) - g(t)|^2 e^{-2\nu t} dt\right)^{\frac{1}{2}}.$$

For $f, g \in H_{\nu,0} \otimes H$ with $\chi_{[0,a]}(m_0)(f-g) = 0$ we find approximating sequences $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}} \in ((\partial_{\nu,0}M_0 + M_1 + A_\nu)[H_{\nu,0} \otimes H])^\mathbb{N}$ with $f_n \to f$ and $g_n \to g$ in $H_{\nu,0} \otimes H$ as $n \to \infty$. For $u_n := (\partial_{\nu,0}M_0 + M_1 + A_\nu)^{-1}(f_n)$ and $v_n := (\partial_{\nu,0}M_0 + M_1 + A_\nu)^{-1}(g_n)$ we define the limits

$$u := \lim_{n \to \infty} u_n$$

and

$$v := \lim_{n \to \infty} v_n,$$

which implies

$$(u, f), (v, g) \in \overline{\partial_{\nu,0}M_0 + M_1 + A_\nu}$$

and conclude, by using the inequality above and the continuity of the cut-off operator $\chi_{[0,a]}(m_0)$ in $L^2(\mathbb{R}_0^+)$

$$\left(\int_0^a |u(t) - v(t)|^2 e^{-2\nu t} dt\right)^{\frac{1}{2}} \leq c^{-1} \left(\int_0^a |f(t) - g(t)|^2 e^{-2\nu t} dt\right)^{\frac{1}{2}} = 0.$$

This shows that $u(t) = v(t)$ for almost every $t \in [0, a]$. Thus, the solution operator is indeed causal.

\[\Box\]
We summarize our observations of the previous and this section in the following theorem.

**Theorem 2.17.** Let $A \subseteq H \oplus H$ be a maximal monotone relation with $0 \in [H]A$ and $M_0, M_1 : H \to H$ be two bounded, linear operators. Moreover, let $M_0$ be selfadjoint and the following conditions should be satisfied: $M_0|_{M_0[H]} \geq c_0 > 0$ and $\text{Re}M_1|_{\{0\}}M_0 \geq c_1 > 0$. Then there exists $\nu_0 \in \mathbb{R}_{>0}$ such that for all $\nu \geq \nu_0$ the relation $(\partial_{\nu,0}M_0 + M_1 + A_{\nu})^{-1}$ is a Lipschitz-continuous, causal mapping, which is defined on $H_{\nu,0} \otimes H$.

In Theorem 2.17 we find a dependence of the solution operator on the parameter $\nu \geq \nu_0$. The proceeding propositions will show the independence of the choice of $\nu \geq \nu_0$.

**Lemma 2.18.** Let $\nu, \mu \in \mathbb{R}_{>0}$ with $\mu \geq \nu$ and $A \subseteq H \oplus H$. Then the following statements hold:

(a) $H_{\nu,0} \hookrightarrow H_{\mu,0}$,
(b) $\partial_{\nu,0} \subseteq \partial_{\mu,0}$,
(c) $A_{\nu} \subseteq A_{\mu}$.

**Proof.** (a) Let $f \in H_{\nu,0}$. We calculate
\[
\int_0^\infty |f(x)|^2 e^{-2\mu x} \, dx = \int_0^\infty |f(x)|^2 e^{-2\nu x} e^{-2(\mu-\nu) x} \, dx \\
\leq \int_0^\infty |f(x)|^2 e^{-2\nu x} \, dx.
\]
Thus, we obtain $f \in H_{\mu,0}$ and the continuity of the embedding.

(b) Let $u \in D(\partial_{\nu,0})$. This implies the existence of a sequence $(\phi_n)_{n \in \mathbb{N}} \in C^\infty_c(\mathbb{R}_{>0})^N$ with
\[
\phi_n \to u \quad (n \to \infty) \text{ in } H_{\nu,0}, \\
\phi_n' \to \partial_{\nu,0}u \quad (n \to \infty) \text{ in } H_{\nu,0}.
\]
By (a) these convergences imply the convergence in $H_{\mu,0}$. Thus, $u \in D(\partial_{\mu,0})$ with $\partial_{\mu,0}u = \partial_{\nu,0}u$.

(c) The inclusion follows easily by (a) and the definition of $A_{\nu}$.

**Proposition 2.19.** Let the assumptions of Theorem 2.17 hold and let $f \in H_{\nu,0} \otimes H$ for a $\nu \geq \nu_0$. Then for all $\mu \geq \nu$ we obtain
\[
\left(\partial_{\nu,0}M_0 + M_1 + A_{\nu}(H_{\nu,0} \otimes H)^2\right)^{-1} (f) = \left(\partial_{\mu,0}M_0 + M_1 + A_{\mu}(H_{\mu,0} \otimes H)^2\right)^{-1} (f).
\]
2.4. Perturbations and 0-analytic material laws

Proof. By Lemma 2.18 (b) and (c) we obtain

\[ \partial_{\nu,0}M_0 + M_1 + A_\nu \subseteq \partial_{\mu,0}M_0 + M_1 + A_\mu. \]

Since by Lemma 2.18 (a) \( H_{\nu,0} \otimes H \hookrightarrow H_{\mu,0} \otimes H \) it follows, that

\[ \partial_{\nu,0}M_0 + M_1 + A_\nu (H_{\nu,0} \otimes H)^2 \subseteq \partial_{\mu,0}M_0 + M_1 + A_\mu (H_{\mu,0} \otimes H)^2 \]

and this implies the assertion.

2.4. Perturbations and 0-analytic material laws

In this section we like to discuss the well-posedness of evolutionary inclusions of the form

\[ (u, f) \in \partial_{\nu,0}M_0 + M_1 + A_\nu + B, \]

where \( M_0, M_1 \) and \( A \) are as in Section 2.2 and \( B \subseteq (H_{\nu,0} \otimes H)^2 \) is a \( c \)-maximal monotone relation. As applications of such problems, we will study so called 0-analytic material laws, which will be introduced later.

**Proposition 2.20.** Let \( M_0, M_1, A \) be as in Theorem 2.17. Moreover, let \( 0 < c < \tilde{c} < c_1 \) such that \( \Re(\partial_{\nu,0}M_0 + M_1) \geq \tilde{c} \) for some \( \nu \in \mathbb{R}_{>0} \). For a bounded, \( c \)-maximal monotone relation \( B \subseteq (H_{\nu,0} \otimes H)^2 \) and for \( f \in H_{\nu,0} \otimes H \) there exists a unique \( u \in H_{\nu,0} \otimes H \) such that

\[ (u, f) \in \partial_{\nu,0}M_0 + M_1 + A_\nu + B. \]

Moreover, the solution mapping \( (\partial_{\nu,0}M_0 + M_1 + A_\nu + B)^{-1} \) is Lipschitz-continuous.

**Proof.** According to Theorem 2.17 the relation \( \partial_{\nu,0}M_0 + M_1 + A_\nu - \tilde{c} \) is maximal monotone and by Proposition 1.22 we conclude, that also

\[ \partial_{\nu,0}M_0 + M_1 + A_\nu - \tilde{c} + (c + B) \]

is maximal monotone. Hence, according to Minty’s Theorem

\[ (\partial_{\nu,0}M_0 + M_1 + A_\nu + B) [H_{\nu,0} \otimes H] = \left( \begin{array}{c} \tilde{c} - c + \partial_{\nu,0}M_0 + M_1 + A_\nu - \tilde{c} + (c + B) \\ >0 \end{array} \right) [H_{\nu,0} \otimes H] = H_{\nu,0} \otimes H, \]

which yields the existence of a solution for every right-hand side \( f \in H_{\nu,0} \otimes H \). Moreover, \( (\partial_{\nu,0}M_0 + M_1 + A_\nu + B)^{-1} \) is a Lipschitz-continuous mapping due to the strict monotonicity of \( \partial_{\nu,0}M_0 + M_1 + A_\nu + B \). \( \square \)

We specialize the situation of Proposition 2.20 by considering a Lipschitz-continuous mapping \( B \), which is a \( c \)-maximal monotone relation according to Example 1.8 for a suitable \( c \in \mathbb{R}_{\geq 0} \).
Corollary 2.21. Let $M_0, M_1$ and $A$ as in Theorem 2.17. For $0 < c < c_1$ there exists $\nu_0 \in \mathbb{R}_{>0}$ such that for all $\nu \geq \nu_0$ the following statement holds: For a Lipschitz-continuous mapping $B : H_{\nu,0} \otimes H \to H_{\nu,0} \otimes H$ with $|B|_{\text{Lip}} < c$ and $f \in H_{\nu,0} \otimes H$ we find a uniquely determined $u \in H_{\nu,0} \otimes H$ such that

$$(u, f) \in \partial_{\nu,0} M_0 + M_1 + A_{\nu} + B.$$ 

The solution operator is Lipschitz-continuous and if $B$ is causal, then so is the solution operator.

Proof. For $c < \tilde{c} < c_1$ we choose $\nu_0 \in \mathbb{R}_{>0}$ according to Lemma 2.3 such that $\text{Re}(\partial_{\nu,0} M_0 + M_1) \geq \tilde{c}$ for each $\nu \geq \nu_0$. Since $B$ is Lipschitz-continuous and $|B|_{\text{Lip}} < c$, we conclude that $B$ is $c-$maximal monotone and bounded. Thus, the well-posedness follows from Proposition 2.20. To show the causality, let $f, g \in H_{\nu,0} \otimes H$ and $a \in \mathbb{R}_{\geq 0}$ such that $\chi_{[0,a]}(m_0)(f - g) = 0$. We set

$$u := (\partial_{\nu,0} M_0 + M_1 + A_{\nu} + B)^{-1}(f)$$

and

$$v := (\partial_{\nu,0} M_0 + M_1 + A_{\nu} + B)^{-1}(g),$$

which is equivalent to

$$(u, f - Bu), (v, g - Bv) \in \partial_{\nu,0} M_0 + M_1 + A_{\nu}$$

or

$$u = (\partial_{\nu,0} M_0 + M_1 + A_{\nu})^{-1}(f - Bu)$$

and

$$v = (\partial_{\nu,0} M_0 + M_1 + A_{\nu})^{-1}(g - Bv).$$

Hence, $u$ and $v$ are fixed points of the mappings $w \mapsto (\partial_{\nu,0} M_0 + M_1 + A_{\nu})^{-1}(f - Bu)$ and $w \mapsto (\partial_{\nu,0} M_0 + M_1 + A_{\nu})^{-1}(g - Bu)$. Since these mappings are Lipschitz-continuous with a Lipschitz-constant less than 1, we obtain from the Contraction Mapping Theorem that the sequences $(u_n)_{n \in \mathbb{N}}, (v_n)_{n \in \mathbb{N}} \in (H_{\nu,0} \otimes H)^\mathbb{N}$ defined recursively by $u_0 := v_0 := 0$ and

$$u_n := (\partial_{\nu,0} M_0 + M_1 + A_{\nu})^{-1}(f - Bu_{n-1})$$

and

$$v_n := (\partial_{\nu,0} M_0 + M_1 + A_{\nu})^{-1}(g - Bv_{n-1})$$

($n \in \mathbb{N}$) converge to $u$ and $v$ respectively. Since clearly $\chi_{[0,a]}(m_0)(u_0 - v_0) = 0$ and $B$ is causal, we conclude that $\chi_{[0,a]}(m_0)(Bu_0 - Bv_0) = 0$ and hence

$$\chi_{[0,a]}(m_0)(f - Bu_0 - (g - Bv_0)) = \chi_{[0,a]}(m_0)(f - g) + \chi_{[0,a]}(m_0)(Bu_0 - Bv_0) = 0.$$ 

By Proposition 2.16 the mapping $(\partial_{\nu,0} M_0 + M_1 + A_{\nu})^{-1}$ is causal and hence $\chi_{[0,a]}(m_0)(u_1 - v_1) = 0$. By induction we obtain $\chi_{[0,a]}(m_0)(u_n - v_n) = 0$ for all $n \in \mathbb{N}$ and since $\chi_{[0,a]}(m_0) \in L(H_{\nu,0} \otimes H)$, it follows that $\chi_{[0,a]}(m_0)(u - v) = 0$. \hfill \Box

In applications the given material law is often of the form (cf. \cite{13, 53})

$$M(\partial_{\nu,0}^{-1}) = \sum_{k=0}^{\infty} \partial_{\nu,0}^{-k} M_k$$

with $M_k \in L(H)$ for $k \in \mathbb{N}_0$ and $M_0$ selfadjoint with $M_0|_{M_0[H]} \geq c_0 > 0$ and $\text{Re} M_1|_{\{0\}} M_0 \geq c_1 > 0$. The convergence of the series is assumed to be absolute with respect to the operator norm. This for instance holds if $\sup_{k \in \mathbb{N}_0} \|M_k\| < \infty$ and $\nu \geq 1$. By comparison we see that we have dealt with the case $M_k = 0$ for $k \geq 2$ in the previous chapters. We now show that also 0-analytic material laws can be studied, by using classical perturbation results. At first we want to give an equivalent formulation of absolute convergence of the series above.
Lemma 2.22. Let \((M_k)_{k \in \mathbb{N}_0} \in L(H)^{\mathbb{N}_0}\) and \(\nu \in \mathbb{R}_{>0}\). Then the series \(M(\partial^{-1}_{\nu,0}) = \sum_{k=0}^{\infty} \partial^{-k}_{\nu,0} M_k\) converges absolutely if and only if
\[
\sum_{k=0}^{\infty} \frac{1}{\nu^k} \|M_k\| < \infty.
\]
In this case \(M(\partial^{-1}_{\nu,0})\) is a continuous, linear operator.

**Proof.** Since \(\partial^{-k}_{\nu,0} M_k = \partial^{-k}_{\nu,0} \otimes M_k\) for all \(k \in \mathbb{N}\) it follows, that
\[
\|\partial^{-k}_{\nu,0} M_k\| = \|\partial^{-k}_{\nu,0}\| \|M_k\|
\]
for all \(k \in \mathbb{N}\) according to Proposition A.13 and by Proposition 2.2 we obtain
\[
\sum_{k=0}^{\infty} \|\partial^{-k}_{\nu,0} M_k\| = \sum_{k=0}^{\infty} \frac{1}{\nu^k} \|M_k\|.
\]
This shows the asserted equivalence. The fact, that \(M(\partial^{-1}_{\nu,0})\) is a bounded, linear operator, follows from the statement that absolutely convergent series in a Banach space (here \(L(H_{\nu,0} \otimes H)\)) converge.

Lemma 2.23. Let \((M_k)_{k \in \mathbb{N}_0} \in L(H)^{\mathbb{N}_0}\) and \(\nu_0 \in \mathbb{R}_{\geq 1}\) such that \(\sum_{k=3}^{\infty} \|\partial^{-k}_{\nu,0} M_k\| =: K < \infty\). Then
\[
\sum_{k=2}^{\infty} \|\partial^{-k+2}_{\nu,0} M_k\| \leq \|M_2\| + K
\]
for all \(\nu \geq \nu_0^3\).

**Proof.** Since \(\nu \geq \nu_0^3\) it follows that for all \(k \in \mathbb{N}\)
\[
\nu_0^{k+2} \leq \nu_0^3 \leq \nu
\]
and thus
\[
\nu_0^{k+2} \leq \nu^k.
\]
We estimate with Proposition 2.2
\[
\sum_{k=2}^{\infty} \|\partial^{-k+2}_{\nu,0} M_k\| = \sum_{k=2}^{\infty} \frac{1}{\nu^{k-2}} \|M_k\|
\]
\[
= \|M_2\| + \sum_{k=3}^{\infty} \frac{1}{\nu^{k-2}} \|M_k\|
\]
\[
\leq \|M_2\| + \sum_{k=3}^{\infty} \frac{1}{\nu_0^0} \|M_k\|
\]
\[
\leq \|M_2\| + K.
\]
This completes the proof. \(\square\)
We use these two observations to prove the following proposition.

**Proposition 2.24.** Let \((M_k)_{k \in \mathbb{N}} \in L(H)^{\mathbb{N}}\) with \(M_0\) selfadjoint, \(M_0|_{M_0[H]} \geq c_0 > 0\) and \(\text{Re}M_1|_{\{0\}, M_0} \geq c_1 > 0\). Let \(A \subseteq H \oplus H\) be maximal monotone with \(0 \in [H]A\). Moreover, let \(\nu_0 \in \mathbb{R}_{\geq 1}\) such that \(K := \sum_{k=3}^{\infty} \nu_0^{-k} \|M_k\| < \infty\). Then there exists \(\nu_1 \in \mathbb{R}_{\geq 1}\) such that

\[
(\partial_{\nu,0}M(\partial_{\nu,0}^{-1}) + A_\nu)^{-1} : H_{\nu,0} \otimes H \to H_{\nu,0} \otimes H
\]

is Lipschitz-continuous and causal for all \(\nu > \nu_1\), where \(M(\partial_{\nu,0}^{-1}) = \sum_{k=0}^{\infty} \partial_{\nu,0}^{-k}M_k\).

**Proof.** According to Lemma 2.23 we obtain for all \(\nu \geq \nu_0^3\)

\[
\sum_{k=2}^{\infty} \|\partial_{\nu,0}^{-k+2}M_k\| \leq \|M_2\| + K
\]

and thus we can define the operator \(B(\partial_{\nu,0}^{-1}) := \sum_{k=2}^{\infty} \partial_{\nu,0}^{-k+2}M_k \in L(H_{\nu,0} \otimes H)\) for all \(\nu \geq \nu_0^3\). Hence,

\[
\partial_{\nu,0}M(\partial_{\nu,0}^{-1}) + A_\nu = \partial_{\nu,0}M_0 + M_1 + \partial_{\nu,0}^{-1}B(\partial_{\nu,0}^{-1}) + A_\nu
\]

with

\[
\|\partial_{\nu,0}^{-1}B(\partial_{\nu,0}^{-1})\| \leq \|\partial_{\nu,0}^{-1}\| \|B(\partial_{\nu,0}^{-1})\| \leq \frac{1}{\nu}(\|M_2\| + K).
\]

According to Lemma 2.3 we find for \(0 < c < c_1\) a constant \(\nu^* \in \mathbb{R}_{>0}\) such that

\[
\text{Re}(\partial_{\nu,0}M_0 + M_1) \geq c
\]

for all \(\nu \geq \nu^*\). We define \(\nu_1 := \max\{\nu^*, \nu_0^3, \frac{\|M_2\| + K}{c}\}\) and obtain for all \(\nu > \nu_1\)

\[
\|\partial_{\nu,0}^{-1}B(\partial_{\nu,0}^{-1})\| < c \text{ and } \text{Re}(\partial_{\nu,0}M_0 + M_1) \geq c.
\]

Thus, it follows by Corollary 2.21 that

\[
(\partial_{\nu,0}M(\partial_{\nu,0}^{-1}) + A_\nu)^{-1} : H_{\nu,0} \otimes H \to H_{\nu,0} \otimes H
\]

is Lipschitz-continuous. Now consider the causality. According to Corollary 2.21 it suffices to prove the causality of \(\partial_{\nu,0}^{-1}B(\partial_{\nu,0}^{-1})\). By Lemma 2.13 b) this follows, if \(\partial_{\nu,0}^{-1}\) and \(B(\partial_{\nu,0}^{-1})\) are causal. The causality of \(\partial_{\nu,0}^{-1}\) follows from Lemma 2.15 (for \(M_0 = 1\) and \(M_1 = 0\)). Then again by Lemma 2.13 b) we know, that \(\partial_{\nu,0}^{-k}\) is causal for all \(k \in \mathbb{N}\). Thus, \(\partial_{\nu,0}^{-k+2}M_k = \partial_{\nu,0}^{-k+2} \otimes M_k\) is causal for every \(k \in \mathbb{N}_{\geq 2}\) and hence, by Lemma 2.13 a) \(\sum_{k=2}^{n} \partial_{\nu,0}^{-k+2}M_k\) is causal for all \(n \in \mathbb{N}_{\geq 2}\). Since \(\sum_{k=2}^{n} \partial_{\nu,0}^{-k+2}M_k \rightarrow B(\partial_{\nu,0}^{-1})\) for \(n \rightarrow \infty\) uniformly, the causality of \(B(\partial_{\nu,0}^{-1})\) follows from Lemma 2.13 c).
2.5. Initial value problems

In the previous sections we did not mention any initial condition on our solution. However, the definition of our derivative operator \( \partial_{\nu} \) implies the initial condition \( Pu(0+) = 0 \). We now want to consider problems of the following type

\[
(u, f) \in \partial_{\nu} M_0 + M_1 + A_{\nu}
\]

\[
Pu(0+) = u_0
\]

where \( P, A_{\nu}, M_0 \) and \( M_1 \) are as before and \( \partial_{\nu} \) is the so-called weak derivative in time. We like to introduce this operator \( \partial_{\nu} \) and want to discuss how the initial condition \( Pu(0+) = u_0 \) should be understood. Again we want to discuss the questions about existence, uniqueness and continuous dependence of solutions. We may assume that the reader is familiar with the Fourier-transform and refer to [20, p. 187 ff] for the definition and some basic properties.

**Definition 2.25.** Let \( \nu \in \mathbb{R}_{>0} \). We define the weak derivative

\[
\partial_{\nu} := -(\partial_{\nu,0})^* + 2\nu.
\]

**Remark 2.26.** By the definition of \( \partial_{\nu,0} \) it follows that \( \partial_{\nu} = -(\partial_{\nu})^* - \nu + 2\nu = -(\partial_{\nu})^* + \nu \) and by definition of \( \partial_{\nu} \) we obtain

\[
-(\partial_{\nu})^* = -e^{\nu m}\left(\partial\right)^* e^{-\nu m} = e^\nu m \partial e^{-\nu m}
\]

where \( \partial = -(\varnothing) \) is the classical weak derivative on \( L_2(\mathbb{R}_{\geq 0}) \) (see [20, Section 5.2]). Summarizing we get

\[
\partial_{\nu} = e^{\nu m} \partial e^{-\nu m} + \nu
\]

and since \( \varnothing \) is skew-symmetric, it follows that

\[
\partial_{\nu,0} \subseteq \partial_{\nu}.
\]

Moreover, since \( C^\infty(\mathbb{R}_{\geq 0}) \cap D(\partial) \) is a core of \( \partial \) (by the famous \( H = W \)–Theorem, see [36]) and since \( e^{-\nu m}, e^{\nu m} \) are bijections on \( C^\infty(\mathbb{R}_{\geq 0}) \) we observe that \( C^\infty(\mathbb{R}_{\geq 0}) \cap D(\partial_{\nu}) \) is a core of \( \partial_{\nu} \) and we obtain for \( u \in C^\infty(\mathbb{R}_{\geq 0}) \cap D(\partial_{\nu}) \):

\[
\partial_{\nu} u = (e^{\nu m} \partial e^{-\nu m} + \nu) u = u'.
\]

Henceforth, we identify \( \partial_{\nu} \) with its extension \( \partial_{\nu} \otimes 1_H \) on the space \( H_{\nu,0} \otimes H \). At first we want to prove a version of the Sobolev Embedding Theorem. For this we need to define a particular space of continuous functions.

**Definition 2.27.** Let \( \nu \in \mathbb{R}_{>0} \). We define

\[
C_{\nu}(\mathbb{R}_{\geq 0}; H) := \{ f \in \mathbb{R}_{\geq 0} \rightarrow H \mid f \text{ is continuous, } |f|_{\nu,\infty} < \infty \},
\]
where
\[ |f|_{\nu,\infty} := \sup_{t \in \mathbb{R}_{\geq 0}} |f(t)e^{-\nu t}|. \]

One can show, that this space is a Banach space with respect to the norm \(|\cdot|_{\nu,\infty}\), which is called the Morgenstern-Norm (see [38]). Moreover, we define the subspace
\[ C_{\nu,0}(\mathbb{R}_{\geq 0}; H) := \{ f \in C_{\nu}(\mathbb{R}_{\geq 0}; H) | \lim_{t \to \infty} e^{-\nu t} f(t) = 0 \}. \]

It is easy to verify that this subspace is closed and thus a Banach space as well.

**Lemma 2.28.** Let \( \psi, \partial \psi \in L_2(\mathbb{R}; H) \). Then \( \hat{\psi} \in L_1(\mathbb{R}; H) \), where \( \hat{\psi} \) denotes the Fourier-transform of \( \psi \) with
\[ |\hat{\psi}|_{L_1(\mathbb{R}; H)} \leq \sqrt{\pi} (|\psi|_{L_2(\mathbb{R}; H)} + |\partial \psi|_{L_2(\mathbb{R}; H)}). \]

**Proof.** By the definition of the Fourier-transform we get
\[ \hat{\partial \psi}(\xi) = (i\xi) \hat{\psi}(\xi) \quad (\xi \in \mathbb{R} \text{ a.e.}). \]

We estimate
\[
\int_{\mathbb{R}} |\hat{\psi}(\xi)| \, d\xi = \int_{\mathbb{R}} |\hat{\psi}(\xi)(1 + i\xi)||1 + i\xi|^{-1} \, d\xi \\
\leq \left( \int_{\mathbb{R}} |\hat{\psi}(\xi)(1 + i\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} \frac{1}{|1 + i\xi|^2} \, d\xi \right)^{\frac{1}{2}} \\
\leq (|\hat{\psi}|_{L_2(\mathbb{R}; H)} + |\hat{\partial \psi}|_{L_2(\mathbb{R}; H)}) \sqrt{\pi},
\]
which shows the assertion, since the Fourier-transform is an isometry on \( L_2(\mathbb{R}; H) \) (cf. [20, Theorem 1, p.187] ).

With this knowledge we are able to state the one-dimensional Sobolev Embedding Theorem in the Hilbert space case, sometimes also called Morrey’s inequality. For the general Banach space situation we refer to [20, p. 266]. However, since we deal with weighted \( L_2 \)-spaces, we have to use the space \( C_{\nu,0} \), defined in Definition 2.27 (see [44]).

**Proposition 2.29 (Sobolev Embedding Theorem).** Let \( \nu \in \mathbb{R}_{>0} \). Then
\[ D(\partial_{\nu}) \hookrightarrow C_{\nu,0}(\mathbb{R}_{\geq 0}; H), \]
where \( D(\partial_{\nu}) \) is equipped with the graph norm of \( \partial_{\nu} \).

**Proof.** Let \( \psi \in C^\infty(\mathbb{R}_{\geq 0}; H) \cap D(\partial_{\nu}) \). Then \( \varphi := e^{-\nu m} \psi \in C^\infty(\mathbb{R}_{\geq 0}; H) \cap D(\partial) \). We extend \( \varphi \) symmetrically by
\[
\tilde{\varphi}(x) := \begin{cases} 
\varphi(x) & \text{if } x \in \mathbb{R}_{\geq 0}, \\
\varphi(-x) & \text{if } x \in \mathbb{R}_{<0}.
\end{cases}
\]
It is not hard to see, that \( \tilde{\varphi} \in C(\mathbb{R}; H) \cap L_2(\mathbb{R}; H) \) and \( \tilde{\varphi} \) is weakly differentiable with

\[
\partial \tilde{\varphi}(x) = \begin{cases} 
\varphi'(x) & \text{if } x \in \mathbb{R}_{>0}, \\
-\varphi'(-x) & \text{if } x \in \mathbb{R}_{<0}
\end{cases}
\]

and thus \( \partial \tilde{\varphi} \in L_2(\mathbb{R}; H) \). According to Lemma 2.28 we get \( \tilde{\varphi} \in L_1(\mathbb{R}; H) \) and thus, by the Lemma of Riemann-Lebesgue (see [2, Theorem 9.9]), \( \lim_{|x| \to \infty} \tilde{\varphi}(x) = 0 \) and hence \( \lim_{x \to \infty} e^{-\nu x} \psi(x) = \lim_{x \to \infty} \varphi(x) = 0 \). Let now \( x \in \mathbb{R} \). We estimate by using the inequality in Lemma 2.28 and the formula for the inverse Fourier transform

\[
|\tilde{\varphi}(x)| = \frac{1}{\sqrt{2\pi}} \left| \int_{\mathbb{R}} \tilde{\varphi}(y) e^{ixy} \, dy \right|
\]

\[
\leq \frac{1}{\sqrt{2\pi}} |\tilde{\varphi}|_{L_1(\mathbb{R}; H)}
\]

\[
\leq \frac{1}{\sqrt{2}} \left( |\tilde{\varphi}|_{L_2(\mathbb{R}; H)} + |\partial \tilde{\varphi}|_{L_2(\mathbb{R}; H)} \right)
\]

\[
= |\varphi|_{L_2(\mathbb{R}_0; H)} + |\varphi'|_{L_2(\mathbb{R}_{>0}; H)}
\]

and thus

\[
|\varphi|_{\infty} \leq |\varphi|_{L_2(\mathbb{R}_0; H)} + |\varphi'|_{L_2(\mathbb{R}_{>0}; H)}.
\]

By the definition of \( \varphi \), it follows that

\[
|\psi|_{\nu, \infty} \leq |\psi|_{H_{\nu, 0} \otimes H} + |\psi' - \nu \psi|_{H_{\nu, 0} \otimes H} \leq (1 + \nu)(|\psi|_{H_{\nu, 0} \otimes H} + |\psi'|_{H_{\nu, 0} \otimes H})
\]

and hence

\[
|\psi|_{\nu, \infty} \leq \sqrt{2}(1 + \nu) \sqrt{|\psi|_{H_{\nu, 0} \otimes H}^2 + |\psi'|_{H_{\nu, 0} \otimes H}^2}.
\]

This proves the continuity of the mapping

\[
C^\infty(\mathbb{R}_{\geq 0}; H) \cap D(\partial_\nu) \subseteq D(\partial_\nu) \rightarrow C_{\nu, 0}(\mathbb{R}_{\geq 0}; H)
\]

\[
\psi \mapsto \tilde{\psi}
\]

and since \( C^\infty(\mathbb{R}_{\geq 0}; H) \cap D(\partial_\nu) \) is dense in \( D(\partial_\nu) \), we get the desired embedding. \( \square \)

**Remark 2.30.** Proposition 2.29 states that each function \( u \in D(\partial_\nu) \) has a representer in \( C_{\nu, 0}(\mathbb{R}_{\geq 0}; H) \). From now on we like to identify the function \( u \) with its continuous representer.

**Proposition 2.31.** Let \( \nu \in \mathbb{R}_{>0} \) and \( u \in D(\partial_\nu) \). Then \( u(0) = 0 \) if and only if \( u \in D(\partial_\nu, 0) \).

**Proof.** Assume first that \( u \in D(\partial_\nu, 0) \). Then we can choose a sequence \( (\phi_n)_{n \in \mathbb{N}} \in C^\infty(\mathbb{R}_{\geq 0}; H) \) with \( \phi_n \rightarrow u \) and \( \phi_n' \rightarrow \partial_\nu u = \partial_\nu u \) in \( H_{\nu, 0} \otimes H \) as \( n \rightarrow \infty \). Thus, by Proposition 2.29 it follows that \( \phi_n(0) \rightarrow u(0) \) in \( H \) as \( n \rightarrow \infty \). Since \( \phi_n \) is supported by a compact set in \( \mathbb{R}_{\geq 0} \) for each \( n \in \mathbb{N} \), we conclude that \( u(0) = 0 \).
Let us now assume that \( u \in D(\partial_\nu) \) with \( u(0) = 0 \). Thus, there exists a sequence 
\((u_n)_{n \in \mathbb{N}} \in (C^\infty(\mathbb{R}_{\geq 0}; H) \cap D(\partial_\nu))^\mathbb{N}\) such that \( u_n \to u \) in \( D(\partial_\nu) \) as \( n \to \infty \). For \( \psi \in C^\infty(\mathbb{R}_{\geq 0}; H) \cap D(\partial_\nu) \) we obtain
\[
\langle u_n | \psi' \rangle_{H_{\nu,0} \otimes H} = \int_{\mathbb{R}_{\geq 0}} \langle u_n(t) | \psi'(t) \rangle e^{-2\nu t} \, dt \\
= \int_{\mathbb{R}_{\geq 0}} \partial(s \mapsto \langle u_n(s) | \psi(s) \rangle)(t)e^{-2\nu t} \, dt - \int_{\mathbb{R}_{\geq 0}} \langle u_n'(t) | \psi(t) \rangle e^{-2\nu t} \, dt \\
= 2\nu \int_{\mathbb{R}_{\geq 0}} \langle u_n(t) | \psi(t) \rangle e^{-2\nu t} \, dt + \lim_{s \to \infty} \langle u_n(s) | \psi(s) \rangle e^{-2\nu s} - \langle u_n(0) | \psi(0) \rangle \\
- \int_{\mathbb{R}_{\geq 0}} \langle u_n'(t) | \psi(t) \rangle e^{-2\nu t} \, dt \\
= \langle -u_n' + 2\nu u_n | \psi \rangle_{H_{\nu,0} \otimes H} + \lim_{s \to \infty} \langle u_n(s) | \psi(s) \rangle e^{-2\nu s} - \langle u_n(0) | \psi(0) \rangle.
\]
Since by Proposition 2.29 \( u_n, \psi \in C_{\nu,0}(\mathbb{R}_{\geq 0}; H) \), it follows that 
\[
|\langle u_n(s) | \psi(s) \rangle e^{-2\nu s}| \leq |u_n(s)|e^{-\nu s}|\psi(s)|e^{-\nu s} \to 0 \quad (s \to \infty).
\]
Therefore
\[
\langle u_n | \psi' \rangle_{H_{\nu,0} \otimes H} = \langle -u_n' + 2\nu u_n | \psi \rangle_{H_{\nu,0} \otimes H} - \langle u_n(0) | \psi(0) \rangle
\]
for every \( n \in \mathbb{N} \). Passing to the limit \( n \to \infty \) leads to (keep in mind, that also \( u_n \to u \) in \( C_{\nu,0}(\mathbb{R}_{\geq 0}; H) \) by Proposition 2.29)
\[
\langle u | \psi' \rangle_{H_{\nu,0} \otimes H} = \langle -\partial_\nu u + 2\nu u | \psi \rangle_{H_{\nu,0} \otimes H}.
\]
Since \( C^\infty(\mathbb{R}_{\geq 0}; H) \cap D(\partial_\nu) \) is a core of \( \partial_\nu \) we conclude, that \( u \in D(\partial_\nu^* ) \) with
\[
\partial_\nu^* u = -\partial_\nu u + 2\nu u.
\]
By definition of \( \partial_\nu \) we see that
\[
\partial_\nu^* = -\partial_\nu + 2\nu
\]
and thus \( u \in D(\partial_\nu^* ) \).

**Remark 2.32.** Proposition 2.31 shows that the initial condition \( u(0) = 0 \) can be formulated by \( u \in D(\partial_\nu^* ) \). Thus, we have implicitly assumed an initial condition \( u(0) = 0 \) in the Sections 2.2 up to 2.4.

**Lemma 2.33.** Let \( v \in D(\partial_\nu^* ) \) and \( u_0 \in H \). Then \( u := v + u_0 \in D(\partial_\nu) \) with \( u(0) = u_0 \) and \( \partial_\nu u = \partial_\nu^* v \).
Proof. First we prove, that $h : t \mapsto u_0$ is a function in $D(\partial_\nu)$. For every $\phi \in \mathbb{C}_c^\infty(\mathbb{R}_{>0}; H)$ we calculate

$$
\langle h|\phi' \rangle_{H_{\nu,0} \otimes H} = \int_{\mathbb{R}_{\geq 0}} (u_0|\phi'(t))e^{-2\nu t} \ dt
$$

$$
= \int_{\mathbb{R}_{\geq 0}} \partial(s \mapsto (u_0|\phi(s)))(t)e^{-2\nu t} \ dt
$$

$$
= \int_{\mathbb{R}_{\geq 0}} 2\nu(u_0|\phi(t))e^{-2\nu t} \ dt
$$

$$
= (2\nu h|\phi)_{H_{\nu,0} \otimes H}.
$$

Thus, $h \in D(\partial_{\nu,0}^*)$ with $\partial_{\nu,0}^* h = 2\nu h$ and therefore $h \in D(\partial_\nu)$ with $\partial_\nu h = -(\partial_{\nu,0})^* h + 2\nu h = 0$. Thus, $u = v + h \in D(\partial_\nu)$ with $\partial_\nu u = \partial_{\nu,0} v$ and $u(0) = v(0) + u_0 = u_0$ by Proposition 2.31.

We now want to study problems of the form

$$
(u, f) \in \partial_\nu M_0 + M_1 + A_\nu, \quad Pu(0) = u_0 \in H. \quad (2.7)
$$

Remark 2.34. Let $u$ be a solution of (2.7). Then by Proposition 2.31 it follows that $Pu - u_0 \in D(\partial_{\nu,0})$ with $\partial_{\nu,0}(Pu - u_0) = \partial_\nu Pu$ and hence

$$
\partial_{\nu,0} M_0 (u - u_0) + M_1 u = \partial_\nu M_0 u + M_1 u
$$

which implies

$$
(u, f) \in \partial_{\nu,0} M_0 (\cdot - u_0) + M_1 + A_\nu. \quad (2.8)
$$

If on the other hand $u$ is a solution of (2.8), we derive that $Pu - u_0 \in D(\partial_{\nu,0})$ and thus, $Pu \in D(\partial_\nu)$ with

$$
\partial_\nu M_0 u + M_1 u = \partial_{\nu,0} M_0 (u - u_0) + M_1 u
$$

and $Pu(0) = u_0$ according to Lemma 2.33. Thus, $u$ satisfies the inclusion (2.7).

This remark yields that we can replace the problem of finding $u \in H_{\nu,0} \otimes H$ satisfying (2.7), by searching for $u \in H_{\nu,0} \otimes H$ with

$$
(u, f) \in \partial_{\nu,0} M_0 (\cdot - u_0) + M_1 + A_\nu.
$$

The central proposition in this chapter shows that for every solution of the initial value problem, we can find a corresponding solution of an evolutionary inclusion with initial condition 0.
Proposition 2.35. Let \( \nu \in \mathbb{R}_{>0} \) and \( A \subseteq (H_{\nu,0} \otimes H)^2 \). For a given function \( f \in H_{\nu,0} \otimes H \) and an initial state \( u_0 \in H \) the following statements are equivalent

(i) \( (u, f) \in \partial_{\nu,0} M_0 (\cdot - u_0) + M_1 + A \),

(ii) \( (u - u_0, f - M_1 u_0) \in \partial_{\nu,0} M_0 + M_1 + B \), where \( B := [A] - (u_0, 0) := \{(x, y) \in (H_{\nu,0} \otimes H)^2 | (x + u_0, y) \in A \} \).

Proof. (i) \( \Rightarrow \) (ii): If \( (u, f) \in \partial_{\nu,0} M_0 (\cdot - u_0) + M_1 + A \) there exists a sequence \( ((u_n, f_n))_{n \in \mathbb{N}} \in (\partial_{\nu,0} M_0 (\cdot - u_0) + M_1 + A)^\mathbb{N} \) such that \( u_n \to u \) and \( f_n \to f \) in \( H_{\nu,0} \otimes H \) as \( n \to \infty \). Thus, for every \( n \in \mathbb{N} \) we obtain

\[
(u_n, f_n - \partial_{\nu,0} M_0 (u_n - u_0) - M_1 u_n) \in A
\]

and hence, by the definition of \( B \)

\[
(u_n - u_0, f_n - M_1 u_0 - \partial_{\nu,0} M_0 (u_n - u_0) - M_1 (u_n - u_0)) \in B,
\]

which turns equivalently into

\[
(u_n - u_0, f_n - M_1 u_0) \in \partial_{\nu,0} M_0 + M_1 + B.
\]

Sending now \( n \to \infty \) shows:

\[
(u - u_0, f - M_1 u_0) \in \partial_{\nu,0} M_0 + M_1 + B.
\]

(ii) \( \Rightarrow \) (i): This follows, by arguing exactly as above.

With this proposition we simply can answer the questions about existence, uniqueness and continuous dependence of a solution on the given data \( f \). First we consider the uniqueness and continuous dependence of the solution for a fixed initial value.

Proposition 2.36. Let \( 0 < c < c_1 \) and \( \nu \in \mathbb{R}_{>0} \) such that \( \text{Re}(\partial_{\nu,0} M_0 + M_1) \geq c \) and \( B \subseteq (H_{\nu,0} \otimes H)^2 \) be monotone. Moreover, let \( u, v, f, g \in H_{\nu,0} \otimes H, u_0 \in H \) are such that

\[
(u, f), (v, g) \in \partial_{\nu,0} M_0 (\cdot - u_0) + M_1 + B.
\]

Then

\[
|u - v|_{H_{\nu,0} \otimes H} \leq c^{-1} |f - g|_{H_{\nu,0} \otimes H}.
\]

Proof. From Proposition 2.35 we know, that for \( C := [B] - (u_0, 0) \) we have

\[
(u - u_0, f - M_1 u_0), (v - u_0, g - M_1 u_0) \in \partial_{\nu,0} M_0 + M_1 + C.
\]

By Proposition 2.4 we obtain

\[
|u - v|_{H_{\nu,0} \otimes H} \leq |(u - u_0) - (v - u_0)|_{H_{\nu,0} \otimes H} \leq c^{-1} |(f - M_1 u_0) - (g - M_1 u_0)|_{H_{\nu,0} \otimes H} = c^{-1} |f - g|_{H_{\nu,0} \otimes H},
\]

which is the asserted inequality. \( \square \)
2.5. Initial value problems

For proving the existence of a solution, we need the following Lemma.

**Lemma 2.37.** Let \( A \subseteq H \oplus H \) be maximal monotone and \( u_0 \in H \). Then \( B := [A] - (u_0,0) \) is maximal monotone. Moreover, for \( \nu \in \mathbb{R}_{>0} : \)

\[
B_{\nu} = [A_{\nu}] - (u_0,0).
\]

**Proof.** Let \((u,v),(x,y) \in B\). Then by definition it follows, that \((u + u_0,v),(x + u_0,y) \in A\) and thus

\[
\text{Re}\langle u - x|v - y \rangle = \text{Re}\langle (u + u_0) - (x + u_0)|v - y \rangle \geq 0.
\]

This shows the monotonicity of \( B \). Let now \((x,y) \in H \oplus H \) such that

\[
\forall (u,v) \in B : \text{Re}\langle x - u|y - v \rangle \geq 0.
\]

This is equivalent to

\[
\forall (u,v) \in A : \text{Re}\langle x - (u - u_0)|y - v \rangle \geq 0.
\]

Since \( A \) is maximal monotone, it follows that \((x + u_0, y) \in A\), which shows \((x,y) \in B\). Therefore \( B \) is also maximal monotone. Let now \( \nu \in \mathbb{R}_{>0} \) and \((x,y) \in B_{\nu}\). This implies, that \((x(t),y(t)) \in B\) and hence, \((x(t) + u_0,y(t)) \in A\) for almost every \( t \in \mathbb{R}_{\geq 0} \). Since \( x + u_0 \in H_{\nu,0} \otimes H \), we conclude \((x + u_0,y) \in A_{\nu}\), which gives \((x,y) \in [A_{\nu}] - (u_0,0)\). Summarizing we have shown that

\[
B_{\nu} \subseteq [A_{\nu}] - (u_0,0)
\]

but we already know that \( B_{\nu} \) is maximal monotone and since \([A_{\nu}] - (u_0,0)\) is monotone, we conclude

\[
B_{\nu} = [A_{\nu}] - (u_0,0).
\]

This completes the proof.

Now, we want to answer the question about the existence of a solution of \((2.7)\).

**Proposition 2.38.** Let \( 0 < c < c_1 \) and \( \nu_1 \in \mathbb{R}_{>0} \) such that \( \partial_{\nu,0} M_0 + M_1 - c \) is maximal monotone for every \( \nu \geq \nu_1 \) and let \( A \subseteq H \oplus H \) be maximal monotone. Moreover, let \( \nu \geq \nu_1 \), \( f \in H_{\nu,0} \otimes H \) and \( u_0 \in [H]A \). Then there exists \( u \in H_{\nu,0} \otimes H \) such that

\[
(u,f) \in \partial_{\nu,0} M_0(\cdot - u_0) + M_1 + A_{\nu}.
\]

**Proof.** Let \( B := [A] - (u_0,0) \). This relation is maximal monotone by Lemma 2.37 and since \( u_0 \in [H]A \) it follows that \( 0 \in [H]B \). Thus, according to Corollary 2.10 we find an element \( v \in H_{\nu,0} \otimes H \) such that

\[
(v,f - M_1 u_0) \in \partial_{\nu,0} M_0 + M_1 + B_{\nu}.
\]

For \( u := v + u_0 \in H_{\nu,0} \otimes H \) we get by Proposition 2.35

\[
(u,f) \in \partial_{\nu,0} M_0(\cdot - u_0) + M_1 + A_{\nu}.
\]

This completes the proof.
We obtain if in $H \subset C$ the mapping where $u$ provides a continuous dependence result on the initial condition in a suitable norm, assuming and thus, we obtain the asserted causality. 

Proposition 2.39. Let $\nu_1 \in \mathbb{R}_{>0}, M_0, M_1$ and $A$ as in Proposition 2.38. Then for $u_0 \in [H]A$ the mapping $\left( \partial_{v,0}M_0(\cdot - u_0) + M_1 + A_v \right)^{-1}$ is causal for all $\nu \geq \nu_1$.

Proof. Let $f, g \in H_{\nu,0} \otimes H$ such that $\chi_{[0,a]}(m_0)(f - g) = 0$ for an $a \in \mathbb{R}_{\geq 0}$. Then $\chi_{[0,a]}(m_0)(f - M_1u_0 - (g - M_1u_0)) = \chi_{[0,a]}(m_0)(f - g) = 0$.

Let $u, v$ be the solutions of the initial value problem for the right-hand sides $f$ and $g$ respectively. Then by Proposition 2.35 we know that $$(u - u_0, f - M_1u_0, v - u_0, g - M_1u_0) \in \partial_{x,0}M_0 + M_1 + A_v,$$ where $B := [A] - (u_0, 0)$. Since the mapping $\left( \partial_{v,0}M_0 + M_1 + B_v \right)^{-1}$ is causal, according to Proposition 2.10 we conclude

$$\chi_{[0,a]}(m_0)(u - v) = \chi_{[0,a]}(m_0)(u - u_0 - (v - u_0)) = 0,$$
and thus, we obtain the asserted causality.

By looking at Proposition 2.36 we have already proved the continuous dependence of the solution on the given right-hand side, if we assume a fixed initial condition $u_0 \in H$. The next proposition provides a continuous dependence result on the initial condition in a suitable norm, assuming that $u_0 \in [H]A^*$, where $A^*$ is the so-called adjoint relation (see [44]), which is defined as follows.

Definition 2.40. Let $H_0, H_1$ be two Hilbert spaces and $C \subseteq H_0 \oplus H_1$. Then the adjoint relation $C^* \subseteq H_1 \oplus H_0$ is defined by

$$C^* = (-C^{-1})^\perp,$$

where $-C := \{(x, y) \in H_0 \oplus H_1 \mid (x, -y) \in C\}$ and the orthogonal complement should be taken in $H_1 \oplus H_0$. Obviously this relation is linear and closed.

Lemma 2.41. Let $H_0, H_1$ be two Hilbert spaces and $C \subseteq H_0 \oplus H_1$. Then $(x, y) \in C^*$ if and only if $\langle u|y \rangle_{H_0} = \langle v|x \rangle_{H_1}$ for all $(u, v) \in C$.

Proof. We obtain

$$(x, y) \in C^* \iff \forall (u, v) \in - (C^{-1}) : \langle (x, y)|(u, v) \rangle_{H_1 \oplus H_0} = 0$$

$$\iff \forall (u, v) \in C^{-1} : \langle (x, y)|(u, -v) \rangle_{H_1 \oplus H_0} = 0$$

$$\iff \forall (u, v) \in C : \langle (x, y)|(v, u) \rangle_{H_1 \oplus H_0} = 0$$

$$\iff \forall (u, v) \in C : \langle u|y \rangle_{H_0} = \langle v|x \rangle_{H_1},$$

which yields the assertion.
Lemma 2.42. Let $H_0, H_1$ be Hilbert spaces, $C \subseteq H_0 \oplus H_1$ and $\nu \in \mathbb{R}_{>0}$. Then $(C^*)_\nu \subseteq (C_\nu)^*$. 

Proof. Let $(u, v) \in (C^*)_\nu$ and $(x, y) \in C_\nu$. Then we compute, using Lemma 2.41,

\[
\langle u|y \rangle_{H_{\nu,0} \otimes H_1} = \int_{\mathbb{R}_{\geq 0}} \langle u(t)|y(t) \rangle_{H_1} e^{-2\nu t} \, dt = \int_{\mathbb{R}_{\geq 0}} \langle v|x(t) \rangle_{H_0} e^{-2\nu t} \, dt = \langle v|x \rangle_{H_{\nu,0} \otimes H_0}.
\]

Since this holds for every $(x, y) \in C_\nu$, we conclude again by Lemma 2.41 $(u, v) \in (C_\nu)^*$. 

Now we want to consider the pre-set of a linear relation and define a norm, which is a generalization of the graph norm in the operator case.

Lemma 2.43. Let $H_0, H_1$ be two Hilbert spaces and $C \subseteq H_0 \oplus H_1$ be linear. Then $[H_1]C$ is a linear subspace and

\[
|.|_C : [H_1]C \to \mathbb{R}_{\geq 0}, \quad u \mapsto \inf_{v \in C\{u\}} \sqrt{|u|_{H_0}^2 + |v|_{H_1}^2}
\]

defines a norm on $[H_1]C$.

Proof. The linearity of $[H_1]C$ is clear, since $C$ is a linear relation. We now show the properties of a norm for our functional $|.|_C$: Let $u, x \in [H_1]C$ and $\lambda \in \mathbb{C}$. Then

\[
|\lambda u|_C = \inf_{v \in C\{\lambda u\}} \sqrt{|\lambda u|_{H_0}^2 + |v|_{H_1}^2} = |\lambda| \inf_{v \in C\{u\}} \sqrt{|u|_{H_0}^2 + |v|_{H_1}^2} = |\lambda||u|_C.
\]

Moreover, we estimate for $y \in C\{x\}$:

\[
|u + x|_C = \inf_{v \in C\{u+x\}} \sqrt{|u + x|_{H_0}^2 + |v|_{H_1}^2} = \inf_{v \in C\{u+x\}} \sqrt{|u + x|_{H_0}^2 + |(v - y) + y|_{H_1}^2} = \inf_{v - y \in C\{u\}} \sqrt{|u + x|_{H_0}^2 + |(v - y) + y|_{H_1}^2}.
\]
Let us assume that \(|u|_C = 0\). Then we get, since \(|u|_{H_0} \leq |u|_C\) that \(u = 0\). This shows that \(|.|_C\) is indeed a norm on \([H_1]C\). □

With this knowledge, we are able to prove a result about the continuous dependence of solutions on their initial values.

**Proposition 2.44.** Let \(0 < c < c_1\) and \(\nu_0 \in \mathbb{R}_{>0}\) such that \(\text{Re}(\partial_{\nu,0}M_0 + M_1) \geq c\) for all \(\nu \geq \nu_0\). Moreover, let \(A \subseteq H \otimes H\) be monotone and \(u, v, f, g \in H_{\nu,0} \otimes H, u_0, v_0 \in [H]A^*\) such that

\[
(u, f) \in \partial_{\nu,0}M_0(\cdot - u_0) + M_1 + A_\nu \quad \text{and} \quad (v, g) \in \partial_{\nu,0}M_0(\cdot - v_0) + M_1 + A_\nu.
\]

Then there exists a constant \(C \in \mathbb{R}_{>0}\) such that:

\[
|u - v|_{H_{\nu,0} \otimes H} \leq C(|f - g|_{H_{\nu,0} \otimes H} + |u_0 - v_0|_{A^*}).
\]

**Proof.** We find \(x, y \in H_{\nu,0} \otimes H\) such that \((u, x), (v, y) \in A_\nu\) and

\[
\partial_{\nu,0}M_0(u - u_0) + M_1u + x = f \quad \text{and} \quad \partial_{\nu,0}M_0(v - v_0) + M_1v + y = g.
\]

Let \(u_0^*, v_0^* \in H\) such that \((u_0, u_0^*), (v_0, v_0^*) \in A^*\). We estimate, using the monotonicity of \(A_\nu\), Lemma 2.41 and Lemma 2.42.

\[
\text{Re}(f - g)(u - u_0) - (v - v_0))_{H_{\nu,0} \otimes H} = \text{Re}(\partial_{\nu,0}M_0(u - u_0) + M_1u + x - (\partial_{\nu,0}M_0(v - v_0) + M_1v + y))(u - u_0) - (v - v_0))_{H_{\nu,0} \otimes H} \\
= \text{Re}(\partial_{\nu,0}M_0 + M_1)((u - u_0) - (v - v_0))_{H_{\nu,0} \otimes H} \\
+ \text{Re}(M_1(u_0 - v_0))(u - u_0) - (v - v_0))_{H_{\nu,0} \otimes H} \\
+ \text{Re}(x - y)(v_0 - u_0)_{H_{\nu,0} \otimes H} \\
\geq c \|(u - u_0) - (v - v_0))_{H_{\nu,0} \otimes H}^2 \\
+ \text{Re}(M_1(u_0 - v_0))(u - u_0) - (v - v_0))_{H_{\nu,0} \otimes H} \\
+ \text{Re}(x - y)(v_0 - u_0)_{H_{\nu,0} \otimes H} \\
\geq \|M_1\||u_0 - v_0|_{H_{\nu,0} \otimes H}|(u - u_0) - (v - v_0))_{H_{\nu,0} \otimes H} \\
+ \text{Re}(x - y)(v_0 - u_0)_{H_{\nu,0} \otimes H} \\
= |u - v|_{H_{\nu,0} \otimes H}^2 \\
\]
By using ab and since this inequality holds for every $H_{c,0} \otimes H$.

Hence, we obtain by applying the Cauchy-Schwarz-Inequality on the left-hand side:

\[
(\langle u - u_0, v_0 \rangle H_{c,0} \otimes H)^2 \leq (|f - g| H_{c,0} \otimes H)^2 + \|M_1\| |u_0 - v_0| H_{c,0} \otimes H + |u_0 - v_0|^2 + |v_0 - v_0|^2.
\]

By using $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ for all $\epsilon > 0, a, b \geq 0$ and $|x| H_{c,0} \otimes H = \frac{1}{2\epsilon} |x|$ for $x \in H$, we find a constant $C > 0$ (just depending on $c, \|M_1\|, \nu$) such that

\[
|u - u_0| H_{c,0} \otimes H \leq C \left( |f - g| H_{c,0} \otimes H + \sqrt{|u_0 - v_0|^2 + |v_0 - v_0|^2} \right).
\]

A simply application of the triangle inequality leads to

\[
|u - v| H_{c,0} \otimes H \leq (C + 1) \left( |f - g| H_{c,0} \otimes H + \sqrt{|u_0 - v_0|^2 + |v_0 - v_0|^2} \right).
\]

Since this inequality holds for every $u_0^* \in A^*\{u_0\}, v_0^* \in A^*\{v_0\}$ we get the desired estimate.

If $A$ is a skew-selfadjoint operator, then the continuous dependence result shows an estimate in terms of the graph norm of $A$ of the initial states. Since in general $|H| A$ is not a linear set, we have to consider the adjoint relation to get an analogous result in terms of the "graph norm" of $A^*$. We like to summarize our results in the following theorem.

**Theorem 2.45.** Let $0 < c < c_1$ and $\nu_1 \in \mathbb{R}_{>0}$ such that $\partial_{\nu,0} M_0 + M_1 - c$ is maximal monotone for every $\nu \geq \nu_1$ and let $A \subseteq H \otimes H$ be maximal monotone. Moreover, let $\nu \geq \nu_1, f \in H_{\nu,0} \otimes H$ and $u_0 \in [H] A \cap [H] A^*$. Then there exists a unique $u \in H_{\nu,0} \otimes H$ such that

\[
(u, f) \in \partial_{\nu,0} M_0(\cdot - u_0) + M_1 + A^._\nu.
\]

Moreover, $u$ depends continuously on the given data $f$ and $u_0$. Furthermore the mapping $\left( \partial_{\nu,0} M_0(\cdot - u_0) + M_1 + A^._\nu \right)^{-1}$ is causal in $H_{\nu,0} \otimes H$.

**2.6. Evolutionary inclusions on the whole real line**

In this last section we want to introduce a solution-theory, where the time variable is no longer restricted to the positive real axis. We like to generalize the situation of Section 2.2 up to 2.5 by assuming a more general structure of our material law. Here we follow the ideas of [13] and

\[
\geq c(\langle u - u_0, - (v - v_0) \rangle H_{c,0} \otimes H
- \|M_1\| |u_0 - v_0| H_{c,0} \otimes H + |u_0 - v_0|^2 + |v_0 - v_0|^2)
\]
define material laws as mapping, which are relation-valued (instead of operator-valued, as in [43]), in fact, taking values in the set of uniformly bounded relations, introduced in Section 1.4. For doing so, we need the concept of the so-called Fourier-Laplace transform ([45, 44]). We begin to define the Hilbert space setting and the derivative analogously to Section 2.1.

**Definition 2.46.** As in Section 2.1 we define for \( \nu \in \mathbb{R}_{>0} \) the measure \( \mu_\nu \) on the Borel-\( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}) \) by

\[
\mu_\nu(A) := \int_A e^{-2\nu t} \, dt \quad (A \in \mathcal{B}(\mathbb{R}))
\]

and set \( H_\nu := L_2(\mathbb{R}, \mu_\nu) \). Again we find a unitary mapping between \( L_2(\mathbb{R}) \) and \( H_\nu \) given by

\[
e^{\nu m} : L_2(\mathbb{R}) \to H_\nu
\]

\[
f \mapsto (x \mapsto e^{\nu x} f(x))
\]

and denote its inverse by \( e^{-\nu m} \). It is well known that the operator \( \partial \mid_{C^\infty_c(\mathbb{R})} : C^\infty_c(\mathbb{R}) \subseteq L_2(\mathbb{R}) \to L_2(\mathbb{R}), \phi \mapsto \phi' \) can be extended to a skew-selfadjoint operator on \( L_2(\mathbb{R}) \), which we denote by \( \partial \) (see [56, p.198 Example 3],[44, 43]). With a slight abuse of notation, we define the derivative \( \partial_\nu \) on \( H_\nu \) by

\[
\partial_\nu = e^{-\nu m} \partial e^{\nu m} + \nu,
\]

which becomes, due to the skew-selfadjointness of \( \partial \), a continuously invertible operator with \( \|\partial_\nu^{-1}\| \leq \nu^{-1} \) (see [43, 44, 53]).

Using the advantage of working on the whole real line, we can state a spectral representation of \( \partial_\nu \) using the Fourier-Laplace transform, which is defined as follows.

**Definition 2.47.** As in Section 2.5 we denote the Fourier-transform on \( L_2(\mathbb{R}) \) by \( \mathcal{F} \). This mapping is known to be unitary (cf. [47], Theorem 9.13). To involve our weighted Hilbert space \( H_\nu \) for \( \nu \in \mathbb{R}_{>0} \) we define the **Fourier-Laplace transform** \( \mathcal{L}_\nu := \mathcal{F} e^{-\nu m} \). Since \( e^{-\nu m} \) is unitary, we conclude that \( \mathcal{L}_\nu \) is unitary as well.

**Remark 2.48.** It is known that for \( u \in D(\partial) \) we have (c.f. [20])

\[
\mathcal{F}(\partial u) = (t \mapsto it(\mathcal{F} u)(t)).
\]

Since \( \partial_\nu = e^{\nu m} \partial e^{-\nu m} + \nu \) we conclude that for \( u \in D(\partial_\nu) \)

\[
\mathcal{L}_\nu(\partial_\nu u)(t) = \mathcal{L}_\nu(e^{\nu m} \partial e^{-\nu m} u)(t) + \mathcal{L}_\nu(\nu u)(t) = \mathcal{F}(\partial e^{-\nu m} u)(t) + \nu(\mathcal{L}_\nu u)(t)
\]
\[
= (it) \mathcal{F}(e^{-\nu m}u)(t) + \nu (\mathcal{L}_\nu u)(t) \\
= (it + \nu) (\mathcal{L}_\nu u)(t) \quad (t \in \mathbb{R} \text{ a.e.})
\]

and hence
\[
\partial_\nu = \mathcal{L}_\nu^*(im + \nu)\mathcal{L}_\nu,
\]

where \( m \) is defined as

\[
m : D(m) \subseteq L_2(\mathbb{R}) \to L_2(\mathbb{R}) \\
f \mapsto (t \mapsto tf(t))
\]

with \( D(m) := \{ f \in L_2(\mathbb{R}) \mid (t \mapsto tf(t)) \in L_2(\mathbb{R}) \} \). Via this spectral representation of the derivative operator \( \partial_\nu \) we define so-called generalized material laws. Like above, we identify the operator \( \mathcal{L}_\nu \) with its extension on the tensor product space \( H_\nu \otimes H \) via \( \mathcal{L}_\nu \otimes 1_H \).

**Definition 2.49.** Let \( r > 0 \) and \( H \) a Hilbert space. We define

\[
M : BC(r, r) \to B_u(H) \\
z \mapsto M(z)
\]

as a continuous mapping into the set of uniformly bounded relations on the space \( H \). We assume the following conditions

\[
C := \sup_{z \in BC(r, r)} c(M(z)) < \infty
\]

and

\[
\exists c \in \mathbb{R}^+_0 \forall z \in BC(r, r) : z^{-1}M(z) - c \text{ is maximal monotone.}
\]

We call such a mapping \( M \) a **generalized \( c \)-material law** (the notion of \( c \)-material laws was introduced in [53]). We like to define the relation \( M\left(\frac{1}{im + \nu}\right) \) for \( \nu > \frac{1}{2r} \) by

\[
(u, v) \in M\left(\frac{1}{im + \nu}\right) \subseteq (L_2(\mathbb{R}) \otimes H)^2 : \iff (u(t), v(t)) \in M\left(\frac{1}{it + \nu}\right) \quad (t \in \mathbb{R} \text{ a.e.})
\]

and

\[
M(\partial_{\nu}^{-1}) := \mathcal{L}_\nu^* M\left(\frac{1}{im + \nu}\right) \mathcal{L}_\nu \subseteq (H_\nu \otimes H)^2.
\]

**Lemma 2.50.** Let \( r, c > 0, M : BC(r, r) \to B_u(H) \) a generalized \( c \)-material law and \( \nu > \frac{1}{2r} \). Then \( M(\partial_{\nu}^{-1}) \) is bounded.

**Proof.** Let \( d \in \mathbb{R}^+_0 \) and \( v \in M(\partial_{\nu}^{-1})[B(0, d)] \). Then we find \( u \in B(0, d) \) such that \( (u, v) \in M(\partial_{\nu}^{-1}) \), which yields

\[
((\mathcal{L}_\nu u)(t), (\mathcal{L}_\nu v)(t)) \in M\left(\frac{1}{it + \nu}\right) \quad (t \in \mathbb{R} \text{ a.e.}).
\]

We estimate

\[
|v|_{H_\nu \otimes H}^2 = |\mathcal{L}_\nu v|_{L_2(\mathbb{R}) \otimes H}^2
\]
2.6. Evolutionary inclusions on the whole real line

\[
\int_{\mathbb{R}} |(\mathcal{L}_\nu v)(t)|^2 \, dt \\
\leq \int_{\mathbb{R}} C^2 |(\mathcal{L}_\nu u)(t)|^2 \, dt \\
= C^2 |u|_{H^\nu \otimes H}^2 \\
< C^2 d^2,
\]

where \( C := \sup_{z \in B_c(r,r)} c(M(z)) \). This shows \( M(\partial_{\nu}^{-1})|B(0,d)| \subseteq B(0,Cd) \). \( \square \)

**Proposition 2.51.** Let \( r, c > 0 \) and \( M : B_c(\nu, r) \to B_u(H) \) a generalized \( c \)-material law. Then 
\((im + \nu)M \left( \frac{1}{im + \nu} \right)\) is \( c \)-monotone for every \( \nu > \frac{1}{2r} \).

**Proof.** Let \((u, v), (x, y) \in (im + \nu)M \left( \frac{1}{im + \nu} \right)\). We compute

\[
\Re \langle u - x | v - y \rangle_{L^2(\mathbb{R}) \otimes H} = \int_{\mathbb{R}} \Re \langle u(t) - x(t) | v(t) - y(t) \rangle_H \, dt \\
\geq c \int_{\mathbb{R}} |u(t) - x(t)|_H^2 \, dt \\
= c |u - x|_{L^2(\mathbb{R}) \otimes H}^2.
\]

**Lemma 2.52.** Let \( d \in \mathbb{R} \). The mapping

\[
I : \{ A \in B_u(H) \mid A - d \text{ is maximal monotone} \} \subseteq B_u(H) \\
\begin{array}{ccc}
A & \mapsto & A^{-1}
\end{array}
\]

is Lipschitz-continuous.

**Proof.** Let \( x \in H \setminus \{0\} \) and \( A, B \in B_u(H) \) with \( A - d, B - d \) maximal monotone. We conclude that \( B^{-1}, A^{-1} \) are Lipschitz-continuous mappings, defined on the whole space \( H \) with Lipschitz-constant less than or equal to \( \frac{1}{d} \). By the monotonicity of \( A - d \) we obtain

\[
\forall y \in A[\{B^{-1}(x)\}] : d|A^{-1}(x) - B^{-1}(x)|^2 \leq \Re \langle x - y | A^{-1}(x) - B^{-1}(x) \rangle
\]

and hence, by applying the Cauchy-Schwarz-Inequality

\[
\forall y \in A[\{B^{-1}(x)\}] : |A^{-1}(x) - B^{-1}(x)| \leq d^{-1} |x - y|.
\]

The latter yields

\[
|A^{-1}(x) - B^{-1}(x)| \leq d^{-1} \text{dist}(x, A[\{B^{-1}(x)\}])
\]

and since \((B^{-1}(x), x) \in B\) we conclude

\[
|A^{-1}(x) - B^{-1}(x)| \leq d^{-1} \mathcal{H}(B[\{B^{-1}(x)\}], A[\{B^{-1}(x)\}]).
\]
Moreover, $B^{-1}(x) \neq 0$, because if $B^{-1}(x) = 0$, then $(0, x) \in B$ and since $B \in B_u(H)$ we conclude that $x = 0$. By the monotonicity of $B - d$ and the fact that $(0, 0) \in B$ it follows

$$|B^{-1}(x)| = |B^{-1}(x) - B^{-1}(0)| \leq \frac{1}{d}|x|.$$ 

Hence, we conclude

$$\frac{1}{|x|}|A^{-1}(x) - B^{-1}(x)| \leq \frac{1}{d}|x| \mathcal{H}(B[\{B^{-1}(x)\}], A[\{B^{-1}(x)\}])$$

$$ \leq \frac{1}{d^2 |B^{-1}(x)|} \mathcal{H}(B[\{B^{-1}(x)\}], A[\{B^{-1}(x)\}])$$

$$ \leq \frac{1}{d^2} d(A, B),$$

which leads to

$$d(A^{-1}, B^{-1}) \leq \frac{1}{d^2} d(A, B).$$

\qed

**Proposition 2.53.** Let $r, c > 0$ and $M : B_\infty(r, r) \to B_u(H)$ a generalized $c$–material law. Furthermore let $g \in C_c(\mathbb{R}; H)$ and define

$$f(t) := \left( (it + \nu)M \left( \frac{1}{it + \nu} \right) \right)^{-1} (g(t)) \quad (t \in \mathbb{R})$$

for $\nu > \frac{1}{2r}$. Then $f \in C_c(\mathbb{R}; H)$.

\begin{proof}
By Proposition 1.28 we know $(it + \nu)M \left( \frac{1}{it + \nu} \right) \in B_u(H)$ and $(it + \nu)M \left( \frac{1}{it + \nu} \right)$ is $c$–maximal monotone for each $t \in \mathbb{R}$. Since $t \mapsto M \left( \frac{1}{it + \nu} \right)$ is continuous, we conclude that also $t \mapsto (it + \nu)M \left( \frac{1}{it + \nu} \right)$ is continuous (compare Proposition 1.29). According to Lemma 2.52 it follows that also

$$t \mapsto \left( (it + \nu)M \left( \frac{1}{it + \nu} \right) \right)^{-1}$$

is continuous. The continuity of $f$ follows, since for a sequence $(t_k)_{k \in \mathbb{N}} \in \mathbb{R}^N$ with $t_k \to t \in \mathbb{R}$ as $k \to \infty$ we estimate

$$|f(t_k) - f(t)| = \left| \left( (it_k + \nu)M \left( \frac{1}{it_k + \nu} \right) \right)^{-1} (g(t_k)) - \left( (it + \nu)M \left( \frac{1}{it + \nu} \right) \right)^{-1} (g(t)) \right|$$

$$\leq \left| \left( (it_k + \nu)M \left( \frac{1}{it_k + \nu} \right) \right)^{-1} (g(t_k)) - \left( (it_k + \nu)M \left( \frac{1}{it_k + \nu} \right) \right)^{-1} (g(t)) \right|$$

$$\quad + \left| \left( (it + \nu)M \left( \frac{1}{it + \nu} \right) \right)^{-1} (g(t)) - \left( (it + \nu)M \left( \frac{1}{it + \nu} \right) \right)^{-1} (g(t)) \right|$$
Hence, \( x \) for almost every \( t \) we have to show that
\[
(\nu \mapsto M \left( \frac{1}{\nu + t} \right) )^{-1} (y(t)) \quad (n \to \infty).
\]
The last term converges to 0, since convergence of mappings in \( B_u(H) \) implies pointwise convergence. Thus, \( f \) is continuous and since \( \left( (it + \nu) M \left( \frac{1}{it + \nu} \right) \right)^{-1} (0) = 0 \), it follows that \( \text{supp} \, f \subseteq \text{supp} \, \nu \).

**Proposition 2.54.** Let \( r, c > 0 \) and \( M : B_C(\nu, 0) \to B_u(H) \) a generalized \( c \)-material law. Then \( (im + \nu) M \left( \frac{1}{im + \nu} \right) \) is \( c \)-maximal monotone for each \( \nu > \frac{1}{2r} \).

**Proof.** The monotonicity was already shown in Proposition 2.53. According to Minty’s Theorem, we have to show that \( (im + \nu) M \left( \frac{1}{im + \nu} \right) [L_2(\mathbb{R}) \otimes H] = L_2(\mathbb{R}) \otimes H \). Let \( y \in L_2(\mathbb{R}) \otimes H \). Then we find a sequence \( y_n \in C_c(\mathbb{R}; H) \) with \( y_n \to y \) in \( L_2(\mathbb{R}; H) \) and \( y_n(t) \to y(t) \) for almost every \( t \in \mathbb{R} \) as \( n \to \infty \). For \( n \in \mathbb{N} \) we define
\[
x_n(t) := \left( (it + \nu) M \left( \frac{1}{it + \nu} \right) \right)^{-1} (y_n(t)) \quad (t \in \mathbb{R}).
\]
By Proposition 2.53 we know that \( x_n \in C_c(\mathbb{R}; H) \). Furthermore
\[
x_n(t) = \left( (it + \nu) M \left( \frac{1}{it + \nu} \right) \right)^{-1} (y_n(t)) \to \left( (it + \nu) M \left( \frac{1}{it + \nu} \right) \right)^{-1} (y(t)) \quad (n \to \infty)
\]
for almost every \( t \in \mathbb{R} \). Thus, \( x := \left( t \mapsto \left( (it + \nu) M \left( \frac{1}{it + \nu} \right) \right)^{-1} (y(t)) \right) \) is measurable and we estimate
\[
\int_\mathbb{R} |x(t)|_H^2 \, dt \leq \frac{1}{c} \int_\mathbb{R} |y(t)|_H^2 \, dt < \infty.
\]
Hence, \( x \in L_2(\mathbb{R}) \otimes H \) and by definition of \( M \left( \frac{1}{im + \nu} \right) \) we get \( (x, y) \in (im + \nu) M \left( \frac{1}{im + \nu} \right) \).

**Corollary 2.55.** Let \( r, c > 0 \) and \( M : B_C(\nu, 0) \to B_u(H) \) a generalized \( c \)-material law. Then \( \partial_v M(\partial_v^{-1}) \) is \( c \)-maximal monotone for \( \nu > \frac{1}{2r} \).

**Proof.** This follows from Proposition 2.54 since \( \partial_v M(\partial_v^{-1}) - c = \mathcal{L}_\nu^* \left( (im + \nu) M \left( \frac{1}{im + \nu} \right) - c \right) \mathcal{L}_\nu \).

Next we want to show well-posedness of evolutionary inclusion of the form
\[
(u, f) \in \partial_v M(\partial_v^{-1}) + A_v,
\]
where $A \subseteq H \oplus H$ is maximal monotone and whose extension on $H_\nu \otimes H$ is again denoted by $A_\nu$. To guarantee the maximal monotonicity of $A_\nu$, we may assume that $(0,0) \in A$ (see Proposition 1.14). The question on uniqueness and continuous dependence of a solution on the given right-hand side is easy to answer.

**Proposition 2.56.** Let $r,c \in \mathbb{R}_{>0}$ and $M : B_C(r,r) \to B_u(H)$ a generalized $c$–material law. Moreover, let $B \subseteq (H_\nu \otimes H)^2$ be a monotone relation for $\nu > \frac{1}{2r}$. Then for all $(u,f),(v,g) \in \partial_\nu M(\partial_\nu^{-1}) + B$ we estimate

$$\text{Re}(f - g|u - v)_{H_\nu \otimes H} \geq c|u - v|_{H_\nu \otimes H}^2$$

and hence

$$|u - v|_{H_\nu \otimes H} \leq \frac{1}{c}|f - g|_{H_\nu \otimes H}.$$

**Proof.** By assumption we find $x,y,\omega,\sigma \in H_\nu \otimes H$ such that $f = x + \omega, g = y + \sigma$ and $(u,x),(v,y) \in \partial_\nu M(\partial_\nu^{-1})$ as well as $(u,\omega),(v,\sigma) \in B$. We estimate by using the monotonicity of $B$ and $\partial_\nu M(\partial_\nu^{-1}) - c$:

$$\text{Re}(f - g|u - v)_{H_\nu \otimes H} = \text{Re}(x - y|u - v)_{H_\nu \otimes H} + \text{Re}(\omega - \sigma|u - v)_{H_\nu \otimes H} \geq c|u - v|_{H_\nu \otimes H}^2.$$

Applying the Cauchy-Schwarz-Inequality on the left-hand side yields:

$$|u - v|_{H_\nu \otimes H} \leq \frac{1}{c}|f - g|_{H_\nu \otimes H}.$$

To guarantee the existence of a solution $u$ for “sufficiently many” right-hand sides $f$, we assume a translation invariance in time of the material law. This is done, by assuming homogeneity of the relation $M(z)$ for each $z \in B_C(r,r)$.

**Definition 2.57.** Let $r,c \in \mathbb{R}_{>0}$. We call $M : B_C(r,r) \to B_u(H)$ a homogeneous generalized $c$–material law, if $M$ is a generalized $c$–material law and

$$\forall z \in B_C(r,r), \lambda \in \mathbb{C} : (u,v) \in M(z) \Rightarrow (\lambda u, \lambda v) \in M(z).$$

The latter can also be written as $\lambda M(z) = M(z)\lambda$ for all $\lambda \in \mathbb{C}, z \in B_C(r,r)$.

**Proposition 2.58.** Let $r,c \in \mathbb{R}_{>0}$ and $M : B_C(r,r) \to B_u(H)$ a homogeneous generalized $c$–material law. Furthermore let $A \subseteq H \oplus H$ be a maximal monotone relation with $(0,0) \in A$. Then we find for $f \in C^\infty_c(\mathbb{R};H)$ and $\nu > \frac{1}{2r}$ an element $u \in H_\nu \otimes H$ such that

$$(u,f) \in \partial_\nu M(\partial_\nu^{-1}) + A_\nu.$$
Proof. For $\lambda \in \mathbb{R}_{>0}$ we denote the Yosida-Approximation of $A_{\nu}$ by $A_{\nu,\lambda}$. According to Corollary 2.55 the relation $\partial_{\nu}M(\partial_{\nu}^{-1}) - c$ is maximal monotone and since $A_{\nu,\lambda}$ is Lipschitz-continuous and monotone, we conclude that $\partial_{\nu}M(\partial_{\nu}^{-1}) + A_{\nu,\lambda} - c$ is maximal monotone as well (see Corollary 1.16). Hence, for each $\lambda \in \mathbb{R}_{>0}$ there exists $u_{\lambda} \in H_{\nu} \otimes H$ with

$$
(u_{\lambda}, f) \in \partial_{\nu}M(\partial_{\nu}^{-1}) + A_{\nu,\lambda}.
$$

(2.9)

For the existence of $u \in \{f\}(\partial_{\nu}M(\partial_{\nu}^{-1}) + A_{\nu})$ it is sufficient to check, whether \(\sup_{\lambda \in \mathbb{R}_{>0}} |A_{\nu,\lambda}(u_{\lambda})|_{H_{\nu} \otimes H} < \infty\) by Lemma 1.17. For $h \in \mathbb{R}$ we define the translation operator $\tau_{h}$ by

$$
\tau_{h} : H_{\nu} \rightarrow H_{\nu},
\quad g \mapsto (x \mapsto g(x + h)).
$$

It is easy to verify that $\tau_{h}$ is a continuous, linear operator, and its adjoint is given by (cf. Lemma 2.5)

$$
\tau_{h}^{\ast} = e^{2\nu h} \tau_{-h}.
$$

We extend this operator on $H_{\nu} \otimes H$ by identify $\tau_{h}$ with $\tau_{h} \otimes 1_{H}$. By (2.9) we find $v_{\lambda} \in D(\partial_{\nu})$ such that

$$
f = \partial_{\nu}v_{\lambda} + A_{\nu,\lambda}(u_{\lambda}) \quad \text{and} \quad (u_{\lambda}, v_{\lambda}) \in M(\partial_{\nu}^{-1}).
$$

Let $h \in \mathbb{R}$. An easy computation yields

$$
\mathcal{L}_{\nu} \tau_{h} g = e^{(im+\nu)h} \mathcal{L}_{\nu} g
$$

for $g \in H_{\nu}$. Since $(u_{\lambda}, v_{\lambda}) \in M(\partial_{\nu}^{-1})$, it follows by definition that $(\mathcal{L}_{\nu} u_{\lambda}, \mathcal{L}_{\nu} v_{\lambda}) \in M \left( \frac{1}{it + \nu} \right)$, which implies

$$
((\mathcal{L}_{\nu} u_{\lambda})(t), (\mathcal{L}_{\nu} v_{\lambda})(t)) \in M \left( \frac{1}{it + \nu} \right) \quad (t \in \mathbb{R} \text{ a.e.}).
$$

By the homogeneity of $M \left( \frac{1}{it + \nu} \right)$ we conclude

$$
(e^{(it+\nu)h}(\mathcal{L}_{\nu} u_{\lambda})(t), e^{(it+\nu)h}(\mathcal{L}_{\nu} v_{\lambda})(t)) \in M \left( \frac{1}{it + \nu} \right) \quad (t \in \mathbb{R} \text{ a.e.}),
$$

which yields

$$
((\mathcal{L}_{\nu} \tau_{h} u_{\lambda})(t), (\mathcal{L}_{\nu} \tau_{h} v_{\lambda})(t)) \in M \left( \frac{1}{it + \nu} \right) \quad (t \in \mathbb{R} \text{ a.e.}),
$$

since $\tau_{h} u_{\lambda}, \tau_{h} v_{\lambda} \in H_{\nu} \otimes H$. It follows, that $(\tau_{h} u_{\lambda}, \tau_{h} v_{\lambda}) \in M(\partial_{\nu}^{-1})$. An easy computation shows that also

$$
A_{\nu,\lambda}(\tau_{h} u_{\lambda}) = \tau_{h} A_{\nu,\lambda}(u_{\lambda}) \quad (\lambda \in \mathbb{R}_{>0}, h \in \mathbb{R})
$$

and hence

$$
(\tau_{h} u_{\lambda}, \tau_{h} f) \in \partial_{\nu}M(\partial_{\nu}^{-1}) + A_{\nu,\lambda}
$$
Moreover, we estimate using the mean value inequality
\[ |\tau_h u_\lambda - u_\lambda|_{H^r_{\nu,H}} \leq \frac{1}{c} |\tau_h f - f|_{H^r_{\nu,H}}. \]

Let \( h \in (0, 1) \). Then the latter yields
\[ \frac{1}{h} |\tau_h u_\lambda - u_\lambda|_{H^r_{\nu,H}} \leq \frac{1}{c h} |\tau_h f - f|_{H^r_{\nu,H}}. \]

Moreover, we estimate using the mean value inequality
\[ \frac{1}{h} |\tau_h f - f|_{H^r_{\nu,H}}^2 = \int_{[\text{supp} f] + [B(0,1)]} \frac{1}{h} |f(t + h) - f(t)|^2 e^{-2\nu t} dt \leq \mu_{\nu}([\text{supp} f] + [B(0,1)]) |f'|^2_\infty. \]

We want to show that \( u_\lambda \in D(\partial_\nu) \) and
\[ \frac{1}{h} (\tau_h u_\lambda - u_\lambda) \to \partial_\nu u_\lambda \ (h \to 0^+). \]

For this purpose let \( \phi \in C^\infty_c(\mathbb{R}; H) \). Then we obtain
\[ h^{-1}(\tau_h u_\lambda - u_\lambda)\phi \in H^r_{\nu,H} \]
\[ = \langle u_\lambda | h^{-1}(e^{2\nu h \tau_h \phi - \phi}) \rangle_{H^r_{\nu,H}} \]
\[ = \langle u_\lambda | e^{2\nu h \tau_h \phi - \phi} \rangle_{H^r_{\nu,H}} + \langle u_\lambda | h^{-1}(e^{2\nu h} - 1)\phi \rangle_{H^r_{\nu,H}}. \]

We compute
\[ |e^{2\nu h \tau_h \phi - \phi}|_{H^r_{\nu,H}}^2 = \int_{[\text{supp} \phi] + [B(0,1)]} \frac{|e^{2\nu h \tau_h \phi - \phi} - \phi(t)|^2 e^{-2\nu t}}{(2e^{4\nu+2})^2 \nu^2 e^{-2\nu t}} dt \to 0 \ (h \to 0) \]

and
\[ |h^{-1}(e^{2\nu h} - 1)\phi - 2\nu \phi|^2_{H^r_{\nu,H}} = \int_{\text{supp} \phi} \frac{|h^{-1}(e^{2\nu h} - 1) - 2\nu |\phi(t)|^2 e^{-2\nu t}}{8e^{4\nu+1})|\phi(t)|^2 e^{-2\nu t}} dt \to 0 \ (h \to 0) \]

by Lebesgue’s dominated convergence Theorem. Therefore
\[ h^{-1}(\tau_h u_\lambda - u_\lambda)\phi \to \langle u_\lambda | \phi \rangle_{H^r_{\nu,H}} + \langle u_\lambda | 2\nu \phi \rangle_{H^r_{\nu,H}} \]
\[ = \langle u_\lambda | \partial_\nu \phi + 2\nu \phi \rangle_{H^r_{\nu,H}} = \langle u_\lambda | \partial_\nu \phi \rangle_{H^r_{\nu,H}}. \]
By (2.10) and (2.11) we know that there exists a nullsequence \((h_n)_{n \in \mathbb{N}} \in (0,1)^\mathbb{N}\) such that \((\frac{1}{h_n}(\tau_{h_n} u_\lambda - u_\lambda))_{n \in \mathbb{N}}\) is weak-convergent. We denote its weak limit by \(w_\lambda\). According to (2.12) we obtain 

\[
\forall \phi \in C_c^\infty(\mathbb{R}; \mathbb{H}) : \langle u_\lambda | \partial_\nu^* \phi \rangle_{\mathbb{H}} = \langle w_\lambda | \phi \rangle_{\mathbb{H}}.
\]

Since \(C_c^\infty(\mathbb{R}; \mathbb{H})\) is a core of \(\partial_\nu^*\) we conclude that \(u_\lambda \in D(\partial_\nu)\) and \(\partial_\nu u_\lambda = w_\lambda\). Since \(w_\lambda\) is uniformly bounded in \(\lambda\) by (2.11), then so is \(\partial_\nu u_\lambda\). We will now show that \((\partial_\nu u_\lambda, \partial_\nu v_\lambda) \in M(\partial_\nu^{-1})\): We already know that 

\[
((L_\nu u_\lambda)(t), (L_\nu v_\lambda)(t)) \in M\left(\frac{1}{it + \nu}\right) \quad (t \in \mathbb{R}\text{ a.e.})
\]

and thus 

\[
((it + \nu)(L_\nu u_\lambda)(t), (it + \nu)(L_\nu v_\lambda)(t)) \in M\left(\frac{1}{it + \nu}\right) \quad (t \in \mathbb{R}\text{ a.e.}).
\]

Since \(u_\lambda, v_\lambda \in D(\partial_\nu)\) it follows that 

\[
((L_\nu \partial_\nu u_\lambda)(t), (L_\nu \partial_\nu v_\lambda)(t)) \in M\left(\frac{1}{it + \nu}\right) \quad (t \in \mathbb{R}\text{ a.e.})
\]

and hence 

\[
(\partial_\nu u_\lambda, \partial_\nu v_\lambda) \in M(\partial_\nu^{-1}).
\]

Since \(\sup_{\lambda \in \mathbb{R}_{>0}} |\partial_\nu u_\lambda| < \infty\) we conclude with Lemma 2.50 that also \(\sup_{\lambda \in \mathbb{R}_{>0}} |\partial_\nu v_\lambda| < \infty\). Summarizing we estimate 

\[
\sup_{\lambda \in \mathbb{R}_{>0}} |A_{\nu,\lambda}(u_\lambda)| = \sup_{\lambda \in \mathbb{R}_{>0}} |f - \partial_\nu v_\lambda| < \infty,
\]

which yields the existence of an element \(u \in H_\nu \otimes H\) with 

\[
(u, f) \in \partial_\nu M(\partial_\nu^{-1}) + A_\nu.
\]

\[\square\]

**Corollary 2.59.** Let the assumptions of Proposition 2.58 hold. Then 

\[
(\partial_\nu M(\partial_\nu^{-1}) + A_\nu)^{-1} : H_\nu \otimes H \rightarrow H_\nu \otimes H
\]

is a Lipschitz-continuous mapping with Lipschitz-constant less than or equal to \(\frac{1}{\varepsilon}\).

Now the question on causality of our solution mapping \((\partial_\nu M(\partial_\nu^{-1}) + A_\nu)^{-1}\) should be answered. For doing so, we use a Paley-Wiener-type result, which can be found for instance in [47] or [56]. At first we give the definition of the so called Hardy-Lebesgue space.
Definition 2.60. We define the vector space
\[
\mathbb{HL} := \{ f : [\mathbb{R}] - i[\mathbb{R}_{>0}] \rightarrow \mathbb{C} | f \text{ analytic, } \forall \varepsilon > 0 : f(\cdot - i\varepsilon) \in L_2(\mathbb{R}), \sup_{\varepsilon > 0} |f(\cdot - i\varepsilon)|_{L_2(\mathbb{R})} < \infty \},
\]
the so-called Hardy-Lebesgue space and equip it with the norm
\[
f \mapsto \sup_{\varepsilon > 0} |f(\cdot - i\varepsilon)|_{L_2(\mathbb{R})}.
\]
We say a function \( g \in L_2(\mathbb{R}) \) belongs to the Hardy-Lebesgue space (denoted by \( g \in \mathbb{HL} \)), if
\[
([\mathbb{R}] - i[\mathbb{R}_{>0}]) \ni x - i\varepsilon \mapsto (\mathcal{L}_c \mathcal{F}^* g)(x) \in \mathbb{HL}.
\]

**Theorem 2.61** (Paley-Wiener,[47]). A function \( g \in L_2(\mathbb{R}) \) satisfies \( \text{supp} g \subseteq \mathbb{R}_{\geq 0} \) if and only if \( \mathcal{F} g \in \mathbb{HL} \).

Note that the Fourier-Laplace transform applied to a function in \( L_2(\mathbb{R}) \) can also be interpreted as a complexification of \( \mathcal{F} \). As a direct consequence of this theorem we obtain the following.

**Corollary 2.62.** A function \( g \in H_\nu \) satisfies \( \text{supp} g \subseteq \mathbb{R}_{\geq 0} \) if and only if \( \mathcal{L}_c g \in \mathbb{HL} \).

We have to find an assumption on our material law, which guarantees the causality of our solution mapping.

**Definition 2.63.** Let \( r, c \in \mathbb{R}_{>0} \). We call a homogeneous \( c \)-material \( M : B_C(r, r) \rightarrow B_u(H) \) \( \mathbb{HL} \)-preserving, if there exists \( \nu_0 \in \mathbb{R}_{>\frac{1}{\mathbb{R}}} \) such that for all \( f, g \in L_2(\mathbb{R}) \otimes H \) with \( f - g \in \mathbb{HL} \otimes H \)
\[
\left( (im + \nu)M \left( \frac{1}{im + \nu} \right) \right)^{-1} (f) - \left( (im + \nu)M \left( \frac{1}{im + \nu} \right) \right)^{-1} (g) \in \mathbb{HL} \otimes H \tag{2.13}
\]
for each \( \nu \geq \nu_0 \).

**Proposition 2.64.** Let \( r, c \in \mathbb{R}_{>0} \) and \( M : B_C(r, r) \rightarrow B_u(H) \) a homogeneous, generalized \( c \)-material law. There is \( \nu_0 \in \mathbb{R}_{>\frac{1}{\mathbb{R}}} \) such that the mapping \( (\partial_c M(\partial_c^{-1}))^{-1} \) is causal for all \( \nu \geq \nu_0 \) in the sense of Definition 2.17 and Remark 2.14 if and only if \( M \) is a \( \mathbb{HL} \)-preserving material law.

**Proof.** Assume first that \( M \) is \( \mathbb{HL} \)-preserving. Then there is \( \nu_0 \in \mathbb{R}_{>\frac{1}{\mathbb{R}}} \) such that
\[
\left( (im + \nu)M \left( \frac{1}{im + \nu} \right) \right)^{-1} \text{ satisfies } (2.13) \text{ for all } \nu \geq \nu_0.
\]
Let \( a \in \mathbb{R}, \nu \geq \nu_0 \) and \( u, v \in H_\nu \otimes H \) such that \( \chi_{(-\infty,a)}(m_0)(u - v) = 0 \). We set
\[
x := (\partial_c M(\partial_c^{-1}))^{-1}(u),
\]
\[
y := (\partial_c M(\partial_c^{-1}))^{-1}(v)
\]
and conclude
\[
(x, \partial_c^{-1}u), (y, \partial_c^{-1}v) \in M(\partial_c^{-1})
\]
which turns into
\[
\left( (\mathcal{L}_\nu x)(t), \frac{1}{it + \nu}(\mathcal{L}_\nu u)(t) \right), \left( (\mathcal{L}_\nu y)(t), \frac{1}{it + \nu}(\mathcal{L}_\nu v)(t) \right) \in M \left( \frac{1}{it + \nu} \right) \quad (t \in \mathbb{R} \text{ a.e.}). \tag{2.14}
\]
We can rewrite \( \chi_{(-\infty,0]}(m_0)(u - v) = 0 \) into \( \chi_{(-\infty,0]}(m_0)(\tau_a u - \tau_a v) = 0 \) and since \( M(z) \) is homogeneous for each \( z \in BC(r,r) \), we obtain from (2.14) that
\[
\left( e^{(it+\nu)a}(\mathcal{L}_\nu x)(t), \frac{1}{it + \nu}e^{(it+\nu)a}(\mathcal{L}_\nu u)(t) \right) \in M \left( \frac{1}{it + \nu} \right) \quad (t \in \mathbb{R} \text{ a.e.}),
\]
\[
\left( e^{(it+\nu)a}(\mathcal{L}_\nu y)(t), \frac{1}{it + \nu}e^{(it+\nu)a}(\mathcal{L}_\nu v)(t) \right) \in M \left( \frac{1}{it + \nu} \right) \quad (t \in \mathbb{R} \text{ a.e.})
\]
and thus
\[
(\tau_a x, \partial_{\nu}^{-1}\tau_a u), (\tau_a y, \partial_{\nu}^{-1}\tau_a v) \in M(\partial_{\nu}^{-1}).
\]
So it suffices to discuss the case \( a = 0 \). According to Corollary 2.62 we have \( \mathcal{L}_\nu(u - v) \in \mathcal{H}L \otimes H \) and by (2.14) we obtain
\[
(\mathcal{L}_\nu x, \mathcal{L}_\nu u), (\mathcal{L}_\nu y, \mathcal{L}_\nu v) \in (im + \nu)M \left( \frac{1}{im + \nu} \right)
\]
or
\[
\mathcal{L}_\nu(x - y) = \left( (im + \nu)M \left( \frac{1}{im + \nu} \right) \right)^{-1}(\mathcal{L}_\nu u) - \left( (im + \nu)M \left( \frac{1}{im + \nu} \right) \right)^{-1}(\mathcal{L}_\nu v).
\]
By assumption \( \mathcal{L}_\nu(x - y) \in \mathcal{H}L \otimes H \) and this implies \( \chi_{(-\infty,0]}(m_0)(x - y) = 0 \) by Corollary 2.62 For showing the converse, assume that the mapping \( (\partial_{\nu}M(\partial_{\nu}^{-1}))^{-1} \) is causal for all \( \nu \geq \nu_0 \in \mathbb{R}_{>\frac{1}{2}} \). Let \( f, g \in L_2(\mathbb{R}) \otimes H \) such that \( f - g \in \mathcal{H}L \otimes H \), which yields, by using Corollary 2.62 \( \supp \mathcal{L}_\nu^*(f - g) \subseteq \mathbb{R}_{\geq 0} \). The assumed causality implies
\[
\supp((\partial_{\nu}M(\partial_{\nu}^{-1}))^{-1}(\mathcal{L}_\nu^*f) - (\partial_{\nu}M(\partial_{\nu}^{-1}))^{-1}(\mathcal{L}_\nu^*g)) \subseteq \mathbb{R}_{\geq 0}
\]
and thus, again according to Corollary 2.62 we conclude
\[
\mathcal{L}_\nu((\partial_{\nu}M(\partial_{\nu}^{-1}))^{-1}(\mathcal{L}_\nu^*f) - (\partial_{\nu}M(\partial_{\nu}^{-1}))^{-1}(\mathcal{L}_\nu^*g)) \in \mathcal{H}L \otimes H,
\]
which yields
\[
\left( (im + \nu)M \left( \frac{1}{im + \nu} \right) \right)^{-1}(f) - \left( (im + \nu)M \left( \frac{1}{im + \nu} \right) \right)^{-1}(g) \in \mathcal{H}L \otimes H.
\]
Hence, \( M \) is a \( \mathcal{H}L \)-preserving material law. \( \square \)

The next proposition gives a useful criterion for the causality of the mapping \( (\partial_{\nu}M(\partial_{\nu}^{-1}))^{-1} \).
Proposition 2.65. Let $r, c \in \mathbb{R}_{>0}$ and $M : B_C(r, r) \to B_u(H)$ a homogeneous, generalized $c-$material law. Then the following statements are equivalent:

(i) $M$ is $\mathbb{H}_L$-preserving,

(ii) There exists $\nu_0 \in \mathbb{R}_{>\frac{1}{2r}}$ such that $(\partial_{\nu}M(\partial_{\nu}^{-1}))^{-1}$ is a causal mapping for each $\nu \geq \nu_0$,

(iii) There exists $\nu_0 \in \mathbb{R}_{>\frac{1}{2r}}$ such that for all $\nu \geq \nu_0$ the inequality

$$c \int_{-\infty}^{a} |u(t) - x(t)|^2 e^{-2\nu t} \, dt \leq \int_{-\infty}^{a} \text{Re} \langle v(t) - y(t) | u(t) - x(t) \rangle e^{-2\nu t} \, dt$$

holds for all $(u, v), (x, y) \in \partial_{\nu}M(\partial_{\nu}^{-1})$.

Proof. The equivalence of (i) and (ii) was already shown in Proposition 2.64. Assume now that (ii) holds and fix $a \in \mathbb{R}, \nu \geq \nu_0$. Let $(u, v), (x, y) \in \partial_{\nu}M(\partial_{\nu}^{-1})$, which can be rewritten as

$$u = (\partial_{\nu}M(\partial_{\nu}^{-1}))^{-1}(v) \text{ and } x = (\partial_{\nu}M(\partial_{\nu}^{-1}))^{-1}(y).$$

We compute by setting $A := \partial_{\nu}M(\partial_{\nu}^{-1})$ and using Lemma 2.12 (replacing $\chi_{[0,a]}(m_0)$ by $\chi_{(-\infty,a]}(m_0)$):

$$\int_{-\infty}^{a} \text{Re} \langle v(t) - y(t) | u(t) - x(t) \rangle e^{-2\nu t} \, dt$$

$$= \int_{-\infty}^{a} \text{Re} \langle v(t) - y(t) | A^{-1}(v)(t) - A^{-1}(y)(t) \rangle e^{-2\nu t} \, dt$$

$$= \text{Re} \langle v - y | \chi_{(-\infty,a]}(m_0) A^{-1}(v) - \chi_{(-\infty,a]}(m_0) A^{-1}(y) \rangle_{\mathbb{H}_L \otimes H}$$

$$= \text{Re} \langle v - y | \chi_{(-\infty,a]}(m_0) A^{-1}(\chi_{(-\infty,a]}(m_0)v) - \chi_{(-\infty,a]}(m_0) A^{-1}(\chi_{(-\infty,a]}(m_0)y) \rangle_{\mathbb{H}_L \otimes H}$$

$$= \text{Re} \langle \chi_{(-\infty,a]}(m_0)v - \chi_{(-\infty,a]}(m_0)y | A^{-1}(\chi_{(-\infty,a]}(m_0)v) - A^{-1}(\chi_{(-\infty,a]}(m_0)y) \rangle_{\mathbb{H}_L \otimes H}.$$

Since $A - c$ is monotone according to Corollary 2.64, we conclude, again by using Lemma 2.12

$$\int_{-\infty}^{a} \text{Re} \langle v(t) - y(t) | u(t) - x(t) \rangle_{\mathbb{H}_L} e^{-2\nu t} \, dt$$

$$= \text{Re} \langle \chi_{(-\infty,a]}(m_0)v - \chi_{(-\infty,a]}(m_0)y | A^{-1}(\chi_{(-\infty,a]}(m_0)v) - A^{-1}(\chi_{(-\infty,a]}(m_0)y) \rangle_{\mathbb{H}_L \otimes H}$$

$$\geq c \text{Re} \langle A^{-1}(\chi_{(-\infty,a]}(m_0)v) - A^{-1}(\chi_{(-\infty,a]}(m_0)y) \rangle_{\mathbb{H}_L \otimes H}^2 e^{-2\nu t} \, dt$$

$$\geq c \int_{\mathbb{R}} \text{Re} \langle A^{-1}(\chi_{(-\infty,a]}(m_0)v) - A^{-1}(\chi_{(-\infty,a]}(m_0)y) \rangle_{\mathbb{H}_L}^2 e^{-2\nu t} \, dt.$$
This shows (iii). For showing the reverse implication, let \( \nu \geq \nu_0, a \in \mathbb{R} \) and \( f, g \in H_{\nu} \otimes H \) such that \( \chi_{(-\infty,a)}(m_0)(f - g) = 0 \). According to (iii) we find out

\[
0 = \int_{-\infty}^{a} \text{Re}\langle f(t) - g(t) | (\partial_{\nu} M(\partial_{\nu}^{-1}))^{-1}(f)(t) - (\partial_{\nu} M(\partial_{\nu}^{-1}))^{-1}(g)(t) \rangle_{H} e^{-2\nu t} \, dt
\]

\[
\geq c \int_{-\infty}^{a} |(\partial_{\nu} M(\partial_{\nu}^{-1}))^{-1}(f)(t) - (\partial_{\nu} M(\partial_{\nu}^{-1}))^{-1}(g)(t) |_{H}^{2} e^{-2\nu t} \, dt
\]

\[
\geq 0.
\]

The latter implies \( (\partial_{\nu} M(\partial_{\nu}^{-1}))^{-1}(f)(t) - (\partial_{\nu} M(\partial_{\nu}^{-1}))^{-1}(g)(g)(t) = 0 \) for almost every \( t \in (-\infty,a] \) and hence, (ii) holds. \( \square \)

With this knowledge we are able to prove the causality of our solution mapping.

**Proposition 2.66.** Let \( r,c \in \mathbb{R}_{>0} \) and \( M : B_{C}(r,r) \to B_{D}(H) \) a \( \mathbb{H}_L \)-preserving, homogeneous generalized \( c \)-material law. Furthermore let \( A \subseteq H \otimes H \) be a maximal monotone relation with \( (0,0) \in A \). Then there is \( \nu_0 \in \mathbb{R}_{>0} \frac{1}{2r} \) such that the mapping \( (\partial_{\nu} M(\partial_{\nu}^{-1}) + A_{\nu})^{-1} \) is causal for all \( \nu \geq \nu_0 \).

**Proof.** Let \( a \in \mathbb{R} \) and \( f, g \in H_{\nu} \otimes H \) with \( \chi_{(-\infty,a)}(m_0)(f - g) = 0 \). First assume \( f, g \in (\partial_{\nu} M(\partial_{\nu}^{-1}) + A_{\nu})[H_{\nu} \otimes H] \) and define

\[
u := (\partial_{\nu} M(\partial_{\nu}^{-1}) + A_{\nu})^{-1}(f),
\]

\[
x := (\partial_{\nu} M(\partial_{\nu}^{-1}) + A_{\nu})^{-1}(g).
\]

Thus, we find \( v, y \in H_{\nu} \otimes H \) such that \( (u,v), (x,y) \in \partial_{\nu} M(\partial_{\nu}^{-1}) \) and

\[
(u,f - v) \in A_{\nu} \text{ as well as } (x,g - y) \in A_{\nu}.
\]

We estimate, by using Proposition 2.65 (iii):

\[
0 = \int_{-\infty}^{a} \text{Re}\langle f(t) - g(t) | u(t) - x(t) \rangle_{H} e^{-2\nu t} \, dt
\]

\[
= \int_{-\infty}^{a} \text{Re}\langle f(t) - v(t) + v(t) - (g(t) - y(t) + y(t)) | u(t) - x(t) \rangle_{H} e^{-2\nu t} \, dt
\]
2.6. Evolutionary inclusions on the whole real line

\[
\geq \int_{-\infty}^{a} \Re\langle v(t) - g(t)|u(t) - x(t)\rangle_{H}e^{-2\nu t} \, dt \\
\geq c \int_{-\infty}^{a} |u(t) - x(t)|_{H}^{2}e^{-2\nu t} \, dt \\
\geq 0.
\]

The latter yields \( \chi_{(-\infty, a]}(u - x)(m_{0}) = 0 \), which shows the causality of \( (\partial_{\nu} M(\partial_{\nu}^{-1}) + A_{\nu})^{-1} \). Let now \( f, g \in H_{\nu} \otimes H \) such that \( \chi_{(-\infty, a]}(m_{0})(f - g) = 0 \). We choose two sequences \((f_{n})_{n \in \mathbb{N}}, (g_{n})_{n \in \mathbb{N}} \in (\partial_{\nu} M(\partial_{\nu}^{-1}) + A_{\nu})[H_{\nu} \otimes H]^{\mathbb{N}}\) with \( f_{n} \to f \) and \( g_{n} \to g \) in \( H_{\nu} \otimes H \) as \( n \to \infty \).

We set \( u_{n} := (\partial_{\nu} M(\partial_{\nu}^{-1}) + A_{\nu})^{-1}(f_{n}) \) and \( x_{n} := (\partial_{\nu} M(\partial_{\nu}^{-1}) + A_{\nu})^{-1}(g_{n}) \) and by the inequality we showed above, we obtain for each \( n \in \mathbb{N} \)

\[
\int_{-\infty}^{a} |u_{n}(t) - x_{n}(t)|_{H}^{2}e^{-2\nu t} \, dt \leq \int_{-\infty}^{a} \Re\langle f_{n}(t) - g_{n}(t)|u_{n}(t) - x_{n}(t)\rangle_{H}e^{-2\nu t} \, dt.
\]

Since \( u_{n} \to u := (\partial_{\nu} M(\partial_{\nu}^{-1}) + A_{\nu})^{-1}(f) = (\partial_{\nu} M(\partial_{\nu}^{-1}) + A_{\nu})^{-1}(f) \) and \( x_{n} \to x := (\partial_{\nu} M(\partial_{\nu}^{-1}) + A_{\nu})^{-1}(g) \) in \( H_{\nu} \otimes H \) as \( n \to \infty \) and since the cut-off operator \( \chi_{(-\infty, a]}(m_{0}) \) is continuous on \( H_{\nu} \otimes H \), the inequality yields

\[
\int_{-\infty}^{a} |u(t) - x(t)|_{H}^{2}e^{-2\nu t} \, dt \leq \int_{-\infty}^{a} \Re\langle f(t) - g(t)|u(t) - x(t)\rangle_{H}e^{-2\nu t} \, dt = 0
\]

and hence \( \chi_{(-\infty, a]}(m_{0})(u - x) = 0 \). \( \square \)

We summarize our results in the following theorem.

**Theorem 2.67.** Let \( r, c \in \mathbb{R}_{>0} \) and \( M : B_{C}(r, r) \to B_{a}(H) \) a HIL-preservation, homogeneous generalized c-material law. Furthermore let \( A \subseteq H \oplus H \) be a maximal monotone relation with \((0, 0) \in A \). Then there exists \( \nu_{0} \in \mathbb{R}_{>\frac{1}{r}} \) such that the relation \( (\partial_{\nu} M(\partial_{\nu}^{-1}) + A_{\nu})^{-1} \) is a causal, Lipschitz-continuous mapping with domain \( H_{\nu} \otimes H \) for each \( \nu \geq \nu_{0} \).
3. Diffusion in poro-plastic media

To show the versatility of the concepts, developed in the previous chapters, we like to apply our solution theory to a system of partial differential equations and differential inclusions, which describes the diffusion process of a fluid through a porous, plastically deforming media. In \[50\] this system was studied by Showalter and Stefanelli and the existence and uniqueness of a solution was shown, by proving the well-posedness of the corresponding stationary problem (cf. \[48\], IV, Theorem 6.1.). However, to show the solvability of the stationary problem, they have to assume coercitivity conditions on the involved differential operators. In our approach there is no need to look for solvability of the stationary problem. Indeed, one can find examples of well-posed equations, whose stationary problem does not posses a solution. Thus, we can relax the assumptions on the differential operators and relations. Also we do not need regularity assumptions on the source terms. In \[50\] Showalter and Stefanelli studied different boundary conditions, even on different frictions of the boundary. For simplicity we may assume these boundary conditions to be valid on the whole boundary and show that the corresponding differential operator gets skew-selfadjoint and hence maximal monotone (cf. Example 1.8).

Before we can consider the equations in detail, we have to introduce the right Hilbert space setting and the needed differential operators to formulate the problem.

3.1. The Hilbert space setting

In general let \(\Omega \subseteq \mathbb{R}^3\) be open.

**Definition 3.1.** We define the following space:

\[
L_2(\Omega)^{3 \times 3} := \{ \Phi = (\Phi_{ij})_{i,j \in \{1,2,3\}} \mid \Phi_{ij} \in L_2(\Omega) \}
\]

and equip this space with a sesquilinear functional, defined by

\[
\langle \Phi | \Psi \rangle_{L_2(\Omega)^{3 \times 3}} := \int_{\Omega} \text{trace}(\Phi^* \Psi) \quad (\Phi, \Psi \in L_2(\Omega)^{3 \times 3}).
\]

Furthermore we define the subspace of symmetric matrix-valued functions by

\[
H_{sym}(\Omega) := \{ \Phi \in L_2(\Omega)^{3 \times 3} \mid \Phi(x) = \Phi(x)^T \quad (x \in \Omega \text{ a.e.}) \}.
\]

**Proposition 3.2.** The space \(L_2(\Omega)^{3 \times 3}\) is a Hilbert space with respect to the inner product \(\langle \cdot | \cdot \rangle_{L_2(\Omega)^{3 \times 3}}\). The space \(H_{sym}(\Omega)\) is a closed subspace of \(L_2(\Omega)^{3 \times 3}\).
3.1. The Hilbert space setting

Proof. Let $\Phi, \Psi \in L_2(\Omega)^{3\times3}$. Then

$$
\langle \Phi | \Psi \rangle_{L_2(\Omega)^{3\times3}} = \int_{\Omega} \text{trace}(\Phi^* \Psi)
$$

$$
= \int_{\Omega} \text{trace} \left( \sum_{k=1}^{3} \Phi_{k,i}^* \Psi_{k,j} \right)_{i,j}
$$

$$
= \sum_{i,k=1}^{3} \int_{\Omega} \Phi_{k,i}^* \Psi_{k,i}.
$$

(3.1)

With this formula the properties of an inner product can be shown easily. Also the completeness of the space follows directly, since for a Cauchy sequence $(\Phi^n)_{n \in \mathbb{N}}$ in $L_2(\Omega)^{3\times3}$ each component of the matrix is a Cauchy sequence in $L_2(\Omega)$. Therefore, for each $i,k \in \{1,2,3\}$ there exists an element $\Phi_{i,k} \in L_2(\Omega)$ with

$$
\Phi^n_{i,k} \to \Phi_{i,k} \quad (n \to \infty) \text{ in } L_2(\Omega).
$$

Thus, for $\Phi := (\Phi_{i,k})_{i,k \in \{1,2,3\}} \in L_2(\Omega)^{3\times3}$ it follows by (3.1)

$$
|\Phi^n - \Phi|^2_{L_2(\Omega)^{3\times3}} = \sum_{i,k=1}^{3} \int_{\Omega} |\Phi^n_{k,i} - \Phi_{k,i}|^2 \to 0 \quad (n \to \infty).
$$

For a convergent sequence $(\Phi^n)_{n \in \mathbb{N}} \in H_{sym}(\Omega)^N$ with limit $\Phi \in L_2(\Omega)^{3\times3}$ we conclude with (3.1)

$$
\Phi^n_{k,i} \to \Phi_{k,i} \quad (n \to \infty) \text{ in } L_2(\Omega)
$$

for each $k,i \in \{1,2,3\}$ and since $\Phi^n_{k,i} = \Phi^n_{i,k}$ for every $n \in \mathbb{N}$, we obtain the symmetry of $\Phi$. \quad \Box

Remark 3.3. We like to consider the adjoint of the operator

$$
\text{trace} : L_2(\Omega)^{3\times3} \to L_2(\Omega),
$$

$$
\Phi \mapsto \sum_{i=1}^{3} \Phi_{i,i}.
$$

Obviously this is a bounded linear operator with $\|\text{trace}\| = \sqrt{3}$. Let $f \in L_2(\Omega)$ and $\Psi \in L_2(\Omega)^{3\times3}$. Then for $F := \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{pmatrix} \in H_{sym}(\Omega)$ we obtain by (3.1):

$$
\langle \text{trace} \Psi | f \rangle_{L_2(\Omega)} = \sum_{i=1}^{3} \int_{\Omega} \Psi_{i,i} f
$$
3.1. The Hilbert space setting

\[= \sum_{i,k=1}^{3} \int_{\Omega} \Psi_{k,i}^* F_{k,i}\]

Thus, \(\text{trace}^* (f) = F\).

We now define the well-known differential operators gradient and divergence in a \(L^2\)–space setting.

**Definition 3.4.** First we define the gradient on \(C^\infty_c(\Omega)\), where \(C^\infty_c(\Omega)\) denotes the space of the infinite times continuously differentiable functions with compact support in \(\Omega\):

\[
\text{grad}|_{C^\infty_c(\Omega)} : C^\infty_c(\Omega) \subseteq L^2(\Omega) \rightarrow L^2(\Omega)^3
\]

\[
\varphi \mapsto (\partial_j \varphi)_{j \in \{1,2,3\}}.
\]

In the same way we define the divergence on the space \(C^\infty_c(\Omega)^3\):

\[
\text{div}|_{C^\infty_c(\Omega)^3} : C^\infty_c(\Omega)^3 \subseteq L^2(\Omega)^3 \rightarrow L^2(\Omega)
\]

\[
(\psi_1, \psi_2, \psi_3)^T \mapsto \sum_{j=1}^{3} \partial_j \psi_j.
\]

Both operators are clearly densely defined and linear.

We now introduce the general concept of formally adjointness of linear operators in Hilbert spaces (cf. [44]).

**Definition 3.5.** Let \(A : D(A) \subseteq H_1 \rightarrow H_2\) and \(B : D(B) \subseteq H_2 \rightarrow H_1\) be two densely defined linear operators between the Hilbert spaces \(H_1\) and \(H_2\). \(A\) and \(B\) are called formally adjoint, if

\[A \subseteq B^*\]

**Lemma 3.6.** The relation, which is induced by the formally adjointness of operators, is symmetric.

**Proof.** Let \(A\) and \(B\) be two formally adjoint operators, i.e. \(A \subseteq B^*\). Then clearly \(B^*\) is also densely defined and so

\[A^* \supseteq B^{**} = \overline{B} \supseteq B.\]

**Remark 3.7.** The operators \(\text{grad}|_{C^\infty_c(\Omega)}\) and \(-\text{div}|_{C^\infty_c(\Omega)^3}\) are formally adjoint, since for \(\varphi \in C^\infty_c(\Omega)\) and \(\psi := (\psi_1, \psi_2, \psi_3)^T \in C^\infty_c(\Omega)^3\) we compute:

\[
(\text{grad} \varphi | \psi)_{L^2(\Omega)^3} = \sum_{j=1}^{3} \int_{\Omega} (\partial_j \varphi)^* \psi_j
\]
\begin{align*}
&= - \sum_{j=1}^{3} \int_{\Omega} \varphi^{*} \partial_{j} \psi_{j} \\
&= - \langle \varphi | \text{div} \psi \rangle_{L^{2}(\Omega)}.
\end{align*}
Thus, \( \psi \in D((\text{grad}|_{C^{\infty}_{c}(\Omega)})^*) \) and \( (\text{grad}|_{C^{\infty}_{c}(\Omega)})^* \psi = -\text{div}\psi \), i.e. \( -\text{div}|_{C^{\infty}_{c}(\Omega)^3} \subseteq (\text{grad}|_{C^{\infty}_{c}(\Omega)})^* \).

**Definition 3.8.** We define the following operators:

\[
\begin{align*}
\bigcirc \text{grad} &:= \text{grad}|_{C^{\infty}_{c}(\Omega)}, \\
\bigcirc \text{div} &:= \text{div}|_{C^{\infty}_{c}(\Omega)^3}, \\
\text{grad} &:= -\bigcirc \text{div}^*, \\
\text{div} &:= -\bigcirc \text{grad}^*.
\end{align*}
\]

By the Remark above, we conclude that \( \text{grad} \subseteq \bigcirc \text{grad} \) and \( \bigcirc \text{div} \subseteq \text{div} \). The difference of the domains of these operators lies in the implicitly assumed boundary conditions. So a function \( u \in D(\bigcirc \text{grad}) \) satisfies a generalized Dirichlet-boundary condition. This concept of boundary values of weakly-differentiable functions is well-known and can be justified by so-called trace operators (c.f. [20]).

In the same way a function \( v \in D(\bigcirc \text{div}) \) satisfies a generalized Neumann-boundary condition.

Now we want to introduce similar differential operators on the space \( H_{\text{sym}}(\Omega) \), by following the same strategy.

**Definition 3.9.** As above we define operators on matrices or vectors of functions, which are infinite times continuously differentiable and have compact support in \( \Omega \). For doing so, we need to define the following vector space:

\[
C^{\infty}_{c,\text{sym}}(\Omega) := \{ \Psi \in C^{\infty}_{c}(\Omega)^{3 \times 3} \mid \Psi = \Psi^{T} \}.
\]

By [3.1] it is easy to see that this space is dense in \( H_{\text{sym}}(\Omega) \). We define:

\[
\begin{align*}
\text{Grad}|_{C^{\infty}_{c}(\Omega)^3} : C^{\infty}_{c}(\Omega)^3 &\subseteq L^{2}(\Omega)^3 \rightarrow H_{\text{sym}}(\Omega) \\
(\varphi_{1}, \varphi_{2}, \varphi_{3})^{T} &\mapsto \left( \frac{\partial_{i} \varphi_{j} + \partial_{j} \varphi_{i}}{2} \right)_{i,j \in \{1,2,3\}}
\end{align*}
\]

and

\[
\begin{align*}
\text{Div}|_{C^{\infty}_{c,\text{sym}}(\Omega)} : C^{\infty}_{c,\text{sym}}(\Omega) &\subseteq H_{\text{sym}}(\Omega) \rightarrow L^{2}(\Omega)^3 \\
(\Psi_{i,j})_{i,j \in \{1,2,3\}} &\mapsto \left( \sum_{j=1}^{3} \partial_{j} \Psi_{i,j} \right)_{i \in \{1,2,3\}}
\end{align*}
\]

which are again densely defined, linear operators.
Remark 3.10. The operators $\text{Grad}|_{C_c^\infty(\Omega)^3}$ and $-\text{Div}|_{C_c^\infty(\Omega)}$ are formally adjoint, since for $\varphi := (\varphi_1, \varphi_2, \varphi_3)^T \in C_c^\infty(\Omega)^3$ and $\Psi := (\Psi_{i,j})_{i,j \in \{1,2,3\}} \in C_c^\infty(\Omega)$ we compute, by using (3.1)

$$\langle \text{Grad}\varphi | \Psi \rangle_{L^2(\Omega)^3 \times 3} = \sum_{i,j=1}^3 \int_\Omega \left( \frac{\partial_i \varphi_j + \partial_j \varphi_i}{2} \right)^* \Psi_{i,j}$$

$$= -\frac{1}{2} \sum_{i,j=1}^3 \int_\Omega \varphi_j^* \partial_i \Psi_{i,j} - \frac{1}{2} \sum_{i,j=1}^3 \int_\Omega \varphi_i^* \partial_j \Psi_{i,j}$$

$$= -\frac{1}{2} \sum_{j=1}^3 \int_\Omega \varphi_j^* \text{div} \Psi_{.,j} - \frac{3}{2} \sum_{i=1}^3 \int_\Omega \varphi_i^* \text{div} \Psi_{i,.}$$

$$= -3 \sum_{i=1}^3 \int_\Omega \varphi_i^* \text{div} \Psi_{i,.}$$

$$= -\langle \varphi | \text{Div}\Psi \rangle_{L^2(\Omega)^3}.$$ 

So we have $\Psi \in D((\text{Grad}|_{C_c^\infty(\Omega)^3})^*)$ and $(\text{Grad}|_{C_c^\infty(\Omega)^3})^* \Psi = -\text{Div}\Psi$, i.e. $-\text{Div}|_{C_c^\infty(\Omega)} \subseteq (\text{Grad}|_{C_c^\infty(\Omega)^3})^*$.

Definition 3.11. We define the following operators:

$$\overset{o}{\text{Grad}} := \text{Grad}|_{C_c^\infty(\Omega)^3},$$

$$\overset{o}{\text{Div}} := \text{Div}|_{C_c^\infty(\Omega)},$$

$$\overset{o}{\text{Grad}} := -(\overset{o}{\text{Div}})^*,$$

$$\overset{o}{\text{Div}} := -(\overset{o}{\text{Grad}})^*.$$ 

Again, by using $\overset{o}{\text{Grad}}$ we describe a generalized Dirichlet-boundary condition and by using $\overset{o}{\text{Div}}$, a generalized Neumann-type-boundary condition.

Now, our aim is to show that a function $u \in D(\text{grad})$ with compact support in $\Omega$ belongs to the domain of $\overset{o}{\text{grad}}$. For this purpose we need the concept of convolutions with mollifiers. For this topic we refer to [1, p. 109 ff]. We state the next results in an $n-$dimensional setting.

Lemma 3.12. Let $\Omega \subseteq \mathbb{R}^n$ be open, $i \in \{1,\ldots,n\}$ and $f \in L^2(\Omega)$, such that $\partial_i f \in L^2(\Omega)$ and $\text{supp} f$ is compact in $\Omega$. Then for

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{otherwise} \end{cases} \quad (x \in \mathbb{R}^n)$$

we get $\partial_i \tilde{f} = \overset{o}{\partial_i} f \in L^2(\mathbb{R}^n)$, where $\overset{o}{\partial_i}$ again denotes the canonical extension of $\partial_i f$ on $\mathbb{R}^n$.

The derivative is meant in the distributional sense.
Proof. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\text{supp}\psi \subseteq \Omega$, $0 \leq \psi \leq 1$ and $\psi = 1$ on $\text{supp}f$. Then for each $\varphi \in C_c^\infty(\mathbb{R}^n)$ we obtain $\psi\varphi|\Omega \in C_c^\infty(\Omega)$ and

$$
- \int_{\mathbb{R}^n} \tilde{f}^* \partial_i \varphi = - \int_{\text{supp}f} f^* \psi \partial_i \varphi \\
= - \int_{\text{supp}f} f^* \partial_i (\psi \varphi) \\
= \int_{\text{supp}f} (\partial_i f)^* \psi \varphi \\
= \int_{\mathbb{R}^n} (\tilde{\partial}_i f)^* \varphi,
$$

where we have used $\text{supp} \partial_i f \subseteq \text{supp}f$. Thus, the distributional derivative of $\tilde{f}$ is given by $\tilde{\partial}_i f$.

Proposition 3.13. Let $\Omega \subseteq \mathbb{R}^n$ be open, $i \in \{1, \ldots, n\}$ and $f \in L_2(\Omega)$ with $\partial_i f \in L_2(\Omega)$ and $\text{supp}f$ compact in $\Omega$. Let $(\rho_k)_{k \in \mathbb{N}} \in C_c^\infty(\mathbb{R}^n)^\mathbb{N}$ denote the Friedrichs mollifier. Then $(\tilde{f} * \rho_k)|\Omega \in C_c^\infty(\Omega)$ for $k$ large enough and

$$
(\tilde{f} * \rho_k)|\Omega \to f \text{ and } \partial_i(\tilde{f} * \rho_k)|\Omega \to \partial_i f \text{ in } L_2(\Omega) \text{ as } k \to \infty.
$$

Proof. Let $\varepsilon := \text{dist}(\text{supp}f, \partial \Omega) > 0$ and $k_0 \in \mathbb{N}$ such that $\frac{1}{k_0} < \varepsilon$. Then it follows, since

$$
\text{supp}(\tilde{f} * \rho_k) \subseteq [\text{supp}\tilde{f}] + [\text{supp}\rho_k] \subseteq [\text{supp}\tilde{f}] + [B(0, k^{-1})]
$$

that for $k \geq k_0$

$$(\tilde{f} * \rho_k)|\Omega \in C_c^\infty(\Omega).$$

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. According to Lemma 3.12 we know that $\partial_i \tilde{f} = \tilde{\partial}_i f$ and thus

$$
\int_{\mathbb{R}^n} (\tilde{f} * \rho_k)^*(x) \partial_i \varphi(x) \, dx = \int_{\mathbb{R}^n} \int_{B(0,k^{-1})} \tilde{f}^*(x-y) \rho_k(y) \, dy \partial_i \varphi(x) \, dx \\
= \int_{B(0,k^{-1})} \rho_k(y) \int_{\mathbb{R}^n} \tilde{f}^*(x-y) \partial_i \varphi(x) \, dx \, dy \\
= \int_{B(0,k^{-1})} \rho_k(y) \int_{\mathbb{R}^n} \tilde{\partial}_i f^*(z) \varphi(z+y) \, dz \, dy \\
= - \int_{B(0,k^{-1})} \rho_k(y) \int_{\mathbb{R}^n} \tilde{\partial}_i f^*(z) \varphi(z+y) \, dz \, dy
$$

2For a definition see for instance [20, Chapter C.4].
\begin{align*}
&= - \int_{B(0,k^{-1})} \rho_k(y) \int_{\mathbb{R}^n} \partial_i f^*(x - y) \varphi(x) \, dx \, dy \\
&= - \int_{\mathbb{R}^n} (\tilde{\partial}_i f \ast \rho_k)(x) \varphi(x) \, dx.
\end{align*}

The latter implies \( \partial_i (\tilde{f} \ast \rho_k) = \tilde{\partial}_i f \ast \rho_k \in L_2(\mathbb{R}^n) \). Obviously for \( k \geq k_0 \) we have

\[ \text{supp} \, \partial_i (\tilde{f} \ast \rho_k) \subseteq \Omega. \]

Hence, we conclude for \( k \geq k_0 \)

\[ \int_{\Omega} |(\tilde{f} \ast \rho_k)| - f |^2 = \int_{\mathbb{R}^n} |\tilde{f} \ast \rho_k - f|^2 \to 0 \quad (k \to \infty) \]

and

\[ \int_{\Omega} |\partial_i (\tilde{f} \ast \rho_k)| - \partial_i f |^2 = \int_{\mathbb{R}^n} |\tilde{\partial}_i f \ast \rho_k - \tilde{\partial}_i f|^2 \to 0 \quad (k \to \infty). \]

This shows the assertion. \( \square \)

**Corollary 3.14.** Let \( \Omega \subseteq \mathbb{R}^3 \) be open and \( f \in D(\text{grad}) \) with \( \text{supp} \, f \) compact in \( \Omega \). Then \( f \in D(\circ \text{grad}) \).

**Proof.** Since \( f \in D(\text{grad}) \), we know that \( \partial_i f \in L_2(\Omega) \) for all \( i \in \{1, 2, 3\} \). Therefore, according to Proposition 3.13, it follows that for \( i \in \{1, 2, 3\} \)

\[ (\tilde{f} \ast \rho_k)|_\Omega \to f \text{ and } \partial_i (\tilde{f} \ast \rho_k)|_\Omega \to \partial_i f \text{ in } L_2(\Omega) \text{ for } k \to \infty, \]

where \( (\rho_k)_{k \in \mathbb{N}} \in (C^\infty(\mathbb{R}^3))^N \) denotes the Friedrichs-mollifier. So we have

\[ |\text{grad}(\tilde{f} \ast \rho_k)|_\Omega - \text{grad} f |_{L_2(\Omega)}|^2 = \sum_{i=1}^{3} |\partial_i (\tilde{f} \ast \rho_k)|_\Omega - \partial_i f |_{L_2(\Omega)}|^2 \to 0 \quad (k \to \infty). \]

Since \( (\tilde{f} \ast \rho_k)|_\Omega \in C^\infty(\Omega) \) for \( k \) large enough, this yields the assertion. \( \square \)

To obtain an analogous result for the divergence, we need the following Lemma.

**Lemma 3.15.** Let \( \Omega \subseteq \mathbb{R}^3 \) be open, \( f \in D(\text{grad}) \) and \( \psi \in C^\infty(\Omega) \). Then \( f \psi \in D(\circ \text{grad}) \) with

\[ \circ \text{grad} (f \psi) = (\text{grad} f)\psi + f \circ \text{grad} \psi. \]
Proof. By Corollary 3.14 it suffices to check \( f\psi \in D(\text{grad}) \). So let \( \varphi \in C_c^\infty(\Omega)^3 \). Then

\[
\langle f\psi | \text{div} \varphi \rangle_{L^2(\Omega)} = \langle f| \psi^* \text{div} \varphi \rangle_{L^2(\Omega)} = \langle f \rangle_{L^2(\Omega)} \text{div} (\psi^* \varphi) - \langle \text{grad} \psi^* \rangle_{L^2(\Omega)} \cdot \varphi_{L^2(\Omega)} \]

Thus, \( f\psi \in D(\text{grad}) \) and

\[
\text{grad}(f\psi) = (\text{grad}f)\psi + f \text{ grad } \psi.
\]

\[\square\]

Corollary 3.16. Let \( \Omega \subseteq \mathbb{R}^3 \) be open and \( f \in D(\text{div}) \) with \( \text{supp} f \) compact in \( \Omega \). Then \( f \in D(\text{grad}^*) = D(\text{grad}^*) \).

Proof. Let \( g \in D(\text{grad}) \) and \( \psi \in C_c^\infty(\Omega) \) with \( \psi = 1 \) on \( \text{supp} f \). Thus, \( \psi g \in D(\text{grad}^*) \), according to Lemma 3.15 and, using \( \text{grad}\psi = 0 \) on \( \text{supp} f \), we compute:

\[
\langle f| \text{grad} g \rangle_{L^2(\Omega)^3} = \sum_{i=1}^{3} \int_{\text{supp} f} f_i^*(x)(\partial_i g)(x)\psi(x)\, dx
\]

\[
= \langle f| \psi(\text{grad} g) \rangle_{L^2(\Omega)^3}
\]

\[
= \langle f| \text{grad}(\psi g) \rangle_{L^2(\Omega)^3}
\]

\[
= \langle f| \psi^* \text{grad} \rangle_{L^2(\Omega)^3}
\]

\[
= -\langle \text{div} f | \psi g \rangle_{L^2(\Omega)}
\]

\[
= -\langle \text{div} f | g \rangle_{L^2(\Omega)}
\]

where we have used \( \text{supp} \text{div} f \subseteq \text{supp} f \). Hence, \( f \in D(\text{grad}^*) = D(\text{grad}^*) \).

\[\square\]
3.2. The equations

The following system of partial differential equations and differential inclusions describes the diffusion of a compressible fluid through a saturated, porous media, which underlies a plastic deformation. These equations go back to the fundamental works of M. A. Biot \[6, 7\] and K. Terzaghi \[51\].

\[
\begin{align*}
\partial_{t,0}(c_0 p + \alpha \text{div} u) - \text{div}(k \text{grad} p) &= h, \quad (3.2) \\
\rho \partial^2_{t,0} u - \text{grad}\lambda \partial_{t,0}(\text{div} u) - \text{Div} \sigma + \alpha \text{grad} p &= g, \quad (3.3) \\
\partial_{t,0} M + L \ni (\sigma, \partial_{t,0} \text{Grad} u). &\quad (3.4)
\end{align*}
\]

The unknown functions are \(p \in H_{t,0} \otimes L_2(\Omega)\), which models the pressure on the fluid, the displacement field \(u \in H_{t,0} \otimes L_2(\Omega)^3\) of the media and the effective stress tensor \(\sigma \in H_{t,0} \otimes H_{sym}(\Omega)\), which goes back to K. Terzaghi \[51\]. Furthermore \(c_0 : \Omega \to \mathbb{R}^+\) is a function, which describes the compressibility of the fluid and the porosity of the medium. The constant \(k \in \mathbb{R}^+\) is related to the viscosity of the fluid as well as to the permeability of the medium. \(\rho : \Omega \to \mathbb{R}^+\) is describing the density of the medium, \(\alpha \in \mathbb{R}\) is a constant coupling term and \(\lambda \in \mathbb{R}^+\). Equation (3.4) describes the material behavior, where \(M : H_{sym}(\Omega) \to H_{sym}(\Omega)\) is a self-adjoint, strictly positive definite operator and \(L \subseteq H_{sym}(\Omega) \oplus H_{sym}(\Omega)\) is maximal monotone. If \(L = 0\), then (3.4) reduces to Hooke’s law and the equations describe the diffusion in poro-elastic media. This case was already studied by Showalter \[49\] and by McGhee and Picard \[35\]. In the case of non-vanishing \(L\) we cover models of plastic hysteresis, like the Prandtl-Reuss model of elastic perfectly plastic materials (see \[17\], p.229 ff.). Further examples of hysteresis models, like the Prandtl-Ishlinskii hysteresis type (see \[52\]), which can be modeled by maximal monotone relations, are given in \[50\]. Equation (3.2) describes the diffusion of the fluid through the media by using Darcy’s law modified by the physical parameter \(k\). The term \(\text{div} u\) describes the additional fluid content, due to the dilation of the medium. The second equation (3.3) describes the deformation and is derived from the classical model of linear elasticity. The phenomena of secondary consolidation (see \[14\]) comes into play by the term \(\text{grad}\lambda \partial_{t,0}(\text{div} u)\) (see \[40, 49\]) and \(\alpha \text{grad} p\) stands for the additional stress, generated by the fluid pressure.

For simplicity we may assume that the initial conditions are set to 0, in the knowledge that other initial conditions could be handled by the strategies developed in Section 2.5.

**Definition 3.17.** We define:

\[
\begin{align*}
\varepsilon &:= \text{Grad} u, \\
v &:= \partial_{t,0} u, \\
q &:= k \text{grad} p.
\end{align*}
\]

Moreover, we define:

\[
\tau := \text{trace}\varepsilon
\]

\(^3\text{All involved operators should be understood as operators in time and space. This is done by taking the tensor products with the identities (see Appendix part A.2).}\)
\[ \tilde{p} := c_0 p + \alpha \tau, \]
\[ \tilde{q} := \partial_{\nu,0}^{-1} k^{-1} q, \]
\[ \tilde{v} := \rho v, \]
\[ \omega := \lambda \partial_{\nu,0} \tau - \alpha p. \]

The last equality can be reformulated by
\[ \tau = \partial_{\nu,0}^{-1} (\lambda^{-1} \omega + \lambda^{-1} \alpha p) \]
and thus we obtain
\[ \tilde{p} = (c_0 + \partial_{\nu,0}^{-1} \alpha \lambda^{-1} \omega) p + \partial_{\nu,0}^{-1} \alpha \lambda^{-1} \omega. \]

**Remark 3.18.** With this definitions we are able to reformulate the problem. We conclude the following equations:

\[ \partial_{\nu,0} \tilde{p} - \text{div} q = h, \]
\[ \partial_{\nu,0} \tilde{v} - \text{Div}(\sigma + \text{trace}^* \omega) = g, \]
\[ \partial_{\nu,0} \varepsilon - \text{Grad} v = 0, \]
\[ \partial_{\nu,0} \tau - \text{trace} \text{Grad} v = 0, \]
\[ \partial_{\nu,0} \tilde{q} - \text{grad} p = 0. \]

So we can summarize the equations (3.2) and (3.3) by

\[ \partial_{\nu,0} \begin{pmatrix} \tilde{p} \\ \varepsilon \\ \tau \\ \tilde{v} \\ \tilde{q} \end{pmatrix} - U^* \begin{pmatrix} 0 & 0 & 0 & 0 & \text{div} \\ 0 & 0 & 0 & \text{Grad} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \text{Div} & 0 & 0 & 0 \\ \text{grad} & 0 & 0 & 0 & 0 \end{pmatrix} U \begin{pmatrix} p \\ \sigma \\ \omega \\ v \\ q \end{pmatrix} = \begin{pmatrix} h \\ \varepsilon \\ \tau \\ \omega \\ g \end{pmatrix}, \quad (3.5) \]

where

\[ U := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \text{trace}^* & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \]

with a new material law given by:

\[ \begin{pmatrix} p \\ \sigma \\ \omega \\ v \\ q \end{pmatrix} , \quad \begin{pmatrix} \tilde{p} \\ \varepsilon \\ \tau \\ \tilde{v} \\ \tilde{q} \end{pmatrix} \in \begin{pmatrix} c_0 & 0 & 0 & 0 & 0 \\ 0 & M & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \partial_{\nu,0}^{-1} \begin{pmatrix} \alpha \lambda^{-1} \sigma^* & \alpha \lambda^{-1} & 0 & 0 \\ 0 & L & 0 & 0 & 0 \\ \lambda^{-1} \alpha^* & \lambda^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k^{-1} \end{pmatrix}. \]
From now on we will use the following notation:

\[
M_0 := \begin{pmatrix}
  c_0 & 0 & 0 & 0 & 0 \\
  0 & M_0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & \rho & 0 \\
  0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
M_1 := \begin{pmatrix}
  \alpha \lambda^{-1} & \alpha \lambda^{-1} & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  \lambda^{-1} & \lambda^{-1} & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & k^{-1}
\end{pmatrix},
\]

\[
B := \begin{pmatrix}
  0 & 0 & 0 & 0 & 0 \\
  0 & L & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\tilde{A} := \begin{pmatrix}
  0 & 0 & 0 & 0 & \text{div} \\
  0 & 0 & 0 & \text{Grad} & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & \text{Div} & 0 & 0 & 0 \\
  \text{grad} & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

This leads to an evolutionary inclusion of the form

\[
(V, F) \in (\partial_{\nu,0} M_0 + M_1 - A + B)
\]

for \(V \in [H_{\nu,0} \otimes H](\partial_0 M_0 + M_1 - A + B), \ F \in H_{\nu,0} \otimes H\) and \(A := U^* \tilde{A} U\), where the Hilbert space \(H\) is given by \(L_2(\Omega) \oplus H_{sym}(\Omega) \oplus L_2(\Omega) \oplus L_2(\Omega)^3 \oplus L_2(\Omega)^3\).

**Remark 3.19.** By the structure of \(M_1\) we see that we can generalize \(\lambda\) and \(k\) to selfadjoint, strictly positive operators on \(L_2(\Omega)\) and \(L_2(\Omega)^3\) respectively. The strict positive definiteness of \(\lambda^{-1}\) and \(k^{-1}\) then holds due to the following general fact.

**Lemma 3.20.** Let \(H\) be a Hilbert space and \(M : H \to H\) be selfadjoint with \(M \geq c\) for some \(c \in \mathbb{R}_{>0}\). Then \(M\) is continuously invertible and the inverse \(M^{-1}\) is strictly positive definite as well. In detail we have

\[
M^{-1} \geq \frac{c}{\|M\|^2}.
\]

**Proof.** Since \(M\) is selfadjoint, it follows that \((\{0\}]M)^{-1} = \overline{M[H]}\) and since \(M\) is injective, by the strict positive definiteness, we conclude that \(M[H]\) is dense in \(H\). Furthermore we have that
$M[H]$ is closed, since $M$ is continuously invertible on its range and thus, $M$ is onto. Moreover, for $u \in H$ we estimate

$$
\langle M^{-1}u|u\rangle_H = \langle M^{-1}u|MM^{-1}u\rangle_H \\
\geq c|M^{-1}u|^2_H \\
\geq \frac{c}{\|M\|^2}u^2_H,
$$

which shows the desired strict positive definiteness of $M^{-1}$.

### 3.3. Boundary conditions and maximal monotonicity

At first we have to verify, whether the operators $M_0$ and $M_1$ satisfy the conditions, which are required for the application of the results in Chapter 2.

**Lemma 3.21.** The operator $M_0$ is selfadjoint and strictly positive definite on its range. $\text{Re}M_1$ is strictly positive definite on the null space of $M_0$.

**Proof.** $M_0$ is selfadjoint, since $M$ is selfadjoint and the functions $c_0, \rho$ are real-valued. The null space of the operator $M_0$ is given by $\{0\}M_0 := \{0\} \oplus \{0\} \oplus L_2(\Omega) \oplus \{0\} \oplus L_2(\Omega)^3$.

Clearly, $M_0$ is strictly positive definite on $M_0[H] = (\{0\}M_0)^\perp = L_2(\Omega) \oplus H_{\text{sym}}(\Omega) \oplus \{0\} \oplus L_2(\Omega)^3 \oplus \{0\}$. We need to ensure that $\text{Re}M_1$ is strictly positive definite on $\{0\}M_0$. Let $\chi := (0, 0, \varphi, 0, \psi) \in \{0\}M_0$. We calculate

$$
\text{Re}(M_1\chi|\chi)_H = \text{Re}(\lambda^{-1}\varphi|\varphi)_{L_2(\Omega)} + \text{Re}(k^{-1}\psi|\psi)_{L_2(\Omega)^3} \geq c(\chi|\chi)_H
$$

for a certain $c \in \mathbb{R}_{>0}$, whose existence is guaranteed by Lemma 3.20.

We have to find out that the relation $-A + B$ is maximal monotone to apply our solution theory to the inclusion

$$(V, F) \in \partial_{\nu, 0}M_0 + M_1 + (-A + B)_\nu$$

for $\nu \in \mathbb{R}_{>0}$ sufficiently large. We can easily verify that $B$ is maximal monotone, due to the maximal monotonicity of $L$.

**Lemma 3.22.** The relation $B \subseteq H \oplus H$ is maximal monotone.

**Proof.** Clearly $(u, v) \in B$ if and only if $(u_2, v_2) \in L$ and $v_i = 0$ for $i \in \{1, 3, 4, 5\}$. We estimate for $(u, v), (x, y) \in B$

$$
\text{Re}(u - x|v - y)_H = \text{Re}(u_2 - x_2|v_2 - y_2)_{L_2(\Omega)^{3 \times 3}} \geq 0.
$$

For showing the maximal monotonicity we fix an arbitrary $y \in H$. According to Minty’s Theorem (Theorem 1.6) there exists $x_2 \in H_{\text{sym}}(\Omega)$ such that $(x_2, y_2) \in 1 + L$. By setting $x_i := y_i$ for $i \in \{1, 3, 4\}$, we conclude $(x, y) \in 1 + B$, which yields the maximal monotonicity according to Theorem 1.6.
Next we want to study different boundary conditions and corresponding versions of the operator $A$, such that $A$ gets skew-selfadjoint and hence maximal monotone (see Example 1.8). In [50], different boundary conditions on frictions of the boundary are studied. For simplicity, we may assume that the variables satisfy the boundary conditions on the whole boundary.

Remark 3.23. By the right use of $\circ \text{grad}$, $\circ \text{div}$, $\circ \text{Div}$ or $\circ \text{Grad}$ in $\tilde{A}$ instead of the operators without boundary conditions in (3.5), we can make $\tilde{A}$ and thus $A$ skew-selfadjoint. This simulates Dirichlet- or Neumann-type-boundary-conditions for the variables $p, \sigma + \text{trace}' \omega$, $v$ and $q$ respectively. More precisely the following versions of $\tilde{A}$ are skew-selfadjoint:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & \circ \text{div} \\
0 & 0 & 0 & \circ \text{Grad} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & \circ \text{Div} & 0 & 0 & 0 \\
\circ \text{grad} & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & \circ \text{div} \\
0 & 0 & 0 & \circ \text{Grad} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & \circ \text{Div} & 0 & 0 & 0 \\
\circ \text{grad} & 0 & 0 & 0 & 0
\end{pmatrix},
$$

Following [50], we want to study the boundary condition:

$$
k \text{grad} p \cdot n - \alpha \beta \partial_{\nu, \partial} u \cdot n = 0 \quad \text{on } \partial \Omega,
$$

$$
\lambda (\text{div} \partial_{\nu, \partial} u) n + \sigma n - \alpha (1 - \beta) p n = 0 \quad \text{on } \partial \Omega.
$$

Here $n$ is the outward normal vector on $\partial \Omega$ and $\beta : \partial \Omega \to [0, 1]$ is a measurable function, which describes the fraction of the pores on $\partial \Omega$. By using our variables $p, q, v, \sigma$ and $\omega$ we get the following conditions:

$$
q \cdot n - \alpha \beta v \cdot n = 0 \quad \text{on } \partial \Omega,
$$

$$
(\sigma + \text{trace}' \omega) n + \alpha \beta p n = 0 \quad \text{on } \partial \Omega. \quad (3.7)
$$

Since we do not want to assume any regularity on $\partial \Omega$, we may assume that $\beta$ can be extended to a real-valued function defined on $\Omega$, which we denote by $\beta$ again and which satisfies the following conditions:

$$
\beta \in L_\infty(\Omega),
$$

$$
\text{grad} \beta \in L_\infty(\Omega)^3,
$$

where $\text{grad} \beta$ is to be understood in the distributional sense. Before we consider the boundary condition (3.7), we want to show some general properties, which will be used later.

\footnote{Note that $U$ is a homeomorphism.}
Lemma 3.24. Let $f \in L_2(\Omega)$ and $g \in L_2(\Omega)^3$.

(a) $f \in D(\overset{\circ}{\text{grad}})$ if and only if $\text{trace}^* f \in D(\overset{\circ}{\text{Div}})$ and then
\[ \overset{\circ}{\text{grad}} f = \overset{\circ}{\text{Div}} (\text{trace}^* f). \]

(b) If $g \in D(\overset{\circ}{\text{Grad}})$, then $g \in D(\overset{\circ}{\text{div}})$ with
\[ \overset{\circ}{\text{div}} g = \text{trace} \overset{\circ}{\text{Grad}} g. \]

Proof. (a) If $f \in D(\overset{\circ}{\text{grad}})$, then there exists a sequence $(f_n)_{n \in \mathbb{N}} \in C_c^\infty(\Omega)^N$ with $f_n \to f$ and $\overset{\circ}{\text{grad}} f_n \to \overset{\circ}{\text{grad}} f$ in $L_2(\Omega)$ and $L_2(\Omega)^3$ respectively as $n \to \infty$. For all $n \in \mathbb{N}$ the equality
\[ \overset{\circ}{\text{grad}} f_n = \overset{\circ}{\text{Div}} (\text{trace}^* f_n) \]
holds trivially. By the continuity of $\text{trace}^*$ we get $\text{trace}^* f_n \to \text{trace}^* f$ in $H_{sym}(\Omega)$ as $n \to \infty$ and thus $\text{trace}^* f \in D(\overset{\circ}{\text{Div}})$ with
\[ \overset{\circ}{\text{Div}} \text{trace}^* f = \lim_{n \to \infty} \overset{\circ}{\text{Div}} \text{trace}^* f_n = \lim_{n \to \infty} \overset{\circ}{\text{grad}} f_n = \overset{\circ}{\text{grad}} f. \]

By using the same argumentation backwards and the fact that $\frac{1}{3} \text{trace} \text{trace}^* = 1$, we get the missing implication.

(b) We find a sequence $(g_n)_{n \in \mathbb{N}} \in (C_c^\infty(\Omega)^3)^N$ with $g_n \to g$ and $\overset{\circ}{\text{Grad}} g_n \to \overset{\circ}{\text{Grad}} g$ in $L_2(\Omega)^3$ and $H_{sym}(\Omega)$ respectively as $n \to \infty$. By using the trivial observation that
\[ \overset{\circ}{\text{div}} g_n = \text{trace} \overset{\circ}{\text{Grad}} g_n \]
holds for all $n \in \mathbb{N}$ and by using the continuity of $\text{trace}$, we get the assertion.

We state an analogous result for the operators without boundary conditions.

Lemma 3.25. The following holds:

(a) For a function $f \in L_2(\Omega)$ we have: $f \in D(\text{grad})$ if and only if $\text{trace}^* f \in D(\text{Div})$. For such $f$ the equality
\[ \text{grad} f = \text{Div}(\text{trace}^* f) \]
holds.
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(b) Let \( g \in D(\text{Grad}) \). Then \( g \in D(\text{div}) \) with

\[
\text{div} g = \text{trace Grad} g.
\]

Proof. (a) First we assume \( f \in D(\text{grad}) \). Then for each \( \varphi \in C_c^\infty(\Omega)^3 \) the following holds:

\[
\langle \text{trace}^* f | \text{Grad} \varphi \rangle_{L^2(\Omega)^{3 \times 3}} = \int_\Omega \sum_{i=1}^3 f^* \partial_i \varphi_i
\]

\[
= \langle f | \text{div} \varphi \rangle_{L^2(\Omega)}
\]

\[
= -\langle \text{grad}f | \varphi \rangle_{L^2(\Omega)^3}.
\]

Therefore \( \text{trace}^* f \in D(\text{Div}) \) and \( \text{Div} \text{trace}^* f = \text{grad}f \). By using the same argumentation backwards, we get the other implication.

(b) For \( \psi \in C_c^\infty(\Omega) \) we have:

\[
\langle g \| \text{grad} \psi \rangle_{L^2(\Omega)^3} = \sum_{i=1}^3 \langle g_i | \partial_i \psi \rangle_{L^2(\Omega)}
\]

\[
= \langle g | \text{Div} \text{trace}^* \psi \rangle_{L^2(\Omega)^3}
\]

\[
= -\langle \text{Grad} g | \text{trace} \psi \rangle_{L^2(\Omega)}.
\]

That is, \( g \in D(\text{div}) \) and \( \text{div} g = \text{trace Grad} g \).

\[\square\]

**Lemma 3.26.** Let \( \beta \in L_\infty(\Omega) \) with \( \text{grad} \beta \in L_\infty(\Omega)^3 \). Then for every \( \varphi \in C_c^\infty(\Omega)^3 \) it follows that \( \beta \varphi \in D(\text{div}) \) and

\[
\text{div} (\beta \varphi) = \beta \text{div} \varphi + (\text{grad} \beta) \cdot \varphi.
\]

Moreover, for every \( \eta \in C_c^\infty(\Omega) \) it holds that \( \beta \eta \in D(\text{grad}) \) and

\[
\text{grad} (\beta \eta) = (\text{grad} \beta) \eta + \beta \text{grad} \eta.
\]

Proof. We show that \( \beta \varphi \in D(\text{div}) \) and by Corollary 3.16 we conclude \( \beta \varphi \in D(\text{div}) \) since \( \text{supp}(\beta \varphi) \) is compact. Let \( \psi \in C_c^\infty(\Omega) \). Then we calculate

\[
\langle \beta \varphi | \text{grad} \psi \rangle_{L^2(\Omega)^3} = \sum_{i=1}^3 \int_\Omega \beta^* \varphi_i^* \partial_i \psi
\]

\[
= \sum_{i=1}^3 \int_\Omega \beta^* (\partial_i (\varphi_i^* \psi) - (\partial_i \varphi_i^*) \psi)
\]
\[ \begin{align*}
\int_{\Omega} \beta^* \text{div}(\varphi^* \psi) - \int_{\Omega} \beta^* \text{div}(\varphi^*) \psi \\
= - \int_{\Omega} \text{grad}\beta \cdot (\psi \varphi^*) - \int_{\Omega} \beta \text{div}(\varphi)^* \psi \\
= - \langle (\text{grad}\beta) \cdot \varphi + \beta \text{div}\varphi | \psi \rangle_{L^2(\Omega)}.
\end{align*} \]

Thus, we get \( \beta \varphi \in D(\text{div}) \) and with Corollary 3.16 we obtain
\[ \text{div}(\beta \varphi) = \text{div}(\beta \varphi) = \beta \text{div}\varphi + (\text{grad}\beta) \cdot \varphi. \]

Let \( \psi \in C_c^\infty(\Omega)^3 \). Then it follows
\[ \langle \beta \eta | \text{div}\psi \rangle_{L^2(\Omega)} = \int_{\Omega} \beta^* \eta^* \text{div}\psi \\
= \int_{\Omega} \beta^* \text{div}(\eta^* \psi) - \int_{\Omega} \beta^* (\text{grad}\eta^*) \cdot \psi \\
= - \int_{\Omega} \text{grad}\beta^* \cdot (\eta^* \psi) - \int_{\Omega} (\text{grad}\eta)^* \cdot \psi \\
= - \langle (\text{grad}\beta) \eta + \beta \text{grad}\eta | \psi \rangle_{L^2(\Omega)^3} \]
and so \( \beta \eta \in D(\text{grad}) \) with
\[ \text{grad}(\beta \eta) = (\text{grad}\beta) \eta + \beta \text{grad}\eta. \]

By Corollary 3.14 we get the assertion. \( \square \)

**Lemma 3.27.** Let \( \beta \in L_\infty(\Omega) \) with \( \text{grad}\beta \in L_\infty(\Omega)^3 \). Then the following holds:

(a) Let \( f \in D(\text{grad}) \). Then \( \beta f \in D(\text{grad}) \) and the following equality holds:
\[ \text{grad}(\beta f) = (\text{grad}\beta) f + \beta \text{grad} f. \]

(b) Let \( g \in D(\text{div}) \). Then \( \beta g \in D(\text{div}) \) with
\[ \text{div}(\beta g) = \beta \text{div} g + (\text{grad}\beta) \cdot g. \]

**Proof.** (a) We know that \( \beta f \in L_2(\Omega) \), since \( \beta \) is bounded. For \( \psi \in C_c^\infty(\Omega)^3 \) we calculate with Lemma 3.20
\[ \langle \beta f | \text{div}\psi \rangle_{L^2(\Omega)} = \langle f | \beta^* \text{div}\psi \rangle_{L^2(\Omega)} \]
\[ = \langle f | \text{div}(\beta^* \psi) - \text{grad}\beta^* \cdot \psi \rangle_{L^2(\Omega)}. \]
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\[ -\langle \text{grad} f|\beta^* \psi \rangle_{L^2(\Omega)} - \langle f|(\text{grad} \beta)^* \cdot \psi \rangle_{L^2(\Omega)} = -\langle \beta \text{grad} f|\psi \rangle_{L^2(\Omega)} - \langle (\text{grad} \beta)f|\psi \rangle_{L^2(\Omega)} = -\langle \beta \text{grad} f + (\text{grad} \beta)f|\psi \rangle_{L^2(\Omega)}. \]

Since \( \beta \) and \( \text{grad} \beta \) are bounded, it follows that \( \beta \text{grad} f + (\text{grad} \beta)f \in L^2(\Omega) \), which shows the assertion.

(b) Like in (a) we know that \( \beta g \in L^2(\Omega) \). For \( \psi \in C^\infty_c(\Omega) \) it follows with Lemma 3.26

\[ -\langle \text{div} g|\beta^* \psi \rangle_{L^2(\Omega)} - \langle (\text{grad} \beta) \cdot g|\psi \rangle_{L^2(\Omega)} = -\langle \beta \text{div} g + (\text{grad} \beta) \cdot g|\psi \rangle_{L^2(\Omega)}. \]

Since \( \beta \) and \( \text{grad} \beta \) are bounded, we get that \( \beta \text{div} g + (\text{grad} \beta) \cdot g \in L^2(\Omega) \), which shows the assertion.

With this knowledge we can define the operator, which allows us to model the boundary conditions (3.7).

**Definition 3.28.** We define the following two operators:

\[ T : D(T) \subseteq L^2(\Omega) \oplus H_{\text{sym}}(\Omega) \to L^2(\Omega)^3 \oplus L^2(\Omega)^3 \]

\[ (p, \tilde{\sigma}) \mapsto (\text{grad} p, \text{Div} \tilde{\sigma}) \]

with

\[ D(T) := \{(p, \tilde{\sigma}) \in D(\text{grad}) \oplus D(\text{Div}) | \tilde{\sigma} + \alpha \beta \text{trace}^* p \in D(\text{Div}) \} \]

and

\[ \tilde{T} : D(\tilde{T}) \subseteq L^2(\Omega)^3 \oplus L^2(\Omega)^3 \to L^2(\Omega) \oplus H_{\text{sym}}(\Omega) \]

\[ (q, v) \mapsto (\text{div} q, \text{Grad} v) \]

with

\[ D(\tilde{T}) := \{(q, v) \in D(\text{div}) \oplus D(\text{Grad}) | q - \alpha \beta v \in D(\text{div}) \}. \]

\( T \) and \( \tilde{T} \) are obviously linear.

**Lemma 3.29.** The operators \( T \) and \( \tilde{T} \) are densely defined and closed.
Proof. For showing the density of \( D(T) \) we take arbitrary functions \( \tilde{\sigma} \in C_{c,sym}^\infty(\Omega) \) and \( p \in C^\infty_c(\Omega) \) and prove that \((\tilde{\sigma}, p) \in D(T)\). For seeing this, we have to show that

\[
\tilde{\sigma} + \alpha \beta \text{trace}^* p \in D(\text{Div}).
\]

Since clearly \( \tilde{\sigma} \in D(\text{Div}) \), we have to conclude that also \( \beta \text{trace}^* p = \text{trace}^*(\beta p) \in D(\text{Div}) \). By Lemma 3.26 we know that \( \beta p \in D(\text{grad}) \) and thus, \( \text{trace}^*(\beta p) \in D(\text{Div}) \) according to Lemma 3.24. Via an analogous argumentation we also get the density of \( D(\bar{T}) \).

Now we show that \( T \) is closed. Let \((p_n, \tilde{\sigma}_n)_{n \in \mathbb{N}} \in D(T) \) with

\[
(p_n, \tilde{\sigma}_n) \to (p, \tilde{\sigma}) \in L^2(\Omega) \oplus H_{\text{sym}}(\Omega) \quad (n \to \infty),
\]

\[
T(p_n, \tilde{\sigma}_n) \to (f, g) \in L^2(\Omega)^3 \oplus L^2(\Omega)^3 \quad (n \to \infty).
\]

This implies that \( p_n \to p \) and \( \text{grad} p_n \to f \) as \( n \to \infty \). Since \( \text{grad} \) is closed by, we know that \( p \in D(\text{grad}) \) and \( f = \text{grad} p \). In the same way we get that \( \tilde{\sigma} \in D(\text{Div}) \) and \( g = \text{Div} \tilde{\sigma} \). So it leaves to prove that

\[
\tilde{\sigma} + \alpha \beta \text{trace}^* p \in D(\text{Div}).
\]

This follows by observing that

\[
|\tilde{\sigma}_n + \alpha \beta \text{trace}^* p_n - (\tilde{\sigma} + \alpha \beta \text{trace}^* p)|_{L^2(\Omega)^3} \leq |\tilde{\sigma}_n - \tilde{\sigma}|_{L^2(\Omega)^3} + |\alpha| \beta |(\text{trace}^* p_n - \text{trace}^* p)|_{L^2(\Omega)^3}
\]

\[
\leq |\tilde{\sigma}_n - \tilde{\sigma}|_{L^2(\Omega)^3} + |\alpha| \beta |(\text{trace}^* p_n - \text{trace}^* p)|_{L^2(\Omega)^3}
\]

\[
= |\tilde{\sigma}_n - \tilde{\sigma}|_{L^2(\Omega)^3} + \sqrt{3} |\alpha| \beta |p_n - p|_{L^2(\Omega)^3}
\]

\[
\to 0,
\]

as \( n \to \infty \) and by Lemma 3.25 (a)

\[
|\text{Div}(\tilde{\sigma}_n + \alpha \beta \text{trace}^* p_n) - \text{Div}(\tilde{\sigma} + \alpha \beta \text{trace}^* p)|_{L^2(\Omega)^3} \leq |\text{Div} \tilde{\sigma}_n - \text{Div} \tilde{\sigma}|_{L^2(\Omega)^3} + |\alpha| |\text{Div} (\text{trace}^* \beta p_n) - \text{Div} (\text{trace}^* \beta p)|_{L^2(\Omega)^3}
\]

\[
= |\text{Div} \tilde{\sigma}_n - \text{Div} \tilde{\sigma}|_{L^2(\Omega)^3} + |\alpha| |\text{grad} (\beta p_n) - \text{grad} (\beta p)|_{L^2(\Omega)^3}.
\]

By Lemma 3.27 (a) we conclude

\[
|\text{grad} (\beta p_n) - \text{grad} (\beta p)|_{L^2(\Omega)^3} \leq |(\text{grad} \beta)(p_n - p)|_{L^2(\Omega)^3} + |\beta \text{grad} (p_n - p)|_{L^2(\Omega)^3}
\]

\[
\leq |\text{grad} \beta| L^\infty_p|p_n - p|_{L^2(\Omega)^3} + |\beta| L^\infty_p |\text{grad} (p_n - p)|_{L^2(\Omega)^3}
\]

\[
\to 0 \quad (n \to \infty).
\]

Since also \( |\text{Div} \tilde{\sigma}_n - \text{Div} \tilde{\sigma}|_{L^2(\Omega)^3} \to 0 \) as \( n \to \infty \), we see that \( \tilde{\sigma} + \alpha \beta \text{trace}^* p \in D(\text{Div}) \), since \( \text{Div} \) is a closed operator. The closedness of \( \bar{T} \) follows analogously. \( \square \)
Proposition 3.30. The operator
\[
\tilde{A} := \begin{pmatrix}
0 & 0 & \tilde{T} \\
0 & 0 & 0 \\
\tilde{T} & 0 & 0
\end{pmatrix}
\]
is skew-selfadjoint.

Proof. We will show that \( T^* = -\tilde{T} \). Let \( (q, v) \in D(\tilde{T}) \). Then for each \( (p, \tilde{\sigma}) \in D(T) \) we compute
\[
\langle T(p, \tilde{\sigma}) | (q, v) \rangle_{L^2(\Omega)^3 \oplus L^2(\Omega)^3} = \langle \text{grad} \tilde{\sigma} | v \rangle_{L^2(\Omega)^3} + \langle \text{div}(\tilde{\sigma} + \alpha \beta v) | q \rangle_{L^2(\Omega)^3} - \langle \text{div}(\alpha \beta \text{trace} p) | v \rangle_{L^2(\Omega)^3}.
\]
Since we know that \( \tilde{\sigma} + \alpha \beta \text{trace} p \in D(\text{div}) \), we get
\[
\langle \text{div}(\tilde{\sigma} + \alpha \beta \text{trace} p) | q \rangle_{L^2(\Omega)^3} = -\langle \tilde{\sigma} + \alpha \beta \text{trace} p | (\text{grad} v) \rangle_{L^2(\Omega)^3 \otimes L^2(\Omega)^3}.
\]
Using Lemma 3.25 (a) and Lemma 3.27 (a) we compute\(^5\)
\[
-\langle \text{div}(\alpha \beta \text{trace} p) | v \rangle_{L^2(\Omega)^3} = -\alpha \langle \text{grad} \beta p | v \rangle_{L^2(\Omega)^3} = -\alpha \langle (\text{grad} \beta) p | v \rangle_{L^2(\Omega)^3} - \alpha \langle \text{grad} (\alpha \beta v) | v \rangle_{L^2(\Omega)^3} = -\alpha \langle p | (\text{grad} \beta) \cdot v \rangle_{L^2(\Omega)^3} - \alpha \langle \text{grad} (\alpha \beta v) | v \rangle_{L^2(\Omega)^3}.
\]
Summarizing we get
\[
\langle T(p, \tilde{\sigma}) | (q, v) \rangle_{L^2(\Omega)^3 \oplus L^2(\Omega)^3} = -\langle \tilde{\sigma} + \alpha \beta \text{trace} p | (\text{grad} v) \rangle_{L^2(\Omega)^3 \otimes L^2(\Omega)^3} - \langle \alpha \beta \text{trace} p | (\text{grad} v) \rangle_{L^2(\Omega)^3} - \langle \alpha \beta \text{trace} p | (\text{grad} v) \rangle_{L^2(\Omega)^3},
\]
Since \( q - \alpha \beta v \in D(\text{div}) \) we conclude
\[
\langle \text{grad} p | q - \alpha \beta v \rangle_{L^2(\Omega)^3} = -\langle p | \text{div}(q - \alpha \beta v) \rangle_{L^2(\Omega)}
\]
and hence, using Lemma 3.25 (b)
\[
\langle T(p, \tilde{\sigma}) | (q, v) \rangle_{L^2(\Omega)^3 \otimes L^2(\Omega)^3} = -\langle \tilde{\sigma} + \alpha \beta \text{trace} p | (\text{grad} v) \rangle_{L^2(\Omega)^3 \otimes L^2(\Omega)^3} - \langle \alpha \beta \text{trace} p | (\text{grad} v) \rangle_{L^2(\Omega)^3} - \langle \alpha \beta \text{trace} p | (\text{grad} v) \rangle_{L^2(\Omega)^3}.
\]
Again by Lemma 3.25 (b) we get
\[
\langle \beta \text{trace} p | (\text{grad} v) \rangle_{L^2(\Omega)^3} = \langle p | \beta \text{trace} \text{grad} v \rangle_{L^2(\Omega)} = \langle p | \beta \text{div} v \rangle_{L^2(\Omega)}
\]
\(^5\)Keep in mind that \( \beta \) was assumed to be real-valued.
and by applying Lemma 3.27 (b) it follows that
\[
\langle p | \operatorname{div}(\beta v) - ((\operatorname{grad}\beta) \cdot v + \beta \operatorname{div} v) \rangle_{L^2(\Omega)} = 0.
\]
So we conclude
\[
\langle T(p, \tilde{\sigma})(q, v) \rangle_{L^2(\Omega) \oplus L^2(\Omega)^3} = -\langle \tilde{\sigma} | \operatorname{Grad} v \rangle_{L^2(\Omega)^3} - \langle p | \operatorname{div} q \rangle_{L^2(\Omega)} = -\langle (p, \tilde{\sigma}) | \tilde{T}(q, v) \rangle_{L^2(\Omega) \oplus L^2(\Omega)^3 \times 3},
\]
which shows \(-\tilde{T} \subseteq T^*\).

Now let \((q, v) \in D(T^*)\). We denote the image of \((q, v)\) under \(T^*\) by \((T_1^*(q, v), T_2^*(q, v)) \in L^2(\Omega) \oplus H^1_{sym}(\Omega)\). Then we obtain
\[
\langle \operatorname{grad} p | q \rangle_{L^2(\Omega)^3} + \langle \operatorname{Div} \tilde{\sigma} | v \rangle_{L^2(\Omega)^3} = \langle T(p, \tilde{\sigma})(q, v) \rangle_{L^2(\Omega) \oplus L^2(\Omega)^3} = \langle p | T_1^*(q, v) \rangle_{L^2(\Omega)} + \langle (p, \tilde{\sigma}) | T_2^*(q, v) \rangle_{L^2(\Omega)^3 \times 3}
\]
for each \((p, \tilde{\sigma}) \in D(T)\). Since \((p, 0) \in D(T)\) for every \(p \in C^\infty_c(\Omega)\), we conclude that \(q \in D(\operatorname{div})\) and \(T_1^*(q, v) = -\operatorname{div} v\). Also \((0, \tilde{\sigma}) \in D(T)\) for every \(\tilde{\sigma} \in C^\infty_{c,sym}(\Omega)\) and so it follows that \(v \in D(\operatorname{Grad})\) with \(T_2^*(q, v) = -\operatorname{Grad} v\).

So it leaves to prove that \(q - \alpha \beta v \in D(\operatorname{div})\). For doing so, we fix an arbitrary \(p \in D(\operatorname{grad})\) and define \(\tilde{\sigma} = -\alpha \beta \operatorname{trace}^* p\). By Lemma 3.27 (a) we know that \(\beta p \in D(\operatorname{grad})\) and so it follows by Lemma 3.25 (a) that \(-\operatorname{trace}^* p \in D(\operatorname{Div})\). Thus, we obtain that \(\tilde{\sigma} \in D(\operatorname{Div})\) and moreover, \((p, \tilde{\sigma}) \in D(T)\). We calculate
\[
\langle \operatorname{grad} p | q - \alpha \beta v \rangle_{L^2(\Omega)^3} = \langle \operatorname{grad} p | q \rangle_{L^2(\Omega)^3} - \alpha \langle \operatorname{grad} p | \beta v \rangle_{L^2(\Omega)^3} + \langle \operatorname{Div} \tilde{\sigma} | v \rangle_{L^2(\Omega)^3}
\]
\[
= \langle T(p, \tilde{\sigma})(q, v) \rangle_{L^2(\Omega)^3} - \alpha \langle \operatorname{grad} p | \beta v \rangle_{L^2(\Omega)^3} - \langle \operatorname{Div} \tilde{\sigma} | v \rangle_{L^2(\Omega)^3}
\]
\[
= -\langle p | \operatorname{div} q \rangle_{L^2(\Omega)} + \langle (p, \tilde{\sigma}) | \operatorname{Grad} v \rangle_{L^2(\Omega)^3} - \alpha \langle \operatorname{grad} p | \beta v \rangle_{L^2(\Omega)^3}
\]
\[
= -\langle p | \operatorname{div} q \rangle_{L^2(\Omega)} + \alpha \langle \beta \operatorname{trace}^* p | \operatorname{Grad} v \rangle_{H^1_{sym}} - \alpha \langle \operatorname{grad} p | \beta v \rangle_{L^2(\Omega)^3}
\]
\[
+ \alpha (\operatorname{grad} \beta p) \langle v | L^2(\Omega) \rangle^3
\]
\[
= -\langle p | \operatorname{div} q \rangle_{L^2(\Omega)} + \alpha \langle p | \beta \operatorname{div} v \rangle_{L^2(\Omega)^3} - \alpha \langle \beta \operatorname{grad} p | v \rangle_{L^2(\Omega)^3}
\]
\[
+ \alpha (\operatorname{grad} \beta p) \langle v | L^2(\Omega) \rangle^3
\]
\[
= -\langle p | \operatorname{div} q \rangle_{L^2(\Omega)} + \alpha (\operatorname{grad} \beta p) \langle v | L^2(\Omega) \rangle^3
\]
\[
+ \alpha (\operatorname{grad} \beta p) \langle v | L^2(\Omega) \rangle^3
\]
\[
= -\langle p | \operatorname{div} q \rangle_{L^2(\Omega)} + \alpha (\operatorname{grad} \beta p) \langle v | L^2(\Omega) \rangle^3
\]
\[
= -\langle p | \operatorname{div} q \rangle_{L^2(\Omega)} + \alpha (\operatorname{grad} \beta p) \langle v | L^2(\Omega) \rangle^3
\]
Here we have used the results of Lemma 3.25 and Lemma 3.27. This shows that \(q - \alpha \beta v \in D(\operatorname{div})\) and so \((q, v) \in D(T\tilde{*})\). Since we have already seen that \(T\tilde{*}(q, v) = (-\operatorname{div} q, -\operatorname{Grad} q)\), we conclude \(T\tilde{*} \subseteq -\tilde{T}\).

By Lemma 3.29 \(T\tilde{*}\) is closed and hence, the skew-selfadjointness of
\[
\begin{pmatrix}
0 & 0 & T \\
0 & 0 & 0 \\
\tilde{T} & 0 & 0
\end{pmatrix}
\]
follows immediately. \(\square\)
Summarizing, we have shown that $-A$ and $B$ are maximal monotone relations. However, it is left to ensure the maximal monotonicity of $-A + B$. Of course this can not hold in general and so we have to assume further properties on our relation $B$, to apply the results of Section 1.3. The easiest assumption would be the boundedness of $B$ and apply Proposition 1.22 or assume a relative boundedness with respect to the operator $A$ (see Proposition 1.24). In [50] we find an assumption of the form $(M + L)^{-1}$ is coercive. However, this condition implies that $L$ and hence, $B$ is bounded due to the following Lemma.

**Lemma 3.31.** Let $H$ be a Hilbert space and $A \subseteq H \oplus H$. Furthermore let $T : H \rightarrow H$ be bounded and linear such that $(T + A)^{-1}$ is coercive. Then $A$ is bounded.

**Proof.** We prove the statement by contradiction. Assume that $A$ is not bounded, i.e. there exists a bounded subset $M \subseteq H$ such that $A[M]$ is unbounded. We may choose a sequence $((u_n, v_n))_{n \in \mathbb{N}} \in A^\mathbb{N}$ such that $u_n \in M$ for each $n \in \mathbb{N}$ and $|v_n| \rightarrow \infty$ as $n \rightarrow \infty$. It follows that $(u_n, Tu_n + v_n) \in T + A$ or equivalently $(Tu_n + v_n, u_n) \in (T + A)^{-1}$ for each $n \in \mathbb{N}$. For $x \in H$ we estimate, by using the Cauchy-Schwarz-Inequality

\[ \limsup_{n \rightarrow \infty} \frac{\text{Re}\langle Tu_n + v_n - x|u_n\rangle}{|Tu_n + v_n|} \leq \sup_{n \in \mathbb{N}} |u_n| < \infty. \]

This contradicts the coercitivity of $(T + A)^{-1}$, since $|Tu_n + v_n| \geq |v_n| - \|T\| \sup_{k \in \mathbb{N}} |u_k| \rightarrow \infty$ as $n \rightarrow \infty$. Thus, $A$ is bounded.

We summarize our findings under the assumption that $L$ is bounded in the following theorem.

**Theorem 3.32.** Let $M_0, M_1, U$ be the operators, defined in Remark 3.18. Let $\tilde{A}$ be one of the operators in Remark 3.23 or in Proposition 3.30 and $L \subseteq H_{\text{sym}}(\Omega) \oplus H_{\text{sym}}(\Omega)$ be maximal monotone and bounded. We set

\[
B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

and assume $D(A) \cap [H]B \neq \emptyset$, where $H = L_2(\Omega) \oplus H_{\text{sym}}(\Omega) \oplus L_2(\Omega) \oplus L_2(\Omega)^3 \oplus L_2(\Omega)^3$ and $A = U^*\tilde{A}U$. Then there exists $\nu_0 \in \mathbb{R}_{>0}$, such that for all $\nu \geq \nu_0$ the following holds:

---

*A relation $A \subseteq H \oplus H$ is called coercive, if there exists $x \in H$ such that for every sequence $((u_n, v_n))_{n \in \mathbb{N}} \in A^\mathbb{N}$ with $|u_n| \rightarrow \infty$ as $n \rightarrow \infty$:

\[ \frac{\text{Re}(u_n - x|v_n)}{|u_n|} \rightarrow \infty \quad (n \rightarrow \infty). \]
(a) For each $F \in H_{\nu,0} \otimes H$ and $V_0 \in D(A) \cap [H]B$, there exists a unique $V \in H_{\nu,0} \otimes H$ satisfy

$$(V,F) \in \partial_{\nu,0}M_0(\cdot - V_0) + M_1 + (A + B)_{\nu}.$$ 

Furthermore the solution $V$ depends continuously on $F$.

(b) If additionally $V_0 \in [H](A + B)^*$, then $V$ depends continuously on $F$ and $V_0$.

(c) The solution operator is causal.
4. Conclusion and Summary

The theory developed in this thesis covers a large field of applications in mathematical physics. The general form of the differential inclusion

\[(u, f) \in \partial_{\nu,0} M(\partial_{\nu,0}^{-1}) + A_{\nu}\]

can be found in the classical linear partial differential equations, like the wave equation, the heat conduction, Maxwell’s equations and the equations of elasticity. In all of these examples, the relation \(A\) is (under suitable boundary conditions) a skew-selfadjoint operator (see [43]). The advantage of studying inclusions, instead of equations lies in its application to hysteresis phenomena as well as to nonlinear ordinary differential equations, even with discontinuous right hand sides (by using the so called Filippov-modification [21]). So, for instance the equation

\[\partial_{\nu,0} u + \text{sgn}(u) = f\]

is in general not solvable, but by extending the signum function \(\text{sgn}\) to a maximal monotone relation, given by

\[(x, y) \in \text{sgn} \iff \begin{cases} y = 1 & \text{if } x > 0, \\ y \in [-1, 1] & \text{if } x = 0, \\ y = -1 & \text{if } x < 0, \end{cases}\]

we obtain well-posedness of the differential inclusion

\[(u, f) \in \partial_{\nu,0} + \text{sgn}.\]

This strategy of modifying discontinuous right-hand sides can also be found in the study of switched dynamical systems (cf. [34]). Also ordinary differential equations of the form

\[\partial_{\nu,0} u = F(u),\]

where \(F\) is a Lipschitz-continuous mapping are covered, since \(F + |F|_{\text{Lip}}\) becomes maximal monotone. This very general type of an ordinary differential equation also covers equations with memory effects (see [27]).

A further mentionable advantage lies in the very general structure of the material laws introduced in Section 2.6. These types of operators allow us to represent equations and inclusions with convolution terms (see [53]), integro-differential equations and inclusions, neutral differential equations and inclusions as well as inclusions with delay.

Starting from the general setting developed in the previous chapters, questions of stability should be addressed in future. Furthermore, in view of [53], limiting processes of evolutionary inclusions should be studied to work out a unified concept of homogenization of such inclusions.
A. Tensor products

In this part of the appendix, we introduce the needed theory of tensor products. For this topic we refer to [54, p. 47 and p. 262], where the case of selfadjoint operators is studied. However we want to deal a more general case, where the operators do not have to be selfadjoint, but only closable (c.f. [44, 42]).

A.1. Tensor products of Hilbert spaces

Throughout let $H_0, H_1$ be complex Hilbert spaces.

**Definition A.1.** For $x \in H_0, y \in H_1$ we define a functional on $H_0 \times H_1$ by

$$x \otimes y : H_0 \times H_1 \rightarrow \mathbb{C}$$

$$(\phi, \psi) \mapsto \langle x|\phi\rangle_{H_0}\langle y|\psi\rangle_{H_1}.$$  

This functional is bilinear and continuous. We define a linear structure on such elements by setting

$$\left(\sum_{i=1}^{n} \alpha_i (x_i \otimes y_i) \right)(\phi, \psi) := \sum_{i=1}^{n} \alpha_i^* (x_i \otimes y_i)(\phi, \psi)$$

for $x_i \in H_0, y_i \in H_1, \alpha_i \in \mathbb{C}$ with $i \in \{1, \ldots, n\}$ and $n \in \mathbb{N}$ and obtain again a bilinear, continuous functional. For subsets $V_0 \subseteq H_0, V_1 \subseteq H_1$ we define the *algebraic tensor product* by

$$V_0 \overset{\text{a}}{\otimes} V_1 := \text{Lin}\{x \otimes y | x \in V_0, y \in V_1\},$$

where the linear hull is taken with respect to the linear structure defined above. For $x, u \in H_0$ and $y, v \in H_1$ we define

$$\langle x \otimes y | u \otimes v \rangle_\otimes := \langle x|u\rangle_{H_0}\langle y|v\rangle_{H_1} = (x \otimes y)(u, v)$$

and denote by $\langle .| . \rangle_\otimes$ the sesquilinear extension of this mapping to $(H_0 \overset{\text{a}}{\otimes} H_1) \times (H_0 \overset{\text{a}}{\otimes} H_1)$.

**Proposition A.2.** $\langle .| . \rangle_\otimes$ defines an inner product on $H_0 \overset{\text{a}}{\otimes} H_1$.

*Proof.* The sesquilinearity of $\langle .| . \rangle_\otimes$ follows by definition. Let $n, m \in \mathbb{N}$. For $i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}$ let $x_i, r_j \in H_0, y_i, s_j \in H_1, \alpha_i, \beta_j \in \mathbb{C}$. We calculate

$$\left(\sum_{i=1}^{n} \alpha_i (x_i \otimes y_i) \right) \left(\sum_{j=1}^{m} \beta_j (r_j \otimes s_j) \right)_\otimes = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i^* \beta_j \langle x_i|r_j\rangle_{H_0}\langle y_i|s_j\rangle_{H_1}$$
and obtain the symmetry. Also we see that

\[ \sum_{j=1}^{m} \sum_{i=1}^{n} \beta_j^* \alpha_i (r_j | x_i) H_0 \langle s_j | y_i \rangle H_1 \]

which can be rewritten, using the Gramian matrices \( G_0 := (\langle x_i | x_j \rangle)_{i,j \in [1, \ldots, n]} \), 
\( G_1 := (\langle y_i | y_j \rangle)_{i,j \in [1, \ldots, n]} \) and their square roots \( A_0 = (a^0_{i,j})_{i,j \in [1, \ldots, n]} \), \( A_1 = (a^1_{i,j})_{i,j \in [1, \ldots, n]} \), as

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i^* \alpha_j \langle x_i | x_j \rangle H_0 \langle y_i | y_j \rangle H_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i^* \alpha_j \sum_{k_0=1}^{n} a^0_{k_0} a^0_{k_0} \sum_{k_1=1}^{n} a^1_{k_1} a^1_{k_1} \\
= \sum_{k_0=1}^{n} \sum_{k_1=1}^{n} \sum_{i=1}^{n} a^*(a^0_{k_0}) (a^1_{k_1})^* \sum_{j=1}^{n} \alpha_j a^0_{k_0} a^1_{k_1} \\
= \sum_{k_0,k_1=1}^{n} \sum_{j=1}^{n} \alpha_j a^0_{k_0} a^1_{k_1} |^2 \\
\geq 0.
\]

Thus, we obtain the positivity of \( \langle | \rangle \otimes \).

Next we prove that \( \langle | \rangle \otimes \) is a mapping. Let \( \sum_{i=1}^{n} \alpha_i (x_i \otimes y_i) = \sum_{j=1}^{m} \beta_j (r_j \otimes s_j) \in H_0 \otimes H_1 \). We get that

\[
\sum_{i=1}^{n} \alpha_i (x_i \otimes y_i) | \phi \otimes \psi \rangle - \sum_{j=1}^{m} \beta_j (r_j \otimes s_j) | \phi \otimes \psi \rangle \\
= \sum_{i=1}^{n} \alpha_i^* (x_i \otimes y_i) (\phi, \psi) - \sum_{j=1}^{m} \beta_j^* (r_j \otimes s_j) (\phi, \psi) \\
= \left( \sum_{i=1}^{n} \alpha_i (x_i \otimes y_i) \right) (\phi, \psi) - \left( \sum_{j=1}^{m} \beta_j (r_j \otimes s_j) \right) (\phi, \psi) \\
= 0
\]

for all \( \phi \otimes \psi \in [H_0] \otimes [H_1] \). Using the linearity in the second argument we conclude

\[
\sum_{i=1}^{n} \alpha_i (x_i \otimes y_i) |u\rangle \otimes \sum_{j=1}^{m} \beta_j (r_j \otimes s_j) |u\rangle \\
= \sum_{j=1}^{m} \beta_j (r_j \otimes s_j) |u\rangle \otimes \sum_{i=1}^{n} \alpha_i (x_i \otimes y_i) |u\rangle.
\]
for all \( u \in H_0 \otimes H_1 \) and by the symmetry we also obtain
\[
\left\langle u \mid \sum_{i=1}^{n} \alpha_i (x_i \otimes y_i) \right\rangle_{\otimes} = \left\langle u \mid \sum_{j=1}^{m} \beta_j (r_j \otimes s_j) \right\rangle_{\otimes}
\]
for all \( u \in H_0 \otimes H_1 \), which shows the right-uniqueness of \( \left\langle \cdot, \cdot \right\rangle_{\otimes} \). So it is left to show the definiteness of \( \left\langle \cdot, \cdot \right\rangle_{\otimes} \). Let \( \sum_{i=1}^{n} \alpha_i (x_i \otimes y_i) \in H_0 \otimes H_1 \) with
\[
\left\langle \sum_{i=1}^{n} \alpha_i (x_i \otimes y_i) \mid \sum_{i=1}^{n} \alpha_i (x_i \otimes y_i) \right\rangle_{\otimes} = 0.
\]
Then for all \( \phi \in H_0, \psi \in H_1 \) we get by the Cauchy-Schwarz-Inequality for symmetric sesquilinear forms:
\[
\left| \left\langle \sum_{i=1}^{n} \alpha_i (x_i \otimes y_i) \mid (\phi, \psi) \right\rangle_{\otimes} \right| = \left| \left\langle \sum_{i=1}^{n} \alpha_i (x_i \otimes y_i) \mid \phi \otimes \psi \right\rangle_{\otimes} \right| \leq \left( \sum_{i=1}^{n} \alpha_i (x_i \otimes y_i) \right) \left( \sum_{i=1}^{n} \alpha_i (x_i \otimes y_i) \right) \sqrt{\left\langle \phi \otimes \psi \mid \phi \otimes \psi \right\rangle_{\otimes}} = 0.
\]
Hence, \( \sum_{i=1}^{n} \alpha_i (x_i \otimes y_i) = 0 \). \( \Box \)

**Definition A.3.** We define the **tensorproduct** \( H_0 \otimes H_1 \) as the completion of \( H_0 \otimes H_1 \) with respect to the inner product \( \left\langle \cdot, \cdot \right\rangle_{\otimes} \). Thus, \( H_0 \otimes H_1 \) is a Hilbert space and we denote its inner product by \( \left\langle \cdot, \cdot \right\rangle_{H_0 \otimes H_1} \).

**Proposition A.4.** Let \( S_0 \subseteq H_0 \) and \( S_1 \subseteq H_1 \) be total\(^1\) in \( H_0 \) and \( H_1 \) respectively. Then \( S_0 \otimes S_1 \) is dense in \( H_0 \otimes H_1 \).

**Proof.** Let \( x \in H_0, y \in H_1 \). Then there exist sequences \( \{x_n\}_{n \in \mathbb{N}} \in (\text{Lin} S_0)^\mathbb{N} \) and \( \{y_n\}_{n \in \mathbb{N}} \in (\text{Lin} S_1)^\mathbb{N} \) such that \( x_n \to x \) in \( H_0 \) and \( y_n \to y \) in \( H_1 \) as \( n \to \infty \). At first we show that for all \( n \in \mathbb{N} \) we have \( x_n \otimes y_n \in S_0 \otimes S_1 \). Let \( n \in \mathbb{N} \). Then we find \( k, m \in \mathbb{N} \) and elements \( s_1^0, \ldots, s_k^0 \in S_0 \) as well as \( s_1^1, \ldots, s_m^1 \in S_1 \) and complex numbers \( \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m \in \mathbb{C} \) such that
\[
x_n = \sum_{i=1}^{k} \alpha_i s_i^0, \quad y_n = \sum_{j=1}^{m} \beta_j s_j^1.
\]

\(^1\)\( S \subseteq H \) is called **total**, if \( \text{Lin} S \) is dense in \( H \).
Hence,
\[ x_n \otimes y_n = \left( \sum_{i=1}^{k} \alpha_i s_i^0 \right) \otimes \left( \sum_{j=1}^{m} \beta_j s_j^1 \right) = \sum_{i=1}^{k} \sum_{j=1}^{m} \alpha_i \beta_j (s_i^0 \otimes s_j^1) \in S_0 \hat{\otimes} S_1. \]

Moreover, we estimate
\[
|x_n \otimes y_n - x \otimes y|_{H_0 \hat{\otimes} H_1} \leq \left| x_n \otimes y_n - x \otimes y_{H_0 \otimes H_1} \right| + \left| x \otimes y_n - x \otimes y \right|_{H_0 \otimes H_1} \\
= \left| (x_n - x) \otimes y_n \right|_{H_0 \otimes H_1} + \left| x \otimes (y_n - y) \right|_{H_0 \otimes H_1} \\
= \left| x_n - x \right|_{H_0} \left| y_n \right|_{H_1} + \left| x \right|_{H_0} \left| y_n - y \right|_{H_1} \\
\rightarrow 0 \quad (n \to \infty).
\]

Thus, decomposable elements of \( H_0 \otimes H_1 \) can be approximated by elements of \( S_0 \hat{\otimes} S_1 \). We now show that each element of \( H_0 \otimes H_1 \) can be approximated by elements of \( S_0 \hat{\otimes} S_1 \). Let \( n \in \mathbb{N} \) and \( x_1, \ldots, x_n \in H_0, y_1, \ldots, y_n \in H_1, \alpha_1, \ldots, \alpha_n \in \mathbb{C} \setminus \{0\} \). Then, according to what we have shown above, we find for every \( \varepsilon > 0 \), \( i \in \{1, \ldots, n\} \) an element \( u_i \in S_0 \hat{\otimes} S_1 \) such that
\[
\left| x_i \otimes y_i - u_i \right|_{H_0 \otimes H_1} < \left( \sum_{i=1}^{n} \left| \alpha_i \right| \right)^{-1} \varepsilon.
\]

Then \( \sum_{i=1}^{n} \alpha_i u_i \in S_0 \hat{\otimes} S_1 \) and
\[
\left| \sum_{i=1}^{n} \alpha_i (x_i \otimes y_i) - \sum_{i=1}^{n} \alpha_i u_i \right|_{H_0 \otimes H_1} \leq \sum_{i=1}^{n} |\alpha_i| \left| x_i \otimes y_i - u_i \right|_{H_0 \otimes H_1} < \varepsilon.
\]

Since \( H_0 \hat{\otimes} H_1 \) is dense in \( H_0 \otimes H_1 \), this yields the assertion. \( \square \)

**Corollary A.5.** Let \( O_0 \subseteq H_0, O_1 \subseteq H_1 \) be two complete orthonormal sets in \( H_0 \) and \( H_1 \) respectively. Then
\[
[O_0] \otimes [O_1] = \{ u \otimes v \mid u \in O_0, v \in O_1 \}
\]
is a complete orthonormal set in \( H_0 \hat{\otimes} H_1 \).

**Proof.** According to Proposition A.4 \( O_0 \hat{\otimes} O_1 = \text{Lin} ([O_0] \otimes [O_1]) \) is dense in \( H_0 \hat{\otimes} H_1 \). Let now \( u \otimes v, x \otimes y \in [O_0] \otimes [O_1] \). Then
\[
\langle u \otimes v | x \otimes y \rangle_{H_0 \hat{\otimes} H_1} = \langle u | x \rangle_{H_0} \langle v | y \rangle_{H_1} = \begin{cases} 
1 & \text{if } u = x, v = y, \\
0 & \text{otherwise}.
\end{cases}
\]

Thus, \( [O_0] \otimes [O_1] \) is orthonormal in \( H_0 \otimes H_1 \). \( \square \)
Example A.6. Let $H$ be a complex Hilbert space. Then $L_2(\mathbb{R}) \otimes H = L_2(\mathbb{R}, H)$ in the sense of a unitary mapping. We begin to define the mapping on elements of the form $\chi_I \otimes \phi$ for $I \subseteq \mathbb{R}$ a bounded interval and $\phi \in H$. We define

$$U(\chi_I \otimes \phi) := (x \mapsto \chi_I(x)\phi).$$

This function is well defined and isometric, since

$$\int |\chi_I(x)\phi|_H^2\, dx = \int I |\phi|_{L_2(\mathbb{R})}^2\, dx = |\chi_I|_{L_2(\mathbb{R})}^2 = |\chi_I \otimes \phi|_{L_2(\mathbb{R}) \otimes H}^2.$$

We extend this mapping linearly to $I(\mathbb{R}) \otimes H$, where $I(\mathbb{R})$ denotes the indicator functions of intervals in $\mathbb{R}$. We have to prove that this extension stays a mapping. For this, let $0 = \sum_{i=1}^n \alpha_i \chi_{I_i} \otimes \phi_i \in I(\mathbb{R}) \otimes H$. Then we get for all intervals $J \subseteq \mathbb{R}$ and $\psi \in H$

$$\left\langle \sum_{i=1}^n \alpha_i \chi_{I_i} \phi_i | \chi_J \psi \right\rangle_{L_2(\mathbb{R}, H)} = \sum_{i=1}^n \alpha_i^* \int_{\mathbb{R}} \langle \chi_{I_i}(x) \phi_i | \chi_J(x) \psi \rangle_H\, dx$$

$$= \sum_{i=1}^n \alpha_i^* \int_{I_i \cap J} dx \langle \phi_i | \psi \rangle_H$$

$$= \sum_{i=1}^n \alpha_i^* \langle \chi_{I_i} | \chi_J \rangle_{L_2(\mathbb{R})} \langle \phi_i | \psi \rangle_H$$

$$= \sum_{i=1}^n \alpha_i^* (\chi_{I_i} \otimes \phi_i)(\chi_J, \psi)$$

$$= \left( \sum_{i=1}^n \alpha_i (\chi_{I_i} \otimes \phi_i) \right) (\chi_J, \psi)$$

$$= 0.$$

Since the set $\{\chi_J \psi | J \subseteq \mathbb{R} \text{ interval}, \psi \in H\}$ is total in $L_2(\mathbb{R}, H)$, we conclude $\sum_{i=1}^n \alpha_i \chi_{I_i} \phi_i = 0$. Thus, we have proved the right uniqueness of $U$. Now we show that $U$ is an isometry. Let $u := \sum_{i=1}^n \alpha_i \chi_{I_i} \otimes \phi_i \in I(\mathbb{R}) \otimes H$ and assume without loss of generality that the intervals $I_i$ are pairwise disjoint. Then on the one hand

$$|u|_{L_2(\mathbb{R}) \otimes H}^2 = \left\langle \sum_{i=1}^n \alpha_i \chi_{I_i} \otimes \phi_i | \sum_{j=1}^n \alpha_j \chi_{I_j} \otimes \phi_j \right\rangle_{L_2(\mathbb{R}) \otimes H}$$

$$= \sum_{i,j=1}^n \alpha_i^* \alpha_j \langle \chi_{I_i} | \chi_{I_j} \rangle_{L_2(\mathbb{R})} \langle \phi_i | \phi_j \rangle_H$$

$$= \sum_{i=1}^n |\alpha_i|^2 |\chi_{I_i}|_{L_2(\mathbb{R})}^2 |\phi_i|_H^2.$$
and on the other hand

\[ \int_{\mathbb{R}} |(U u)(x)|^2_H \, dx = \int_{\mathbb{R}} \left| \sum_{i=1}^{n} \alpha_i \chi_{I_i}(x) \phi_i \right|^2_H \, dx \]
\[ = \sum_{i=1}^{n} \int_{I_i} |\alpha_i|^2 |\phi_i|^2_H \, dx \]
\[ = \sum_{i=1}^{n} |\alpha_i|^2 |\chi_{I_i}|^2_{L^2(\mathbb{R})} |\phi_i|^2_H \]
\[ = |u|^2_{L^2(\mathbb{R}) \otimes H}. \]

Thus, U is an isometric, linear operator. It is left to prove that U has a dense range. For this purpose let \((x \mapsto \sum_{i=1}^{n} \alpha_i \chi_{I_i}(x) \phi_i) \in L^2(\mathbb{R}, H)\) be a simple function. Then

\[ U \left( \sum_{i=1}^{n} \alpha_i \chi_{I_i} \otimes \phi_i \right) = \sum_{i=1}^{n} \alpha_i U(\chi_{I_i} \otimes \phi_i) \]
\[ = (x \mapsto \sum_{i=1}^{n} \alpha_i \chi_{I_i}(x) \phi_i). \]

By the density of simple functions in \(L^2(\mathbb{R}; H)\) the assertion follows. Since \(I(\mathbb{R}) \overset{\phi}{\otimes} H\) is dense in \(L^2(\mathbb{R}) \otimes H\) by Proposition A.4, we can extend U to a unitary, linear operator from \(L^2(\mathbb{R}) \otimes H\) into \(L^2(\mathbb{R}, H)\).

At last we want to show that the tensor product of Hilbert spaces is associative.

**Proposition A.7.** Let \(H_0, H_1, H_2\) be complex Hilbert spaces. Then

\( (H_0 \otimes H_1) \otimes H_2 = H_0 \otimes (H_1 \otimes H_2) =: H_0 \otimes H_1 \otimes H_2 \)

in the sense of a unitary mapping.

**Proof.** For \(\phi \in H_0, \psi \in H_1, \chi \in H_2\) we set

\[ U((\phi \otimes \psi) \otimes \chi) := \phi \otimes (\psi \otimes \chi). \]

We extend this mapping to \((H_0 \overset{\phi}{\otimes} H_1) \overset{\phi}{\otimes} H_2\) by

\[ \sum_{j=1}^{m} \beta_j \left( \sum_{i=1}^{n_i} \alpha_i^j (x_i^j \otimes y_i^j) \right) \otimes z_j \quad \Rightarrow \quad \sum_{j=1}^{m} \beta_j \sum_{i=1}^{n_i} \alpha_i^j (x_i^j \otimes (y_i^j \otimes z_j)). \]

At first we prove that this extension stays right-unique.
Let \( \phi \in (H_0 \otimes H_1) \otimes H_2 \) with
\[
\phi = \sum_{j=1}^{m} \beta_j \left( \sum_{i=1}^{n_j} \alpha_i^j (x_i^j \otimes y_i^j) \right) \otimes z_j,
\]
\[
= \sum_{j=1}^{p} \delta_j \left( \sum_{i=1}^{k_j} \gamma_i^j (u_i^j \otimes v_i^j) \right) \otimes w_j.
\]

Then for all \( a \in H_0, b \in H_1, c \in H_2 \):
\[
U \left( \sum_{j=1}^{m} \beta_j \left( \sum_{i=1}^{n_j} \alpha_i^j (x_i^j \otimes y_i^j) \right) \otimes z_j \right) (a, b \otimes c) = \left( \sum_{j=1}^{m} \beta_j \left( \sum_{i=1}^{n_j} \alpha_i^j (x_i^j \otimes y_i^j \otimes z_j) \right) \right) (a, b \otimes c)
\]
\[
= \sum_{j=1}^{m} \beta_j^* \sum_{i=1}^{n_j} \left( \alpha_i^j \right)^* (x_i^j | a \rangle_{H_0} \langle y_i^j | b \rangle_{H_1} \langle z_j | c \rangle_{H_2})_{H_0 \otimes H_1 \otimes H_2}
\]
\[
= \sum_{j=1}^{m} \beta_j^* \sum_{i=1}^{n_j} \left( \alpha_i^j \right)^* (x_i^j \otimes y_i^j | a \otimes b \rangle_{H_0 \otimes H_1} \langle z_j | c \rangle_{H_2})_{H_0 \otimes H_1 \otimes H_2}
\]
\[
= \sum_{j=1}^{m} \beta_j^* \left( \sum_{i=1}^{n_j} \alpha_i^j (x_i^j \otimes y_i^j) \right) (a \otimes b \otimes c)_{H_0 \otimes H_1 \otimes H_2}
\]
\[
= \phi (a \otimes b, c).
\]

The same augmentation yields
\[
U \left( \sum_{j=1}^{p} \delta_j \left( \sum_{i=1}^{k_j} \gamma_i^j (u_i^j \otimes v_i^j) \right) \otimes w_j \right) (a, b \otimes c) = \phi (a \otimes b, c).
\]

Since these functionals are bilinear, continuous and since \( H_0 \otimes (H_1 \otimes H_2) \) is dense in \( H_0 \otimes (H_1 \otimes H_2) \), we conclude the right-uniqueness of \( U \). Moreover, \( U \) is linear, since for \( \kappa \in \mathbb{C}, \phi, \psi \in (H_0 \otimes H_1) \otimes H_2 \) with
\[
\phi = \sum_{j=1}^{m} \beta_j \left( \sum_{i=1}^{n_j} \alpha_i^j (x_i^j \otimes y_i^j) \right) \otimes z_j,
\]
\[
\psi = \sum_{j=1}^{p} \delta_j \left( \sum_{i=1}^{k_j} \gamma_i^j (u_i^j \otimes v_i^j) \right) \otimes w_j.
\]
Next we prove that 

\[ \kappa \phi + \psi = \kappa \sum_{j=1}^{m} \beta_j \left( \sum_{i=1}^{n_j} \alpha^j_i (x^j_i \otimes y^j_i) \right) \otimes z_j + \sum_{j=1}^{p} \delta_j \left( \sum_{i=1}^{k_j} \gamma^j_i (u^j_i \otimes v^j_i) \right) \otimes w_j \]

\[ = \sum_{j=1}^{m+p} \zeta_j \left( \sum_{i=1}^{l_j} \eta^j_i (\theta^j_i \otimes \lambda^j_i) \right) \otimes \mu_j \]

where

\[ \zeta_j := \begin{cases} \kappa \beta_j & \text{if } j \in \{1, \ldots, m\}, \\ \delta_{j-m} & \text{if } j \in \{m+1, \ldots, m+p\}, \end{cases} \]

\[ l_j := \begin{cases} n_j & \text{if } j \in \{1, \ldots, m\}, \\ k_{j-m} & \text{if } j \in \{m+1, \ldots, m+p\}, \end{cases} \]

\[ \eta^j_i := \begin{cases} \alpha^j_i & \text{if } j \in \{1, \ldots, m\}, \\ \gamma^j_{i-m} & \text{if } j \in \{m+1, \ldots, m+p\}, \end{cases} \]

\[ \theta^j_i := \begin{cases} x^j_i & \text{if } j \in \{1, \ldots, m\}, \\ u^j_{i-m} & \text{if } j \in \{m+1, \ldots, m+p\}, \end{cases} \]

\[ \lambda^j_i := \begin{cases} y^j_i & \text{if } j \in \{1, \ldots, m\}, \\ v^j_{i-m} & \text{if } j \in \{m+1, \ldots, m+p\}, \end{cases} \]

\[ \mu_j := \begin{cases} z_j & \text{if } j \in \{1, \ldots, m\}, \\ w_{j-m} & \text{if } j \in \{m+1, \ldots, m+p\}. \end{cases} \]

Then we compute

\[ U(\kappa \phi + \psi) = \sum_{j=1}^{m+p} \zeta_j \sum_{i=1}^{l_j} \eta^j_i \left( \theta^j_i \otimes (\lambda^j_i \otimes \mu_j) \right) \]

\[ = \sum_{j=1}^{m} \zeta_j \sum_{i=1}^{l_j} \eta^j_i \left( \theta^j_i \otimes (\lambda^j_i \otimes \mu_j) \right) + \sum_{j=m+1}^{m+p} \zeta_j \sum_{i=1}^{l_j} \eta^j_i \left( \theta^j_i \otimes (\lambda^j_i \otimes \mu_j) \right) \]

\[ = \kappa \sum_{j=1}^{m} \beta_j \sum_{i=1}^{n_j} \alpha^j_i \left( x^j_i \otimes (y^j_i \otimes z_j) \right) + \sum_{j=1}^{p} \delta_j \sum_{i=1}^{k_j} \gamma^j_i \left( u^j_i \otimes (v^j_i \otimes w_j) \right) \]

\[ = \kappa U(\phi) + U(\psi). \]

Next we prove that \( U \) is an isometry. Let \( \phi = \sum_{j=1}^{m} \beta_j \left( \sum_{i=1}^{n_j} \alpha^j_i (x^j_i \otimes y^j_i) \right) \otimes z_j \in (H_0 \otimes H_1) \otimes H_2. \) Then

\[ |U \phi|^2_{H_0 \otimes (H_1 \otimes H_2)} = \sum_{j,k=1}^{m} \beta_j \beta_k \sum_{i=1}^{n_j} \sum_{l=1}^{n_k} \left( \alpha^j_i \alpha^k_l \right) \left( x^j_i \otimes (y^j_i \otimes z_j) \right) \left( x^k_l \otimes (y^k_l \otimes z_k) \right) |(H_0 \otimes (H_1 \otimes H_2)) \]
The assertion follows by the density of $H$ unitary operator. Let $A.2. Tensor products of linear operators on Hilbert spaces

Throughout this section let $H_{00}, H_{01}, H_{10}, H_{11}$ be complex Hilbert spaces and $A \subseteq H_{00} \oplus H_{10}, B \subseteq H_{01} \oplus H_{11}$ linear operators.

If we show that the range of $U$ is dense, the proof is finished, since then we can extend $U$ to a unitary operator. Let $\psi := \sum_{j=1}^{m} \beta_j (x_j \otimes (\sum_{i=1}^{n_j} \alpha_i^j (y_i^j \otimes z_i^j))) \in H_0 \overset{a}{\otimes} (H_1 \otimes H_2)$ and define

$\phi := \sum_{j=1}^{m} \beta_j \sum_{i=1}^{n_j} \alpha_i^j ((x_j \otimes y_i^j) \otimes z_i^j) \in (H_0 \overset{a}{\otimes} H_1) \overset{a}{\otimes} H_2.$

Since $U$ is linear we get

$U \phi = \sum_{j=1}^{m} \beta_j \sum_{i=1}^{n_j} \alpha_i^j U((x_j \otimes y_i^j) \otimes z_i^j) = \sum_{j=1}^{m} \beta_j \sum_{i=1}^{n_j} \alpha_i^j (x_j \otimes (y_i^j \otimes z_i^j)) = \sum_{j=1}^{m} \beta_j \left( x_j \otimes (\sum_{i=1}^{n_j} \alpha_i^j (y_i^j \otimes z_i^j)) \right) = \psi.$

The assertion follows by the density of $H_0 \overset{a}{\otimes} (H_1 \overset{a}{\otimes} H_2)$ in $H_0 \otimes (H_1 \otimes H_2)$ according to Proposition A.4.

A.2. Tensor products of linear operators on Hilbert spaces
Proposition A.8. We define

$$(A \hat{\otimes} B)|_{[D(A)] \otimes [D(B)]} : [D(A)] \otimes [D(B)] \subseteq H_{00} \otimes H_{01} \to H_{10} \otimes H_{11}$$

$$(\phi \otimes \psi) \mapsto A\phi \otimes B\psi$$

and set

$$A \hat{\otimes} B := \text{Lin}(A \hat{\otimes} B)|_{[D(A)] \otimes [D(B)]},$$

where the linear hull is taken in the space $$(H_{00} \otimes H_{01}) \oplus (H_{10} \otimes H_{11})$$. Then $A \hat{\otimes} B$ is right-unique.

Proof. Since $A \hat{\otimes} B$ is linear, it suffices to consider the case $$(0, w) \in A \hat{\otimes} B$$. Thus,

$$w = \sum_{i=1}^{n} \alpha_i (A\phi_i \otimes B\psi_i),$$

where

$$\sum_{i=1}^{n} \alpha_i (\phi_i \otimes \psi_i) = 0.$$

Let us first assume that $\{\phi_i \mid i \in \{1, \ldots, n\}\}$ and $\{\psi_i \mid i \in \{1, \ldots, n\}\}$ are linear independent. Then we conclude for all $a \in H_{00}, b \in H_{01}$

$$0 = \left( \sum_{i=1}^{n} \alpha_i (\phi_i \otimes \psi_i) \right) (a, b)$$

$$= \sum_{i=1}^{n} \alpha_i^* \langle \phi_i | a \rangle_{H_{00}} \langle \psi_i | b \rangle_{H_{01}}$$

$$= \left\langle \sum_{i=1}^{n} \alpha_i \langle b | \psi_i \rangle_{H_{01}} \phi_i \right| a \right\rangle_{H_{00}}.$$

Since this holds for every $a \in H_{00}$, it follows that $\sum_{i=1}^{n} \alpha_i \langle b | \psi_i \rangle_{H_{01}} \phi_i = 0$ and by the linear independence we get

$$0 = \sum_{i=1}^{n} \alpha_i \langle b | \psi_i \rangle_{H_{01}} = \left\langle b | \sum_{i=1}^{n} \alpha_i \psi_i \right\rangle_{H_{01}}.$$

By the same reason we observe $\sum_{i=1}^{n} \alpha_i \psi_i = 0$ and hence, $\alpha_i = 0$ for all $i \in \{1, \ldots, n\}$, which shows $w = 0$. If $\{\phi_i \mid i \in \{1, \ldots, n\}\}$ and $\{\psi_i \mid i \in \{1, \ldots, n\}\}$ are not linear independent, we choose linear independent sets $\{x_i \mid i \in \{1, \ldots, k\}\} \subseteq H_{00}$ and $\{y_i \mid i \in \{1, \ldots, m\}\} \subseteq H_{01}$ such that

$$\phi_i = \sum_{j=1}^{k} \beta_j x_j,$$

$$\psi_i = \sum_{j=1}^{m} \gamma_j y_j.$$
for $i \in \{1, \ldots, n\}$. Then we obtain by the linearity of $A$ and $B$

\[
\sum_{i=1}^{n} \alpha_i (A\phi_i \otimes B\psi_i) = \sum_{i=1}^{n} \alpha_i \left( A \left( \sum_{j=1}^{k} \beta_j^i x_j \right) \otimes B \left( \sum_{j=1}^{m} \gamma_j^i y_j \right) \right) 
= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{k} \sum_{l=1}^{m} \beta_j^i \gamma_l^i (Ax_j \otimes By_l)
\]

and

\[
0 = \sum_{i=1}^{n} \alpha_i (\phi_i \otimes \psi_i) = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{k} \sum_{l=1}^{m} \beta_j^i \gamma_l^i (x_j \otimes y_l).
\]

By the linear independence, it follows that $\alpha_i \beta_j^i \gamma_l^i = 0$ for all $i, j, l$. Hence, $w = 0$ and thus, $A \otimes B$ is right-unique. \qed

**Definition A.9.** If $A \otimes B$ is closable, then we define the tensorproduct $A \otimes B$ as the closure of $A \otimes B$.

**Lemma A.10.** Let $A, B$ be densely defined, closable, linear operators. Then $A \otimes B$ is closable and

\[
A \otimes B = A \overline{\otimes} B \subseteq (A^* \otimes B^*)^*.
\]

**Proof.** Let $\xi \in D(A^*) \overline{\otimes} D(B^*)$, i.e. there exist $n \in \mathbb{N}, x_1, \ldots, x_n \in D(A^*), y_1, \ldots, y_n \in D(B^*)$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ such that

\[
\xi = \sum_{i=1}^{n} \alpha_i (x_i \otimes y_i).
\]

Moreover, let $\eta \in D(A) \overline{\otimes} D(B)$, i.e. there exist $m \in \mathbb{N}, u_1, \ldots, u_m \in D(A), v_1, \ldots, v_m \in D(B)$ and $\beta_1, \ldots, \beta_m \in \mathbb{C}$ such that

\[
\eta = \sum_{i=1}^{m} \beta_i (u_i \otimes v_i).
\]

We calculate

\[
\langle (A \overline{\otimes} B)\eta | \xi \rangle_{H_{10} \otimes H_{11}} = \sum_{i=1}^{m} \sum_{j=1}^{n} \beta_i^* \alpha_j \langle Au_i | x_j \rangle_{H_{10}} \langle Bv_i | y_j \rangle_{H_{11}}
= \sum_{i=1}^{m} \sum_{j=1}^{n} \beta_i^* \alpha_j \langle u_i | A^* x_j \rangle_{H_{00}} \langle v_i | B^* y_j \rangle_{H_{01}}
= \langle \eta | (A^* \overline{\otimes} B^*)\xi \rangle_{H_{00} \otimes H_{11}}
\]

and hence $A^* \overline{\otimes} B^* \subseteq (A \overline{\otimes} B)^*$. Thus, we conclude

\[
A \overline{\otimes} B \subseteq \overline{A \otimes B} = A^{**} \overline{\otimes} B^{**} \subseteq (A^* \overline{\otimes} B^*)^*.
\]
Since $A$ and $B$ are closable, $A^\ast \hat{\otimes} B^\ast$ is densely defined by Proposition A.4 and therefore, $(A^\ast \hat{\otimes} B^\ast)^\ast$ is a closed linear operator. This yields the assertion.

Proposition A.11. Let $A$ and $B$ be densely defined and closable. Then

$$A \otimes B = \overline{A} \otimes \overline{B}.$$ 

Proof. Clearly

$$A^\ast \hat{\otimes} B \subseteq \overline{A}^\ast \hat{\otimes} \overline{B} \subseteq A \otimes B$$

and hence $A \otimes B \subseteq \overline{A} \otimes \overline{B}$. Let $x = \sum_{i=1}^n \alpha_i (\xi_i \otimes \eta_i) \in D(\overline{A}^\ast \otimes D(\overline{B})) = D(\overline{A}^\ast \otimes \overline{B})$ with $x \neq 0$ and $\varepsilon > 0$. Then we find for each $i \in \{1, \ldots, n\}$ an element $x_i \in D(A)$ such that

$$|x_i - \xi_i|_{H_{01}} < \frac{\varepsilon}{2\sum_{j=1}^n |\alpha_j||\eta_j|_{H_{01}}}$$

and

$$|Ax_i - \overline{\alpha} \xi_i|_{H_{10}} < \frac{\varepsilon}{2\sum_{j=1}^n |\alpha_j||\overline{B} \eta_j|_{H_{11}}}.$$

Also we find an element $y_i \in D(B)$ for every $i \in \{1, \ldots, n\}$ with

$$|y_i - \eta_i|_{H_{01}} < \frac{\varepsilon}{2\sum_{j=1}^n |\alpha_j||x_j|_{H_{00}}}$$

and

$$|B y_i - \overline{B} \eta_i|_{H_{11}} < \frac{\varepsilon}{2\sum_{j=1}^n |\alpha_j||A x_j|_{H_{10}}}.$$

We set $y := \sum_{i=1}^n \alpha_i (x_i \otimes y_i) \in D(\overline{A}^\ast \otimes B)$ and estimate

$$|y - x|_{H_{00} \otimes H_{01}} = \left| \sum_{i=1}^n \alpha_i (x_i \otimes y_i) - \sum_{i=1}^n \alpha_i (\xi_i \otimes \eta_i) \right|_{H_{00} \otimes H_{01}}$$

$$= \left| \sum_{i=1}^n \alpha_i (x_i \otimes y_i - x_i \otimes \eta_i + x_i \otimes \eta_i - \xi_i \otimes \eta_i) \right|_{H_{00} \otimes H_{01}}$$

$$= \left| \sum_{i=1}^n \alpha_i (x_i \otimes (y_i - \eta_i) + (x_i - \xi_i) \otimes \eta_i) \right|_{H_{00} \otimes H_{01}}$$

$$\leq \sum_{i=1}^n |\alpha_i||x_i|_{H_{00}}|y_i - \eta_i|_{H_{01}} + \sum_{i=1}^n |\alpha_i||x_i - \xi_i|_{H_{00}}|\eta_i|_{H_{01}}$$

$$< \varepsilon$$

and

$$|(A^\ast \hat{\otimes} B)y - (\overline{A}^\ast \hat{\otimes} \overline{B})x|_{H_{10} \otimes H_{11}} = \left| \sum_{i=1}^n \alpha_i (Ax_i \otimes B y_i) - \sum_{i=1}^n \alpha_i (\overline{A} \xi_i \otimes \overline{B} \eta_i) \right|_{H_{10} \otimes H_{11}}.$$
\[
\left| \sum_{i=1}^{n} \alpha_i (Ax_i \otimes By_i - Ax_i \otimes B\eta_i + Ax_i \otimes \overline{B}\eta_i - \overline{A}\xi_i \otimes B\eta_i) \right|_{H_{10} \otimes H_{11}} 
\]
\[
\leq \sum_{i=1}^{n} |\alpha_i||Ax_i|_{H_{10}}|By_i - B\eta_i|_{H_{11}} + \sum_{i=1}^{n} |\alpha_i||Ax_i - \overline{A}\xi_i|_{H_{10}}|B\eta_i|_{H_{11}} < \varepsilon.
\]

Thus, we conclude that \(x \in D(A \otimes B)\) and \((A \otimes B)x = (\overline{A} \otimes \overline{B})x\), which shows
\[
\overline{A} \otimes \overline{B} \subseteq A \otimes B
\]
and thus
\[
A \otimes B \subseteq A \otimes B.
\]

**Corollary A.12.** Let \(H_{ij}\) be complex Hilbert spaces for \(i \in \{0,1\}\) and \(j \in \{0,1,2\}\). Moreover, let \(A \subseteq H_{00} \oplus H_{10}, B \subseteq H_{01} \oplus H_{11}, C \subseteq H_{02} \oplus H_{12}\) be densely defined, closable, linear operators. Then
\[
(A \otimes B) \otimes C = A \otimes (B \otimes C) =: A \otimes B \otimes C.
\]

**Proof.** It is clear that
\[
(A \otimes B) \otimes C = A \otimes (B \otimes C)
\]
in the virtue of Proposition A.7. Then it follows by Proposition A.11 that
\[
\begin{align*}
(A \otimes B) \otimes C &= (A \otimes B) \otimes C \\
&= (A \otimes B) \otimes C \\
&= A \otimes (B \otimes C) \\
&= A \otimes (B \otimes C).
\end{align*}
\]

**Proposition A.13** (\cite{[4, p. 25, Theorem 2.1]}. Let \(A \in L(H_{00}, H_{10})\) and \(B \in L(H_{01}, H_{11})\). Then \(A \otimes B \in L(H_{00} \otimes H_{01}, H_{10} \otimes H_{11})\) and \(\|A \otimes B\| = \|A\|\|B\|\).

**Proof.** We denote by \(S_{ij}\) a complete orthonormal set in the Hilbert space \(H_{ij}\) for \(i, j \in \{0,1\}\). Let \(x = \sum_{j=1}^{l} \kappa_j (\varphi_j \otimes \psi_j) \in H_{00} \otimes H_{01}\). Then we find for each \(j \in \{1,\ldots,l\}\) sequences \((\alpha_n^j)_{n \in \mathbb{N}}, (\beta_n^j)_{n \in \mathbb{N}} \in \mathbb{C}^N, (\zeta_n)_{n \in \mathbb{N}} \in S_{00}^N, (\xi_n)_{n \in \mathbb{N}} \in S_{01}^N\) such that
\[
\kappa_j \varphi_j = \sum_{n \in \mathbb{N}} \alpha_n^j \zeta_n \quad \text{and} \quad \psi_j = \sum_{n \in \mathbb{N}} \beta_n^j \xi_n
\]
and hence

\[ x = \sum_{j=1}^{l} (\kappa_j \varphi_j) \otimes \psi_j = \sum_{n,m \in \mathbb{N}} \sum_{j=1}^{l} \alpha_n^j \beta_m^j \,(\zeta_n \otimes \xi_m). \]

We set \( \gamma_{nm} := \sum_{j=1}^{l} \alpha_n^j \beta_m^j \) for \( n, m \in \mathbb{N} \) and obtain

\[ x = \sum_{n,m \in \mathbb{N}} \gamma_{nm} \,(\zeta_n \otimes \xi_m). \]

By the continuity of \( A \) and \( B \) we conclude

\[ (A \otimes B)(x) = \sum_{j=1}^{l} (A \kappa_j \varphi_j) \otimes (B \psi_j) \]

\[ = \sum_{j=1}^{l} \left( A \left( \sum_{n \in \mathbb{N}} \alpha_n^j \zeta_n \right) \right) \otimes \left( B \left( \sum_{m \in \mathbb{N}} \beta_m^j \xi_m \right) \right) \]

\[ = \sum_{n,m \in \mathbb{N}} \sum_{j=1}^{l} \alpha_n^j \beta_m^j \,(A \zeta_n \otimes B \xi_m) \]

\[ = \sum_{n,m \in \mathbb{N}} \gamma_{nm} \,(A \zeta_n \otimes B \xi_m). \]

Let \( y = \sum_{i=1}^{k} \lambda_i (\sigma_i \otimes \tau_i) \in H_{10} \otimes H_{11} \). Then we find for each \( i \in \{1, \ldots, k\} \) sequences \((\mu_n^i)_{n \in \mathbb{N}}, (\rho_n^i)_{n \in \mathbb{N}} \in \mathbb{C}^\infty, (\eta_n)_{n \in \mathbb{N}} \in S_{10}^\infty \) and \((\theta_n)_{n \in \mathbb{N}} \in S_{11}^\infty \) such that

\[ \lambda_i \sigma_i = \sum_{n \in \mathbb{N}} \mu_n^i \eta_n \text{ and } \tau_i = \sum_{n \in \mathbb{N}} \rho_n^i \theta_n \]

and thus

\[ y = \sum_{n,m \in \mathbb{N}} \sum_{i=1}^{k} \mu_n^i \rho_m^i \,(\eta_n \otimes \theta_m) \]

and again by setting \( \delta_{nm} := \sum_{i=1}^{k} \mu_n^i \rho_m^i \) for \( m, n \in \mathbb{N} \) we get

\[ y = \sum_{n,m \in \mathbb{N}} \delta_{nm} \,(\eta_n \otimes \theta_m). \]

We calculate

\[ \langle (A \otimes B)x | y \rangle_{H_{10} \otimes H_{11}} = \sum_{n,m,s,t \in \mathbb{N}} \gamma_{nm}^* \delta_{st} \langle A \zeta_n \otimes B \xi_m | \eta_s \otimes \theta_t \rangle_{H_{10} \otimes H_{11}} \]

\[ = \sum_{n,m,s,t \in \mathbb{N}} \gamma_{nm}^* \delta_{st} \langle A \zeta_n | \eta_s \rangle_{H_{10}} \langle B \xi_m | \theta_t \rangle_{H_{12}} \]

\[ = \sum_{n,m,s,t \in \mathbb{N}} \gamma_{nm}^* \delta_{st} \langle A \zeta_n | \eta_s \rangle_{H_{10}} \langle \xi_m | B^* \theta_t \rangle_{H_{01}} \]
\[
\sum_{n,m,s,t \in \mathbb{N}} \gamma_{nm}^* \langle A \zeta_n | \eta_s \rangle_{H_{10}} \delta_{st} \langle B^* \theta_t | \xi_m \rangle_{H_{01}}^* \\
= \sum_{m,s \in \mathbb{N}} \left\langle A \left( \sum_{n \in \mathbb{N}} \gamma_{nm} \zeta_n \right) | \eta_s \right\rangle_{H_{10}} \left( \left\langle B^* \left( \sum_{t \in \mathbb{N}} \delta_{st} \theta_t \right) | \xi_m \right\rangle_{H_{01}} \right)^*.
\]

Hence, we estimate by the Cauchy-Schwarz-Inequality
\[
|\langle (A \otimes B)x | y \rangle_{H_{10} \otimes H_{11}}|^2 \leq \left( \sum_{m \in \mathbb{N}} \left| \sum_{n \in \mathbb{N}} \gamma_{nm} \zeta_n \right|^2_{H_{10}} \right) \left( \sum_{s \in \mathbb{N}} \left| \sum_{t \in \mathbb{N}} \delta_{st} \theta_t \right|^2_{H_{01}} \right).
\]

By the orthonormality of \((\zeta_n)_{n \in \mathbb{N}}\) and \((\theta_t)_{t \in \mathbb{N}}\) and using \(\|B\| = \|B^*\|\), we conclude
\[
|\langle (A \otimes B)x | y \rangle_{H_{10} \otimes H_{11}}|^2 \leq \|A\|^2 \|B\|^2 \left( \sum_{m,n \in \mathbb{N}} |\gamma_{nm}|^2 \right) \left( \sum_{s,t \in \mathbb{N}} |\delta_{st}|^2 \right).
\]

Now using Parseval’s equality (cf. [55, Theorem V.4.9]) we finally get
\[
|\langle (A \otimes B)x | y \rangle_{H_{10} \otimes H_{11}}|^2 \leq \|A\|^2 \|B\|^2 \|x\|_{H_{00} \otimes H_{01}}^2 \|y\|_{H_{10} \otimes H_{11}}^2.
\]

By choosing \(y = (A \otimes B)x \in H_{10} \otimes H_{11}\), we derive from this last inequality
\[
|\langle (A \otimes B)x | H_{10} \otimes H_{11} \rangle| \leq \|A\| \|B\| \|x\|_{H_{00}}^2,
\]

which shows the continuity (since \(H_{00} \otimes H_{01}\) is dense in \(H_{00} \otimes H_{01}\)) and we estimate
\[
\|A \otimes B\| \leq \|A\| \|B\|.
\]

To show the equality of the operator norms, we choose sequences \((x_n)_{n \in \mathbb{N}} \in B_{H_{00}}(0,1)^N\) and \((y_n)_{n \in \mathbb{N}} \in B_{H_{01}}(0,1)^N\) with
\[
|Ax_n|_{H_{10}} \to \|A\| \text{ and } |By_n|_{H_{11}} \to \|B\| \quad (n \to \infty).
\]
Then \((x_n \otimes y_n)_{n \in \mathbb{N}} \in B_{H_0 \otimes H_1}(0, 1)^\mathbb{N}\) and
\[
|(A \otimes B)(x_n \otimes y_n)|_{H_0 \otimes H_1} = |Ax_n|_{H_0}|By_n|_{H_1} \to \|A\|\|B\| \quad (n \to \infty),
\]
which leads to
\[
\|A \otimes B\| \geq \|A\|\|B\|.
\]
This finishes the proof. \(\square\)

**Proposition A.14.** Let \(H_0, H_1\) and \(H_2\) be complex Hilbert spaces. We denote the identity on \(H_0\) by \(1_{H_0}\). Let \(A \subseteq H_1 \oplus H_2\) be a densely defined, closable, linear operator. Then
\[
(1_{H_0} \otimes A)^* = 1_{H_0} \otimes A^*
\]
and
\[
(A \otimes 1_{H_0})^* = A^* \otimes 1_{H_0}.
\]

**Proof.** From Lemma A.10 and Proposition A.11 we get
\[
(1_{H_0} \otimes A^*) \subseteq (1_{H_0} \otimes \overline{A})^* = (1_{H_0} \otimes A)^*.
\]
To show the missing inclusion we take \(x \in D((1_{H_0} \otimes A)^*) \subseteq H_0 \otimes H_2\). Let \(S_i\) be a complete orthonormal set in \(H_i\) for \(i \in \{0, 1, 2\}\). According to Corollary A.5 \([S_0] \otimes [S_2]\) is a complete orthonormal set in \(H_0 \otimes H_j\) for \(j \in \{1, 2\}\). Hence, we find sequences \((\xi_n)_{n \in \mathbb{N}} \in S_0^\mathbb{N}\), \((\eta_n)_{n \in \mathbb{N}} \in S_2^\mathbb{N}\) and \((\eta_n)_{n \in \mathbb{N}} \in S_2^\mathbb{N}\) such that
\[
x = \sum_{n \in \mathbb{N}} \langle \xi_n \otimes \eta_n | x \rangle_{H_0 \otimes H_2} (\xi_n \otimes \eta_n),
\]
\[
(1_{H_0} \otimes A)^* x = \sum_{n \in \mathbb{N}} \langle \xi_n \otimes \zeta_n | (1_{H_0} \otimes A)^* x \rangle_{H_0 \otimes H_1} (\xi_n \otimes \zeta_n).
\]
For \(s \in S_0\) and \(u \in D(A)\) we have
\[
\langle (1_{H_0} \otimes A)(s \otimes u) | x \rangle_{H_0 \otimes H_2} = \langle s \otimes u | (1_{H_0} \otimes A)^* x \rangle_{H_0 \otimes H_1}
\]
and by using the representations above as well as the linearity and the continuity of the inner product in the second argument, we obtain
\[
\sum_{n \in \mathbb{N}} \langle \xi_n \otimes \eta_n | x \rangle_{H_0 \otimes H_2} ((1_{H_0} \otimes A)(s \otimes u)) \langle \xi_n \otimes \eta_n \rangle_{H_0 \otimes H_2}
\]
\[
= \sum_{n \in \mathbb{N}} \langle \xi_n \otimes \zeta_n | (1_{H_0} \otimes A)^* x \rangle_{H_0 \otimes H_1} \langle s \otimes u | \xi_n \otimes \zeta_n \rangle_{H_0 \otimes H_1}.
\]
Let \(i \in \mathbb{N}\) and set \(s = \xi_i\). Then the latter reads as
\[
\langle Au | \xi_i \otimes \eta_i | x \rangle_{H_0 \otimes H_2} \eta_i | x \rangle_{H_0 \otimes H_2} \eta_i \in D(A^*)
\]
for all \(u \in D(A)\). This shows \(\langle \xi_i \otimes \eta_i | x \rangle_{H_0 \otimes H_2} \eta_i \in D(A^*)\) and
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for all \( i \in \mathbb{N} \). This implies

\[
\sum_{n=1}^{m} \langle \xi_n \otimes \eta_n | x \rangle_{H_0 \otimes H_2} (\xi_n \otimes \eta_n) \in H_0 \otimes D(A^*) \subseteq D(1_{H_0} \otimes A^*)
\]

and

\[
(1_{H_0} \otimes A^*) \left( \sum_{n=1}^{m} \langle \xi_n \otimes \eta_n | x \rangle_{H_0 \otimes H_2} (\xi_n \otimes \eta_n) \right) = \sum_{n=1}^{m} \langle \xi_n \otimes \zeta_n | (1_{H_0} \otimes A^*)x \rangle_{H_0 \otimes H_1} (\xi_n \otimes \zeta_n)
\]

for all \( m \in \mathbb{N} \). Since \((1_{H_0} \otimes A^*)\) is closed, we conclude \( x \in D(1_{H_0} \otimes A^*) \) with \( (1_{H_0} \otimes A^*)x = (1_{H_0} \otimes A)^*x \). The second equality follows by an analogous argument.

\[\Box\]

**Proposition A.15.** Let \( \{0\} \neq H_0, H_1, H_2 \) be complex Hilbert spaces. We denote the identity on \( H_0 \) by \( 1_{H_0} \). Let \( A \subseteq H_1 \oplus H_2 \) be a linear, densely defined, closable operator. Then \( 1_{H_0} \otimes A \) is continuously invertible if and only if \( A \) is continuously invertible. Furthermore

\[
(1_{H_0} \otimes A)^{-1} = (1_{H_0} \otimes A^{-1}).
\]

**Proof.** First let \( A \) be continuously invertible. Then \( 1_{H_0} \otimes A^{-1} \in L(H_0 \otimes H_2, H_0 \otimes H_1) \) according to Proposition A.13. We will prove that \( 1_{H_0} \otimes A^{-1} \) is the left- and right-inverse of \( 1_{H_0} \otimes A \). Let

\[
x = \sum_{i=1}^{n} \alpha_i (\zeta_i \otimes \xi_i) \in H_0 \otimes D(A).
\]

Then

\[
(1_{H_0} \otimes A^{-1})(1_{H_0} \otimes A)x = (1_{H_0} \otimes A^{-1}) \left( \sum_{i=1}^{n} \alpha_i (\zeta_i \otimes A \xi_i) \right)
\]

\[
= \sum_{i=1}^{n} \alpha_i (\zeta_i \otimes A^{-1} A \xi_i)
\]

\[
= \sum_{i=1}^{n} \alpha_i (\zeta_i \otimes A^{-1} \xi_i)
\]

Hence,

\[
(1_{H_0} \otimes A^{-1})(1_{H_0} \otimes A) \big|_{D(1_{H_0} \otimes A)} = 1_{D(1_{H_0} \otimes A)}. \]

Let now \( x \in D(1_{H_0} \otimes A) \). Then we find a sequence \((x_n)_{n \in \mathbb{N}} \in D(1_{H_0} \otimes A)^{\mathbb{N}} \) such that \( x_n \to x \) in \( H_0 \otimes H_1 \) and \( (1_{H_0} \otimes A)(x_n) \to (1_{H_0} \otimes A)(x) \) in \( H_0 \otimes H_2 \) as \( n \to \infty \). Using the continuity of \((1_{H_0} \otimes A^{-1})\) we obtain

\[
x_n = (1_{H_0} \otimes A^{-1})(1_{H_0} \otimes A)(x_n) \to (1_{H_0} \otimes A^{-1})(1_{H_0} \otimes A)(x) \quad \text{in} \quad H_0 \otimes H_1 \quad (n \to \infty)
\]

and since the limit is unique, we conclude \( x = (1_{H_0} \otimes A^{-1})(1_{H_0} \otimes A)(x) \), which shows \( (1_{H_0} \otimes A^{-1})(1_{H_0} \otimes A) = 1_{D(1_{H_0} \otimes A)} \). Analogously one sees that for \( y = \sum_{i=1}^{m} \beta_i (\phi_i \otimes \psi_i) \in H_0 \otimes H_2 \) the following holds

\[
(1_{H_0} \otimes A)(1_{H_0} \otimes A^{-1})y = (1_{H_0} \otimes A) \left( \sum_{i=1}^{m} \beta_i (\phi_i \otimes A^{-1} \psi_i) \right)
\]
Let now $y \in H_0 \otimes H_2$ and choose a sequence $(y_n)_{n \in \mathbb{N}} \in (H_0 \otimes H_2)^{\mathbb{N}}$ with $y_n \to y$ in $H_0 \otimes H_2$ as $n \to \infty$. Then by continuity we get $(1_{H_0} \otimes A^{-1})(y_n) \to (1_{H_0} \otimes A^{-1})(y)$ in $H_0 \otimes H_1$ and since $(1_{H_0} \otimes A^{-1})(y_n) \in D(1_{H_0} \otimes A)$ with $(1_{H_0} \otimes A)(1_{H_0} \otimes A^{-1})(y_n) = y_n$ for all $n \in \mathbb{N}$, we obtain by the closedness of $1_{H_0} \otimes A$ that $(1_{H_0} \otimes A^{-1})y \in D(1_{H_0} \otimes A)$ and $(1_{H_0} \otimes A)(1_{H_0} \otimes A^{-1})y = \lim_{n \to \infty} y_n = y$. Thus,$$
abla (1_{H_0} \otimes A)^{-1} = (1_{H_0} \otimes A^{-1}) \in L(H_0 \otimes H_2, H_0 \otimes H_1).$$

We now assume that $1_{H_0} \otimes A$ is continuously invertible. We fix an element $x_0 \in H_0$ with $x_0 \neq 0$. Let $y \in D(A)$ with $Ay = 0$. Then $(1_{H_0} \otimes A)(x_0 \otimes y) = x_0 \otimes Ay = 0$ and since $1_{H_0} \otimes A$ is injective, we conclude $x_0 \otimes y = 0$. This means that for all $(\phi, \psi) \in H_0 \times H_1$,$$(x_0 | \phi)_H (\psi | y)_H = (x_0 \otimes y)(\phi, \psi) = 0.$$In particular for $\phi = x_0$ and $\psi = y$ this yields $|x_0|_{H_0}|y|_{H_1} = 0$ and since $x_0 \neq 0$, we get $y = 0$. Thus, $A$ is injective. Next we show the density of the range of $A$. Let $y \in A[H_1]^{\perp}$. Then for all $z \in D(A)$ we have $(A|z||y|H_2) = 0$ and thus$$\forall x \in H_0, z \in D(A) : \langle (1_{H_0} \otimes A)(x \otimes z), (x_0 \otimes y) \rangle_{H_0 \otimes H_2} = 0.$$By the conjugate linearity of the inner product in the first argument, we conclude$$\forall \xi \in H_0 \overset{\text{a}}{\otimes} D(A) : \langle (1_{H_0} \otimes A)\xi, (x_0 \otimes y) \rangle_{H_0 \otimes H_2} = 0$$and by the continuity of the inner product$$\forall \xi \in D(1_{H_0} \otimes A) : \langle (1_{H_0} \otimes A)\xi, (x_0 \otimes y) \rangle_{H_0 \otimes H_2} = 0.$$Hence, $x_0 \otimes y \in ((1_{H_0} \otimes A)[H_0 \otimes H_1])^{\perp}$ and thus $x_0 \otimes y = 0$. Like above, this implies $y = 0$, since $x_0 \neq 0$, which yields the asserted density of $A[H_1]$. It is left to show the continuity of the inverse of $A$. By using $(1_{H_0} \otimes A)^{-1}|_{[H_0] \otimes [A[H_1]]} = (1_{H_0} \otimes A^{-1})|_{[H_0] \otimes [A[H_1]]}$ we estimate for $y \in A[H_1]$:

$$|A^{-1}y|_{H_1} = \frac{|x_0|_{H_0}|A^{-1}y|_{H_1}}{|x_0|_{H_0}} = \frac{1}{|x_0|_{H_0}}|A^{-1}y|_{H_0 \otimes H_1} = \frac{1}{|x_0|_{H_0}}|(1_{H_0} \otimes A)^{-1}(x_0 \otimes y)|_{H_0 \otimes H_1} \leq \frac{1}{|x_0|_{H_0}}\|(1_{H_0} \otimes A)^{-1}\||x_0 \otimes y|_{H_0 \otimes H_1} = \|(1_{H_0} \otimes A)^{-1}\||y|_{H_1}.$$This completes the proof. \qed
B. The Hausdorff metric

In this part of the appendix we want to recall some well-known results concerning the Hausdorff distance of closed, bounded sets (cf. [22, 18, 1]). In [18] a different definition of the metric is given, which, however, can be shown to be equivalent to our definition (see [1, p. 34]). Throughout, let \((X, d)\) be a metric space.

**Definition B.1.** We define the distance between an element of \(X\) and a subset of \(X\) by

\[
\text{dist} : X \times \mathcal{P}(X) \rightarrow [0, \infty) \\
(x, A) \mapsto \inf\{d(x, y) | y \in A\}.
\]

Moreover, we define the diameter of a subset of \(X\) by

\[
\text{diam} A := \sup\{d(x, y) | x, y \in A\} \quad (A \subseteq X).
\]

A subset is called **bounded**, if its diameter is finite. Next we introduce the so-called **Hausdorff distance** between two subsets of \(X\) by

\[
\tilde{H} : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow [0, \infty] \\
(A, B) \mapsto \max\left\{\sup_{x \in A} \text{dist}(x, B), \sup_{y \in B} \text{dist}(y, A)\right\}.
\]

We denote the set of all closed, bounded and nonempty subsets of \(X\) by \(BC(X)\) and the restriction of \(\tilde{H}\) to \(BC(X) \times BC(X)\) by \(H\).

**Lemma B.2.** Let \(A, B \subseteq X\) and \(\varepsilon > 0\). Then for each \(x \in A\) there exists \(y \in B\) such that

\[
d(x, y) \leq \tilde{H}(A, B) + \varepsilon.
\]

**Proof.** If \(\tilde{H}(A, B) = \infty\) the assertion holds trivially. So let \(\tilde{H}(A, B) < \infty\). By definition of the Hausdorff distance we conclude

\[
\forall x \in A : \text{dist}(x, B) \leq \tilde{H}(A, B).
\]

Let \(x \in A\). We choose a sequence \((y_n)_{n \in \mathbb{N}} \in B^\mathbb{N}\) with \(d(x, y_n) \searrow \text{dist}(x, B)\) as \(n \to \infty\). Thus, we find \(n \in \mathbb{N}\) with

\[
d(x, y_n) - \text{dist}(x, B) \leq \varepsilon
\]

and hence

\[
d(x, y_n) = d(x, y_n) - \text{dist}(x, B) + \text{dist}(x, B) \leq \varepsilon + \tilde{H}(A, B).
\]

\hfill \Box
Proposition B.3. \((BC(X), \mathcal{H})\) is a metric space.

Proof. At first we prove the finiteness of \(\mathcal{H}(A, B)\) for \(A, B \in BC(X)\): Since \(A\) and \(B\) are bounded, there exists \(r \in \mathbb{R}_{>0}\) such that \(\text{diam} A \leq r\) and \(\text{diam} B \leq r\). We fix \(x_0 \in A, y_0 \in B\) and obtain for all \(x \in A, y \in B\)

\[
d(x, y) \leq d(x, x_0) + d(x_0, y) \leq r + d(x_0, y)
\]

and thus

\[
\text{dist}(x, B) \leq r + \text{dist}(x_0, B)
\]

for all \(x \in A\). Analogously we conclude

\[
\text{dist}(y, A) \leq r + \text{dist}(y_0, A)
\]

for all \(y \in B\) and read off

\[
\mathcal{H}(A, B) \leq r + \text{dist}(x_0, B) + \text{dist}(y_0, A) < \infty.
\]

The symmetry of \(\mathcal{H}\) follows directly from the definition. Let \(A, B \in BC(X)\). Then \(\text{dist}(x, A) = 0\) for all \(x \in A\) and hence \(\mathcal{H}(A, A) = 0\). Assume now that \(\mathcal{H}(A, B) = 0\). This implies

\[
\forall x \in A : \text{dist}(x, B) = 0.
\]

Then there exists \((y_n)_{n \in \mathbb{N}} \in B^\mathbb{N}\) with \(d(x, y_n) \searrow \text{dist}(x, B) = 0\) for \(n \to \infty\) and thus \(x = \lim_{n \to \infty} y_n \in \overline{B} = B\). Since \(x \in A\) was arbitrary we get \(A \subseteq B\). Since also \(\text{dist}(y, A) = 0\) for all \(y \in B\) we conclude \(B \subseteq A\) as well and hence \(A = B\). It is left to prove the triangle inequality. For doing so let \(A, B, C \in BC(X)\) and \(x \in A, y \in B\). Then for all \(z \in C\) we obtain

\[
d(x, z) \leq d(x, y) + d(y, z)
\]

and hence, after taking the infimum over all \(z \in C\),

\[
\text{dist}(x, C) \leq d(x, y) + \text{dist}(y, C) \leq d(x, y) + \mathcal{H}(B, C).
\]

This implies, since \(y \in B\) was chosen arbitrarily,

\[
\text{dist}(x, C) \leq \text{dist}(x, B) + \mathcal{H}(B, C) \leq \mathcal{H}(A, B) + \mathcal{H}(B, C)
\]

and thus

\[
\sup_{x \in A} \text{dist}(x, C) \leq \mathcal{H}(A, B) + \mathcal{H}(B, C).
\]

Via an analogous argument (start by taking the infimum over all \(x \in A\)) we also obtain

\[
\sup_{z \in C} \text{dist}(z, A) \leq \mathcal{H}(A, B) + \mathcal{H}(B, C)
\]

and hence

\[
\mathcal{H}(A, C) \leq \mathcal{H}(A, B) + \mathcal{H}(B, C).
\]

\(\square\)
Proposition B.4. If \((X,d)\) is complete, then so is \((BC(X),\mathcal{H})\).

Proof. Let \((A_n)_{n \in \mathbb{N}} \in (BC(X))^\mathbb{N}\) be a Cauchy-sequence. We define the following set:

\[
A := \left\{ \lim_{k \to \infty} x_{n_k} \mid (n_k)_{k \in \mathbb{N}} \text{ strictly monotone increasing,} \right. \\
\left. (x_{n_k})_{k \in \mathbb{N}} \in X^\mathbb{N} \text{ convergent with } x_{n_k} \in A_{n_k} \right\}.
\]

At first we want to show that \(A \in BC(X)\).

- \(A\) is nonempty: There exists \((n_k)_{k \in \mathbb{N}} \in \mathbb{N}^\mathbb{N}\) strictly monotone increasing such that \(H(A_{n_k}, A_{n_k+1}) \leq \frac{1}{2^{k+1}}\) for all \(k \in \mathbb{N}\). Let \(x_{n_1} \in A_{n_1}\). According to Lemma B.2, there exists \(x_{n_2} \in A_{n_2}\) with

\[
d(x_{n_1}, x_{n_2}) \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.
\]

Again, by applying Lemma B.2, we find \(x_{n_3} \in A_{n_3}\) with

\[
d(x_{n_2}, x_{n_3}) \leq \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.
\]

In that way we define a sequence \((x_{n_k})_{k \in \mathbb{N}}\) with \(x_{n_k} \in A_{n_k}\) and \(d(x_{n_k}, x_{n_{k+1}}) \leq \frac{1}{2^k}\) for all \(k \in \mathbb{N}\). We show, that \((x_{n_k})_{k \in \mathbb{N}}\) is a Cauchy-sequence. Let \(\varepsilon > 0\). Then there exists \(K \in \mathbb{N}\) such that \(\frac{1}{2^{K+1}} \leq \varepsilon\). We estimate for all \(k, l \geq K, l > k\):

\[
d(x_{n_k}, x_{n_l}) \leq \sum_{i=0}^{l-k-1} d(x_{n_{k+i}}, x_{n_{k+i+1}}) \leq \sum_{i=0}^{l-k-1} \frac{1}{2^{k+i}} \leq \frac{1}{2^{k-1}} \leq \varepsilon. \tag{B.1}
\]

Hence, \((x_{n_k})_{k \in \mathbb{N}}\) is a Cauchy-sequence and thus convergent. Its limit belongs to \(A\) by definition.

- \(A\) is bounded: We first prove that there exists a constant \(c \in \mathbb{R}_{>0}\) such that \(\text{diam} A_n \leq c\) for all \(n \in \mathbb{N}\). Let \(N \in \mathbb{N}\) such that \(H(A_n, A_m) \leq 1\) for all \(n, m \geq N\). Let \(n \geq N, x_1, x_2 \in A_n\). By Lemma B.2, there exists \(y_1, y_2 \in A_N\) such that

\[
d(x_1, y_1) + d(x_2, y_2) \leq 2H(A_n, A_N) + 1 \leq 3.
\]

Hence,

\[
d(x_1, x_2) \leq d(x_1, y_1) + d(y_1, y_2) + d(y_2, x_2) \leq 3 + \text{diam} A_N.
\]

This implies, that for all \(n \geq N\) we have

\[
\text{diam} A_n \leq 3 + \text{diam} A_N.
\]
We set \( c := 3 + \max\{\diam(A_m) \mid 1 \leq m \leq N\} \) and get the desired estimate. Let now \( x, y \in A \). Then we find \( n, m \in \mathbb{N}_{\geq N}, \tilde{x} \in A_n, \tilde{y} \in A_m \) such that \( d(x, \tilde{x}) \leq 1 \) and \( d(y, \tilde{y}) \leq 1 \). According to Lemma \[ \text{B.2} \] we find \( \tilde{z} \in A_m \) with \( d(\tilde{x}, \tilde{z}) \leq \mathcal{H}(A_n, A_m) + 1 \leq 2 \). Hence, we estimate

\[
\begin{align*}
d(x, y) &\leq d(x, \tilde{x}) + d(\tilde{x}, \tilde{z}) + d(\tilde{z}, \tilde{y}) + d(\tilde{y}, y) \\
&\leq 1 + 2 + \diam(A_m) + 1 \\
&\leq 4 + c.
\end{align*}
\]

Since \( x, y \in A \) were chosen arbitrarily, it follows that

\[
\diam A \leq 4 + c < \infty.
\]

- \( A \) is closed: Let \( (x_k)_{k \in \mathbb{N}} \in A^\mathbb{N} \) with \( x_k \to x \) in \( X \) as \( k \to \infty \). For \( x_1 \) we find \( n_1 \in \mathbb{N} \) and \( x_{n_1} \in A_{n_1} \) with \( d(x_1, x_{n_1}) \leq \frac{1}{2} \). For \( x_2 \) we find \( n_2 \in \mathbb{N}_{\geq n_1} \) and \( x_{n_2} \in A_{n_2} \) with \( d(x_2, x_{n_2}) \leq \frac{1}{3} \). In that way we define recursively a strictly monotone increasing sequence \( (n_k)_{k \in \mathbb{N}} \in \mathbb{N}^\mathbb{N} \) and a corresponding sequence \( (x_{n_k})_{k \in \mathbb{N}} \) with \( x_{n_k} \in A_{n_k} \) and

\[
d(x_{n_k}, x_k) \leq \frac{1}{2^k}
\]

for all \( k \in \mathbb{N} \). Let \( \varepsilon > 0 \) and \( K_1 \in \mathbb{N} \) such that \( \frac{1}{2^{K_1}} \leq \frac{\varepsilon}{2} \). There exists \( K \in \mathbb{N}_{\geq K_1} \) such that \( d(x, x_k) \leq \frac{\varepsilon}{2} \) for all \( k \in \mathbb{N}_{\geq K} \). We estimate for all \( k \in \mathbb{N}_{\geq K} \)

\[
\begin{align*}
d(x_{n_k}, x) &\leq d(x_{n_k}, x_k) + d(x_k, x) \\
&\leq \frac{1}{2^k} + \frac{\varepsilon}{2} \\
&\leq \varepsilon
\end{align*}
\]

and hence \( x \in A \).

Next we show that \( A_n \to A \) as \( n \to \infty \). Let \( \varepsilon > 0 \) and \( N \in \mathbb{N} \) such that \( \mathcal{H}(A_n, A_m) \leq \frac{\varepsilon}{2} \) for all \( n, m \in \mathbb{N}_{\geq N} \). Let \( x \in A \). Then we find \( m \in \mathbb{N}_{\geq N} \) and an element \( x_m \in A_m \) such that \( d(x, x_m) \leq \frac{\varepsilon}{2} \). For every \( x_n \in A_n \) with \( n \geq N \) we estimate

\[
d(x, x_n) \leq d(x, x_m) + d(x_m, x_n)
\]

and hence

\[
dist(x, A_n) \leq d(x, x_m) + dist(x_m, A_n) \leq \frac{\varepsilon}{2} + \mathcal{H}(A_m, A_n) \leq \varepsilon.
\]

Since \( x \in A \) was arbitrary we get for all \( n \geq N \)

\[
\sup_{x \in A} dist(x, A_n) \leq \varepsilon. \tag{B.2}
\]

Using the Cauchy-property of \( (A_n)_{n \in \mathbb{N}} \) we find a strictly monotone increasing sequence \( (n_k)_{k \in \mathbb{N}} \in \mathbb{N}^\mathbb{N} \) with \( n_1 := N \) and

\[
\mathcal{H}(A_{n_k}, A_{n_{k+1}}) \leq \frac{\varepsilon}{2^k} \quad (k \in \mathbb{N}).
\]
We take \( x_1 \in A_N = A_{n_1} \) and find, by using Lemma [B.2] an element \( x_2 \in A_{n_2} \) with
\[
d(x_1, x_2) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
We define recursively a sequence \((x_k)_{k \in \mathbb{N}}\) with \( x_k \in A_{n_k} \) and
\[
d(x_k, x_{k+1}) \leq \frac{\varepsilon}{2^{k-1}}
\]
for all \( k \in \mathbb{N} \). Like in (B.1) we find out that \((x_k)_{k \in \mathbb{N}}\) is a Cauchy-sequence and hence convergent with limit \( x \in A \). Let \( k_0 \in \mathbb{N} \) such that \( d(x_k, x) \leq \varepsilon \) for all \( k \geq k_0 \). Then
\[
d(x_1, x) \leq \sum_{i=1}^{k_0-1} d(x_i, x_{i+1}) + d(x_{k_0}, x)
\leq \sum_{i=1}^{k_0-1} \frac{\varepsilon}{2^{i-1}} + \varepsilon
\leq \varepsilon \left( \sum_{i=0}^{k_0-2} \frac{1}{2^i} + 1 \right)
\leq 3\varepsilon.
\]
This implies
\[
dist(x_1, A) \leq 3\varepsilon\]
and since \( x_1 \in A_N \) was chosen arbitrarily, we get
\[
\sup_{y \in A_N} \text{dist}(y, A) \leq 3\varepsilon.
\]
Together with (B.2) this shows
\[
\mathcal{H}(A_N, A) \leq 3\varepsilon.
\]
Let now \( n \geq N \). Then the triangle inequality yields
\[
\mathcal{H}(A_n, A) \leq \mathcal{H}(A_n, A_N) + \mathcal{H}(A_N, A) \leq 4\varepsilon.
\]
This shows \( A_n \to A \) as \( n \to \infty \), which was the desired result. \( \square \)
Bibliography


Versicherung

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

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