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On some Banach Algebra Tools in Operator Theory

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Introduction

This work is devoted to sequences $\mathcal{A} = \{A_n\}$ of operators $A_n$ with a specific asymptotic structure. Such sequences usually arise from certain approximation methods applied to bounded linear operators. The fundamental tool in the present approach is to take so-called snapshots $W^t(\mathcal{A})$, that are operators which capture some aspects of the asymptotics of the sequence $\mathcal{A}$ under consideration and which are derived by a transformation of $\mathcal{A}$ and a limiting process. We are particularly interested in criteria for the stability of $\mathcal{A}$. Here, $\mathcal{A}$ is called stable if there is an $N \in \mathbb{N}$ such that the operators $A_n$ are invertible for all $n > N$ and their inverses are uniformly bounded, i.e. $\sup_{n > N} \|A_n^{-1}\| < \infty$.

We are further interested in the so-called Fredholm property of sequences $\mathcal{A} = \{A_n\}$, a weaker analogon of stability, which provides surprisingly deep connections between the Fredholm properties of the entries $A_n$ and certain characteristics of their snapshots $W^t(\mathcal{A})$, such as their indices, kernels and cokernels. To be a bit more precise, Fredholmness of $\mathcal{A}$ means invertibility of its respective coset in a certain quotient algebra, in analogy to the characterization of the operator Fredholm property as invertibility in the Calkin algebra. Moreover, we will derive results on spectral approximation linking the (pseudo)spectra of the $A_n$ with the (pseudo)spectra of the snapshots $W^t(\mathcal{A})$, and describing the convergence of the norms $\|A_n\|$, $\|A_n^{-1}\|$ as well as the condition numbers $\text{cond}(A_n)$. For this, a second essential tool in the present text is given by local principles, which replace the invertibility problem for an element in an algebra by the problem of invertibility of local representatives of this element in a family of (hopefully simpler) local algebras.

These Banach algebra techniques and the structure of the outcoming results are not new at all, but possess a long history and have been developed for and applied to many concrete classes of operators and approximation methods. Throughout this text we will try to convey an impression of the diversity of ideas, methods, papers and authors who contributed to their evolution. For the moment we only mention some of the most important milestones:

The story has begun with the approach of Silbermann in his groundbreaking paper [83] where the finite section sequences of Toeplitz operators have been studied with the help of two snapshots, and it initiated an evolution which lead to comprehensive monographs such as [12], [56] or [29] among others. In the paper [10] Böttcher, Krupnik and Silbermann collected the local principles of Simonenko, Allan/Douglas and Gohberg/Krupnik, and even presented a framework which permits to relate the norm of an element under consideration to its local norms. The pioneering work on the asymptotics of norms and pseudospectra (at least for the elements in certain sequence algebras which we have in mind) is due to Böttcher [6], [4]. A rather complete and abstract presentation of the whole theory in the case of $C^*$-algebras, the so-called Standard Model, is subject of the monograph [30] of Hagen, Roch and Silbermann. In this setting the Fredholm theory for operator sequences and the splitting phenomenon of their singular values has been studied. After Böttcher [5] introduced the approximation numbers as substitutes for the singular values into that business in order to extend the observation on the splitting phenomenon
for Toeplitz operators on $\ell^2$ also to the Banach spaces $l^p$. Rogozhin and Silbermann presented a Banach space based Fredholm theory for matrix sequences [75]. A modification of this approach which is applicable to the finite section sequences of convolution type operators on cones, i.e. to a class of sequences of operators acting on infinite dimensional Banach spaces, was presented by Mascarenhas and Silbermann in [50].

Notice that (almost) all of these variants are based on strong or even $\star$-strong convergence, and many of them are restricted to sequences $A_n$ of operators $A_n$ acting on finite dimensional spaces. Particularly, approximation methods for operators on spaces like $l^\infty$ stay out of reach.

The infinite dimensional applications require additional assumptions like $\text{ind} A_n = 0$. The aim of the present text is a generalization and unification which covers most versions of the mentioned algebraic framework for structured operator sequences and extends them to the infinite dimensional case (without additional conditions) and to non-strongly converging (so called $\mathcal{P}$-strongly converging) sequences. This will put us in a position to close a couple of gaps in the “big picture”, and to include more exotic settings (such as operators on spaces with a sup-norm). We prove several results on a much more abstract level than before, such that they become available not only for certain specific classes of operators but help to tackle many applications at one sweep.

Actually, the mentioned $\mathcal{P}$-strong convergence and the machinery behind this notion will prove to be the key which opens the door to the generalization and unification we have in mind. Perhaps, after having read this text the reader joins us in believing that this concept is in impressive harmony with the stability- and Fredholm theory, and is rather the natural choice than an unpretentious substitute for the classical notions of strong convergence and Fredholmness.

The first part of the thesis is devoted to basic notions and results for bounded linear operators, to their $(N, \epsilon)$-pseudospectra, and at its heart we develop the concept of approximate projections $\mathcal{P}$. This already appeared in [86] and [73] in elementary forms and was thoroughly revised in [63]. Here we look at these results from a different point of view, contribute some new proofs, and answer a long-standing question on the characterization of the $\mathcal{P}$-Fredholm property, with important consequences.

The Fredholm theory for sequences $A_n$ is presented in Part 2. Condensed to one sentence the outcome is: Much substantial information on the asymptotics of the entries $A_n$ of an operator sequence $A_n = \{A_n\}$ is stored in the snapshots $W^t(A_n)$ and in the Fredholmness of $A_n$.

In the 3rd Part we even achieve a characterization of the Fredholm property of sequences belonging to a certain class of sequence algebras solely in terms of their snapshots. This is done via localization.

Throughout the development of the general theory we successively apply the achieved results to band-dominated operators and their finite sections, and conversely we employ this prototypic example as a motivation and a guide through the following steps on the abstract level. Notice that the class of band-dominated operators is subject of many publications [42], [62], [61], [60], [57], [70], [63], [44], [59], [58], [69], [15], mainly done by Rabinovich, Roch, Silbermann and Lindner, and has been the engine for the development of the theory on approximate projections $\mathcal{P}$, in a sense.

Finally, Part 4 is devoted to several applications. Many of them are well known and have been intensively studied. Nevertheless, having the abstract tools of the previous parts available, we will recover the known results in an impressive conciseness and clarity, and we will be able to harmonize and to extend them into several directions.

Parts of the content have already been published or have been submitted for publication in [80], [81], [82] and [78].

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1Here, $(B_n)$ is said to converge $\star$-strongly, if both $(B_n)$ and $(B_n^*)$ converge strongly.
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Part 1

Bounded linear operators on Banach spaces

In the beginning of this first part we recall several elementary definitions and results, particularly the notions of compactness, Fredholmness and convergence in operator algebras. This is, in large parts, just for the sake of completeness, and the informed reader may skip it.

Since these classical notions are too rigid for our purposes, we introduce a fabric of generalizations in Section 1.2, based on so-called approximate projections $\mathcal{P}$, and show how to embed the classical setting into this $\mathcal{P}$-framework. The main contributions of the present text to this approach, which was developed in [86], [73] and [63], are given by Theorems 1.14 and 1.27.

Section 1.3 is devoted to the discussion of approximation numbers and their relations to other well known geometric characteristics of linear operators like the singular values and furthermore to the injection and surjection modulus. The latter also play an important role in Section 1.4 where we deal with the spectrum of a bounded linear operator and its approximation via pseudospectra.
1.1 Preliminaries

1.1.1 Banach spaces, linear operators and Banach algebras

In this text we will exclusively deal with complex Banach spaces, that are normed vector spaces over the field \( \mathbb{C} \) of complex numbers which are complete with respect to their norms. Therefore we drop this word complex and simply call them Banach spaces. As usual, we denote by \( \mathbb{N}, \mathbb{Z} \) and \( \mathbb{R} \) the sets of natural, integer and real numbers, respectively. Furthermore, \( \mathbb{Z}_+ \) (\( \mathbb{Z}_- \)) stands for the sets of all non-negative (non-positive) integers. Analogously, we define \( \mathbb{R}_+ \) and \( \mathbb{R}_- \).

For two Banach spaces \( X, Y \) the set of all bounded linear operators \( A : X \to Y \) is denoted by \( \mathcal{L}(X, Y) \), can be equipped with pointwise defined linear operations

\[
(\alpha A + \beta B)x := \alpha Ax + \beta Bx, \quad A, B \in \mathcal{L}(X, Y), \quad \alpha, \beta \in \mathbb{C}, \quad x \in X
\]

and the so called operator norm

\[
\|A\|_{\mathcal{L}(X, Y)} := \sup\{\|Ax\|_Y : x \in X, \|x\|_X \leq 1\},
\]

and forms a Banach space, as is well known from every text book on functional analysis. In the case \( X = Y \) we shortly write \( \mathcal{L}(X) \) for \( \mathcal{L}(X, X) \) and note that the composition of two operators, given by \( (AB)x := A(Bx) \) with \( x \in X \), turns \( \mathcal{L}(X) \) into a Banach algebra.

Here a Banach space \( \mathcal{A} \) with another binary operation \( \cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \), usually called the multiplication in \( \mathcal{A} \), is said to be a Banach algebra if \( (\mathcal{A}, +, \cdot) \) forms a ring and the multiplication is related to the norm by the following inequality:

\[
\|AB\|_{\mathcal{A}} \leq \|A\|_{\mathcal{A}} \|B\|_{\mathcal{A}} \quad \text{for all} \quad A, B \in \mathcal{A}.
\]

If there is an element \( I \in \mathcal{A} \) which fulfills \( AI = IA = A \) for every \( A \in \mathcal{A} \) and which is of norm one then \( \mathcal{A} \) is referred to as a unital Banach algebra. In this case, an element \( A \in \mathcal{A} \) is invertible from the left (right) if there is a \( B \in \mathcal{A} \) with \( BA = I \) (\( AB = I \), respectively), and it is said to be invertible if it is invertible from both sides. The inverse of an invertible element \( A \) is uniquely determined and is usually denoted by \( A^{-1} \).

A Banach subalgebra \( \mathcal{B} \) of a unital Banach algebra \( \mathcal{A} \) is said to be inverse closed in \( \mathcal{A} \) if, whenever an element \( A \in \mathcal{B} \) is invertible in \( \mathcal{A} \), also its inverse \( A^{-1} \) belongs to \( \mathcal{B} \). Recall that if an element \( A \) in a unital Banach algebra has norm less than one then the Neumann series \( \sum_{k \in \mathbb{Z}_+} A^k \) converges in \( \mathcal{A} \) and its sum is the inverse of the element \( I - A \).

A linear subspace \( \mathcal{J} \) of \( \mathcal{A} \) is called an ideal if \( AJ \in \mathcal{J} \) and \( JA \in \mathcal{J} \) for all \( A \in \mathcal{A} \) and all \( J \in \mathcal{J} \). For a closed ideal \( \mathcal{J} \) of a Banach algebra \( \mathcal{A} \), the set

\[
\mathcal{A}/\mathcal{J} := \{ A + \mathcal{J} : A \in \mathcal{A} \}
\]

of cosets \( A + \mathcal{J} := \{ A + J : J \in \mathcal{J} \} \), provided with the operations

\[
\alpha (A + \mathcal{J}) + \beta (B + \mathcal{J}) := (\alpha A + \beta B) + \mathcal{J}, \quad (A + \mathcal{J}) \cdot (B + \mathcal{J}) := (AB) + \mathcal{J},
\]

for all \( A, B \in \mathcal{A} \) and \( \alpha, \beta \in \mathbb{C} \), and the norm

\[
\|A + \mathcal{J}\|_{\mathcal{A}/\mathcal{J}} := \inf\{\|A + J\|_{\mathcal{A}} : J \in \mathcal{J}\},
\]

is a Banach algebra again, referred to as the quotient algebra of \( \mathcal{A} \) modulo \( \mathcal{J} \). Clearly, if \( \mathcal{A} \) has a unit element \( I \) then \( I + \mathcal{J} \) is a unit in \( \mathcal{A}/\mathcal{J} \). Analogously, we get the quotient space \( X/\mathbb{Z} \) of a Banach space \( X \) modulo a closed subspace \( \mathbb{Z} \).
1.1. PRELIMINARIES

Given a Banach space $X$, the Banach space $X^* := L(X, \mathbb{C})$ is referred to as the dual space of $X$ and its elements are the so-called linear functionals on $X$. For $A \in L(X, Y)$, the adjoint operator $A^* \in L(Y^*, X^*)$ is defined by the relation $(A^* f)x = f(Ax)$ for every $f \in Y^*$ and $x \in X$. Notice that $\|A^*\| = \|A\|$ for every $A \in L(X, Y)$, and that $A$ is invertible if and only if $A^*$ is so.

Finally, let $H$ be a complex vector space with an inner product $\langle \cdot, \cdot \rangle_H$ such that the norm $\|x\|_H := \sqrt{\langle x, x \rangle_H}$ turns $H$ into a complete normed space. Then $H$ is said to be a Hilbert space. This special case of the more general Banach spaces only plays a minor part in this text. Nevertheless, most ideas and results in the theory which is subject of the forthcoming sections firstly appeared in this much more comfortable setting, and many (or even most) applications involve problems in Hilbert spaces. Therefore we will meet them when talking about the roots, former approaches and important applications.

We point out that every element $y \in H$ defines a bounded linear functional $f_y \in H^*$ by the rule $f_y(x) := \langle x, y \rangle_H$ for every $x \in H$. The famous Riesz representation theorem even states the converse: Every bounded linear functional $f \in H^*$ is of the form $f = f_z$ with some $z \in H$. Moreover, the mapping $U_H : H \to H^*$, $y \mapsto f_y$ is an isometric and antilinear bijection.

In the sense of this identification of Hilbert spaces with their duals one usually introduces the Hilbert adjoint $A^*$ of an operator $A \in L(H_1, H_2)$ in a slightly different form, namely as the operator $A^* \in L(H_2^*, H_1^*)$ which fulfills

$$\langle Ax, y \rangle_{H_2} = \langle x, A^* y \rangle_{H_1}$$

for all $x \in H_1$ and all $y \in H_2$.

In other words we have $A^* = U_H^{-1} A^* U_{H_2}$ with $A^*$ being the adjoint operator as introduced above for the Banach space setting.

In all what follows we will denote the norms and inner products without the subscripts indicating the respective spaces, if the situation is unambiguous.

### 1.1.2 Fredholm operators

This section also contains and summarizes material which is well known and can be found (together with proofs) in [25], [12] or in many textbooks on functional analysis.

For a bounded linear operator $A \in L(X, Y)$ we define its kernel, its range and its cokernel by

$$\ker A := \{ x \in X : Ax = 0 \}$$
$$\text{im} A := A(X) = \{ Ax : x \in X \}$$
$$\text{coker} A := Y / \text{im} A.$$

The kernel of a bounded linear operator $A$ always forms a closed subspace of $X$, and its range is a subspace of $Y$ but, in general, it is not necessarily closed. We say that $A$ is normally solvable if $\text{im} A$ is a closed subspace of $Y$. One usually refers to rank $A := \dim \text{im} A$ as the rank of $A$.

Further, $A \in L(X, Y)$ is called compact if the closure of the set $\{ Ax : x \in X, \|x\| \leq 1 \}$ in $Y$ is compact. Denote the set of all compact operators $A \in L(X, Y)$ by $K(X, Y)$. In case $Y = X$ we again abbreviate $K(X, X)$ by $K(X)$. It is well known that $K(X)$ forms a closed ideal in $L(X)$, and the quotient algebra $L(X)/K(X)$ is usually referred to as the Calkin algebra.

The adjoint operator $A^*$ is compact if and only if $A$ is compact. Similarly, $A^*$ is normally solvable if and only if $A$ is so, and in this case

$$\dim \ker A^* = \dim \text{coker} A, \quad \dim \text{coker} A^* = \dim \ker A.$$

**Definition 1.1.** An operator $A \in L(X, Y)$ is called Fredholm operator if $\dim \ker A < \infty$ and $\dim \text{coker} A < \infty$. If $A$ is Fredholm then the integer $\text{ind} A := \dim \ker A - \dim \text{coker} A$ is referred to as the index of $A$. 
In the following theorems we record the most important properties of the class of Fredholm operators and several equivalent characterizations of the Fredholm property. Actually, the subsequent sections will reveal that these characterizations constitute a corner stone and a guide throughout the whole text.

**Theorem 1.2.** Let \( \Phi(X, Y) \) denote the set of all Fredholm operators in \( \mathcal{L}(X, Y) \). Then:

- \( \Phi(X, Y) \) is an open subset of \( \mathcal{L}(X, Y) \) and for \( A \in \Phi(X, Y) \) we have
  \[
  \dim \ker(A + C) \leq \dim \ker A, \quad \dim \ker(A + C) \leq \dim \ker A
  \]
  whenever \( C \) has sufficiently small norm. The mapping \( \text{ind} : \Phi(X, Y) \to \mathbb{Z} \) is constant on the connected components of \( \Phi(X, Y) \).
- Let \( A \in \Phi(X, Y) \) and \( K \in \mathcal{K}(X, Y) \). Then \( A + K \in \Phi(X, Y) \) and \( \text{ind}(A + K) = \text{ind} A \).
- (Atkinson’s theorem). Let \( Z \) be a further Banach space, \( A \in \Phi(Y, Z) \) and \( B \in \Phi(X, Y) \). Then we have \( AB \in \Phi(X, Z) \) and \( \text{ind}(AB) = \text{ind} A + \text{ind} B \).

**Theorem 1.3.** Let \( A \in \mathcal{L}(X, Y) \). Then the following conditions are equivalent:

1. \( A \) is Fredholm.
2. \( A \) is normally solvable, \( \dim \ker A < \infty \), and \( \dim \ker A^* < \infty \).
3. \( A^* \) is Fredholm.
4. There exist projections \(^1 P \in \mathcal{K}(X) \) and \( P' \in \mathcal{K}(Y) \) s.t. \( \text{im} P = \ker A \) and \( \ker P' = \text{im} A \).
5. The following is NOT true: For each \( l \in \mathbb{N} \) and each \( \epsilon > 0 \) there exists a projection \( Q \in \mathcal{K}(X) \) or a projection \( Q' \in \mathcal{K}(Y) \) with \( \text{rank} Q, \text{rank} Q' \geq l \) such that \( \|AQ\| < \epsilon \) or \( \|Q'A\| < \epsilon \).

In case \( Y = X \) the list can be completed by

6. The coset \( A + \mathcal{K}(X) \) is invertible in the Calkin algebra \( \mathcal{L}(X)/\mathcal{K}(X) \).

**Proof.** The equivalence of the Assertions 1. till 3. and 6. is well known, and can again be found in the mentioned textbooks. For 4. \( \Rightarrow \) 1. notice that compact projections are of finite rank.

The role of compact projections is subject of the next section and the considerations there provide the proof of the remaining implications. In particular, 2. \( \Rightarrow \) 4., 5. as well as NOT2. \( \Rightarrow \) NOT5., and hence 5. \( \Rightarrow \) 2., follow from Proposition 1.6. \( \Box \)

### 1.1.3 Auerbach’s lemma and the versatility of projections

One of the most valuable instruments in this work is the characterization of subspaces via projections, that are operators \( P \in \mathcal{L}(X) \) with \( P^2 = P \).

To be more precise, we say that a closed subspace \( X_1 \) of a Banach space \( X \) is complemented if there is another closed subspace \( X_2 \subset X \) (a so-called complement of \( X_1 \)) such that \( X \) decomposes into \( X_1 \) and \( X_2 \) in the sense that \( X = X_1 + X_2 \) and \( X_1 \cap X_2 = \{0\} \). In this case, every \( x \in X \) admits a unique decomposition \( x = x_1 + x_2 \) with \( x_i \in X_i, \ i = 1, 2 \). It is well known that the mapping \( P : x \mapsto x_1 \) is a (bounded) projection onto \( X_1 \) and \( I - P \) is a (bounded) projection onto \( X_2 \). One further says that \( P \) maps parallel to \( X_2 \). Conversely, for every projection \( P \in \mathcal{L}(X) \) the space ker \( P \) is complemented and im \( P \) is a complement.

\(^1\)An operator \( P \in \mathcal{L}(X) \) is called projection if \( P^2 = P \).
Proposition 1.4. (Auerbach’s lemma 2)
Let $X$ be of dimension $n$. Then there exist \{x_i\}_{i=1}^n \subset X$ and functionals \{f_i\}_{i=1}^n \subset X^*$ such that, for every $i, k$,
$$
\|x_i\| = \|f_k\| = 1 \quad \text{and} \quad f_k(x_i) = \begin{cases} 1 & : i = k \\ 0 & : i \neq k \end{cases}.
$$
This immediately yields the existence of bounded projections onto finite dimensional and finite codimensional subspaces of Banach spaces. For details see [54], B.4.9f.

Proposition 1.5. Let $X$ be a Banach space. If $X_1 \subset X$ is an $m$-dimensional subspace then there is a projection $P \in \mathcal{L}(X)$ ($P' \in \mathcal{L}(Y)$) of rank $k$ and with norm less than $k+2$ such that $AP = 0$ ($P'A = 0$, respectively).

Now we can capture all Fredholm properties of bounded linear operators in terms of projections, as the next well known proposition shows. For completeness we give its short proof.

Proposition 1.6. Let $X, Y$ be Banach spaces and $A \in \mathcal{L}(X, Y)$.

1. If $A$ is normally solvable and $k \leq \dim \ker A$ ($k \leq \dim \text{coker} A$) then there is a projection $P \in \mathcal{L}(X)$ ($P' \in \mathcal{L}(Y)$) of rank $k$ and with norm less than $k+2$ such that $AP = 0$ ($P'A = 0$, respectively).

2. If $A$ is not normally solvable then, for every $k \in \mathbb{N}$ and every $\epsilon > 0$, there are projections $P \in \mathcal{L}(X)$ and $P' \in \mathcal{L}(Y)$ of rank $k$ and with norm less than $k+2$ such that $\|AP\| < \epsilon$ and $\|P'A\| < \epsilon$.

Proof. In order to prove the second assertion, assume first that $\text{im} A$ is not closed. We want to show that for every $l \in \mathbb{N}$ and for every $\delta > 0$ there exists an $l$-dimensional subspace $X_l$ of $X$ such that $\|A|X_l\| < \delta$. Fix $\delta > 0$. Then there exists $x_l \in X$, $\|x_l\| = 1$ such that $\|Ax_l\| < \delta$, otherwise the range of $A$ would be closed. Assume that the assertion is true for $l-1$. By Proposition 1.5 there exist a complement $Z$ of $X_{l-1}$ and a projection $Q$ onto $X_{l-1}$ with $\ker Q = Z$ and $\|Q\| \leq l$. $A(Z)$ is not closed (otherwise $A(X) = A(X_{l-1}) + A(Z)$ would be the sum of a finite dimensional and a closed subspace of $Y$ and hence closed, as is known). Now we can choose $x_l \in Z$ with $\|x_l\| = 1$ and $\|Ax_l\| < \delta$, and for arbitrary scalars $\alpha, \beta$ and elements $x \in X_{l-1}$ we see

$$
\|A(\alpha x + \beta x_l)\| \leq \|A(\alpha x)\| + \|A(\beta x_l)\| < \delta(\|\alpha x\| + \|\beta x_l\|)
$$

$$
\|\alpha x\| = \|Q(\alpha x)\| \leq \|Q\|\|\alpha x\| \leq \|Q\|\|\alpha x + \beta x_l\|
$$

$$
\|\beta x_l\| = \|\text{Im} Q(\beta x_l)\| \leq \|Q\|\|\alpha x + \beta x_l\|
$$

$$
\leq (1 + \|Q\|)\|\alpha x + \beta x_l\|,
$$

i.e. $\|A(\alpha x + \beta x_l)\| < \delta(1 + 2l)\|\alpha x + \beta x_l\|$. Since $\delta > 0$ is arbitrary the assertion follows by induction. Consequently, for $k \in \mathbb{N}$ and $\epsilon > 0$ we can choose $0 < \delta < \epsilon/k$, a $k$-dimensional subspace of $X$ and an appropriate projection $P$ onto this subspace with $\|P\| \leq k$, such that $\|AP\| \leq \delta\|P\| < \epsilon$.

To find $P'$ in the second assertion we recall that $A^*$ is not normally solvable whenever $A$ is so, hence there is a projection $Q \in \mathcal{L}(X^*)$ of rank $k$ such that $\|A^*Q\| < \frac{\epsilon}{k+2}$ by what we have proved before. Let $\hat{Y}$ be the intersection of the kernels of all functionals in im $Q$. Obviously, $\hat{Y}$ has codimension $k$ and we can choose a rank $k$ projection $P'$ of norm less than $k + 2$ (see Proposition 1.5) such that $I - P'$ is onto $\hat{Y}$. Then $f \circ (I - P') = 0$ for all $f \in \text{im} Q$, that is

2For a proof see [54], B.4.8.
onto a subspace of \( Q \) and \( P \). Now, for the first assertion apply Proposition 1.5 to find \( \text{Proposition 1.7.} \)

1.1.4 Notions of convergence. Remember that an operator \( A \in \mathcal{L}(X, Y) \) is said to be generalized invertible, if there is an operator \( B \in \mathcal{L}(Y, X) \) such that \( A = ABA \) and \( B = BAB \).

**Proposition 1.7.** Let \( A \in \mathcal{L}(X, Y) \) and \( B \in \mathcal{L}(Y, X) \). Then the following conditions are equivalent

- \( ABA = A \).
- \( I_X - BA \) is a bounded projection with \( \text{im}(I_X - BA) = \ker A \).
- \( I_Y - AB \) is a bounded projection with \( \ker(I_Y - AB) = \text{im} A \).

Moreover, if \( A \in \mathcal{L}(X, Y) \) is an operator and \( P \in \mathcal{L}(X), P' \in \mathcal{L}(Y) \) are projections with \( \ker P = \ker A \) and \( \ker P' = \text{im} A \) then there exists an operator \( C \in \mathcal{L}(Y, X) \) with \( A = ACA \) and \( C = CAT \) and \( P = I_X - CA, P' = I_Y - AC \).

**Proof.** Let \( ABA = A \). Then

\[
(I_X - BA)^2 = I_X - BA - BA + BABA = I_X - BA - BA + BA = I_X - BA \quad \text{and} \\
(I_Y - AB)^2 = I_Y - AB - AB + BAB = I_Y - AB - AB + AB = I_Y - AB
\]

are projections which fulfill

\[
\text{im} A = \text{im} ABA \subset \text{im} AB \subset \text{im} A, \text{ hence } \text{im} A = \text{im} AB = \ker(I_Y - AB), \text{ and} \\
\ker A \subset \ker BA \subset \ker ABA = \ker A, \text{ hence } \ker A = \ker BA = \text{im}(I_X - BA).
\]

The rest of the first part is obvious.

Now let \( A, P \) and \( P' \) be given as stated. Then \( \ker P \) and \( \ker P' \) are closed. As a consequence of the Banach inverse mapping theorem, the operator \( A|_{\ker P} : \ker P \to \ker P' \) is invertible. Let \( A^{-1} \) be its inverse. Then \( C := (I - P)A^{-1}(I - P') \) does the job.

1.1.4 Notions of convergence

**Definition 1.8.** A sequence \( (A_n) \) of operators \( A_n \in \mathcal{L}(X, Y) \) is said to converge to \( A \in \mathcal{L}(X, Y) \)

- **strongly** if \( \|A_nx - Ax\|_Y \to 0 \) for each \( x \in X \) (we write \( A = \lim_{n \to \infty} A_n \)),

- **uniformly** if \( \|A_n - A\|_{\mathcal{L}(X, Y)} \to 0 \).

Let \( (A_n) \subset \mathcal{L}(X, Y) \) be a bounded sequence of operators. The Banach Steinhaus Theorem states that if \( A_nx \) converges for every \( x \in X \) then \( Ax := \lim_n A_nx \) defines a bounded linear operator \( A \in \mathcal{L}(X, Y) \), and

\[
\|A\| \leq \lim_{n \to \infty} \|A_n\|.
\]

It is well known that compact operators turn strong convergence into uniform convergence:
Proposition 1.9. Let $X$ be a Banach space and $A, A_n \in \mathcal{L}(X)$. Then

- $A_n \to A$ strongly $\iff$ $A_n K \to AK$ uniformly for every $K \in \mathcal{K}(X)$.
- $A_n^* \to A^*$ strongly $\iff$ $KA_n \to KA$ uniformly for every $K \in \mathcal{K}(X)$.

We say that a sequence of operators $A_n$ converges *-strongly, if $(A_n)$ as well as $(A_n^*)$ converge strongly.

1.1.5 Example: $l^p$-spaces and the finite section method

Now, it’s time to illustrate the introduced notions and their interactions with an example.

Let $X$ stand for a fixed complex Banach space and, for $1 \leq p < \infty$, let $l^p = l^p(\mathbb{Z}, X)$ denote the Banach space of all sequences $x = (x_i)_{i \in \mathbb{Z}}$ of elements $x_i \in X$ such that

$$\|x\|_p := \sum_{i \in \mathbb{Z}} \|x_i\|_X < \infty. \quad (1.1)$$

Note that the linear structure is given by entrywise defined addition and scalar multiplication, and (1.1) defines the norm. We further introduce $l^\infty = l^\infty(\mathbb{Z}, X)$ as the Banach space of all $x = (x_i)$ with

$$\|x\|_\infty := \sup_{i \in \mathbb{Z}} \|x_i\|_X < \infty.$$ 

In all what follows we again omit the subscript of the norm if the situation is unambiguous. For $1 \leq p < \infty$ the dual space of $l^p(\mathbb{Z}, X)$ can be identified with $l^q(\mathbb{Z}, X^*)$ where $1/p + 1/q = 1$.

Here, the dual product is given by

$$\langle (x_i), (y_i) \rangle := \sum_{i \in \mathbb{Z}} y_i(x_i).$$

Let, for a moment, $p = 2$ and $X$ be a Hilbert space of finite dimension. Then $l^2$ becomes a separable Hilbert space, provided with the scalar product defined by $\langle (x_i), (y_i) \rangle := \sum_{i \in \mathbb{Z}} \langle x_i, y_i \rangle X$.

Moreover, we can introduce projections $P_m \in \mathcal{L}(l^2)$, $m \in \mathbb{N}$, by the rule

$$P_m : (x_i) \mapsto (\ldots, 0, x_{-m}, \ldots, x_m, 0, \ldots) \quad (1.2)$$

and we may easily check that the sequence $(P_m)_{m \in \mathbb{N}}$ converges *-strongly to the identity.

For an equation $Ax = b$ with given $A \in \mathcal{L}(l^2)$, $b \in l^2$ and the unknown $x \in l^2$, the so-called finite section method uses the (finite dimensional) linear equations $P_n A P_n x_n = P_n b$ as substitutes for $Ax = b$. One hopes that they provide solutions $x_n$ (at least for sufficiently large $n$) which converge to the desired $x$. The following result, which is due to Polski [55], justifies this approach.

Proposition 1.10. Let $X$ be a Banach space and suppose that $A \in \mathcal{L}(X)$ is invertible, the sequence $(A_n) \subset \mathcal{L}(X)$ converges strongly to $A$, $(b_n) \subset X$ converges to $b \in X$, and $(A_n)$ is stable, that is

$$\exists N \text{ such that } A_n \text{ is invertible for } n > N \text{ and } \sup_{n > N} \|A_n^{-1}\| < \infty.$$ 

Then $A_n x_n = b_n$ are uniquely solvable for $n > N$ and the solutions $x_n$ converge to the (unique) solution $x$ of $Ax = b$.

---

3One usually says that the sequence $x$ is $p$-summable.
1.2 Approximate projections and the $\mathcal{P}$-setting

Looking back at the development and also at the modern proofs in the Banach algebra techniques which are used for the theory of approximation methods it becomes apparent that the interactions between compactness, Fredholmness, and strong convergence\(^6\) constitute a large part of the core.

We now want to modify this triple such that these fruitful relations survive and in this way also the familiar proof strategies still work, but a larger flexibility is achieved, such that we can also cover approximation methods which do not converge strongly.

In a way, we turn the table and we no longer hunt for methods which suit to the classical notions of compactness, Fredholmness and convergence, but we take the method itself as the starting point and construct a suitable triple.

1.2.1 Substitutes for compactness, Fredholmness and convergence

**Approximate projections** Let $X$ be a Banach space and let $\mathcal{P} = (P_n)_{n \in \mathbb{N}}$ be a bounded sequence of operators in $\mathcal{L}(X)$ with the following properties:

- $P_n \neq 0$ and $P_n \neq I$ for all $n \in \mathbb{N}$,
- For every $m \in \mathbb{N}$ there is an $N_m \in \mathbb{N}$ such that $P_n P_m = P_m P_n = P_m$ if $n \geq N_m$.

\(^4\)Notice that for an operator $A \in \mathcal{L}(l^p)$ the finite section sequence $(P_n A P_n)$ is stable (where the operators $P_n A P_n$ are considered as operators in $\mathcal{L}(\text{im } P_n)$, respectively) if and only if the sequence $(A_n) \subset \mathcal{L}(l^p)$ with $A_n = P_n A P_n + (I - P_n)$ is stable.

\(^5\)See, for example, the pioneering paper [83] of Silbermann, the monograph [12] or our applications in Part 4.

\(^6\)See Theorem 1.3 and Proposition 1.9

So, we see that for an invertible $A$ the stability of the finite section sequence $(P_n A P_n)$ provides the applicability of the finite section method to approximate the solution of $Ax = b$.\(^4\)

It is well known that the projections $P_n$ generate the ideal $\mathcal{K}(l^2)$ of all compact operators in the sense, that the smallest closed ideal in $\mathcal{L}(l^2)$ which contains all $P_n$ coincides with $\mathcal{K}(l^2)$. This observation conveys an impression that also compact operators will occur in the treatment of the stability, and, indeed, it has turned out during the last decades that one has to deal with ideals of “compact” sequences of the form\(^5\)

$$\{(P_n K P_n) + (G_n) : K \in \mathcal{K}(l^2), \|G_n\| \to n \to \infty 0\}.$$  

Now, we want to relax our assumptions on $p$ and $X$ and we ask what happens if $1 \leq p \leq \infty$ and $X$ is an arbitrary Banach space. The definition of the $P_n$ extends correctly to all cases, the Polski result covers all spaces $l^p$ with $1 \leq p < \infty$, but $p = \infty$ stays out of reach, because there the sequence $(P_n)$ does not converge strongly anymore. Since the classical approaches for the stability even require *-strong convergence in general, also the case $p = 1$ is unattainable. One reason for this is the fact that in the cases $p \in \{1, \infty\}$ the ideal which is generated by the operators $P_n$ does not coincide with the ideal of the compact operators anymore. Moreover, since the projections $P_n$ are non-compact if $\dim X = \infty$, the picture also gets dramatically worse in the infinite dimensional case.

Therefore, the next section is devoted to a slightly modified framework, which will again provide certain notions of compactness, Fredholmness and convergence having similar properties and interactions, which will allow to restate and to extend the Polski result and the theory which is necessary to characterize the stability, but which is more flexible. In particular, we aspire to an equal treatment of all cases $1 \leq p \leq \infty$ and all spaces $X$. 

---

\(^4\)Notice that for an operator $A \in \mathcal{L}(l^p)$ the finite section sequence $(P_n A P_n)$ is stable (where the operators $P_n A P_n$ are considered as operators in $\mathcal{L}(\text{im } P_n)$, respectively) if and only if the sequence $(A_n) \subset \mathcal{L}(l^p)$ with $A_n = P_n A P_n + (I - P_n)$ is stable.

\(^5\)See, for example, the pioneering paper [83] of Silbermann, the monograph [12] or our applications in Part 4.

\(^6\)See Theorem 1.3 and Proposition 1.9
Then $\mathcal{P}$ is referred to as an approximate projection.\footnote{This notion was introduced in [73].} In all what follows we set $Q_n := I - P_n$ and we write $m \ll n$ if $P_kQ_l = Q_lP_k = 0$ for all $k \leq m$ and all $l \geq n$.

**$\mathcal{P}$-compactness** Let $\mathcal{P}$ be an approximate projection. A bounded linear operator $K$ is called $\mathcal{P}$-compact if $\|KP_n - K\|$ and $\|P_nK - K\|$ tend to zero as $n \to \infty$. By $\mathcal{K}(X, \mathcal{P})$ we denote the set of all $\mathcal{P}$-compact operators on $X$ and by $\mathcal{L}(X, \mathcal{P})$ the set of all bounded linear operators $A$ for which $AK$ and $KA$ are $\mathcal{P}$-compact whenever $K$ is $\mathcal{P}$-compact.

**Theorem 1.11.** Let $\mathcal{P}$ be an approximate projection on the Banach space $X$. $\mathcal{L}(X, \mathcal{P})$ is a closed subalgebra of $\mathcal{L}(X)$, it contains the identity operator, and $\mathcal{K}(X, \mathcal{P})$ is a proper closed ideal of $\mathcal{L}(X, \mathcal{P})$. An operator $A \in \mathcal{L}(X)$ belongs to $\mathcal{L}(X, \mathcal{P})$ if and only if, for every $k \in \mathbb{N}$, 

$$\|P_kAQ_n\| \to 0 \text{ and } \|Q_nAP_k\| \to 0 \text{ as } n \to \infty.$$ 

The “$\mathcal{P}$-concept” which we are introducing here will be developed later on in Part 3 also in more abstract Banach algebras. Therefore we will elaborate the details there and, for the moment, we refer to Theorem 3.25.

**$\mathcal{P}$-Fredholmness and invertibility at infinity** Here are two possible generalizations of Fredholmness based on $\mathcal{P}$-compact operators instead of compact ones which are motivated by Theorem 1.3.

**Definition 1.12.** We say that $A \in \mathcal{L}(X)$ is invertible at infinity (with respect to $\mathcal{P}$) if there is an operator $B \in \mathcal{L}(X)$ with $I - AB, I - BA \in \mathcal{K}(X, \mathcal{P})$. In this case $B$ is referred to as a $\mathcal{P}$-regularizer for $A$.

An operator $A \in \mathcal{L}(X, \mathcal{P})$ is said to be $\mathcal{P}$-Fredholm if the coset $A + \mathcal{K}(X, \mathcal{P})$ is invertible in the quotient algebra $\mathcal{L}(X, \mathcal{P})/\mathcal{K}(X, \mathcal{P})$.

Notice that invertibility at infinity is defined in $\mathcal{L}(X)$, whereas for $\mathcal{P}$-Fredholmness we are restricted to $\mathcal{L}(X, \mathcal{P})$, since we need that $\mathcal{K}(X, \mathcal{P})$ forms a closed ideal in $\mathcal{L}(X, \mathcal{P})$. Both notions have been known and have been studied for a long time. The monographs by Rabinovich, Roch and Silbermann [63] and Lindner [44] already contain a comprehensive theory and many applications of this approach. But until now it was an open problem if these two notions coincide for operators $A \in \mathcal{L}(X, \mathcal{P})$. Here we answer this question affirmatively, under a natural condition on $\mathcal{P}$ which has been in the business since the inverse closedness of $\mathcal{L}(X, \mathcal{P})$ in $\mathcal{L}(X)$ is known, based on a proof of Simonenko [86].

**Definition 1.13.** Given a Banach space $X$ with an approximate projection $\mathcal{P} = (P_n)$, we set $S_1 := P_1$ and $S_n := P_n - P_{n-1}$ for $n > 1$. Further, for every finite subset $U \subset \mathbb{N}$, we define $P_U := \sum_{k \in U} S_k$. $\mathcal{P}$ is said to be uniform if $C_\mathcal{P} := \sup \|P_U\| < \infty$, the supremum over all finite $U \subset \mathbb{N}$.

**Theorem 1.14.** Let $\mathcal{P}$ be a uniform approximate projection on $X$. An operator $A \in \mathcal{L}(X, \mathcal{P})$ is $\mathcal{P}$-Fredholm if and only if it is invertible at infinity. In this case every $\mathcal{P}$-regularizer of $A$ belongs to $\mathcal{L}(X, \mathcal{P})$. Particularly, $\mathcal{L}(X, \mathcal{P})$ is inverse closed in $\mathcal{L}(X)$.

This main result (except the inverse closedness) is really new and due to the author, closes a long-standing gap in the theory of approximate projections and has been published in [81] for the first time. It is a special case of Theorem 3.28 whose proof is essentially based on Theorem 3.27 which is new as well, interesting on its own and, in the present context, reads as follows.
Theorem 1.15. Let \( \mathcal{P} \) be a uniform approximate projection on \( X \) and \( A \in \mathcal{L}(X, \mathcal{P}) \). Then there is an equivalent uniform approximate projection \( \hat{\mathcal{P}} = (\hat{F}_n) \) on \( X \) (depending on \( A \)) with \( C_\mathcal{P} \leq C_{\hat{\mathcal{P}}} \) such that
\[
\|F_nA\| = \|AF_n - F_nA\| \to 0 \quad \text{as} \quad n \to \infty.
\]
Here, two approximate projections \( \mathcal{P} = (P_n) \) and \( \hat{\mathcal{P}} = (F_n) \) on \( X \) are said to be equivalent if for every \( m \in \mathbb{N} \) there is an \( n \in \mathbb{N} \) such that
\[
P_mF_n = F_nP_m = P_m \quad \text{and} \quad F_nP_n = P_nF_n = F_m.
\]
Note that in this case we have \( \mathcal{K}(X, \mathcal{P}) = \mathcal{K}(X, \hat{\mathcal{P}}) \). Indeed, for \( K \in \mathcal{K}(X, \mathcal{P}) \),
\[
\|K(I - F_n)\| \leq \|KQ_m\|\|I - F_n\| + \|K\|\|P_m(I - F_n)\|,
\]
where the first summand tends to zero as \( m \to \infty \) and, for every fixed \( m \), the second one equals zero for an \( n_0 \) by the definition, and then also for all \( n \gg \mathcal{P} n_0 \). Analogously, we check \( \|(I - F_n)K\| \to 0 \) as \( n \to \infty \), hence \( K \in \mathcal{K}(X, \hat{\mathcal{P}}) \). Reversing the roles of \( \mathcal{P} \) and \( \hat{\mathcal{P}} \) gives the claim. Consequently, also \( \mathcal{L}(X, \mathcal{P}) = \mathcal{L}(X, \hat{\mathcal{P}}) \) and the notions of \( \mathcal{P} \)-compactness and \( \mathcal{P} \)-Fredholmness coincide with the respective \( \hat{\mathcal{P}} \)-notions. The same holds true for the so-called \( \mathcal{P} \)-strong convergence, a substitute for the third part of the triple, the strong convergence, which we introduce in the next paragraph.

\( \mathcal{P} \)-strong convergence Let \( \mathcal{P} = (P_n) \) be an approximate projection and, for each \( n \in \mathbb{N} \), let \( A_n \in \mathcal{L}(X) \). The sequence \( (A_n) \) converges \( \mathcal{P} \)-strongly to \( A \in \mathcal{L}(X) \) if, for all \( K \in \mathcal{K}(X, \mathcal{P}) \), both \( \|(A_n - A)K\| \) and \( \|K(A_n - A)\| \) tend to zero as \( n \to \infty \). In this case we write \( A_n \to A \) \( \mathcal{P} \)-strongly or \( A = \mathcal{P} \text{-lim}_n A_n \). We mention that this definition is based on the similar properties of strong convergence as stated in Proposition 1.9 and was introduced in [73].

Proposition 1.16. A bounded sequence \( (A_n) \) in \( \mathcal{L}(X) \) converges \( \mathcal{P} \)-strongly to \( A \in \mathcal{L}(X) \) iff
\[
\|(A_n - A)P_m\| \to 0 \quad \text{and} \quad \|P_m(A_n - A)\| \to 0 \quad \text{for every fixed} \quad P_m \in \mathcal{P}.
\] (1.3)

This easily follows from \( \mathcal{P} \subset \mathcal{K}(X, \mathcal{P}) \) and the estimates
\[
\|(A_n - A)K\| \leq \|K\|(A_n - A)P_m\| + \|A_n - A\|\|Q_nK\|
\]
\[
\|K(A_n - A)\| \leq \|K\|P_m(A_n - A)\| + \|A_n - A\|\|KQ_n\|.
\]

Unfortunately, the \( \mathcal{P} \)-strong limit is not unique in general. Consider, for instance, a projection \( P \notin \{0, 1\} \) and \( \mathcal{P} = (P_n) \) given by \( P_n := P \) for all \( n \). Then the sequence \( (P_n) \) converges \( \mathcal{P} \)-strongly to both \( P \) and \( 1 \). Therefore we adopt a further condition on \( \mathcal{P} \) from [63] to force uniqueness.

Definition 1.17. An approximate projection \( \mathcal{P} \) is called approximate identity if for every \( x \in X \) the estimate \( \sup_n \|P_n x\| \geq \|x\| \) holds.

By \( \mathcal{F}(X, \mathcal{P}) \) we denote the set of all bounded sequences \( (A_n) \subset \mathcal{L}(X) \), which possess a \( \mathcal{P} \)-strong limit in \( \mathcal{L}(X, \mathcal{P}) \).

Remark 1.18. Let \( \mathcal{P} \) be an approximate identity. Then \( \|A\| \leq \sup_n \|P_n A\| \) holds for every bounded linear operator \( A \). To check this, fix \( \epsilon > 0 \) and choose \( x_0 \in X \), \( \|x_0\| = 1 \) such that \( \|Ax_0\| \geq \|A\| - \epsilon \). Furthermore, take \( n_0 \) such that \( \|P_n A x_0\| \geq \|Ax_0\| - \epsilon \) to get
\[
\|A\| \leq \|P_{n_0} A x_0\| + 2\epsilon \leq \|P_{n_0} A\| + 2\epsilon \leq \sup_{m \in \mathbb{N}} \|P_m A\| + 2\epsilon.
\]

This gives the claim, since \( \epsilon \) was chosen arbitrarily.
Besides the uniqueness of $\mathcal{P}$-strong limits this provides several further structural properties.

**Theorem 1.19.** Let $\mathcal{P}$ be an approximate identity on $X$. Then \(^8\)

- The algebra $\mathcal{L}(X, \mathcal{P})$ is closed with respect to $\mathcal{P}$-strong convergence, this means if a sequence $(A_n) \subset \mathcal{L}(X, \mathcal{P})$ converges $\mathcal{P}$-strongly to $A$ then $A \in \mathcal{L}(X, \mathcal{P})$. Moreover, $(A_n)$ is bounded in this case, and hence belongs to $\mathcal{F}(X, \mathcal{P})$.
- The $\mathcal{P}$-strong limit of every $(A_n) \in \mathcal{F}(X, \mathcal{P})$ is uniquely determined.
- Provided with the operations $\alpha(A_n) + \beta(B_n) := (\alpha A_n + \beta B_n)$, $(A_n)(B_n) := (A_n B_n)$, and the norm $\|(A_n)\| := \sup_n \|A_n\|$, $\mathcal{F}(X, \mathcal{P})$ becomes a Banach algebra with identity $1 := (I)$. The mapping $\mathcal{F}(X, \mathcal{P}) \to \mathcal{L}(X, \mathcal{P})$ which sends $(A_n)$ to its limit $A = \mathcal{P}$-lim$_n A_n$ is a unital algebra homomorphism and (with $B_P := \sup_n \|P_n\|$)

$$\|A\| \leq B_P \liminf_{n \to \infty} \|A_n\|. \quad (1.4)$$

**Proof.** For the details of the proof we again refer to the more general result given with Theorem 3.34 and note that we can set $D_P := B_P$ due to Remark 1.18 and Remark 3.33. \(\square\)

**Some remarks on inclusions** It is obvious that for arbitrary Banach spaces $X$ with an approximate projection $\mathcal{P}$ the inclusions

$$\mathcal{K}(X, \mathcal{P}) \subset \mathcal{L}(X, \mathcal{P}) \subset \mathcal{L}(X)$$

hold. To disclose how the set of compact operators fits into that picture we borrow the following result from [44], Proposition 1.24.

**Proposition 1.20.** The inclusion $\mathcal{K}(X, \mathcal{P}) \subset \mathcal{K}(X)$ holds if and only if $\mathcal{P} \subset \mathcal{K}(X)$.

If $\mathcal{P}$ is an approximate identity on $X$ then we also have

$$\mathcal{L}(X, \mathcal{P}) \cap \mathcal{K}(X) \subset \mathcal{K}(X, \mathcal{P}).$$

**Proof.** At first, recall that every operator $A \in \mathcal{K}(X, \mathcal{P})$ is the norm limit of the sequence $(AP_n)$. Thus it is compact if all $AP_n$ are compact. If, conversely, one $P_n$ is not compact, then it belongs to $\mathcal{K}(X, \mathcal{P}) \setminus \mathcal{K}(X)$. Now let $\mathcal{P}$ be an approximate identity, let $A \in \mathcal{L}(X, \mathcal{P})$ be compact, and assume that $\|AQ_n\|$ does not tend to zero. Then there is a bounded sequence $(x_n) \subset X$ such that $AQ_n x_n$ does not tend to zero as well, but with the help of Theorem 1.11 we see that at least $P_k AQ_n x_n \to 0$ as $n \to \infty$ for every fixed $k$. Since $A$ is compact, it is always true that each subsequence of $(AQ_n x_n)$ has a convergent subsequence and the respective limit $y$ fulfills $P_k y = 0$ for every $k$, hence $y = 0$ since $\mathcal{P}$ is an approximate identity. This contradicts $AQ_n x_n \not\to 0$.

To show that also $\|Q_n A\| \to 0$, we mention that

$$\|Q_n A\| \leq \|Q_n A P_k\| + \|Q_n\|\|A P_k\|,$$

where the last term at the right hand side can be made as small as desired by choosing $k$ large and the first one tends to zero for every fixed $k$ by Theorem 1.11. \(\square\)

\(^8\)These results are already known from [63], Section 1.1.4 et seq.
1.2.2 On approximation methods for linear equations

Let \( X \) be a Banach space with an approximate projection \( P \). In accordance with the definition for operator sequences we say that a sequence \( (x_n) \) of elements \( x_n \in X \) converges \( P \)-strongly to \( x \in X \) if
\[
\|K(x_n - x)\| \to 0 \quad \text{as} \quad n \to \infty, \quad \text{for every} \quad K \in \mathcal{K}(X, P).
\]
It is obvious from the definition that a bounded sequence \( (x_n) \subset X \) converges \( P \)-strongly, iff
\[
\|P_m(x_n - x)\| \to 0 \quad \text{as} \quad m \to \infty, \quad \text{for every} \quad m \in \mathbb{N}.
\]

**Definition 1.21.** A Banach space \( X \) with an approximate projection \( P \) is called complete w.r.t. \( P \)-strong convergence if the following holds: Every bounded sequence \( (x_n) \subset X \) for which \( (P_n x_n)_{n \in \mathbb{N}} \) is a Cauchy sequence for every \( m \in \mathbb{N} \) has a \( P \)-strong limit \( x \in X \).

Here comes the announced generalized version of the Polski result (Proposition 1.10) which was discovered by Roch and Silbermann in [73].

**Proposition 1.22.** Let \( P \) be a uniform approximate projection on the Banach space \( X \). Suppose that \( (b_n) \subset X \) is bounded and converges \( P \)-strongly to \( b \in X \), \( A \in \mathcal{L}(X, P) \) is invertible, and the sequence \( (A_n) \subset \mathcal{L}(X) \) is stable and converges \( P \)-strongly to \( A \). Then the solutions \( x_n \) of \( A_n x_n = b_n \) converge \( P \)-strongly to the solution \( x \) of \( Ax = b \).

**Proof.** Let \( K \in \mathcal{K}(X, P) \), \( n \) be sufficiently large such that \( A_n \) is invertible, and consider
\[
\|K(x_n - x)\| = \|K(A_n^{-1}b_n - A^{-1}b)\| = \|K((A_n^{-1} - A^{-1})b_n + A^{-1}(b_n - b))\|
\]
\[
= \|KA^{-1}(A - A_n)A_n^{-1}b_n + KA^{-1}(b_n - b)\|
\]
\[
\leq \|KA^{-1}(A - A_n)\|\|A_n^{-1}b_n\| + \|KA^{-1}(b_n - b)\|.
\]

The operator \( A^{-1} \) belongs to \( \mathcal{L}(X, P) \) by Theorem 1.14, hence \( KA^{-1} \) is \( P \)-compact. This obviously yields that \( \|KA^{-1}(A - A_n)\| \) and \( \|KA^{-1}(b_n - b)\| \) tend to zero as \( n \to \infty \). Since \( \|A_n^{-1}b_n\| \) are uniformly bounded w.r.t. \( n \), the assertion follows. \( \square \)

1.2.3 Example: \( l^p \)-spaces

Let us again rest for a minute and ask, what this means for the spaces \( l^p = l^p(\mathbb{Z}, X) \) which were introduced above in Section 1.1.5 and for the finite section method applied to bounded linear operators on these spaces.

First of all we note that the sequence \( P = (P_n) \) as defined in (1.2) gives a uniform approximate identity with \( B_P = C_P = 1 \) for all \( 1 \leq p \leq \infty \) and all Banach spaces \( X \). Therefore all results of Section 1.2 can be applied. That is, the algebra \( \mathcal{L}(l^p, P) \) is closed under passing to inverses or \( P \)-regularizers (by Theorem 1.14), the notions “\( P \)-Fredholm” and “invertible at infinity” are equivalent, and \( P \)-strong convergence is well defined (see Theorem 1.19). We will even see that Fredholm operators in \( \mathcal{L}(l^p, P) \) are automatically \( P \)-Fredholm (Corollary 1.28).

Figure 1.1 illustrates the relations between the respective operator algebras \( \mathcal{L}(l^p), \mathcal{K}(l^p), \mathcal{L}(l^p, P) \), and \( \mathcal{K}(l^p, P) \) for the various configurations of \( p \) and \( \dim X \). To substantiate the asserted proper inclusions there, we consider two examples which we again take from [44].

- Let \( a \in X \) and \( f \in X^* \) be non-zero elements and let \( A \in \mathcal{L}(l^1) \) be given by the rule
\[
A : (x_i) \mapsto \left( \ldots, 0, 0, \sum_i f(x_i)a, 0, 0, \ldots \right).
\]
### 1.2. APPROXIMATE PROJECTIONS AND THE \( P \)-SETTING

<table>
<thead>
<tr>
<th>( 1 &lt; p &lt; \infty )</th>
<th>( \dim X &lt; \infty )</th>
<th>( \dim X = \infty )</th>
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<tr>
<td>( \mathcal{L}(l^p) = \mathcal{L}(l^p, P) )</td>
<td>( \mathcal{K}(l^p) = \mathcal{K}(l^p, P) )</td>
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<tr>
<th>( p = 1, \quad p = \infty )</th>
<th>( \mathcal{L}(l^p) )</th>
<th>( \mathcal{K}(l^p) )</th>
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<td>( \mathcal{L}(l^p, P) )</td>
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\( L^p(\mathbb{Z}, X) \) 

Figure 1.1: Venn diagrams of \( \mathcal{L}(l^p) \), \( \mathcal{K}(l^p) \), \( \mathcal{L}(l^p, P) \), and \( \mathcal{K}(l^p, P) \) depending on \( l^p = L^p(\mathbb{Z}, X) \).

This operator is compact, but does not belong to \( \mathcal{L}(l^1, P) \) due to Theorem 1.11. The adjoint of \( A \) provides the same outcome for the case \( p = \infty \).

- Let \( X \) be the space \( L^p[0,1] \) of all \( p \)-Lebesgue integrable functions over the interval \( [0,1] \) and define \( B : L^p(\mathbb{Z}, X) \to X \), \( (u_i) \mapsto v \) by

\[
v(x) := \begin{cases} 
  u_k(x) & : \text{if } x \in \left( 1 - \frac{1}{2^{k+1}}, 1 - \frac{1}{2^k} \right) \text{ for one } k \in \mathbb{N} \\
  0 & : \text{otherwise.}
\end{cases}
\]

(As customary we let \( L^\infty[0,1] \) stand for the space of all Lebesgue measurable functions that are essentially bounded.) Then the linear operator \( B : (u_i) \mapsto (\ldots, 0, 0, B(u_i), 0, 0, \ldots) \) acts boundedly on \( l^p = L^p(\mathbb{Z}, X) \) for every \( p \), but does not belong to \( \mathcal{L}(l^p, P) \) in any case.

For an operator \( A \in \mathcal{L}(l^p, P) \) the finite section sequence \( (P_n A P_n) \) is stable (where the operators \( P_n A P_n \) are considered as operators in \( \mathcal{L}(\operatorname{im} P_n) \), respectively) if and only if \( (A_n) \subset \mathcal{L}(l^p, P) \) with
A_n = P_n A P_n + Q_n is a stable sequence. Moreover, for b ∈ ℓ^p, the sequence (P_n b) is always bounded and converges ℋ-strongly to b. Thus, Proposition 1.22, applied to this concrete setting, reads as follows.

**Corollary 1.23.** Let A ∈ ℒ(ℓ^p, ℋ) be invertible and its finite section sequence be stable. Then the solutions x_n of the truncated equations P_n A P_n x_n = P_n b exist for sufficiently large n and they converge ℋ-strongly to the solution x of the equation Ax = b. \(^9\)

### 1.2.4 The ℋ-dichotomy, or how to grasp Fredholmness in the ℋ-setting

From Theorem 1.3 we know that the usual Fredholm property of bounded linear operators can also be described in terms of compact projections. Here comes the analogon based on ℋ-compact projections which was firstly studied in [80] and refined to the present shape in [81].

**Definition 1.24.** An operator A ∈ ℒ(X) is said to be properly ℋ-Fredholm, if there exist projections P, P′ ∈ K(X, ℋ) such that im P = ker A and ker P′ = im A.

An operator A ∈ ℒ(X) is called properly ℋ-deficient from the right (left) if, for each ε > 0 and each k ∈ ℤ, there is a projection R ∈ K(X, ℋ) of rank at least k such that ∥AR∥ < ε (∥RA∥ < ε, respectively).

Of course, it would be a great benefit to capture the usual Fredholm property (and the related attributes like the dimensions of the kernel and cokernel, the index, or the normal solvability) by ℋ-compact projections. Concerning this let us first get a short overview, the proofs and a more detailed discussion will follow afterwards. We initially condense our aim to a definition and give a first result.

**Definition 1.25.** An operator A ∈ ℒ(X) is said to have the ℋ-dichotomy if it is either Fredholm and properly ℋ-Fredholm, or it is properly ℋ-deficient from at least one side.

Further, we say that X has the ℋ-dichotomy, if every operator A ∈ ℒ(X, ℋ) has the ℋ-dichotomy.

**Proposition 1.26.** Let ℋ be an approximate projection and let A ∈ ℒ(X) be invertible at infinity. Then A has the ℋ-dichotomy.

Notice that, in case rank P_n < ∞ for all n, every ℋ-compact operator is compact. Hence, invertibility at infinity implies Fredholmness, and by the previous Proposition also proper ℋ-Fredholmness. If, on the other hand, not all P_n are of finite rank then there are operators which are not Fredholm but invertible at infinity and hence properly ℋ-deficient. A quite trivial example is the operator A = Q_1 on ℓ^2(ℤ, X) with dim X = ∞.

Nevertheless, we need practicable criteria to ensure this ℋ-dichotomy for A, even if we do not know anything about its invertibility at infinity. As in the preceding sections also here the situation becomes much more comfortable if we tighten the assumptions on ℋ.

**Theorem 1.27.** Let ℋ = (P_n) be a uniform approximate identity on the Banach space X. Then X has the ℋ-dichotomy if one of the following conditions is fulfilled:

- ℋ∗ = (P_n*) is an approximate identity on X∗.
- X is complete w.r.t. ℋ-strong convergence.

With these sufficient conditions we have everything what we (and the reader) need to know about the ℋ-dichotomy to cope with the material in the forthcoming sections. In particular, for our prototypic example ℓ^p we get

\(^9\)Actually, the demand on the invertibility of A is even redundant since it follows from the stability as we will see in the next step.
Corollary 1.28. All $l^p$-spaces have the $\mathcal{P}$-dichotomy.

Proof. With the identifications $(l^p(Z,X))^* = l^q(Z,X^*)$ for $1 < p < \infty$ and $1/p + 1/q = 1$, as well as $(l^1(Z,X))^* = l^\infty(Z,X^*)$ we easily deduce that $\mathcal{P}^*$ is an approximate identity in all the cases except for $p = \infty$. Furthermore, the spaces $l^\infty$ are complete w.r.t. $\mathcal{P}$-strong convergence. Thus, Theorem 1.27 gives the claim. 

Moreover, we mention that, for every operator $A \in \mathcal{L}(l^p,\mathcal{P})$, the stability of its finite section sequence $(P_nAP_n)$ already implies the invertibility of $A$, as the following Proposition shows. This simplifies Corollary 1.23.

Proposition 1.29. Let $\mathcal{P}$ be a uniform approximate identity, $(A_n) \in \mathcal{F}(X,\mathcal{P})$ be stable, and $A := \mathcal{P}\lim_n A_n$ have the $\mathcal{P}$-dichotomy. Then $A$ is invertible and the inverses $A_n^{-1}$ converge $\mathcal{P}$-strongly to $A^{-1}$.

Proof. For large $n$ and every $K \in \mathcal{K}(X,\mathcal{P})$

$$
\|A_nK\| = \frac{\|A_n^{-1}\|}{\|A_n\|} \|A_nK\| \geq \frac{1}{\|A_n^{-1}\|} \|K\|.
$$

For $n \to \infty$, we obtain $\|AK\| \geq C\|K\|$ and analogously $\|KA\| \geq C\|K\|$ for every $\mathcal{P}$-compact $K$, where $C \geq (\sup_n \|A_n^{-1}\|)^{-1} > 0$ is constant. Thus, $A$ is not $\mathcal{P}$-deficient, hence properly $\mathcal{P}$-Fredholm. Suppose that the kernel of $A$ is not trivial. Then there is a non-trivial projection $P \in \mathcal{K}(X,\mathcal{P}), P \neq 0$ such that $0 = \|AP\| \geq C\|P\| \geq C$, yielding a contradiction. Thus $A$ is injective. Analogously one shows that $A$ is surjective and hence invertible, due to the Banach inverse mapping theorem. From Theorem 1.14 we even get $A^{-1} \in \mathcal{L}(X,\mathcal{P})$. Further, we know for large $n$ and every $K \in \mathcal{K}(X,\mathcal{P})$

$$(A_n^{-1} - A^{-1})K = A_n^{-1}(I - A_nA^{-1})K = A_n^{-1}(A - A_n)A^{-1}K,$$

which obviously converges to zero in the norm as $n \to \infty$. In the same way we find that $K(A_n^{-1} - A^{-1}) \to 0$ in the norm, hence the $\mathcal{P}$-strong convergence $A_n^{-1} \to A^{-1}$ follows. 

Proofs and technical details Let us try to gather more understanding of the somewhat mysterious $\mathcal{P}$-dichotomy. First of all, from Proposition 1.7 we immediately conclude

Corollary 1.30. An operator $A \in \mathcal{L}(X)$ is properly $\mathcal{P}$-Fredholm iff there is a $\mathcal{P}$-regularizer $B \in \mathcal{L}(X)$ for $A$ which is also a generalized inverse, that is $I - AB, I - BA \in \mathcal{K}(X,\mathcal{P})$ and $ABA = A, BAB = B$.

If $A \in \mathcal{L}(X,\mathcal{P})$ and $\mathcal{P}$ is even uniform then the situation becomes more relaxed.

Corollary 1.31. Let $B$ be a uniform approximate projection. Then $A \in \mathcal{L}(X,\mathcal{P})$ is properly $\mathcal{P}$-Fredholm if and only if it is $\mathcal{P}$-Fredholm and has a generalized inverse in $\mathcal{L}(X,\mathcal{P})$. In this case there is a $B \in \mathcal{L}(X,\mathcal{P})$ which is $\mathcal{P}$-regularizer and generalized inverse at the same time.

Proof. Let $A \in \mathcal{L}(X,\mathcal{P})$ be properly $\mathcal{P}$-Fredholm. Then, by Corollary 1.30, there is an operator $B \in \mathcal{L}(X)$ such that $A = ABA, B = BAB$ and $I - AB, I - BA \in \mathcal{K}(X,\mathcal{P})$. From Theorem 1.14 we obtain that $B \in \mathcal{L}(X,\mathcal{P})$.

Conversely, let $A$ be $\mathcal{P}$-Fredholm with $\mathcal{P}$-regularizer $C \in \mathcal{L}(X,\mathcal{P})$ and let $B \in \mathcal{L}(X,\mathcal{P})$ be a generalized inverse for $A$. The projections $P := I - BA, P' := I - AB$ are contained in $\mathcal{L}(X,\mathcal{P})$. Moreover, $P = (I - CA)P + CAP = (I - CA)P$ and $P' = PA^*(I - AC) + P'AC = P'(I - AC)$ even show that $P, P' \in \mathcal{K}(X,\mathcal{P})$, that is $B$ is a $\mathcal{P}$-regularizer for $A$. In view of Corollary 1.30, this yields the proper $\mathcal{P}$-Fredholmness. 

\[\Box\]
Proposition 1.32. \textit{Let }A \in \mathcal{L}(X) \text{ be invertible at infinity (w.r.t. an approximate projection } P). \\
If \( A \) is normally solvable then for every \( k \in \mathbb{N} \) with \( k \leq \dim \ker A \ (k \leq \dim \ker A) \) there is a }\( P\)-\textit{compact projection }\( P \text{ of rank } k \text{ such that } AP = 0 \ (PA = 0). \)

\text{If } A \text{ is not normally solvable then } A \text{ is properly } P\text{-deficient from both sides.}

\textbf{Proof.} \textit{Let }B \in \mathcal{L}(X) \text{ be a } P\text{-regularizer for the operator } A. \text{ For every } x \in \ker A \text{ we find that } Q_n x = Q_n(I - BA)x + Q_nBAx = Q_n(I - BA)x \text{ tends to zero as } n \to \infty. \text{ Let } X_1 \text{ be a finite dimensional subspace of } \ker A. \text{ Then we can fix } m \in \mathbb{N} \text{ s.t. } \sup \{ \|Q_m x\| : x \in X_1,\|x\| = 1 \} < 1/2 \text{ and deduce that the operator } P_m : X_1 \to X_2 := P_m(X_1) \text{ is invertible and its inverse has norm less than } 2. \text{ Let } S \text{ denote its inverse and let } \tilde{m} \gg m. \text{ Since } X_2 \text{ is a finite dimensional subspace of the Banach space } \ker Q_{\tilde{m}}, \text{ there is a bounded projection } R \in \mathcal{L}(\ker Q_{\tilde{m}}) \text{ onto } X_2 \text{ with } \|R\| \leq \dim X_1, \text{ by Proposition 1.5. Now, we define } P := S R P_m \text{ and we easily check that } \text{im } P = X_1 \text{ as well as } P^\perp = S R P_m P = S P_m P = P, \text{ hence } P \text{ is a projection onto } X_1 \text{ which obviously belongs to } K(X, P).

Assume now that } A \text{ is not normally solvable and fix } \epsilon > 0 \text{ and } k \in \mathbb{N}. \text{ Proposition 1.6 provides us with a rank-k-projection } T \text{ such that } \|AT\| \leq \epsilon. \text{ Further, denote by } d \text{ the finite number } \sup_n \|Q_n\| \text{ and check that }

\[
\|Q_n T\| \leq \|Q_n(I - BA)\| + \|Q_n BAT\| \\
\leq \|Q_n(I - BA)\|\|T\| + d\|B\|\|A\| \leq 2d\|B\|\|A\| \epsilon
\]

for sufficiently large \( n \). Further, for \( x \in \text{im } T, \)

\[
\frac{\|AP_n x\|}{\|P_n x\|} \leq \frac{\|Ax\| + \|A\|\|Q_n x\|}{\|x\| - \|Q_n x\|} \leq \frac{1 + 2d\|B\|\|A\|}{1 - 2d\|B\|\|A\|} \epsilon.
\]

This shows that for sufficiently small \( \epsilon \) and sufficiently large \( l \) the space }\( X_3 := \text{im } P T \text{ is of dimension } k \) \text{ and } \|Az\| \leq 4d\|B\|\|A\|\|z\| \text{ holds for all } z \in X_3. \text{ Since } X_3 \subset \ker Q_l \subset \ker Q_{\tilde{l}} \text{ for } \tilde{l} \gg l \text{ we can again choose a projection } R \in \mathcal{L}(\ker Q_{\tilde{l}}) \text{ onto } X_3 \text{ with norm at most } k \text{ and define } P := RP_l. \text{ Obviously, } P \text{ is a } P\text{-compact projection of rank } k \text{ and we have the estimate } \|AP\| \leq 4kd(d + 1)\|B\|\|A\|\|A\|\|z\| \leq 2k(k + 1)(d + 1) \text{ such that } \text{im } (P^\perp) = \text{im } T, \text{ and hence }

\[
\|P^\perp A\| = \|A^\perp (P^\perp)\| = \|A^\perp T(P^\perp)\| \leq 2k(k + 1)(d + 1)\epsilon.
\]

Then, since } \epsilon \text{ was chosen arbitrarily, } A \text{ proves to be properly } P\text{-deficient from the left. For this, let } l \text{ be such that } \|Q_l T\| < 1/2. \text{ Then, for every } f \in \text{im } T \text{ we have }

\[
\|f \circ P_l\| = \|P_l f\| \geq \|f\| - \|Q_l f\| = \|f\| - \|(Q_l T)f\| \geq \frac{1}{2}\|f\|, \quad (1.5)
\]

that is }\{ f |_{\text{im } P_l} : f \in \text{im } T \} \text{ forms a linear space of dimension } k. \text{ Hence, for } \tilde{l} \gg l, \text{ }

\[
X_1 := \bigcap_{f \in \text{im } T} \ker f |_{\ker Q_l} \subset \ker Q_l
\]

has codimension } k \text{ in } \ker Q_l. \text{ Due to Proposition 1.5 we can choose a projection } R \text{ parallel to } X_1 \text{ onto a certain complement } X_2 \text{ of } X_1 \text{ in } \ker Q_l \text{ with the norm not greater than } k + 1. \text{ Since the}
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set of all restrictions $g = f|_{X_2}$ of the functionals $f \in \text{im } T$ to $X_2$ forms a $k$-dimensional space we conclude that each functional on $X_2$ is of the form $g = f|_{X_2}$ with $f \in \text{im } T$. Auerbach’s Lemma (Proposition 1.4) provides bases $x_1, \ldots, x_k \in X_2$ and $f_1, \ldots, f_k \in \text{im } T$ such that $\|x_i\| = \|g_j\| = 1$ and $g_j(x_i) = \delta_{ij}$ for all $i, j = 1, \ldots, k$, where $g_j := f_j|_{X_2}$ (here $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise). Due to (1.5) the norms $\|f_j\|$ can be estimated by

$$\|f_j\| \leq 2\|f_j \circ R\| \leq 2\|f_j \circ R \circ P_l\| + 2\|f_j \circ (I - R) \circ P_l\| \leq 2\|g_j\| \|R\| \|P_l\| < 2(k + 1)(d + 1),$$

thus, defining $P^x := \sum_{i=1}^k f_i(x)x_i$, we obtain a $\mathcal{P}$-compact projection onto $X_2$ with the norm less than $2k(k + 1)(d + 1)$. Since $f_j(P^x) = \sum_{i=1}^k f_i(x)f_j(x_i) = f_j(x)$ for all $j$ and $x$ we find that $\text{im}(P^x) = \text{im } T$.

It remains to consider operators $A$ which are normally solvable, hence $\text{dim } \ker A = \text{dim } \ker A^*$, and to check that for every finite $k \leq \text{dim } \ker A$ there is a $\mathcal{P}$-compact projection $P'$ of rank $k$ with $P'A = 0$. Since $A^*$ is invertible at infinity (w.r.t. $\mathcal{P}^*$) we can apply the first part of this proof to find a $\mathcal{P}^*$-compact projection $T$ of rank $k$ onto a respective subspace of $\ker A^*$. Then we simply apply the above argument with $\epsilon = 0$ to construct $P'$.

For the proof of Theorem 1.27 we need some preliminary results and we introduce a useful and important closed subspace $X_0$ of $X$ by

$$X_0 := \{x \in X : \|Q_nx\| \to 0 \text{ as } n \to \infty\}.$$

**Proposition 1.33.**

1. For each $A \in \mathcal{L}(X, \mathcal{P})$ the restriction $A|_{X_0}$ to $X_0$ is contained in $\mathcal{L}(X_0, \mathcal{P})$ and if $K \in \mathcal{K}(X, \mathcal{P})$ then $K|_{X_0} \in \mathcal{K}(X_0, \mathcal{P})$.

2. Let $X_1 \subset X_0$ be a finite dimensional subspace. Then there is a projection $P \in \mathcal{K}(X, \mathcal{P})$ onto $X_0$ with the norm not greater than $2B_{\mathcal{P}} \dim X_1$, where $B_{\mathcal{P}} := \sup_n \|P_n\|$.

3. If $\mathcal{P} = (P_n)$ is an approximate identity on $X$ and $A \in \mathcal{L}(X, \mathcal{P})$ then

$$B_{\mathcal{P}}^{-2}\|A\|_{\mathcal{L}(X)} \leq \|A|_{X_0}\|_{\mathcal{L}(X_0)} \leq \|A\|_{\mathcal{L}(X)},$$

Furthermore, for every $T \in \mathcal{K}(X_0, \mathcal{P})$ there is a lifting $K \in \mathcal{K}(X, \mathcal{P})$ of $T$, that is $K|_{X_0} = T$. Restrictions and liftings of $\mathcal{P}$-compact projections are again projections of the same rank.

**Proof.** 1. Let $A \in \mathcal{L}(X, \mathcal{P})$ and $x \in X_0$. Since $\|Q_nAx\| \leq \|Q_nA P_l\| \|x\| + \|Q_n\| \|A\| \|Q_lx\|$, where the latter term is smaller than any prescribed $\epsilon > 0$ if $l$ is large enough, and the first term tends to zero for any fixed $l$ and $n \to \infty$, we find that $Ax \in X_0$. Hence $A|_{X_0} \in \mathcal{L}(X_0, \mathcal{P})$. If $K \in \mathcal{K}(X, \mathcal{P})$ then it is also clear by the definition that $K|_{X_0} \in \mathcal{K}(X_0, \mathcal{P})$.

2. Recall the projection $P := \text{SRP}_m$ from the proof of Proposition 1.32 which obviously fulfills $\|P\| \leq \|S\| \|R\| \|P\|_{\mathcal{P}} \leq 2B_{\mathcal{P}} \dim R$.

3. For every $\epsilon > 0$ there is an $m \in \mathbb{N}$ such that $\|A\| \leq \|P_m\| \|A\| + \epsilon$ (see Remark 1.18) and for all sufficiently large $n$ we have $\|P_nA Q_n\| \leq \epsilon$. Now (1.6) easily follows by

$$\|A|_{X_0}\| \leq \|A\| \leq \|P_nA P_n\| + 2\epsilon \leq B_{\mathcal{P}}^2 \|A|_{X_0}\| + 2\epsilon.$$

Further, let $T \in \mathcal{K}(X_0, \mathcal{P})$. From $\|T - P_n TP_n\|_{\mathcal{L}(X_0)} \to 0$ as $n \to \infty$ we conclude for the sequence $(P_n TP_n) \subset \mathcal{K}(X, \mathcal{P})$ and $n, m$ large that

$$\|P_n TP_n - P_m TP_m\|_{\mathcal{L}(X_0)} \leq B_{\mathcal{P}}^2 \|P_n TP_n - P_m TP_m\|_{\mathcal{L}(X_0)}$$

$$\leq B_{\mathcal{P}}^2 (\|P_n TP_n - P\|_{\mathcal{L}(X_0)} + \|T - P_n TP_m\|_{\mathcal{L}(X_0)}) \to n, m \to \infty 0.$$

Hence $(P_n TP_n) \subset \mathcal{K}(X, \mathcal{P})$ is a Cauchy sequence. For its limit $K \in \mathcal{K}(X, \mathcal{P})$ we easily check that the compression $K|_{X_0}$ coincides with $T$. The rest is easy to prove. \qed
Here now comes an extended version of Theorem 1.27.

**Theorem 1.27***. Let $\mathcal{P}$ be a uniform approximate identity. Then $A \in \mathcal{L}(X, \mathcal{P})$ has the $\mathcal{P}$-dichotomy if one of the following conditions is fulfilled:

- \( \mathcal{P}^* \) is an approximate identity on $X^*$.
- There are a Banach space $Y$ and operators $R_n, B \in \mathcal{L}(Y)$ such that $Y^* = X$, $R_n^* = P_n$ for all $n \in \mathbb{N}$ and $B^* = A$.
- For every properly $\mathcal{P}$-Fredholm operator $B \in \mathcal{L}(X, \mathcal{P})$ the sequence $(BP_n)$ has a $\mathcal{P}$-strong limit in $\mathcal{L}(X)$.
- $X$ is complete w.r.t. $\mathcal{P}$-strong convergence.

**Proof.** We prepare the proof with some basic observations in the first and a technical result in the second step.

1st step. Let $\mathcal{P} = (P_n)$ be a uniform approximate projection given by Theorem 1.15. Notice first that for every fixed $k$ and sufficiently large $n$, $\|P_k x\| = \|P_k F_n x\| \leq C_\mathcal{P} \|F_n x\|$. Hence, we deduce that

$$
\limsup_{n \to \infty} \|F_n x\| \geq C_\mathcal{P}^{-1} \|x\| \quad \text{for all } x \in X.
$$

Moreover, for each $x \in X$ we have $\|Q_k x\| \to 0$ if and only if $\|(I - F_n)x\| \to 0$ as $n \to \infty$. Further we write $m \ll n$ if $F_k(I - F_l) = (I - F_l)F_k = 0$ for all $k \leq m$ and all $l \geq n$.

2nd step. Suppose that, for given $\epsilon, \delta > 0$ and $x \in X$ with $\|x\| = 1$, we have $\|Ax\| \leq \epsilon$ and

$$
\limsup_{n \to \infty} \|(I - F_n)x\| \geq \delta.
$$

Choose $n_1 \gg N_0$ such that $\|(I - F_n_1)x\| \geq \delta/2$ and $m_1 \gg n_1$ such that, due to (1.7),

$$
\|(F_{m_1} - F_{n_1})x\| = \|F_{m_1}(I - F_{n_1})x\| = (2C_\mathcal{P})^{-1} \|(I - F_{n_1})x\| \geq (4C_\mathcal{P})^{-1} \delta.
$$

Set $x_1 := \frac{(F_{m_1} - F_{n_1})x}{\|(F_{m_1} - F_{n_1})x\|}$ and fix $N_1 \gg m_1$.

We iterate this procedure as follows: Suppose that $n_k, m_k, N_k, x_k$ for $k = 1, \ldots, l - 1$ are given. Then we analogously choose $n_l \gg N_{l-1}$ such that $\|(I - F_{n_l})x\| \geq \delta/2$, and $m_l \gg n_l$ such that $\|(F_{m_l} - F_{n_l})x\| \geq (4C_\mathcal{P})^{-1} \delta$, and set $x_l := \frac{(F_{m_l} - F_{n_l})x}{\|(F_{m_l} - F_{n_l})x\|}$ as well as $N_l \gg m_l$. By doing this, we obtain a set $\{x_1, \ldots, x_l\} \subset X$ and integers $N_0, \ldots, N_l$ such that $(F_{N_k} - F_{N_{k-1}})x_i = \delta_k x_i$. For $y = \sum_{i=1}^l \alpha_i x_i$ and every $k = 1, \ldots, l$ we find

$$
\|y\| = \left\| \sum_{i=1}^l \alpha_i x_i \right\| \geq \frac{1}{\|F_{N_{k-1}} - F_{N_k}\|} \left\| (F_{N_k} - F_{N_{k-1}}) \sum_{i=1}^l \alpha_i x_i \right\| \geq \frac{1}{C_\mathcal{P}} \|\alpha_k x_k\| = \frac{1}{C_\mathcal{P}} |\alpha_k|
$$

and hence

$$
\|Ay\| \leq \sum_{i=1}^l |\alpha_i| \|Ax_i\| \leq \sum_{i=1}^l C_\mathcal{P} \|y\| \frac{4C_\mathcal{P}}{\delta} \delta \leq \frac{16C_\mathcal{P}^4}{\delta} \epsilon \|y\|.
$$

Let $N_{l+1} \gg N_l$ and choose a projection $R \in \mathcal{L}(\ker(I - F_{N_{l+1}}))$ of norm at most $l$ onto span\{x_1, \ldots, x_l\} \subset \ker(I - F_{N_l}) \subset \ker(I - F_{N_{l+1}})$, due to Proposition 1.5. Then $P := RF_{N_l}$ is a projection of rank $l$ and $\|AP\| \leq \frac{16C_\mathcal{P}^4}{\delta} \epsilon$. Moreover, $P$ is $\mathcal{P}$-compact, hence $\mathcal{P}$-compact.
1.2. APPROXIMATE PROJECTIONS AND THE \( \mathcal{P} \)-SETTING

3rd step. If there is an \( x \in \ker A \) such that the norms \( \|Q_n x\| \) do not tend to zero then the second step yields that for every \( k \) and every \( \gamma > 0 \) there is a \( \mathcal{P} \)-compact projection \( P \) of rank \( k \) such that \( \|AP\| < \gamma \), hence \( A \) is properly \( \mathcal{P} \)-deficient from the right.

If, on the other hand, \( \|Q_n x\| \to 0 \) as \( n \to \infty \) for all \( x \in \ker A \), i.e. \( \ker A \subset X_0 \) then, due to Proposition 1.33, 2, we have that for Fredholm operators \( A \) there is always a \( \mathcal{P} \)-compact projection onto its kernel, and if \( A \) has an infinite dimensional kernel then it is properly \( \mathcal{P} \)-deficient.

4th step. Suppose that \( A|_{X_0} \) is not normally solvable, that is for every fixed \( \epsilon > 0 \) and every \( k \in \mathbb{N} \) there is a subspace \( X_1 \subset X_0 \) of dimension \( k \) such that \( \|A|_{X_1}\| \leq \epsilon/(2kBP) \) (see Proposition 1.6).

By Proposition 1.33, 2, we can choose a projection \( P \in \mathcal{K}(X, \mathcal{P}) \) onto \( X_1 \) with the norm not greater than \( 2kB \). Then \( \|AP\| \leq \|A|_{X_1}\| \|P\| \leq \epsilon. \) Since \( \epsilon \) and \( k \) are arbitrary, we deduce the proper \( \mathcal{P} \)-deficiency of \( A \) from the right.

Now suppose that \( A|_{X_0} \) is normally solvable and \( A \) has a finite dimensional kernel contained in \( X_0 \) but \( A \) is not normally solvable. Let \( X_2 \) denote a complement of \( \ker A \) in \( X \). Then the operator \( A|_{X_2} : X_2 \to X \) is still not normally solvable, but the compression \( A|_{X_2} : X_3 \to X_0 \) to the space \( X_3 := \{x \in X_2 : \|Q_n x\| \to 0 \text{ as } n \to \infty\} \) which is a complement of \( \ker A \) in \( X_0 \) is normally solvable, hence \( C := \inf\{\|Ax\| : x \in X_3, \|x\| = 1\} > 0. \) Then for every \( x \in X_2 \), \( \|x\| = 1 \) with \( \text{dist}(x, X_3) < 1/4 \min\{1, C/\|A\|\} \) we have \( \|Ax\| \geq C/2. \) Consequently, there is a \( \delta > 0 \) such that for every \( \epsilon > 0 \) there is an \( x \in X_2 \) with \( \|x\| = 1, \limsup_n \|I - F_n x\| \geq \delta \) and \( \|Ax\| \leq \epsilon \), and the second step again yields the proper \( \mathcal{P} \)-deficiency from the right.

5th step. It remains to consider operators \( A \) which are normally solvable and \( \dim \ker A < \infty. \)

Notice that \( \dim \ker A = \dim \ker A^* \) in this case, and due to the above considerations \( A|_{X_0} \) is normally solvable, too, hence \( \dim \ker A|_{X_0} = \dim \ker(A|_{X_0})^*. \)

Obviously, \( \mathcal{P} \) is a uniform approximate identity on \( X_0 \), which tends strongly to the identity, and the sequence \( \mathcal{P}^* = (P_n^*) \) still forms a uniform approximate identity on \( (X_0)^\circ \). Indeed, for a functional \( f \in (X_0)^\circ \) and for \( \epsilon > 0 \) let \( x \in X_0 \), \( \|x\| = 1 \) be such that \( |f(x)| \geq \|f\| - \epsilon. \) Then

\[
\|P_n^* f\| \geq |P_n^* f(x)| = |f(P_n x)| \geq |f(x)| - |f(Q_n x)| \geq \|f\| - \epsilon - \|f\|\|Q_n x\|,
\]

hence \( \sup_n \|P_n^* f\| \geq \|f\|. \)

Thus, we can apply the previous steps to \( (A|_{X_0})^* \) and find, for every \( k \leq \dim \ker A|_{X_0} \), a \( \mathcal{P}^* \)-compact rank-\( k \)-projection \( T \) such that \( (A|_{X_0})^* T = 0 \). As in the proof of Proposition 1.32 this yields a \( \mathcal{P} \)-compact projection \( P' \) of rank \( k \) such that \( P' A|_{X_0} = 0 \). For its lifting \( \hat{P}' \) which is given by Proposition 1.33 we find that also \( \hat{P}' A = 0 \) by the estimate

\[
\|\hat{P}' A\| \leq \|\hat{P}' A P_n\| + \|\hat{P}' A^* Q_n\| \leq \|P' A|_{X_0}\| \|P_n\| + \|\hat{P}' A Q_n\|,
\]

(1.8)

where the first term equals zero and the second one tends to zero as \( n \to \infty. \) Thus, we conclude that if \( \dim \ker A|_{X_0} = \infty \) then \( A|_{X_0} \) and \( A \) are properly \( \mathcal{P} \)-deficient from the left. Moreover, if \( \dim \ker A|_{X_0} < \infty \) then \( A|_{X_0} \) is properly \( \mathcal{P} \)-Fredholm, and if, additionally,

\[
\dim \ker A = \dim \ker A|_{X_0}
\]

(1.9)

would be true then \( A \) would be properly \( \mathcal{P} \)-Fredholm as well.

6th step. If \( \mathcal{P}^* \) is an approximate identity then the idea of the 5th step can be directly applied to \( A \) instead of \( A|_{X_0} \), and the assertion of the theorem is proved for this case.

7th step. If there is a predual space \( Y \) and operators \( (R_n) \) as supposed in the second condition of the theorem then notice that \( (R_n) \) is a uniform approximate identity on \( Y \). Indeed, assume that there is a \( y \in Y \) with \( \sup_n \|R_n y\| < \|y\| \). Then \( y \notin Y_0. \) Thus, a bounded linear functional \( x \in X \) exists with \( |x(y)| > 0 \) and \( x(Y_0) = \{0\} \), that is \( x \neq 0 \) but \( P_n x = x \circ R_n^* = 0 \) for all \( n \), a contradiction. Moreover, \( \|R_n (x - I) R_m\| = \|P_m (R_n - I)\|, \|R_m (R_n - I)\| = \|(P_n - I) P_m\| \), and
The preadjoint operator $B$ has the $(R_n)$-dichotomy by what we have already proved and the assertion is now also obvious for $A$.

8th step. We show that the third condition in the theorem implies (1.9). Let $\dim \ker A < \infty$ as well as $\dim \coker A|_{X_0} < \infty$. Then $A|_{X_0} : X_0 \to X_0$ is properly $\mathcal{P}$-Fredholm and we can choose a $\mathcal{P}$-regularizer $B$ such that $P' := I - AB \in \mathcal{K}(X_0, \mathcal{P})$ is a projection parallel to the image of $A|_{X_0}$. Let $\tilde{B}$ denote the $\mathcal{P}$-strong limit of $(B_P^n)_n$ which is in $\mathcal{L}(X, \mathcal{P})$ (see Theorem 1.19). Further, let $\tilde{\tilde{P}}$ denote the lifting of $P'$. With the help of Remark 1.18 we find an $m$ such that

$$\|I - \tilde{\tilde{P}} - A\tilde{\tilde{B}}\| \leq 2\|P_m(I - \tilde{\tilde{P}} - A\tilde{\tilde{B}})\| \leq 2\|P_m((I - P')P_n - A\tilde{\tilde{B}})\| + 2\|P_m(I - \tilde{\tilde{P}}')Q_n\|$$

Since both items tend to zero as $n \to \infty$ we see that $\tilde{\tilde{P}} = I - A\tilde{\tilde{B}}$. Together with $\tilde{\tilde{P}}'A = 0$ due to (1.8) this yields that $P'$ is a projection parallel to $\im A$. Hence $A$ is Fredholm, together with step 3 also $\mathcal{P}$-Fredholm, and the assertion of the theorem is proved as well.

9th step. We finally show that the last condition of the theorem implies the third one. For this, let $\tilde{B} \in \mathcal{L}(X_0, \mathcal{P})$ be properly $\mathcal{P}$-Fredholm. Then $(B_P^n)_n$ is bounded and $(P_kBP_n)_n$ is a Cauchy sequence in $\mathcal{L}(X, \mathcal{P})$ for every fixed $k$, since

$$\|P_kBP_n - P_kBP_m\| \leq \|P_kBQ_l\|\|P_n - P_m\|$$

for $n, m \gg l$, and $\|P_kBQ_l\| \to 0$ as $l \to \infty$, hence there are uniform limits $B_k \in \mathcal{L}(X, \mathcal{P})$ of $(P_kBP_n)_{n \in \mathbb{N}}$ for each $k$. Moreover, due to our assumption, for each $x \in X$, there is a uniquely determined $\mathcal{P}$-strong limit

$$\tilde{B}x := \mathcal{P}\lim_{n \to \infty} B_P^n x \quad \text{in} \quad X,$$

and the mapping $x \mapsto \tilde{B}x$ defines a bounded linear operator $\tilde{B}$. Obviously, $\tilde{B}|_{X_0} = B$ and by

$$\|(P_k\tilde{B} - B_k)x\| \leq \|P_k(\tilde{B} - BP_n)x\| + \|P_kBP_n - B_k\||x|| \to 0 \quad \text{as} \quad n \to 0,$$

for every $x \in X$, we see that $P_k\tilde{B} = B_k$. Hence

$$\|P_k(\tilde{B} - BP_n)\| = \|B_k - P_kBP_n\| \to 0 \quad \text{as} \quad n \to \infty,$$

$$\|(\tilde{B} - BP_n)P_k\| = \|(\tilde{B} - B)P_k\| = 0 \quad \text{for} \quad n \gg k.$$

Thus, $(BP_n)$ converges $\mathcal{P}$-strongly to $\tilde{B}$. \qed

In summary, we see that in the case of uniform approximate identities every operator $A \in \mathcal{L}(X, \mathcal{P})$ has the $\mathcal{P}$-dichotomy, if $X$ is small (in the sense that $\mathcal{P}^+$ should be an approximate identity) or large (in the sense of being complete w.r.t. $\mathcal{P}$-strong convergence). Until now it is not clear if there really is a gap between these two extremal cases. The exact formulation of this open question is as follows:

*Are there a Banach space $X$, a uniform approximate identity $\mathcal{P}$ and an operator $A \in \mathcal{L}(X, \mathcal{P})$ which is normally solvable, and fulfills $\dim \ker A < \infty$ and $\dim \coker A|_{X_0} < \dim \coker A$?*

In the Fredholm theory for band-dominated operators on $l^\infty$ the existence of a predual setting played an important role to ensure that the Fredholm properties of $A$ and $A|_{X_0}$ coincide.\textsuperscript{10} To integrate this aspect into the picture which arises from the present considerations we note that, by the previous theorem, a predual setting always implies the $\mathcal{P}$-dichotomy. On the other hand, the $\mathcal{P}$-dichotomy guarantees the required connection between $A$ and $A|_{X_0}$.\textsuperscript{11}

\textsuperscript{10}See, for instance, [44] or [15].

\textsuperscript{11}This particularly clarifies the open problem No. 4 formulated in [15].
Proposition 1.34. Let \( \mathcal{P} \) be a uniform approximate projection on the Banach space \( X \) and let \( A \in \mathcal{L}(X, \mathcal{P}) \) have the \( \mathcal{P} \)-dichotomy. Then \( A \) is Fredholm if and only if \( A|_{X_0} \) is Fredholm. In this case

\[
\dim \ker A = \dim \ker A|_{X_0}, \quad \dim \coker A = \dim \coker A|_{X_0}, \quad \text{and} \quad \text{ind} A = \text{ind} A|_{X_0}.
\]

Proof. If \( A \) is not Fredholm then it is properly \( \mathcal{P} \)-deficient, which also yields the \( \mathcal{P} \)-deficiency of \( A|_{X_0} \in \mathcal{L}(X_0, \mathcal{P}) \) (with \( \mathcal{P} \) regarded as approximate projection on \( X_0 \)) by Proposition 1.33. Hence \( A|_{X_0} \) can not be Fredholm as well, as shown by the fourth condition in Theorem 1.3. Vice versa, let \( A \) be Fredholm and properly \( \mathcal{P} \)-Fredholm. Corollary 1.31 provides an operator \( B \) in \( \mathcal{L}(X, \mathcal{P}) \) such that \( ABA = A, BAB = B \) and \( P = I - BA, P' = I - AB \in \mathcal{K}(X, \mathcal{P}) \). Then \( P \) is a projection onto the kernel of \( A \) and parallel to the range of \( B \) (see Proposition 1.7). Analogously, \( P' \) is a projection onto the kernel of \( B \) and parallel to the range of \( A \). Thus, \( \ker B \) is a complement of \( \text{im} A \), \( \ker A = \text{im} P = \text{im} P|_{X_0} = \ker A|_{X_0} \), and \( \ker B = \text{im} P' = \text{im} P'|_{X_0} = \ker B|_{X_0} \). This proves \( \dim \ker A = \dim \ker A|_{X_0} \). By Proposition 1.33 we find that the compressions also fulfill \( A|_{X_0}B|_{X_0}A|_{X_0} = A|_{X_0} \) and the projection \( P'|_{X_0} = I - A|_{X_0}B|_{X_0} \) is onto \( \ker B|_{X_0} \) and parallel to \( \text{im} A|_{X_0} \). Consequently, \( \ker B|_{X_0} \) is a complement of \( \text{im} A|_{X_0} \) and we conclude that \( \dim \coker A = \dim \coker A|_{X_0} \). The remaining statements now easily follow.

\[\square\]

1.3 Geometric characteristics of bounded linear operators

1.3.1 One-sided approximation numbers and their relatives

Lower approximation numbers For Banach spaces \( X, Y \) and an operator \( A \in \mathcal{L}(X, Y) \) the \( m \)-th approximation number from the right \( s^r_m(A) \) and the \( m \)-th approximation number from the left \( s^l_m(A) \) of \( A \) are defined as

\[
s^r_m(A) := \inf \{ \| A - F \|_{\mathcal{L}(X, Y)} : F \in \mathcal{L}(X, Y), \dim \ker F \geq m \},
\]

\[
s^l_m(A) := \inf \{ \| A - F \|_{\mathcal{L}(X, Y)} : F \in \mathcal{L}(X, Y), \dim \coker F \geq m \},
\]

\((m = 0, 1, 2, \ldots)\), respectively. It is clear that \( 0 = s^r_0(A) \leq s^r_1(A) \leq s^r_2(A) \leq \ldots \) and that the same holds true for the sequence \((s^l_m(A))\). Notice that in the case \( Y = X \) and \( \dim X < \infty \) the right- and left-sided variants coincide.

Lower Bernstein and Mityagin numbers Besides the approximation numbers there are further similar geometric characteristics for bounded linear operators \( A \in \mathcal{L}(X, Y) \) on Banach spaces \( X, Y \). Denote by \( B_X \) the closed unit ball in \( X \) and by

\[
j(A) := \sup \{ \tau \geq 0 : \| Ax \| \geq \tau \| x \| \text{ for all } x \in X \},
\]

\[
q(A) := \sup \{ \tau \geq 0 : A(B_X) \supset \tau B_Y \}
\]

the injection modulus and the surjection modulus, respectively. Due to [54], B.3.8 we have

\[
j(A) = \inf \{ \| Ax \| : x \in X, \| x \| = 1 \}, \quad j(A^*) = q(A) \quad \text{and} \quad q(A^*) = j(A). \quad (1.10)
\]

Furthermore, for given closed subspaces \( V \subset X \) and \( W \subset Y \) we let \( J_V \) denote the embedding map of \( V \) into \( X \) and by \( Q_W \) the canonical map of \( Y \) onto the quotient \( Y/W \). We define the lower Bernstein and Mityagin numbers by

\[
B_m(A) := \sup \{ j(AJ_V) : \dim X/V < m \},
\]

\[
M_m(A) := \sup \{ q(Q_W A) : \dim W < m \}.
\]

The content of Section 1.3 was already published, up to some minor modifications, in [81].
These characteristics have been discussed in [65], for instance. We will show that there are estimates that connect them to the approximation numbers defined above. Furthermore, these estimates will allow to relate the approximation numbers to the Fredholm property and the norm of the inverse of an operator as well as to the lower singular values in case of $X$ being a Hilbert space. Note also that the sequences $(B_m(A))$, $(M_m(A))$ are monotonically increasing and the following auxiliary result holds.

**Proposition 1.35.** Let $X, Y$ be Banach spaces, $A \in \mathcal{L}(X, Y)$ and $R \in \mathcal{L}(Y)$ be a projection. Then $q(Q_{\ker R}A) = j(A^*J_{im R^*})$. In particular, $M_m(A) = B_m(A^*)$ holds for every $m \in \mathbb{N}$.

**Proof.** Define an operator $T : \text{im } R^* \to (Y/\ker R)^*$ by $Tf = g$, $g(y + \ker R) := f(y)$. $T$ is well defined and surjective since every $f \in \text{im } R^*$ vanishes on $\ker R$ and since every functional $g \in Y/\ker R$ induces a functional $g(\mathcal{Q}_{\ker R}y)$ on $Y$ which, due to $Q_{\ker R} = Q_{\ker R}\circ R + Q_{\ker R}\circ (I - R) = Q_{\ker R}\circ R$, even fulfills the relation $g \circ Q_{\ker R} = g \circ Q_{\ker R} \circ R = R^*(g \circ Q_{\ker R})$, i.e., it belongs to $\text{im } R^*$. Furthermore, $T$ is an isometry since

$$
\|Tf\| = \sup_{y \in Y/\ker R} \frac{|Tf| y(\mathcal{Q}_{\ker R}y)}{\|\mathcal{Q}_{\ker R}y\|} = \sup_{y \in Y/\ker R} \frac{|f(y)|}{\inf_{z \in \ker R} \|y + z\|} = \sup_{y \in Y/\ker R} \sup_{z \in \ker R} \frac{|f(y)|}{\|y + z\|} = \sup_{y \in Y/\ker R} \frac{|f(y)|}{\|y + z\|} = \|f\|,
$$

and $(\mathcal{Q}_{\ker R}(Tf))(y) = (Tf)(Q_{\ker R}ry) = f(y)$ holds for all $f \in \text{im } R^*$ and all $y \in Y$. Hence

$$
q(Q_{\ker R}A) = j((Q_{\ker R}A)^*) = \inf\{\|A^*Q_{\ker R}g\| : g \in (Y/\ker R)^*, \|g\| = 1\}
$$

$$
= \inf\{\|A^*Q_{\ker R}(Tf)\| : f \in \text{im } R^*, \|f\| = 1\}
$$

$$
= \inf\{\|A^*f\| : f \in \text{im } R^*, \|f\| = 1\} = j(A^*J_{im R^*})
$$

follows with (1.10) which gives the first assertion.

Now, notice that, by Proposition 1.5, every subspace $W \subset Y$ with $\dim W < m$ is the kernel of a certain projection $R$ which provides a subspace $U := \text{im } R^*$ of $Y^*$. Vice versa, every subspace $U \subset Y^*$ with $\dim Y^*/U < m$ induces a subspace $W := \{y \in Y : f(y) = 0 \forall f \in U\} \subset Y$ and for a projection $S$ onto $W$ we see that $S^*f = f \circ S = 0$ for every $f \in U$ which easily yields that $R := I - S$ fulfills $W = \ker R$ and $U = \text{im } R^*$. Thus, the coincidence of $M_m(A)$ and $B_m(A^*)$ follows from their definitions.

**Proposition 1.36.** Let $X, Y$ be Banach spaces and $A \in \mathcal{L}(X, Y)$. Then, for all $m \in \mathbb{N}$,

$$
\frac{s_m(A)}{2^m - 1} \leq B_m(A) \leq s_m(A), \quad \text{as well as} \quad \frac{s_m(A)}{2^m - 1} \leq M_m(A) \leq s_m(A).
$$

**Proof.** Let $F \in \mathcal{L}(X, Y)$ with $\dim \ker F \geq m$ and let $V \subset X$ be a subspace with the codimension $\dim V = \dim X/V < m$. Then $\ker F \cap V$ is at least 1-dimensional. Thus

$$
\|A - F\| = \sup\{\|x\| : x \in \ker F, \|x\| = 1\} \geq \inf\{\|Ax\| : x \in \ker F, \|x\| = 1\}
$$

$$
\geq \inf\{\|Ax\| : x \in (\ker F \cap V), \|x\| = 1\} \geq \inf\{\|Ax\| : x \in V, \|x\| = 1\} = j(AJ_{\ker V}).
$$

Since $V$ was chosen arbitrarily, it follows that $\|A - F\| \geq B_m(A)$, and since $F$ is arbitrary, too, we find that $s_m(A) \geq B_m(A)$. Now, let $\epsilon > 0$. We inductively construct rank-$k$-projections $S_k$ ($k = 1, \ldots, m$) and respective elements $x_k \in \ker S_{k-1}$ with $\|x_k\| = 1$ such that

$$
c_{k-1} \leq c_k := \|Ax_k\| < j(AJ_{\ker S_{k-1}}) + \epsilon.
$$
as well as functionals $f_k \in X^*$ as follows: Set $S_0 := 0$ and $c_0 := 0$. If the projections $S_0, \ldots, S_{k-1}$ together with their corresponding elements and functionals are already given, we choose $x_k$ in ker $S_{k-1}$ and a functional $f_k$ on ker $S_{k-1}$ with $\|f_k\| = 1$ and $f_k(x_k) = 1$. Furthermore, we define the functional $f_k := f_k \circ (I - S_{k-1})$ and the projection $S_kx := \sum_{i=1}^k f_i(x)x_i$ on $X$. Notice that we obviously have $f_i(x_j) = \delta_{ij}$ \footnote{\(\delta_{ij} = 1 \text{ if } i = j, \text{ and } \delta_{ij} = 0 \text{ otherwise}\)} for all $i, j \leq k$ and from

$$\|f_k\| \leq \|I - S_{k-1}\| \leq 1 + \sum_{i=1}^{k-1} \|f_i\|$$

it follows that $\|f_k\| \leq 2^{k-1}$, hence

$$\|AS_kx\| = \left\| \sum_{i=1}^k f_i(x)Ax_i \right\| \leq \sum_{i=1}^k c_i \|f_i\|\|x\| \leq c_k \sum_{i=1}^k 2^{i-1}\|x\| = c_k(2^k - 1)\|x\|.$$ 

Defining $V := \ker S_{m-1}$, we find

$$s^r_m(A) = \inf\{\|A - F\| : \dim ker F \geq m\} \leq \|A - A(I - S_m)\|$$

$$= \|AS_m\| \leq c_m(2^m - 1) < (2^m - 1)(j(AJ_Y) + \epsilon) \leq (2^m - 1)(B_m(A) + \epsilon),$$

which completes the proof of the first estimate, since $\epsilon$ is arbitrary.

From Proposition 1.35 and the above applied to $A^*$ we know that $M_m(A) = B_m(A^*) \leq s^r_m(A^*)$. Fix $\epsilon > 0$ and an operator $F \in C(X, Y)$ with dim coker $F \geq m$ such that $\|A - F\| \leq s^r_m(A^*) + \epsilon$. Choose a rank $m$ projection $Q$ such that $\|QF\| < \epsilon$ (see Proposition 1.6). Then the kernel of $F^*(I - Q^*)$ is at least $m$-dimensional and

$$s^r_m(A^*) \leq \|A^* - F^*(I - Q^*)\| \leq \|A^* - F^*\| + \|F^*Q^*\| = \|A - F\| + \|QF\| \leq s^r_m(A) + 2\epsilon$$

gives the asserted estimate $M_m(A) \leq s^r_m(A)$ since $\epsilon$ was chosen arbitrarily.

For the remaining part of the second estimate in the assertion we use a construction similar to the one above. Fix $\epsilon > 0$ and set $R_0 := 0$, $d_0 := 0$. For $k = 1, \ldots, m$ we gradually choose functionals $g_k \in \ker R_{k-1}$ with $\|g_k\| = 1$ such that

$$d_{k-1} \leq d_k := \|A^*g_k\| < j(A^*J_{\ker R_{k-1}}) + \epsilon,$$

as well as elements $\tilde{y}_k \in Y$ with $\|\tilde{y}_k\| = 1$ and $|g_k(\tilde{y}_k)| \geq 1 - \epsilon$, respectively. Furthermore, we define $y_k := (g_k(\tilde{y}_k))^{-1}(I - R_{k-1})\tilde{y}_k$ and an operator $R_ky := \sum_{i=1}^k g_i(y)y_i$ on $Y$ each time. We easily check that $g_i(y_j) = \delta_{ij}$ for all $i, j \leq k$ hence $R_k$ is a projection of rank $k$, and from

$$\|y_k\| \leq \frac{1}{1 - \epsilon}\|I - R_{k-1}\| \leq \frac{1}{1 - \epsilon}\left(1 + \sum_{i=1}^{k-1} \|y_i\|\right)$$

we conclude $\|y_k\| \leq \frac{2^k}{(1 - \epsilon)^2}$. Thus, for all $g \in Y^*$ and all $x \in X$,

$$\|(A^*R_k^*g)(x)\| = \|g(R_kAx)\| = \left\| \sum_{i=1}^k g_i(Ax)g(y_i) \right\| = \left\| \left( \sum_{i=1}^k g_i(y_i)A^*g_i \right)(x) \right\|$$

$$\leq \sum_{i=1}^k \|g\|\|y_i\|d_k\|x\| \leq d_k \sum_{i=1}^k \frac{2^{i-1}}{(1 - \epsilon)^i}\|g\|\|x\| \leq d_k \frac{2^k - 1}{(1 - \epsilon)^k}\|g\|\|x\|.$$
The assertion then follows by
\[ s^l_m(A) = \inf \{ \| A - F \| : \text{dim coker } F \geq m \} \leq \| R_mA \| = \| A^* R_m^* \| \leq d_m \frac{2^m - 1}{(1 - \epsilon)^m} \]
\[ \leq \frac{2^m - 1}{(1 - \epsilon)^m} (B_m(A^*) + \epsilon) = \frac{2^m - 1}{(1 - \epsilon)^m} (M_m(A) + \epsilon) \]
where \( \epsilon > 0 \) is arbitrary.

Now we have \( s^l_1(A) = B_1(A) = j(A) \) and \( s^l_1(A) = M_1(A) = q(A) = j(A^*) \), hence we deduce

**Corollary 1.37.** Let \( A \in \mathcal{L}(X, Y) \). Then
\[ s_{1,1}^l(A) = \begin{cases} \| A^{-1} \|^{-1} & \text{if } A \text{ is invertible} \\ 0 & \text{if } A \text{ is not invertible from the left/right.} \end{cases} \]

With the help of Proposition 1.6 we find even more.

**Corollary 1.38.** Let \( A \in \mathcal{L}(X, Y) \).

1. If \( A \) is normally solvable and \( k \) is greater than the dimension of the kernel (cokernel) of \( A \) then \( s^k_k(A) (s^k_k(A), \text{resp.}) \) is non-zero. Otherwise \( s^k_k(A) (s^k_k(A)) \) is equal to zero.

2. If \( A \) is not normally solvable then all approximation numbers are equal to zero.

In particular, \( A \) is Fredholm if and only if the number of vanishing approximation numbers of \( A \) is finite.

### 1.3.2 Hilbert spaces and lower singular values

Suppose now that \( X, Y \) are Hilbert spaces and \( A \in \mathcal{L}(X, Y) \). We denote by \( A^* \) the Hilbert adjoint as introduced in Section 1.1.1 and consider the essential spectrum of the operator \( A^*A \) which is a subset of \( \mathbb{R}_+ \), the set of all non-negative real numbers. Furthermore, let \( d := \sqrt{\inf \sigma_{\text{ess}}(A^*A)} \) and
\[ \sigma_1(A) \leq \sigma_2(A) \leq \ldots \]
be the sequence of the non-negative square roots of the eigenvalues of \( A^*A \) less than \( d \), counted according to their multiplicities. If there are only \( N^A (= 0, 1, 2, \ldots) \) such eigenvalues, we put \( \sigma_{N^A+1}(A) = \sigma_{N^A+2}(A) = \ldots = d \). These numbers may be called the lower singular values of \( A \).

**Corollary 1.39.** Let \( X, Y \) be Hilbert spaces and \( A \in \mathcal{L}(X, Y) \). Then, for all \( m \),
\[ s^l_m(A) = B_m(A) = \sigma_m(A), \quad \text{as well as} \quad s^l_m(A) = M_m(A) = \sigma_m(A^*). \]

**Proof.** Firstly, suppose that \( A \) is normally solvable and has a trivial kernel. Set \( B := A^*A \) and for a subspace \( U \subset X \) let \( U^\perp \) denote its orthogonal complement. Then
\[ (B_m(A))^2 = \sup \{ \inf \{ \| Ax \|^2 : x \in U^\perp, \| x \| = 1 \} : \text{dim } U < m \} \]
\[ = \sup \{ \inf \{ \langle x, Bx \rangle : x \in U^\perp, \| x \| = 1 \} : \text{dim } U < m \} \]
which equals \( (\sigma_m(A))^2 \) by the Min-Max Principle (see [66], Theorem XIII.1 or below).

\[ \epsilon^{14} \text{See the definition in Section 1.4 and Theorem 1.40, and note that } (A^*A)^* = A^*A. \]
If $A$ is not normally solvable or dim ker $A = \infty$ then $s_n^r(A) = B_m(A) = 0$ for all $m$, due to Corollary 1.38. Moreover, $B$ is not Fredholm in this case, that is $0 \in \sigma_{\text{ess}}(B)$ hence $s_n(A) = 0$ for all $m$.

In the case that $A$ is normally solvable and $k = \dim \ker A < \infty$ we set $\hat{A} := AJ_{(\ker A)^\perp}$ and find that

$$B_{m+k}(A) = B_m(\hat{A}) = \sigma_m(\hat{A}) = \sigma_{m+k}(A)$$

for all $m$, as well as $B_l(A) = \sigma_l(A) = 0$ for all $l \leq k$.

Moreover, for a subspace $U$ of dimension $m$ and $P_U$ the orthogonal projection onto $U$,

$$\left(s_n(A)^r\right)^2 \leq \|AP_U\|^2 = \sup\{\langle Ax, Ax \rangle : x \in U, \|x\| \leq 1\} \leq \sup\{\|x\|\|Bx\| : x \in U, \|x\| \leq 1\} \leq \|BJ_U\|.$$  \hspace{1cm} (1.12)

If $m \leq N^A$, then there is an orthonormal system $\{x_i\}_{i=1}^m$ of eigenvectors, i.e. $Bx_i = (\sigma_i(A))^2 x_i$.

Set $U := \text{span}\{x_i\}_{i=1}^m$ and deduce from (1.12) that

$$\left(s_n(A)^r\right)^2 \leq \sup \left\{ \sum_{i=1}^m \alpha_i^2(\sigma_i(A))^2 : x = \sum_{k=1}^m \alpha_k x_k, \|x\| \leq 1 \right\} \leq (\sigma_m(A))^2.$$  

For the case $m > N^A$ we note that $C := B - d^2 I$ is not Fredholm, since $d^2 \in \sigma_{\text{ess}}(B)$. That is, $C$ is not normally solvable or dim ker $C = \dim \text{coker} C = \infty$. Thus, by Proposition 1.6, for every $\epsilon > 0$ there is a subspace $U$ of dimension $m$ such that $\|CJ_U\| < \epsilon$, hence $(s_n^r(A))^2 \leq \|BJ_U\| < d^2 + \epsilon$.

Together with Proposition 1.36 this finishes the proof of the first assertion, and the second one follows by duality, Proposition 1.35 and since $s_n^r(A) = s_n^r(A^*)$ in the case of Hilbert spaces.

For completeness we state the Min-Max-Principle from [66], Theorem XIII.1 here, where we call $H \in \mathcal{L}(X)$ self-adjoint, if $H = H^*$, that means $\langle Hx, y \rangle = \langle x, Hy \rangle$ for all $x, y \in X$.

**Theorem 1.40.** Let $H \in \mathcal{L}(X)$ be a normally solvable, injective, and self-adjoint operator on the Hilbert space $X$ and define $\mu_n(H) := \sup_U \inf\{\langle x, Hx \rangle : x \in U^\perp, \|x\| = 1\}$, the supremum over all subspaces $U$ of dimension at most $n - 1$. Then, for each fixed $n$, exactly one of the following hold:

- There are $n$ eigenvalues (counted according to their algebraic multiplicities) below the bottom of the essential spectrum, and $\mu_n(H)$ is the $n$th eigenvalue.
- $\mu_n(H)$ is the bottom of the essential spectrum.

**Remark 1.41.** Besides the lower approximation, Bernstein and Mityagin numbers there are also their much more famous “upper relatives”. For example, the upper approximation numbers $s_k(A)$ of an operator $A$ are given by

$$s_k(A) := \inf\{\|A - F\|_{\mathcal{L}(X,Y)} : F \in \mathcal{L}(X,Y), \text{rank } F < k\}.$$  

These numbers form a decreasing sequence $\|A\| = s_0(A) \geq s_1(A) \geq \ldots \geq 0$ and, roughly speaking, they sound out the spectrum of an operator from above, and not from below as the lower approximation numbers do. For the other characteristics the definitions are similar. Notice that all of these “big brothers” are so-called $s$-numbers, for which a well developed theory is available that includes many results on the relations between them. The modern books [53] and [22] may provide a good overview and introduction to that business. Unfortunately, as far as we know, there are no results in the literature for the lower versions in the case of (infinite dimensional) Banach spaces, except those of [65], from where we borrowed the equalities $B_m(A) = \sigma_m(A)$ and $M_m(A) = \sigma_m(A^*)$. In particular, note that the estimates in Proposition 1.36 seem hardly to be optimal. Nevertheless, they suffice for the aims of the present text.
1.4 Spectral properties

1.4.1 The spectrum

Let $A$ be a Banach algebra with identity $I$ and $A \in A$. The spectrum $\text{sp}A$ of $A$ is the set of all complex numbers $z \in \mathbb{C}$ such that $A - zI$ is not invertible in $A$. From any textbook on functional analysis or spectral theory it is well known that the spectrum is a non-empty compact set. Hence, the spectral radius $\rho(A)$ of $A$

$$\rho(A) := \sup \{|z| : z \in \text{sp}A\}$$

is well defined and the Spectral Radius Formula states that

$$\rho(A) = \inf_{n \in \mathbb{N}} \|A^n\|^{1/n} = \lim_{n \to \infty} \|A^n\|^{1/n}.$$ We particularly get the spectrum of a bounded linear operator $A \in \mathcal{L}(X)$ on a Banach space $X$:

$$\text{sp}A := \{z \in \mathbb{C} : A - zI \text{ is not invertible}\}.$$ Further, the essential spectrum $\text{sp}_{\text{ess}}A$ of $A$ is defined as the spectrum of $A + \mathcal{K}(X)$ in the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$ or, equivalently due to Theorem 1.3,

$$\text{sp}_{\text{ess}}A := \{z \in \mathbb{C} : A - zI \text{ is not Fredholm}\}.$$  

1.4.2 The $(N, \epsilon)$-pseudospectrum

In [30] the authors write “A computer working with finite accuracy cannot distinguish between a non-invertible matrix and an invertible matrix the inverse of which has a very large norm”. Therefore one replaces the spectrum by so-called pseudospectra which reflect finite accuracy. For some pioneering work we refer to Landau [40], [41], Reichel, Trefethen [67] and Böttcher [6], and for a (or more striking: the) comprehensive source we recommend the monograph [87] of Trefethen and Embree.

Definition 1.42. For $N \in \mathbb{Z}_+$ and $\epsilon > 0$ the $(N, \epsilon)$-pseudospectrum of a bounded linear operator $A$ on a Banach space $X$ is defined as the set

$$\text{sp}_{N, \epsilon}A := \{z \in \mathbb{C} : \|(A - zI)^{2N}\|^{1/2N} \geq 1/\epsilon\}.\quad 15$$

Remark 1.43. Notice that (for $N = 0$) this definition of the $(N, \epsilon)$-pseudospectrum includes the definition of the (classical) $\epsilon$-pseudospectrum

$$\text{sp}_\epsilon A := \{z \in \mathbb{C} : \|(A - zI)^{-1}\| \geq 1/\epsilon\}.$$ The $\epsilon$-pseudospectra have gained attention after [67] and [6] disclosed that, on the one hand they approximate the spectrum but are less sensitive to perturbations, and on the other hand the $\epsilon$-pseudospectra of discrete convolution operators mimic exactly the $\epsilon$-pseudospectrum of an appropriate limiting operator, which is in general not true for the “usual” spectrum. See also [4], [13], [30], [11] and the references cited there.

Later on, Hansen [32], [33] introduced the $(N, \epsilon)$-pseudospectra for linear operators on separable Hilbert spaces and pointed out that they share several nice properties with case $N = 0$, but offer a better insight into the approximation of the spectrum. Furthermore, it was shown how the spectrum can be approximated numerically, based on the consideration of singular values of certain finite matrices.

We now recover and extend these results in what follows and in Section 3.2.3.

15Here we use the convention $\|B^{-1}\| = \infty$ if $B$ is not invertible.
Within this section we want to show that the \((N, \epsilon)\)-pseudospectra provide a nice tool for the approximation of the spectrum of a bounded linear operator \(A\), even in the Banach space case. More precisely, we will show that

**Theorem 1.44.** Let \(A \in \mathcal{L}(X)\). For every \(\delta > \epsilon > 0\) there is an \(N_0\) such that, for all \(N \geq N_0\),

\[
B_\epsilon(\text{sp } A) \subset \text{sp}_{N, \epsilon} A \subset B_\delta(\text{sp } A),
\]

where \(B_\epsilon(S) := \{ z \in \mathbb{C} : \text{dist}(z, S) \leq \epsilon \}\) denotes the closed \(\epsilon\)-neighborhood of the set \(S\).

In a very recent preprint [34] Hansen and Nevanlinna mentioned that this result is in force even in the Banach space context, but the Hilbert space approach for the approximate determination of the spectrum via singular values of finite matrices cannot be extended to the Banach case since there is no involution available anymore. Therefore we propose a modification of Hansens idea which replaces the singular values by the injection and surjection modulus. Here comes the precise description:

If there is a sequence \((L_n)\) of compact projections which converge \(*\)-strongly to the identity then we will obtain an approximation of the spectrum of \(A\) based on much simpler operators: For this define the functions

\[
\gamma_{N, \epsilon}^{m,n}(z) := \left( \min \{ j((L_n(A - zI)L_n)^2)^{1/2} \} \right)^{2^{-N}}
\]

where the operators \(J, Q\) are defined as in Section 1.3.1. We are interested in their sublevel sets

\[
\Gamma_{N, \epsilon}^{m,n} := \{ z \in \mathbb{C} : \gamma_{N, \epsilon}^{m,n}(z) \leq \epsilon \}
\]

because for every fixed \(\beta > \epsilon > \alpha > 0\) and sufficiently large \(N, m, n\) we will get

\[
B_\alpha(\text{sp } A) \subset \Gamma_{N, \epsilon}^{m,n} \subset B_\beta(\text{sp } A).
\]

Notice that the projections \(L_n\) are of finite rank, thus the operators which have to be considered for the evaluation of \(\gamma_{N, \epsilon}^{m,n}(z)\) are operators acting on finite dimensional spaces. For the Hilbert space case this particularly means that we only have to consider the smallest singular values of certain finite matrices in order to determine \(\Gamma_{N, \epsilon}^{m,n}\) (by Corollary 1.39). The paper [31] contains an extensive discussion of this case including explicit algorithms and examples. So, we omit a detailed repetition here and focus on the theoretical treatment of the Banach space case instead.

**Remark 1.45.** Actually, we prove the latter for the case that \(X\) possesses a uniform approximate identity \(\mathcal{P}\) with \(B_\mathcal{P} = 1\) such that \(X\) has the \(\mathcal{P}\)-dichotomy, and the sequence \((L_n)\) of \(\mathcal{P}\)-compact projections converges \(\mathcal{P}\)-strongly to the identity. In such a setting the asserted relation (1.15) is true for \(A \in \mathcal{L}(X, \mathcal{P})\). Anyway, the proofs work in both cases. In [78] the author already published these results, proofs and discussions of Section 1.4 for the “classical” setting without \(\mathcal{P}\) and under the additional condition \(\|L_n\| = 1\) for all \(n\).

### A level function for the \((N, \epsilon)\)-pseudospectrum

Define

\[
\gamma_N(z) := \begin{cases} 
\| (A - zI)^{-2^N} \|^{-2^{-N}} & \text{if } z \notin \text{sp } A \\
0 & \text{if } z \in \text{sp } A
\end{cases}, \quad \text{and} \quad \gamma(z) := \text{dist}(z, \text{sp } A).
\]

The function \(\gamma\) is continuous everywhere, equals zero in all points \(z \in \text{sp } A\), and

\[
\gamma(z) = \frac{1}{\rho((A - zI)^{-1})} = \lim_{N \to \infty} \| (A - zI)^{-2^N} \|^{-2^{-N}} = \lim_{N \to \infty} \gamma_N(z)
\]
for every $z \notin \text{sp} A$. The last equation is the definition of $\gamma_N(z)$, the second one is given by the Spectral Radius Formula, and for the first one consider
\[(A - zI)^{-1} - yI = (A - zI)^{-1} [I - y(A - zI)] = (A - zI)^{-1} y \left( z + \frac{1}{y} \right) I - A.\]
We see that $(A - zI)^{-1} - yI$ is not invertible if and only if $z + \frac{1}{y}$ belongs to the spectrum of $A$, that is $\text{sp}((A - zI)^{-1}) = \left\{ \frac{1}{x - z} : x \in \text{sp} A \right\}$, and thus
\[\rho((A - zI)^{-1}) = \sup \left\{ \left| \frac{1}{x - z} \right| : x \in \text{sp} A \right\} = \frac{1}{\inf \left\{ |x - z| : x \in \text{sp} A \right\}} = \frac{1}{\text{dist}(z, \text{sp} A)}.\]
The $\gamma_N$ are continuous in every $z \notin \text{sp} A$ and (pointwise) monotonically increasing w.r.t. $N$ since
\[\|(A - zI)^{-2^{N+1}}\|^{2^{-(N+1)}} \leq \left( \|(A - zI)^{-2^N}\|^2 \right)^{2^{-N}} = \|(A - zI)^{-2^N}\|^{2^{-N}}.\]
Combining these observations we see that $0 \leq \gamma_N(z) \leq \gamma(z)$, the functions $\gamma_N$ are continuous everywhere, and $\gamma_N(z)$ converges increasingly to $\gamma(z)$ for every $z \in \mathbb{C}$. By Dini’s Theorem this even gives uniform convergence on every compact subset of $\mathbb{C}$.

Fix $\delta > \epsilon > 0$. It is clear from the definition that $z \in \text{sp}_{N,\epsilon} A$ if and only if $\gamma_N(z) \leq \epsilon$. Choose $r > 0$ large enough to guarantee that $\gamma_N(z) > \epsilon$ for all $z \in \mathbb{C} \setminus U_r(0)$ and all $N$. This is possible by a Neumann series argument since $A$ is bounded. Then we have uniform increasing convergence of $\gamma_N(z)$ to $\gamma(z)$ on $B_r(0)$. Thus, there is an $N_0$ such that for every $N \geq N_0$ and every $z \in \mathbb{C}$
\[\gamma(z) \leq \epsilon \Rightarrow \gamma_N(z) \leq \epsilon \Rightarrow \gamma(z) \leq \delta,\]
which yields (1.13) and finishes the proof of Theorem 1.44.

**Uniform approximations for $\gamma_N(z)$** Notice that by Corollary 1.37
\[\gamma_N(z) = \left( \min \left\{ j((A - zI)^{2^N}), q((A - zI)^{2^N}) \right\} \right)^{2^{-N}}.\]
We now use this representation as a starting point for the definition of approximating substitutes: Let $\mathcal{P}$ be a uniform approximate identity on the Banach space $X$ with $B_{\mathcal{P}} = 1$ and suppose that $X$ has the $\mathcal{P}$-dichotomy. Further suppose that $(L_n)$ is a sequence of $\mathcal{P}$-compact projections which converge $\mathcal{P}$-strongly to the identity. For $A \in \mathcal{L}(X, \mathcal{P})$ set
\[\gamma_N^n(z) := \left( \min \left\{ j((A - zI)^{2^N} J_{im} L_m), q(Q_{ker} L_m (A - zI)^{2^N}) \right\} \right)^{2^{-N}}.\]

**Proposition 1.46.** For every $N$ and $m$, the functions $\gamma_N^n(z)$ are continuous w.r.t. $z \in \mathbb{C}$. Further, for every fixed point $z \in \mathbb{C}$, the sequence $(\gamma_N^n(z))_m$ is bounded below by $\gamma_N(z)$ and converges to $\gamma_N(z)$. The convergence is even uniform on every compact subset of $\mathbb{C}$.

We first state an auxiliary result.

**Proposition 1.47.** Let $X, Y, Z$ be Banach spaces and $A \in \mathcal{L}(X, Y)$ as well as $B \in \mathcal{L}(Y, Z)$. Then $j(BA) \geq j(B) j(A)$ and $q(BA) \geq q(B) q(A)$.
Proof. If one of the numbers \( j(A), j(B), q(A), q(B) \) equals zero then the respective assertion is obviously true. So, let these four numbers be strictly positive and check that

\[
 j(BA) = \inf \{ \| BAx \| : x \in X, \| x \| = 1 \} = \inf \left\{ \frac{\| BAx \|}{\| Ax \|} : x \in X, \| x \| = 1 \right\}
\]

\[
 \geq \inf \{ \| By \| : y \in Y, \| y \| = 1 \} \inf \{ \| Ax \| : x \in X, \| x \| = 1 \} = j(B)j(A).
\]

Further, fix \( \sigma < q(B) \) and \( \tau < q(A) \). Then \( \sigma B_Z \subset B(B_Y) \) and \( \tau B_Y \subset A(B_X) \). Consequently,

\[
 \sigma \tau B_Z \subset \tau B(B_Y) = B(\tau B_Y) \subset B(A(B_X)) = BA(B_X)
\]

which shows that \( \sigma \tau \leq q(BA) \) and finishes the proof.

Proof of Proposition 1.46. The continuity is obvious by the relations (1.10), and we immediately conclude the estimate \( j(BJ_{im L_m}) \geq j(B)j(J_{im L_m}) = j(B) \) for every \( m \) from the previous result. With Proposition 1.35 also \( q(Q_{ker L_m} B) \geq q(Q_{ker L_m})q(B) = j(J_{im L_m})q(B) = q(B) \) holds. These observations particularly hold for all \( B := (A - zI)^N \) and hence, in every \( z \), we get the lower bound \( \gamma_N(z) \) for \( (\gamma_N^m(z))^m \).

If \( B \in L(X, P) \) is invertible then, due to Theorem 1.14, \( B^{-1} \) belongs to \( L(X, P) \) and we have \( j(B) = q(B) = \| B^{-1} \|^{-1} = \| B^{-1} |X_0|^{-1} \) by Proposition 1.33, since \( B_P = 1 \). Given \( \delta > 0 \) choose \( x \in X_0, \| x \| = 1 \), such that \( \| B^{-1} |X_0|^{-1} > \| B^{-1} x \|^{-1} - \delta \) and set \( y := B^{-1}x, w := \| y \|^{-1}y \). Then

\[
 j(B) = q(B) = \| B^{-1} |X_0|^{-1} > \frac{\| x \|}{\| B^{-1} x \|} - \delta = \frac{\| B y \|}{\| y \|} - \delta = \| B w \| - \delta.
\]

The latter can be further estimated by

\[
 \| B w \| \geq \| B L_m w \| - \| B(I - L_m)w \| = \| B L_m w \| \frac{1 - \| L_m w \|}{\| L_m w \|} \| B L_m w \| - \| B(I - L_m)w \| \geq j(BJ_{im L_m}) - \frac{\| (I - L_m)w \|}{1 - \| (I - L_m)w \|} \| B L_m \| - \| B(I - L_m)w \| (1.17)
\]

where \( \| (I - L_m)w \| \to 0 \) as \( m \to \infty \) since \( w \in X_0 \) and \( (L_m) \) converges \( P \)-strongly to the identity. Also note that \( (L_m) \) is bounded, due to Theorem 1.19. Since \( \delta \) was chosen arbitrarily, we find that \( j(BJ_{im L_m}) \to j(B) \) as \( m \to \infty \). Plugging this in the estimate

\[
 \left(j((A - zI)^2N_{J_{im L_m}})\right)^{2N} \geq \gamma_N^m(z) \geq \gamma_N(z) = \left(j((A - zI)^2N)\right)^{2N}, \quad z \notin \text{sp} A
\]

we deduce that \( \gamma_N^m(z) \) converges to \( \gamma_N(z) \) for every \( z \notin \text{sp} A \).

Now, let \( B \) be \( P \)-deficient from the right, or let the kernel of \( B \) be non-trivial. Fix \( \delta > 0 \) and a \( P \)-compact rank-one-projection \( P \) such that \( \| BP \| < \delta \). Let \( w \in \text{im} P \subset X_0, \| w \| = 1 \) and apply (1.17) to deduce

\[
 \delta > \| Bw \| \geq j(BJ_{im L_m}) - \frac{\| (I - L_m)w \|}{1 - \| (I - L_m)w \|} \| BL_m \| - \| B(I - L_m)w \|,
\]

which yields \( j(BJ_{im L_m}) \to 0 \) as \( m \to \infty \) also in this case, since \( \delta > 0 \) was chosen arbitrarily.
If $B$ is $\mathcal{P}$-deficient from the left, or the cokernel of $B$ is non-trivial then, for given $\delta > 0$, we choose a $\mathcal{P}$-compact rank-one-projection $P$ such that $\|PB\| < \delta$ as well as a functional $f \in \text{im } P^*$, $\|f\| = 1$, and find again by (1.17) and Proposition 1.35 that

$$\delta > \|B^* f\| \geq j(B^* J_{im L_m}) - \frac{\|(I - L_m^*)f\|}{1 - \|(I - L_m^*)f\|} \|B^* L_m^*\| - \|B^*(I - L_m^*)f\| \geq q(Q_{ker L_m} B) - \frac{\|(I - L_m^*)P\|}{1 - \|(I - L_m^*)P\|} \|B^* L_m^*\| - \|B^*(I - L_m^*)P^*\|.$$  

The $\mathcal{P}$-strong convergence of $(L_m)$ together with the $\mathcal{P}$-compactness of $P$ prove $q(Q_{ker L_m} B) \to 0$ as $m \to \infty$ in that case, too. Since $X$ has the $\mathcal{P}$-dichotomy, there are no further possible cases, hence we get pointwise convergence of the continuous functions $\gamma^n_N(z)$ to the continuous function $\gamma_N(z)$ in the whole complex plane.

Let $M \subset \mathbb{C}$ be a compact set. It remains to show the uniform convergence of $\gamma^n_M$ to $\gamma_N$ on $M$. For this we construct a sequence $(f^n_M)_m$ of continuous functions defined on $M$ such that $f^n_M(z) \geq \gamma^n_M(z) \geq \gamma_M(z)$ and $f^n_M(z)$ converge decreasingly to $\gamma_M(z)$ for every $z \in M$. Then Dini’s Theorem applies to $(f^n_M)$, provides its uniform convergence, and hence also the uniform convergence of $(\gamma^n_M)$. So, let us construct the desired functions $(f^n_M)$. Let $k \in \mathbb{N}$. For each $B \in \mathcal{R} := \{(A - zI)^{2N} : z \in M\}$ there is a $w \in \text{im } L_k$, $\|w\| = 1$ such that, using the estimate (1.17) again,

$$j(BJ_{im L_k}) = \inf\{\|Bx\| : x \in \text{im } L_k, \|x\| = 1\} \geq \|Bw\| - \frac{1}{2k} \geq j(BJ_{im L_m}) - \frac{\|(I - L_m)w\|}{1 - \|(I - L_m)w\|} \|BL_m\| - \|B(I - L_m)w\| - \frac{1}{2k} \geq j(BJ_{im L_m}) - \frac{\|(I - L_m)w\|}{1 - \|(I - L_m)w\|} \|BL_m\| - \|B^*\|(I - L_m)w\| - \frac{1}{2k}.$$  

Recall that the operator $L_k$ is $\mathcal{P}$-compact and the sequence $(L_m)$ converges $\mathcal{P}$-strongly to the identity. Hence, there is an $m_k \in \mathbb{N}$ such that, for every operator $B$ in the bounded set $\mathcal{R}$ and every $m \geq m_k$, the estimate $j(BJ_{im L_k}) \geq j(BJ_{im L_m}) - 1/k$ holds. Without loss of generality we can choose these $m_k$ such that $m_k \geq 2k$ for every $k$, and we conclude

$$j(BJ_{im L_k}) + \frac{2}{k} \geq j(BJ_{im L_m}) + \frac{2}{m} \quad \text{for all } k \in \mathbb{N}, \ m \geq m_k \text{ and } B \in \mathcal{R}.$$  

The same arguments applied to $j(B^* J_{im L_k})$, with Proposition 1.35, yield numbers $\tilde{m}_k$ s.t.

$$q(Q_{ker L_k} B) + \frac{2}{k} \geq q(Q_{ker L_m} B) + \frac{2}{m} \quad \text{for all } k \in \mathbb{N}, \ m \geq \tilde{m}_k \text{ and } B \in \mathcal{R}.$$  

Consequently, there is a strictly increasing sequence $(l_n) \subset \mathbb{N}$ such that

$$\min\{j(BJ_{im L_{l_n}}), q(Q_{ker L_{l_n}} B)\} + \frac{2}{l_n} \geq \min\{j(BJ_{im L_s}), q(Q_{ker L_s} B)\} + \frac{2}{s}$$  

for all $n \in \mathbb{N}$, $B \in \mathcal{R}$ and $s \geq l_{n+1}$. For $l_{n+1} \leq m < l_{n+2}$ we now define the functions $f^n_N$ by

$$f^n_N(z) := \min\{j((A - zI)^{2N} J_{im L_{l_n}}), q(Q_{ker L_{l_n}} (A - zI)^{2N})\} + \frac{2}{l_n} 2^{-N}$$  

and straightforwardly check that they meet all requirements: They can be written in the form $f^n_N = ((\gamma^n_N)^{2N} + \frac{2}{l_n})^{2^{-N}}$, hence they are continuous on $M$ and converge pointwise to $\gamma_N$ by what
1.4. SPECTRAL PROPERTIES

we have already proved. Due to the special choice of the \((l_n)\), this convergence is monotonically decreasing and the functions even fulfill \(\int_0^1 \gamma_N^m(z) \geq \gamma_N^m(z)\) on \(M\).

Until now, we have seen that the function \(\gamma\) can be approximated in a sense by the functions \(\gamma_N^m\), and further the functions \(\gamma_N^m\) are approximations of \(\gamma_N\). As a third step we finally approximate \(\gamma_N^m\) by the functions \(\gamma_N^{m,n}\) given in (1.14):

\[
\gamma_N^{m,n}(z) = \left( \min\{j((L_n(A-zI)L_n)^2 J_{in} L_m), q(Q_{ker} L_m (L_n(A-zI)L_n)^2)) \} \right)^{2^{-N}}.
\]

For \(A \in \mathcal{L}(X, \mathcal{P})\) it is clear that the sequence \((L_n(A-zI)L_n)^2\) converges \(\mathcal{P}\)-strongly, hence the arguments of \(j(\cdot)\) and \(q(\cdot)\) converge in the norm as \(n \to \infty\) for every \(z\) since \(L_m\) is \(\mathcal{P}\)-compact and \(Q_{ker} L_m = Q_{ker} L_m \circ L_m\) holds. This convergence is even uniform with respect to \(z\) on every compact set, since these arguments are polynomials in \(z\) whose (operator-valued) coefficients converge in the norm. This provides the uniform convergence \(\gamma_N^{m,n}(z) \to \gamma_N^m(z)\) as \(n \to \infty\) on every compact subset of \(\mathbb{C}\).

Regard the compact set \(M := D_{2\|A\|+4\|\beta\|}(0)\). For given \(\beta > \epsilon > \alpha > 0\) we choose \(N_0\) such that \(\gamma(z) - \gamma_N(z) > (\beta - \epsilon)/2\) for all \(N \geq N_0\) and \(z \in M\). Furthermore, for \(N \geq N_0\), we choose \(m_0(N)\) such that \(\gamma_N^m(z) - \gamma_N(z) < (\epsilon - \alpha)/2\) for all \(z \in M\) and all \(m \geq m_0(N)\). Finally, we take \(m_0(N, m)\) large enough to guarantee that for all \(n \geq m_0(N, m)\) and \(z \in M\) the estimate \(\gamma_N^m(z) - \gamma_N(z) < \min\{(\epsilon - \alpha), (\beta - \epsilon)/2\}\) and, additionally, \(j(L_n J_{in} L_m), q(Q_{ker} L_m \circ L_m) \geq 1/2\) hold. The latter is again possible since \((L_n)\) converges \(\mathcal{P}\)-strongly to the identity and \(L_m\) is \(\mathcal{P}\)-compact. Then, for every \(z \in M\),

\[
\gamma(z) \leq \alpha \Rightarrow \gamma_N(z) \leq \alpha \Rightarrow \gamma_N^m(z) \leq \alpha + \frac{\epsilon - \alpha}{2} \Rightarrow \gamma_N^{m,n}(z) \leq \alpha + 2\frac{\epsilon - \alpha}{2} = \epsilon
\]

and

\[
\gamma_N^{m,n}(z) \leq \epsilon \Rightarrow \gamma_N^m(z) \leq \epsilon + \frac{\beta - \epsilon}{2} \Rightarrow \gamma_N(z) \leq \epsilon + \frac{\beta - \epsilon}{2} \Rightarrow \gamma(z) \leq \epsilon + 2\frac{\beta - \epsilon}{2} = \beta.
\]

Applying Proposition 1.4.7 we get for arbitrary bounded linear operators \(B\) that

\[
j(BL_n J_{in} L_m) = j((BL_n)(L_n J_{in} L_m)) \geq j(BJ_{in} L_m) j(L_n J_{in} L_m) \geq j(BJ_{in} L_m)/2
\]

and analogously \(q(Q_{ker} L_m \circ L_m) \geq q(Q_{ker} L_m) \geq 2/2\). Thus, it is immediate from the definitions and an estimate similar to (1.16) that, for \(z \notin M\),

\[
2\gamma_N^{m,n}(z) \geq \gamma_N^m(z) \geq \gamma_N(z) = |z|\|(L_n - z^{-1} L_n A L_n)^{-1}\| L_{inf} L_m^{-1} > 2\beta.
\]

The estimate \(\gamma(z) > \beta\) for \(z \notin M\) is also obvious. Consequently, for suitably chosen \(N, m\) and \(n\), we arrive at (1.15). Let us condense this outcome into a theorem.

**Theorem 1.48.** Let \(\mathcal{P}\) be a uniform approximate identity on the Banach space \(X\) with the \(\mathcal{P}\)-dichotomy, \(B_{\mathcal{P}} = 1\), \((L_n)\) be a sequence of \(\mathcal{P}\)-compact projections tending \(\mathcal{P}\)-strongly to the identity, and let \(A \in \mathcal{L}(X, \mathcal{P})\). For every fixed \(\beta > \epsilon > \alpha > 0\) and all sufficiently large \(N \geq N_0, m \geq m_0(N)\) and \(n \geq n_0(N, m)\) the set

\[
\Gamma_N^{m,n} = \left\{ z \in \mathbb{C} : \min\{j((L_n(A-zI)L_n)^2 J_{in} L_m), q(Q_{ker} L_m (L_n(A-zI)L_n)^2)) \} \leq \epsilon^{2^n} \right\}
\]

fulfills

\[
B_{\alpha}(sp A) \subset \Gamma_N^{m,n} \subset B_{\beta}(sp A).
\]

\(^{16}m_0(N)\) means that \(m_0\) depends on \(N\).
Notice that, if $X$ is a separable Hilbert space and the $L_n$ are compact, orthogonal projections which are nested in the sense that $L_nL_{n+1} = L_{n+1}L_n = L_n$ for all $n \in \mathbb{N}$ and converge strongly to the identity, then $\mathcal{P} = (L_n)$ satisfies the conditions of this section. Corollary 1.39 further yields

$$\Gamma_{N,\epsilon}^{m,n} = \left\{ z \in \mathbb{C} : \min\{\sigma_1((L_n(A - zI)L_n)^{2^N}J_{\text{im } L_n}), \sigma_1((L_n(A^* - \bar{z}I)L_n)^{2^N}J_{\text{im } L_n})\} \leq \epsilon^{2^N} \right\}.$$  

We again note that the classical Hilbert space setting and the idea of approximating the spectrum via the consideration of certain “rectangular matrices” and their smallest singular value is subject of [31].

### 1.4.3 Convergence of sets

**Definition 1.49.** The Hausdorff distance of two compact sets $S, T \subset \mathbb{C}$ is defined as

$$d_H(S, T) := \max \left\{ \max_{s \in S} \text{dist}(s, T), \max_{t \in T} \text{dist}(t, S) \right\}.$$  

This function $d_H$ forms actually a metric on the set of all non-empty compact subsets of $\mathbb{C}$, and offers an alternative view on the present results. Remember that the sets $\text{sp } A, B_\epsilon(\text{sp } A), \text{sp}_{N,\epsilon} A$ and $\Gamma_{N,\epsilon}^{m,n}$ are compact.

**Corollary 1.50.** Let $A \in \mathcal{L}(X)$. Then

$$\lim_{\epsilon \to 0} d_H(\text{sp } A, B_\epsilon(\text{sp } A)) = 0 \quad \text{and} \quad \lim_{N \to \infty} d_H(B_\epsilon(\text{sp } A), \text{sp}_{N,\epsilon}(A)) = 0.$$  

Thus, the $(N, \epsilon)$-pseudospectra converge to the spectrum of $A$ with respect to the Hausdorff distance if $N \to \infty$ and $\epsilon \to 0$. Moreover,

$$\lim_{N \to \infty} \limsup_{m \to \infty} \limsup_{n \to \infty} d_H(B_\epsilon(\text{sp } A), \Gamma_{N,\epsilon}^{m,n}) = 0,$$

that is, the spectrum of $A$ can be approximated w.r.t. the Hausdorff metric even by the sets $\Gamma_{N,\epsilon}^{m,n}$. 

**Proof.** Firstly, note that for compact sets $S, T, U$ with $S \subset T \subset U$ we have

$$d_H(S, T) = \max_{t \in T} \text{dist}(t, S) \leq \max_{u \in U} \text{dist}(u, S) = d_H(S, U).$$  

Clearly, $d_H(\text{sp } A, B_\epsilon(\text{sp } A)) \leq \epsilon$ which proves the first assertion. For the second one apply Theorem 1.44 and the estimate

$$d_H(B_\epsilon(\text{sp } A), \text{sp}_{N,\epsilon}(A)) \leq d_H(B_\epsilon(\text{sp } A), B_\delta(\text{sp } A)) \leq \delta - \epsilon,$$

which holds for every fixed $\delta > \epsilon > 0$ and sufficiently large $N$. For the last assertion we employ Theorem 1.48 in the same way. 

**Remark 1.51.** We want to point out that the $\epsilon$-pseudospectrum does not behave continuously with respect to the parameter $\epsilon$ in general, that is the inclusion

$$\text{clos}\{z \in \mathbb{C} : \|(A - zI)^{-1}\| > 1/\epsilon\} \subset \text{sp}_\epsilon A$$

\[17\]The sequence $(L_n)$ is called a filtration in that case.
can be proper. Shargorodsky addressed a paper [84] to this fact, in which an explanation and appropriate examples are given in great detail. The point is that \(\|(A - zI)^{-1}\|\), the resolvent norm, can take constant values on open sets. Actually, the history of the investigation of this phenomenon is much longer. Globevnik [26] posed the question “Can \(\|(\lambda e - a)^{-1}\|\) be constant on an open subset of the resolvent set” for an element \(a\) of a complex Banach algebra. He could only derive a partial answer, but he was able to show that the answer is No for the resolvent norm of a bounded linear operator on a complex uniformly convex Banach space.\(^{18}\) Unfortunately, it remained rather unnoticed, and so this question emerged again in the 90’s where, independently from the earlier, Böttcher asked this and, together with Daniluk, he tackled the case of bounded linear operators on Hilbert spaces in [6], Proposition 6.1 (see also [18]), and later on the \(L^p\)-spaces with \(1 < p < \infty\) as well. Shargorodsky combined all these observations and completed the picture in [84], where he pointed out that Hilbert spaces and the \(L^p\)-spaces are complex uniformly convex by [17], and that the outcome even extends to Banach spaces which have a complex uniformly convex dual space. This particularly permits to cover all \(L^p\)-spaces with \(p \in [1, \infty]\).

Of course, the phenomenon of jumping pseudospectra (if it exists in the underlying setting) is reflected in the behavior of the \((N, \epsilon)\)-pseudospectra as well but, as Theorem 1.44 and Corollary 1.50 show, it gets less significant in any case, since the difference of the respective sets

\[
\text{clos}\{z \in \mathbb{C} : \|(A - zI)^{-2N}\|^{2^{-N}} > 1/\epsilon\} \subset \text{sp}_{N,\epsilon} A
\]

becomes small with growing \(N\).

### 1.4.4 Quasi-diagonal operators

The above observation that we can approximate the spectrum with the help of “rectangular finite sections” raises hope that this may even be possible by the usual finite sections. In fact, this is not true as a simple and well known example demonstrates: Let \(V\) denote the shift operator \((x_i)_{i \in \mathbb{Z}} \mapsto (x_{i+1})_{i \in \mathbb{Z}}\) on \(l^p\). It is invertible, has norm 1 and its inverse has norm 1 as well. Hence \(V - \alpha I\) is invertible whenever \(|\alpha| \neq 1\) by a Neumann series argument. On the other hand, the spectrum of every finite section equals \(\{0\}\) and also their \((N, \epsilon)\)-pseudospectra contain 0, whereas the \((N, \epsilon)\)-pseudospectra of \(V\) approximate the unit circle.

Actually, there is an explanation and a solution to this dilemma, and we will take up this matter again, after we have the required tools of the Parts 2 and 3 available. For the moment we turn our attention to a class of operators which accomplish our desire.

**Definition 1.52.** Let \((L_n)\) be a sequence of \(\mathcal{P}\)-compact projections tending \(\mathcal{P}\)-strongly to the identity, where \(\mathcal{P} = (P_n)\) is a uniform approximate identity on \(X\), which equips \(X\) with the \(\mathcal{P}\)-dichotomy and fulfills \(B_{\mathcal{P}} = 1\). An operator \(A \in \mathcal{L}(X, \mathcal{P})\) is called quasi-diagonal with respect to \((L_n)\) (or \((L_n)\) is said to quasi-diagonalize \(A\)) if

\[
\|[A, L_n]\| := \|AL_n - L_nA\| \to 0 \quad \text{as} \quad n \to \infty.
\]

Note that by a result of Berg [3] every normal operator \(A\) on a separable Hilbert space, hence also every self-adjoint operator, has a sequence of orthogonal projections which quasi-diagonalizes \(A\). We will pay some more attention to such operators and discuss further details in Section 4.1. At this stage we restrict our considerations solely to their pseudospectra.

\(^{18}\)For a definition see Section 3.2.3 where we will have a closer look at this notion.
**Theorem 1.53.** Let $A$ be quasi-diagonal with respect to $(L_n)$. Then

$$0 = \lim_{\epsilon \to 0} d_H(sp A, B_\epsilon(sp A)) = \lim_{\epsilon \to 0} \lim_{N \to \infty} d_H(sp A, sp_{N,\epsilon} A) = \lim_{\epsilon \to 0} \lim_{N \to \infty} \limsup_{m \to \infty} d_H(sp A, sp_{N,\epsilon}(L_m A L_m)).$$

**Proof.** The first and second equation are done by Corollary 1.50. For the last one note that

$$\Gamma_{m,N,\epsilon} = sp_{N,\epsilon}(L_m A L_m)$$

and that

$$\gamma_{m,N,\epsilon} - \gamma_{m,m,N}$$

already converges uniformly to zero on every compact subset of $C$ as $m \to \infty$ since $A$ is quasi-diagonal.

We mention that Brown [14] already proved the convergence of the classical $\epsilon$-pseudospectra in the Hilbert case, based on the results of the $C^*$-algebra approach of Hagen, Roch and Silbermann [30]. For those we have the relation

$$\bigcap_{\epsilon > 0} sp_{\epsilon} A = sp A$$

to the usual spectrum. Also Hansen [32] considered the spectral approximation of self-adjoint operators, taking their quasi-diagonality as well as their $(N,\epsilon)$-pseudospectra into account.

1.5 Example: operators on $l^p$-spaces

Let us summarize what we have already found for the spaces $l^p = l^p(Z, X)$ and the sequence $P = (P_n)$ of projections given by

$$P_m : (x_i) \mapsto (\ldots, 0, x_{-m}, \ldots, x_m, 0, \ldots).$$

For every $1 \leq p \leq \infty$ and every Banach space $X$, $P$ forms a uniform approximate identity and $l^p$ has the $P$-dichotomy (cf. Section 1.2.3 and Corollary 1.28).

Let $A \in L(l^p, P)$ and suppose that the finite section sequence $(P_n A P_n)$ is stable. Then, for every $b \in l^p$, the equation $Ax = b$ has a unique solution $x \in l^p$, and also the equations $P_n A P_n x_n = P_n b$ have unique solutions $x_n$ (for sufficiently large $n$), which converge $P$-strongly to $x$ (Corollary 1.23 and Proposition 1.29).

So, all we are left with is the question how to prove the stability of the finite section sequence. We want to discuss this for a more specific class of operators.

1.5.1 Band-dominated operators

Every sequence $a = (a_i) \in l^\infty(Z, L(X))$ gives rise to an operator $aI \in L(l^p)$ (a so-called multiplication operator) via

$$(ax)_i = a_i x_i, \quad i \in Z.$$  

Evidently, $\|aI\|_{L(l^p)} = \|a\|_\infty$. By this means, the sequences in $l^\infty(Z, C)$ induce multiplication operators as well.

**Definition 1.54.** A band operator $A$ is a finite sum of the form $A = \sum_\alpha a_\alpha V_\alpha$, where $\alpha \in Z$, $a_\alpha$ are multiplication operators, and $V_\alpha$ denote the shift operators

$$(V_\alpha f)(x) := f(x - \alpha), \quad x \in Z, \quad f \in l^p.$$  

An operator is called band-dominated if it is the uniform limit of a sequence of band operators. We denote the class of all band-dominated operators by $A_{l^p}$.  

1.5. EXAMPLE: OPERATORS ON $L^p$-SPACES

Clearly, in case $X = \mathbb{C}$, the matrix representation of a multiplication operator $aI$ with respect to the canonical basis in $l^p$ is an infinite diagonal matrix with the numbers $a_i$ along the main diagonal. The matrix representations of band operators are matrices having a finite bandwidth, that is there are only finitely many non-zero (sub- or super-)diagonals. This is why we call the uniform limits in $A_{lp}$ band-dominated. To state a collection of important properties of $A_{lp}$ and its elements we let $\text{BUC}$ denote the algebra of all bounded and uniformly continuous functions on the real line. For $\varphi \in \text{BUC}$, $t > 0$ and $r \in \mathbb{R}$ we define with $\varphi_t(x) := \varphi(tx)$ and $\varphi_{t,r}(x) := \varphi_t(x-r)$ certain inflated and shifted versions of $\varphi$. Furthermore, let $\hat{\varphi}$ stand for the sequence $(\varphi(n))_{n \in \mathbb{Z}}$ which obviously belongs to $l^\infty$, and therefore induces a multiplication operator $\hat{\varphi}I$.

The following results are well known, and can be found in [63], Theorem 2.1.6 and Propositions 2.1.7ff, and in [44], Theorem 1.42.

**Theorem 1.55.**

1. The set $A_{lp}$ forms a closed and inverse closed subalgebra of $\mathcal{L}(l^p, \mathcal{P})$ and $\mathcal{K} := \mathcal{K}(l^p, \mathcal{P})$, the set of all $\mathcal{P}$-compact operators, is a closed ideal in $A_{lp}$. Furthermore, $A_{lp}/\mathcal{K}$ is inverse closed in the quotient algebra $\mathcal{L}(l^p, \mathcal{P})/\mathcal{K}$.

2. For an operator $A \in \mathcal{L}(l^p)$ the following are equivalent:
   - $A$ is band-dominated.
   - For every $\epsilon > 0$, there exists an $M > 0$ such that whenever $F,G$ are subsets of $\mathbb{Z}$ with $\text{dist}(F,G) := \inf \{\|x - y\| : x \in F, y \in G\} > M$, then $\|\chi_F \chi_G I\|_{\mathcal{L}(l^p)} < \epsilon$.
   - For every $\varphi \in \text{BUC}$,
     \[
     \lim_{t \to 0} \sup_{r \in \mathbb{R}} \|[A, \hat{\varphi}_t I]\| = 0.
     \]
   - For every $\varphi \in \text{BUC}$,
     \[
     \lim_{t \to 0} \|[A, \hat{\varphi}_t I]\| = 0.
     \]

We want to add another characterization of band-dominated operators. Although there is no known reference in the literature where these observations can be found in the present form, their proofs are based on well-known ideas. We consider the continuous piecewise linear splines $\varphi$ and $\psi$ of norm one defined by

$$
\varphi(x) = \begin{cases} 
1 & : |x| \leq \frac{1}{3} \\
0 & : |x| \geq \frac{2}{3}
\end{cases}, \quad \psi(x) = \begin{cases} 
1 & : |x| \leq \frac{4}{5} \\
0 & : |x| \geq \frac{4}{5}.
\end{cases}
$$

and their inflated and shifted copies $\varphi^{\alpha,R}$ and $\psi^{\alpha,R}$, where $f^{\alpha,R}(x) := f(x/R - \alpha)$.

**Proposition 1.56.** The mapping $S_R : \mathcal{L}(l^p, \mathcal{P}) \to \mathcal{L}(l^p, \mathcal{P})$, given by

$$
S_R(A) := \sum_{\alpha \in \mathbb{Z}} \varphi^{\alpha,R} A \psi^{\alpha,R} I,
$$

is well defined and linear. The series (1.18) converges $\mathcal{P}$-strongly, and $S_R(A)$ is a band operator with $\|S_R(A)\| \leq 2\|A\|$ for every $A \in \mathcal{L}(l^p, \mathcal{P})$ and $R \in \mathbb{N}$.

**Proof.** For the spaces $l^p$ with $1 \leq p < \infty$ and for $(l^\infty)_0$ see [63], Proposition 2.2.2 or [44], Lemma 3.14. In order to prove the $\mathcal{P}$-strong convergence on $l^\infty$ we take $S_R(A) \in \mathcal{L}(l^\infty)_0, \mathcal{P}$ and note that the sequence $(S_R(A)P_n)_{n \in \mathbb{N}}$ has a $\mathcal{P}$-strong limit in $\mathcal{L}(l^\infty, \mathcal{P})$ since $l^\infty$ is complete w.r.t. $\mathcal{P}$-strong convergence, and by the arguments of the 9th step in the proof of Theorem 1.27 on page 20.

Since $\chi_F S_R(A) \chi_G I = 0$ for arbitrary subsets $F, G$ of $\mathbb{Z}$ with $\text{dist}(F,G) \geq 2R$, we find that $S_R(A)$ is banded (see [44], Proposition 1.36).
Notice that in case of a band operator $A$, the equality $S_R(A) = A$ holds for sufficiently large $R$ due to the fact that $\hat{\varphi}^\alpha.R A \hat{\psi}^\alpha.R I = \hat{\varphi}^\alpha.R A$ for all $\alpha$ and large $R$.

Let $A$ be a band-dominated operator. Then, by definition, there is a sequence $(A^m)$ of band operators which tends to $A$ in the norm. One might assume that such $A^m$ can be constructed by taking the restrictions of the matrix representation of $A$ to a finite number of diagonals. In general, this is not true, as [44], Remark 1.40 shows. Nevertheless, the sequence $(S_R(A))_{R \in \mathbb{N}}$ provides a canonical approximation of $A$ by band operators, in analogy to the Fejér-Cesàro means for the approximation of continuous functions by trigonometric polynomials. Indeed, for every fixed $m$ and sufficiently large $R$,

$$
\|S_R(A) - A\| \leq \|S_R(A - A^m)\| + \|S_R(A^m) - A^m\| + \|A^m - A\| \leq 3\|A^m - A\|,
$$

where the latter becomes as small as desired if $m$ is chosen sufficiently large.

Conversely, if for an operator $A \in \mathcal{L}(l^p)$ the band operators $S_R(A)$ exist and converge to $A$ in the norm, then $A$ is band-dominated. Thus we have

**Corollary 1.57.** An operator $A \in \mathcal{L}(l^p)$ is band-dominated, if and only if the operators $S_R(A)$ exist for every $R \in \mathbb{N}$ and $\|S_R(A) - A\| \to 0$ as $R \to \infty$.

Concerning the stability of a finite section sequence $(A_n) = (P_n AP_n + Q_n)$ of $A \in \mathcal{L}(l^p)$ we already know that the invertibility of $A$ is a necessary condition. One may say that the single operator $A$, which is just the $\mathcal{P}$-strong limit of $(A_n)$, captures one facet of the asymptotic behavior of this sequence. This seems to be a quite facile observation at a first glance, but nevertheless this is basically what we are going to do for the complete characterization of stability: Take snapshots of $(A_n)$ by passing to certain limit operators and draw on their invertibility.

Of course, this simplest snapshot $A$ describes the behavior at the center of the truncation intervals $[-n, n]$. The much more interesting points where the projections $P_n$ have their major impact are at the boundary $\{-n, n\}$, and it seems to be natural to observe the behavior there as well. To do this one might replace $(A_n)$ by the shifted copies $(A_n^\ell) := (V_n A_n V_{-n})$ and $(A_n^s) := (V_{-n} A_n V_n)$ which “fix one endpoint of the truncation intervals at the origin”. Figure 1.2 illustrates the movement of the focus. Clearly, the stability is invariant under these translations, and it is also obvious that every subsequence of a stable sequence is stable again. So, if $(A_n^\ell)$, $(A_n^s)$, or at least certain subsequences of them converge $\mathcal{P}$-strongly then the invertibility of these limits is also necessary for the stability of the sequence $(A_n)$.

Now, this immediately imposes the question if, with these additional snapshots, we have enough for the characterization of the stability of $(A_n)$, in the sense that their simultaneous invertibility gives a sufficient condition, or if we need any more. The answer will be “Yes, under a natural condition, it suffices to observe the sequence at its center and at the boundaries of the truncation interval.”

Also for the asymptotics of the pseudospectra of $A_n$ we already know from the basic example $A = V$ (see Section 1.4.4) that the operator $A$ alone is not enough to capture the limiting set, and we may guess that the additional snapshots could bring more light into the darkness.

Incited by these problems we are going to examine the notion of rich operators in the next step, and to elaborate this sequence algebra approach in Part 2, of course in a more general and more abstract setting. There, we also discuss a Fredholm theory for sequences which extends the connection between stability and invertibility to certain connections between “almost stability” of a sequence and the Fredholm properties of its snapshots. Part 3 will treat also the question on spectral approximation.

\footnote{Cf. Proposition 1.29}
1.5. Example: Operators on $L^p$-spaces

Figure 1.2: $P$-strong limits of the sequence $(A_n)$ and its shifted copy $(V_n A_n V_{-n})$.

1.5.2 Rich operators

We already discovered that it might be fruitful to examine the asymptotics of shifted copies $(V_{-g_n} A V_{g_n})$ of $A \in \mathcal{L}(l^p, P)$, where $g_n$ is a sequence of integers whose absolute values tend to infinity. If such a sequence converges $P$-strongly (let’s say, to $A_g$) then we call $A_g$ the limit operator of $A$ w.r.t. $g$. The set $\sigma_{op}(A)$ of all limit operators of $A$ is referred to as its operator spectrum. Further, we say that $A$ is a rich operator if every sequence $g$ of integers whose absolute values tend to $\infty$ has a subsequence $h$ such that $A_h$ exists.

The theory of limit operators has been intensively studied during the last years and has a wide range of applications. We only touch this concept here but, nevertheless, we state some of its highlights. For a nice introduction and a comprehensive discussion consult the work of Lindner (e.g. [44], [15]) or the fundamental book of Rabinovich, Roch and Silbermann [63].

Theorem 1.58. Let $A \in \mathcal{A}_{l^p}$ be rich. Then $A$ is $P$-Fredholm if and only if all limit operators of $A$ are invertible and their inverses are uniformly bounded.

This theorem has a long history and we particularly mention the pioneering paper [42] of Lange and Rabinovich. The proof of the if part is based on a construction of a $P$-regularizer, which goes back to Simonenko [86] and can be found in [62] and [44], for example. 20 The only if part was discussed in [62] and [63], Theorem 2.2.1 (for $1 < p < \infty$), in [44] (all $p$ and with an additional assumption on the existence of a predual setting in the case $p = \infty$), and in [15], Theorem 6.28 (all $p$). We mention that the present approach provides this implication even for every rich operator $A \in \mathcal{L}(l^p, P)$:

Proof. Let $g$ be such that $A_g$ exists and let $B$ be a $P$-regularizer for $A$. It is quite obvious from the definition that the operator spectrum of every $P$-compact operator $K$ is trivial: $\sigma_{op}(K) = \{0\}$. Thus, besides $V_{-g_n} A V_{g_n} \to A_g$, we also have $V_{-g_n} (AB - I) V_{g_n} \to 0$ and $V_{-g_n} (BA - I) V_{g_n} \to 0$ $P$-strongly. Then, for every $T \in K(l^p, P)$

$$\|T\| = \|V_{-g(n)} I V_{g(n)} T\| \leq \|V_{-g(n)} B V_{g(n)}\| \|V_{-g(n)} A V_{g(n)} T\| + \|V_{-g(n)} (I - BA) V_{g(n)} T\|,$$

and consequently (for a certain constant $D > 0$ independent of $g$, and $n \to \infty$)

$$\|T\| \leq D \|A_g T\| \quad \text{for all} \quad T \in K(l^p, P).$$

The dual estimate $\|T\| \leq D \|T A_g\|$ for all $T \in K(l^p, P)$ follows analogously. Due to the $P$-dichotomy, this implies the invertibility of $A_g$ and Theorem 1.14 yields that $A_g^{-1}$ belongs to $\mathcal{L}(l^p, P)$.

20 Actually, also our mappings $S_R$ are inspired by this construction.
For every $y \in (l^p)_0$ there is a projection $R_y \in \mathcal{K}(l^p, \mathcal{P})$ with norm not greater than 2 onto $\text{span}(y)$ (by Proposition 1.33), and therefore the first of the above estimates yields for the operator $A_{y}^{-1}R_y \in \mathcal{K}(l^p, \mathcal{P})$ that $\|A_{y}^{-1}R_y\| \leq D\|A_y A_{y}^{-1}R_y\| \leq 2D$. Hence, $\|A_{y}^{-1}\|_{(l^p)_0} \leq 2D$ and with (1.6) we get $\|A_{y}^{-1}\| \leq 2D$ with $D$ independent of $y$. This yields the uniform boundedness of the inverses.

We also note that every limit operator of a band-dominated operator is again band-dominated (cf. [44], Proposition 3.6).

**Proposition 1.59.** The set $\mathcal{L}^\delta(l^p, \mathcal{P})$ of all rich operators is a closed subalgebra of $\mathcal{L}(l^p, \mathcal{P})$ and contains $\mathcal{K}(l^p, \mathcal{P})$ as a closed ideal. Every $\mathcal{P}$-regularizer of a rich $\mathcal{P}$-Fredholm operator is rich. In particular, $\mathcal{L}^\delta(l^p, \mathcal{P})$ is inverse closed in $\mathcal{L}(l^p, \mathcal{P})$ and $\mathcal{L}(l^p, \mathcal{P})$.

**Proof.** It is obvious from the definition that $\mathcal{L}^\delta(l^p, \mathcal{P})$ is a subalgebra of $\mathcal{L}(l^p, \mathcal{P})$. Now, let $(A^g) \subset \mathcal{L}^\delta(l^p, \mathcal{P})$ be a sequence which tends uniformly to an operator $A \in \mathcal{L}(l^p, \mathcal{P})$, and let $h$ be a sequence of integers whose absolute values tend to $\infty$. Choose a subsequence $g^1 < h$ such that $A_{g^1}^1$ exists, a subsequence $g^2 < g^1$ such that $A_{g^2}^2$ exists, a subsequence $g^3 < g^2$ for $A_{g}^3$, and so on. Then define a new sequence $g = (g_n)$ by $g_n := g^k_n$. Evidently, $g$ is a subsequence of $h$ and all limit operators $A_{g^m}^m$, $m \in \mathbb{N}$ exist. One now easily checks that $(A_{g^m}^m)_m$ converges uniformly, and its limit $A_g$ is the limit operator of $A$ w.r.t. $g$. Thus, $\mathcal{L}^\delta(l^p, \mathcal{P})$ is closed.

Let $A \in \mathcal{L}^\delta(l^p, \mathcal{P})$ be $\mathcal{P}$-Fredholm, $B$ be a $\mathcal{P}$-regularizer, and $g$ such that $A_g$ exists. From the previous proof we already know that $A_g$ is invertible, and we claim that $B_g$ exists and equals $A_g^{-1}$. Then it is clear that $B$ is rich. Indeed, since

$$V_{-g(n)} BV_{g(n)} - A_g^{-1}$$

$$= V_{-g(n)} BV_{g(n)} (I - V_{-g(n)} BV_{g(n)} A_g^{-1}) + V_{-g(n)} (BA - I) V_{g(n)} A_g^{-1}$$

$$= V_{-g(n)} BV_{g(n)} [Ah - V_{-g(n)} BV_{g(n)} A_g^{-1}] + V_{-g(n)} (BA - I) V_{g(n)} A_g^{-1}$$

we get $\|(V_{-g(n)} BV_{g(n)} - A_g^{-1})T\| \to 0$ for every $T \in \mathcal{K}(l^p, \mathcal{P})$. Analogously, we can show that $\|T(V_{-g(n)} BV_{g(n)} - A_g^{-1})\| \to 0$ and obtain the $\mathcal{P}$-strong convergence of $V_{-g(n)} BV_{g(n)}$ to $A_g^{-1}$.

**Corollary 1.60.** If $\dim X < \infty$ then every band-dominated operator on $l^p = l^p(\mathbb{Z}, X)$ is rich.

**Proof.** It suffices to consider operators $aI$ of multiplication by $a = (a_n) \in l^\infty(\mathbb{Z}, \mathcal{L}(X))$, since the shift operators are obviously rich, and $\mathcal{L}^\delta(l^p, \mathcal{P})$ is a closed algebra.

With $k := \dim X$ and a fixed basis in $X$ we see that the matrix representations of the (uniformly bounded) operators $a_n$ are $k \times k$-matrices and from the Bolzano-Weierstrass theorem we deduce the existence of a convergent subsequence.

For general $X$ we have

**Theorem 1.61.** (see [63], Theorem 2.1.16)

The operator $aI$ of multiplication by $a = (a_n) \in l^p(\mathbb{Z}, \mathcal{L}(X))$ is rich if and only if the set of its values $\{a_n : n \in \mathbb{Z}\}$ is relatively compact (with respect to the operator norm on $\mathcal{L}(X)$).

---

21 The notion of rich operators can be introduced in much more general settings $\mathcal{L}(X, \mathcal{P})$, where a certain family of “shift-like” operators is available (see [63], Section 1.2 for details). If $X$ has the $\mathcal{P}$-dichotomy then this proposition is valid again and generalizes the results of [63], Section 1.2.1.
Part 2

Fredholm sequences and approximation numbers

We now turn our attention to algebras of operator sequences $\mathcal{A} = \{A_n\}$ which have a certain asymptotic structure in common. More precisely, for every sequence there shall be a family of operators $W^t(\mathcal{A})$ which appear as $T$-strong limits (of the sequence $\mathcal{A}$ or slightly modified copies). The goal of Sections 2.1 and 2.2 will be a Fredholm theory for such sequences, which provides a couple of relations between the properties of $\mathcal{A}$ (such as stability or Fredholmness), properties of its entries $A_n$ (e.g. the asymptotic behavior of the approximation numbers or indices), and the Fredholm properties of its so-called snapshots $W^t(\mathcal{A})$.

Section 2.3 is devoted to a notion of Fredholm sequences which does not require any asymptotic structure and which can be regarded as a generalization of the first concept. Finally, we deal with sequences which are somehow in between of these two approaches, so-called rich sequences. They do not have an asymptotic structure as considered in the first section, but they have at least “sufficiently many” subsequences of that structured type. We apply the established tools to band-dominated operators in Section 2.5.

Such concepts of Fredholmness for operator sequences have their origin in the consideration of the finite section method for Toeplitz operators [83], are well sophisticated in the framework of Standard Algebras [30], that are classes of $C^*$-algebras, have been studied in the case of matrix sequences $\mathcal{A}$ with $*$-strong limits $W^t(\mathcal{A})$ [75] and there are also variants for special classes of operator sequences in a Banach space context, again with the classical notions of compactness and convergence [50]. The fusion with the concept of approximate projections was first done by the author in [80] for matrix and in [81] for operator sequences. Clearly, these latest releases are heavily influenced by its forerunners and include or adapt several former ideas. The reader will particularly rediscover the golden thread of [75] in the presentation of the Sections 2.1 and 2.2. The content of the Sections 2.1 to 2.3 and 2.5 is subject of the mentioned publications [80], [81], but here in a complete and stand-alone form, whereas the notion of rich sequences appears here for the first time and generalizes the notion of rich operators $^1$ and concrete tools for band-dominated operators, in a sense.

$^1$Cf. Section 1.5.2.
2.1 Sequence algebras

Let \((E_n)\) be a sequence of Banach spaces and let \(L_n\) stand for the identity operator on \(E_n\), respectively. We denote by \(F\) the set of all bounded sequences \(\{A_n\}\) of bounded linear operators \(A_n \in L(E_n)\). One easily verifies that, provided with the operations

\[
\alpha \{A_n\} + \beta \{B_n\} := \{\alpha A_n + \beta B_n\}, \quad \{A_n\} \{B_n\} := \{A_n B_n\},
\]

and the supremum norm \(\|\{A_n\}\|_F := \sup_n \|A_n\|_{L(E_n)} < \infty\), \(F\) becomes a Banach algebra with identity \(\mathbb{1} := \{L_n\}\). The set

\[
G := \{\{G_n\} \in F : \|G_n\|_{L(E_n)} \to 0 \text{ as } n \to \infty\}
\]

forms a closed ideal in \(F\).

Further, let \(T\) be a (possibly infinite) index set and suppose that, for every \(t \in T\), there is a Banach space \(E^t\) with a uniform approximate identity \(P^t\) such that \(E^t\) has the \(P^t\)-dichotomy, and a bounded sequence \((L_n^t)\) of projections \(L_n^t \in L(E^t, P^t)\) tending \(P^t\)-strongly to the identity \(I^t\) on \(E^t\). Set

\[
c^t := \sup\{\|L_n^t\|_{L(E^t)} : n \in \mathbb{N}\} < \infty \quad \text{for every} \quad t \in T.
\]

Suppose that, for every \(t \in T\), there is a sequence \((E_n^t)\) of algebra isomorphisms

\[
E_n^t : L(\text{im} L_n^t) \to L(E_n),
\]

such that (for brevity, we write \(E_n^t\) instead of \((E_n^t)^{-1}\))

\[
M^t := \sup\{\|E_n^t\|, \|E_n^{-t}\| : n \in \mathbb{N}\} < \infty. \tag{I}
\]

We denote by \(F^T\) the collection of all sequences \(\mathbb{A} = \{A_n\} \in F\), for which there exist operators \(W^t(\mathbb{A}) \in L(E^t, P^t)\) for all \(t \in T\) such that

\[
A_n^{(t)} := E_n^{-t} A_n L_n^t \to W^t(\mathbb{A}) \quad \text{\(P^t\)-strongly.}
\]

These limits are uniquely determined and with the help of Theorem 1.19 it is easy to show that \(F^T\) is a closed subalgebra of \(F\) which contains the identity and the ideal \(G\). Both, the mappings \(E_n^t\) and \(W^t : F^T \to L(E^t, P^t)\), \(\mathbb{A} \mapsto W^t(\mathbb{A})\) are unital homomorphisms for every \(t \in T\).

Roughly speaking, the mappings \(E_n^t\) allow us to transform a given sequence \(\mathbb{A} \in F^T\) and to generate snapshots \(W^t(\mathbb{A})\) from different angles which outline several aspects of the asymptotic behavior of \(\mathbb{A}\). In what follows, we will examine the connections between the properties of \(\mathbb{A}\) and the properties of its “snapshots at infinity”.

Remark 2.1. The results and proofs of the subsequent sections remain true, if for some or all \(t \in T\) the sequence \((L_n^t)\) of projections converges \(\ast\)-strongly to the identity, and if we replace \(P^t\)-Fredholmness by Fredholmness, \(P^t\)-compactness by compactness, and \(P^t\)-strong convergence by \(\ast\)-strong convergence. An approximate projection \(P^t\) is not needed in this case.

Thus, the results for so-called Standard Algebras (see [30]) and for Banach algebras of matrix sequences in [75] are completely covered by the present considerations. Furthermore, the generalization of the approach of [75] to sequences of operators acting on infinite dimensional Banach spaces in [50] is a special case of the theory in the present text as well.

\footnote{We continue to use \((\cdot)\) for sequences of elements in one common space, whereas the sequences in \(F\) which consist of elements coming from different spaces \(E_n\) in general are denoted by \({\cdot}\).}
2.2. $\mathcal{J}^T$-Fredholm sequences

Here we introduce the notion of Fredholm sequences. A sequence will be called Fredholm, if it is “invertible modulo an ideal of compact sequences”, where in our first approach these “compact sequences” will be generated by lifting $\mathcal{P}^t$-compact operators. For this, it turns out to be reasonable to suppose that the perspectives from which a sequence can be looked at, and which are given by the homomorphisms $W^t$, are different from each other. Therefore, and in all what follows, we suppose that the separation condition

$$ W^t \{ E_n(L_n K^t) \} = \begin{cases} K^t & \text{if } t = \tau \\ 0 & \text{if } t \neq \tau \end{cases} \quad (\text{II}) $$

holds for all $\tau, t \in T$ and every $K^t \in \mathcal{K}^t := \mathcal{K}(\mathcal{E}^t, \mathcal{P}^t)$. Metaphorically speaking, this means that the directions from which one can look at a sequence are separated in the sense that the $\mathcal{P}^t$-compact operators $K^t$ and their liftings $\{ E_n(L_n K^t) \}$ which arise from one point of view $t \in T$ are invisible from every other direction.

**Definition 2.2.** We put \(^3\)

$$ \mathcal{J}^t := \{ E_n(L_n K^t) \} + \{ G_n : K^t \in \mathcal{K}^t, \| G_n \| \to 0 \} \quad (\forall t \in T), $$

$$ \mathcal{J}^T := \text{clos}_{\mathcal{F}^T} \left\{ \sum_{i=1}^{m} \{ J^t_{n_i} \} : m \in \mathbb{N}, t_i \in T, \{ J^t_{n_i} \} \in \mathcal{J}^{t_i} \right\}. $$

One may refer to the elements of these sets as compact sequences. Actually, we will see that this is entirely justified since the algebraic structure is similar to the notions of compact or $\mathcal{P}$-compact operators. We start with the following observation.

**Proposition 2.3.** All $\mathcal{J}^t$, $t \in T$ as well as $\mathcal{J}^T$ are closed ideals in $\mathcal{F}^T$.

\(^3\)The consideration of such ideals goes back to the approach of Silbermann for the finite section method applied to Toeplitz operators in [83]. See also Theorem 2.17 and the annotations there.
Proof. For \( t \in T, K \in \mathcal{K}^t \) and \( \mathcal{A} = \{ A_n \} \in \mathcal{F}^T \) we have
\[
A_n E_n^t( L_n^t K) = E_n^t(A_n^t K) = E_n^t(L_n^t W^t(\mathcal{A}) K) + E_n^t(L_n^t (A_n^t - W^t(\mathcal{A})) K),
\]
\[
E_n^t(L_n^t K) A_n = E_n^t(L_n^t K A_n^t) = E_n^t(L_n^t K W^t(\mathcal{A})) + E_n^t(L_n^t K (A_n^t - W^t(\mathcal{A}))),
\]
where \( W^t(\mathcal{A}) \in \mathcal{L}(\mathcal{E}^t, \mathcal{P}^t) \) and hence \( W^t(\mathcal{A}) K, K W^t(\mathcal{A}) \in \mathcal{K}^t \). Due to the Condition (I) and since \( (A_n^t - W^t(\mathcal{A})) \) tends \( \mathcal{P}^t \)-strongly to 0, the last summands tend to zero in the norm as \( n \to \infty \) in both cases. Consequently, all \( \mathcal{J}^t \) as well as \( \mathcal{J}^T \) are ideals in \( \mathcal{F}^T \), where \( \mathcal{J}^T \) is closed by its definition.
Let \( \{ J_n^t \}_{n \in \mathbb{N}} = \{ (E_n^t(L_n^t K_n^t)) + \{ G_n^k \}_{k \in \mathbb{N}} \}_{n \in \mathbb{N}} \subset \mathcal{J}^t \) be a Cauchy sequence. From Theorem 1.19 we obtain (with \( C^t := c^t \mathcal{M}^t \mathcal{B}^t_{\mathcal{P}^t} \))
\[
\| K_k^t - K_l^t \| = \| W^t \left( \{ E_n^t(L_n^t K_n^t) \} + \{ G_n^k \} - \{ G_n^k \} \right) \|
\]
\[
\leq C^t \| \{ E_n^t(L_n^t K_n^t) \} + \{ G_n^k \} - \{ E_n^t(L_n^t K_n^t) \} + \{ G_n^k \} \|
\]
\[
= C^t \| \{ J_n^t \} - \{ J_n^t \} \|,
\]
therefore the sequence \( \{ K_k^t \}_{k \in \mathbb{N}} \) is a Cauchy sequence in \( \mathcal{K}^t \), and since \( \mathcal{K}^t \) is closed it possesses a limit \( K^t \in \mathcal{K}^t \). Analogously, the estimate
\[
\| \{ G_n^k \} - \{ G_n^k \} \| \leq \| \{ E_n^t(L_n^t K_n^t) \} + \{ G_n^k \} - \{ E_n^t(L_n^t K_n^t) \} + \{ G_n^k \} \|
\]
\[
\leq \| \{ J_n^t \} - \{ J_n^t \} \| + M^t c^t \| K^t - K^t \|
\]
shows that \( \{ G_n^k \} \) converges to a certain \( \{ G_n^k \} \in \mathcal{G} \). Now it’s easy to see that the sequence \( \{ E_n^t(L_n^t K^t) \} + \{ G_n^k \} \in \mathcal{J}^t \) is the limit of \( \{ J_n^t \}_{k} \) and the closedness of \( \mathcal{J}^t \) is proved. \( \square \)

**Definition 2.4.** A sequence \( \mathcal{A} \in \mathcal{F}^T \) is said to be \( \mathcal{J}^T \)-Fredholm, or regularizable with respect to \( \mathcal{J}^T \), if the coset \( \mathcal{A} + \mathcal{J}^T \) is invertible in the quotient algebra \( \mathcal{F}^T / \mathcal{J}^T \).

Notice that this property depends on the underlying algebra \( \mathcal{F}^T \) and the ideal \( \mathcal{J}^T \). It is obvious that the set of \( \mathcal{J}^T \)-Fredholm sequences is open in \( \mathcal{F}^T \), the sum of a \( \mathcal{J}^T \)-Fredholm sequence and a sequence from the ideal \( \mathcal{J}^T \) is \( \mathcal{J}^T \)-Fredholm and that the product of two \( \mathcal{J}^T \)-Fredholm sequences is \( \mathcal{J}^T \)-Fredholm again. We will see that there are much deeper similarities with the usual Fredholm property of operators. In particular, there are analogues to the kernel and cokernel dimension as well as the index. Let us start with a basic result on the regularization of a \( \mathcal{J}^T \)-Fredholm sequence.

**Proposition 2.5.** Let \( \mathcal{A} \in \mathcal{F}^T \) be \( \mathcal{J}^T \)-Fredholm. Then there exist finite subsets \( \{ t_1, ..., t_m \} \) and \( \{ \tau_1, ..., \tau_l \} \) of \( T \) and a \( \delta > 0 \) such that the following holds:
For each \( \mathcal{A} \in \mathcal{F}^T \) with \( \| \mathcal{A} - \mathcal{A} \| < \delta \) there are sequences \( \mathcal{B}, \mathcal{C} \in \mathcal{F}^T \) and \( \mathcal{G}, \mathcal{H} \in \mathcal{G} \) as well as operators \( K^t_1 \in \mathcal{K}^t \) and \( K^t_2 \in \mathcal{K}^t \) such that
\[
\mathcal{B} \mathcal{A} = \mathcal{I} + \sum_{i=1}^{m} \{ E_n^t(L_n^t K^t_1) \} + \mathcal{G}, \quad (2.1)
\]
\[
\hat{\mathcal{A}} \mathcal{C} = \mathcal{I} + \sum_{i=1}^{l} \{ E_n^t(L_n^t K^t_2) \} + \mathcal{H}. \quad (2.2)
\]

**Proof.** By the definitions of \( \mathcal{J}^T \)-Fredholmness and of the ideal \( \mathcal{J}^T \), there exist a sequence \( \mathcal{A} \in \mathcal{F}^T \), finite subsets \( \{ t_1, ..., t_m \} \) and \( \{ \tau_1, ..., \tau_l \} \) of \( T \) and sequences \( \{ \mathcal{A}_i \} \in \mathcal{J}^{t_i}, \{ \mathcal{K}_i^{\tau_i} \} \in \mathcal{J}^{\tau_i} \) as well as
\[ \hat{\mathcal{A}} \in \mathcal{J}^T \text{ with } ||\hat{\mathcal{A}}||, ||\hat{\mathcal{K}}|| < 1/4 \text{ such that} \]
\[ \hat{\mathcal{A}} \hat{\mathcal{A}} = I + \sum_{i=1}^{m} \hat{\mathcal{J}}^i + \hat{\mathcal{J}} \quad \text{and} \quad \hat{\mathcal{A}} \hat{\mathcal{A}} = I + \sum_{i=1}^{l} \hat{\mathcal{K}}^\tau_i + \hat{\mathcal{K}}. \]

For \( \hat{\mathcal{A}} \in \mathcal{J}^T \) with \( ||\hat{\mathcal{A}} - \hat{\mathcal{K}}|| < \delta := 1/(4||\hat{\mathcal{A}}||) \) we have
\[ \hat{\mathcal{A}} \hat{\mathcal{A}} = I + \sum_{i=1}^{m} \hat{\mathcal{J}}^i + \hat{\mathcal{J}} + \hat{\mathcal{A}}(\hat{\mathcal{A}} - \hat{\mathcal{K}}) \]

where \( ||\hat{\mathcal{J}} + \hat{\mathcal{A}}(\hat{\mathcal{A}} - \hat{\mathcal{K}})|| < 1/2 \). Hence, the sequence \( I + \hat{\mathcal{J}} + \hat{\mathcal{A}}(\hat{\mathcal{A}} - \hat{\mathcal{K}}) \) is invertible in the Banach algebra \( \mathcal{J}^T \) and we can define
\[ \mathcal{B} := [I + \hat{\mathcal{J}} + \hat{\mathcal{A}}(\hat{\mathcal{A}} - \hat{\mathcal{K}})]^{-1} \hat{\mathcal{A}} \in \mathcal{J}^T \quad \text{and} \quad \mathcal{J}^i := [I + \hat{\mathcal{J}} + \hat{\mathcal{A}}(\hat{\mathcal{A}} - \hat{\mathcal{K}})]^{-1} \hat{\mathcal{J}}^i \in \mathcal{J}^i, \]

for \( i = 1, \ldots, m \). Due to the definition of the ideals \( \mathcal{J}^i \), there are operators \( \mathcal{K}^\tau_i \in \mathcal{K}^\tau_i \) and a sequence \( \mathcal{G} \in \mathcal{G} \) such that
\[ \mathcal{B} \hat{\mathcal{A}} = I + \sum_{i=1}^{m} \mathcal{J}^i = I + \sum_{i=1}^{m} \{ \mathcal{L}^i_n (\mathcal{L}^i_n \mathcal{K}^\tau_i) \} + \mathcal{G}. \]

Analogously, one obtains sequences \( \mathcal{C} \in \mathcal{F}^T \) and \( \mathcal{H} \in \mathcal{G} \) as well as operators \( \mathcal{K}^\tau_i \in \mathcal{K}^\tau_i \) such that Eq. (2.2) holds. \( \square \)

Applying the homomorphisms \( \mathcal{W}^i, t \in T \) to the Equations (2.1) and (2.2), and utilizing the separation condition, we see that the following theorem is in force.

**Theorem 2.6.** If a sequence \( \mathcal{A} \in \mathcal{F}^T \) is \( \mathcal{J}^T \)-Fredholm, then all corresponding operators \( \mathcal{W}^i(\mathcal{A}) \) are \( \mathcal{P}^i \)-Fredholm on \( \mathcal{E}^i \) and the number of the non-invertible operators among them is finite.

### 2.2.1 Systems of operators

For fixed \( t \in T \) let \( P^t \in \mathcal{K}^t \) be a \( \mathcal{P}^i \)-compact operator. We set \( \hat{P}^t := I^t - P^t \) and obtain
\[ I^t - (I^t - L^t_n) P^t = L^t_n P^t + \hat{P}^t. \]

Since \( P^t \in \mathcal{K}^t \), there is an \( n_t \in \mathbb{N} \) such that \( ||(I^t - L^t_n) P^t||, ||P^t (I^t - L^t_n)|| < 1 \) for all \( n \geq n_t \). Thus, for \( n \geq n_t \), we obtain the invertibility of \( I^t - (I^t - L^t_n) P^t \) and, moreover, the inverses are
\[ (I^t - (I^t - L^t_n) P^t)^{-1} = \sum_{k=0}^{\infty} ((I^t - L^t_n) P^t)^k \in \mathcal{L}(\mathcal{E}^t, \mathcal{P}^t). \]

This justifies the following definition for \( n \geq n_t \):
\[ I^t = L^t_n P^t \sum_{k=0}^{\infty} ((I^t - L^t_n) P^t)^k + \hat{P}^t \sum_{k=0}^{\infty} ((I^t - L^t_n) P^t)^k. \]
For the operators $P^t_n$, we have $\text{im} \ P^t_n \subset \text{im} \ P^t$ and $\|P^t - P^t_n\|$ tends to 0 as $n \to \infty$ since

$$\|P^t - P^t_n\| = \|P^t - I^t + I^t - P^t_n\| = \|\hat{P}^t_n - \hat{P}^t\| = \left\| \hat{P}^t \sum_{k=1}^{\infty} ((I^t - L^t_n)P^t)^k \right\| \leq \|\hat{P}^t\| \left( \frac{\| (I^t - L^t_n)P^t \|}{1 - \| (I^t - L^t_n)P^t \|} \right) \to 0.$$  

Applying the homomorphisms $E^t_n$, we can lift them to a sequence $\{R^t_n\} \in \mathcal{F}^t$ by the rule

$$R^t_n := \begin{cases} E^t_n(P^t_n) : n \geq n_t, \\ 0 : n < n_t. \end{cases}$$

Indeed, it is easy to check that $\{R^t_n\}$ is bounded, and it belongs to $\mathcal{F}^t$ because of

$$\|R^t_n - E^t_n(L^t_nP^t)\| = \|E^t_n(P^t_n - L^t_nP^t)\| \leq M^t\|P^t_n - P^t\| + \| (I^t - L^t_n)P^t \| \to 0.$$  

One can further show that, for large $n$, $\dim \text{im} \ R^t_n = \dim \text{im} \ P^t_n \leq \dim \text{im} \ P^t$ and, for every $k < \dim \text{im} \ P^t$, there is an $N_k^t$ with $\dim \text{im} \ R^t_n > k$ whenever $n \geq N_k^t$.

**Definition 2.7.** For $t \in T$ let $P^t \in \mathcal{K}^t$. The system $(P^t, \hat{P}^t, P^t_n, \hat{P}^t_n, R^t_n)$ of operators (with $n \geq n_t$), which we can construct as above, is called a $P^t$-corresponding system of operators.

The following proposition shows that for sequences $\mathcal{A} \in \mathcal{F}^t$, applying such $P^t$-corresponding systems, certain properties of boundedness of the operators $W^t(\mathcal{A})$ can be devolved on the sequence $\mathcal{A}$.

**Proposition 2.8.** Let $\mathcal{A} = \{A_n\} \in \mathcal{F}^t$, $P^t \in \mathcal{K}^t$ be a $P^t$-compact operator with the $P^t$-corresponding system $(P^t, \hat{P}^t, P^t_n, \hat{P}^t_n, R^t_n)$. Then

$$\limsup_{n \to \infty} \|A_nR^t_n\| \leq M^t\|W^t(\mathcal{A})P^t\|, \quad \text{and} \quad \limsup_{n \to \infty} \|R^t_nA_n\| \leq M^t\|P^tW^t(\mathcal{A})\|.$$  

**Proof.** From the above considerations we get

$$\|A_nR^t_n\| = \|A_nE^t_n(P^t_n)\| = \|E^t_n(A_nP^t_n)\| \leq M^t\|A_nP^t_n\| \leq M^t\|P^t_nP^t - P^t\| \to M^t\|W^t(\mathcal{A})P^t\|,$$

since $A_n^{(t)} \to W^t(\mathcal{A})$ $\mathcal{P}^t$-strongly and $P^t$ is $\mathcal{P}^t$-compact. The dual assertion can be checked analogously.

Moreover, if $P^t$ is a projection, then the relations

$$\hat{P}^t_n \hat{P}^t_n = \hat{P}^t_n \hat{P}^t \sum_{k=0}^{\infty} ((I^t - L^t_n)P^t)^k = \hat{P}^t \sum_{k=0}^{\infty} (I^t - L^t_n)^k = \hat{P}^t_n,$$

$$\hat{P}^t_n \hat{P}^t = \hat{P}^t_n \hat{P}^t \sum_{k=0}^{\infty} ((I^t - L^t_n)P^t)^k = \hat{P}^t_n (I^t \hat{P}^t + 0) = \hat{P}^t_n,$$

$$\hat{P}^t_n \hat{P}^t_n = \hat{P}^t_n (\hat{P}^t_n \hat{P}^t_n) \hat{P}^t_n = \hat{P}^t_n \hat{P}^t_n = \hat{P}^t_n$$

imply that also the operators $P^t_n \in \mathcal{K}^t$ and $R^t_n \in \mathcal{L}(E_n)$ are projections and one can easily check that $\dim \text{im} \ R^t_n = \dim \text{im} \ P^t_n = \dim \text{im} \ P^t$ for sufficiently large $n$.  


2.2. \( J^T \text{-FREDHOLM SEQUENCES} \)

2.2.2 The approximation numbers of \( A_n \) and the snapshots \( W^t \{ A_n \} \)

Here is a further connection between the operators \( A_n \) of a sequence \( \{ A_n \} \in \mathcal{F}^T \) and its snapshots \( W^t \{ A_n \} \).

**Theorem 2.9.** Let \( k = \{ A_n \} \in \mathcal{F}^T \), \( m \in \mathbb{N} \) and \( t_1, \ldots, t_m \in T \) be such that all \( W^{t_i}(k) \) are \( \mathcal{P}^{t_i} \text{-Fredholm} \). Then \( s^k(N) \rightarrow 0 \) for all \( k \leq \sum_{i=1}^m \dim \ker W^{t_i}(k) \), and \( s^k(N) \rightarrow 0 \) for all \( k \leq \sum_{i=1}^m \dim \text{coker} W^{t_i}(k) \) as \( n \rightarrow \infty \).

If, for one \( t \in T \), the snapshot \( W^t(k) \) is properly \( \mathcal{P}^t \text{-deficient from the right (from the left) then} \( s^k(N) \rightarrow 0 \) (or \( s^k(N) \rightarrow 0 \), respectively) for every \( k \in \mathbb{N} \) as \( n \rightarrow \infty \).

**Proof.** Proposition 1.32 reveals that, if all of these \( W^{t_i}(k) \) are normally solvable then, for each \( i = 1, \ldots, m \) and every respective non-negative integer \( k_i \leq \dim \ker W^{t_i}(k) \), we can fix a \( \mathcal{P}^{t_i} \text{-compact projection} P^{t_i} \) onto a \( k_i \)-dimensional subspace of \( \ker W^{t_i}(k) \) and choose the respective system \( \langle P^{t_1}, P^{t_2}, P^{t_3}, \ldots, P^{t_m} \rangle \) of projections. Moreover, for each \( i \) and every \( n \geq \max_j n_{t_j} \) we choose a normed basis \( \{ x_i^{n_{t_j}} \}_{j=1}^{k_i} \) of \( \text{im} \, R_n^{t_j} \), such that for arbitrary scalars \( \alpha_{i,j}^{n} \) the following holds:

\[
|\alpha_{i,j}^{n}| \leq \left\| \sum_{j=1}^{k_i} \alpha_{i,j}^{n} x_i^{n_{t_j}} \right\| \quad \text{for all } p = 1, \ldots, k_i. \tag{2.3}
\]

It is a simple consequence of Auerbach’s Lemma (see Proposition 1.4) that such a basis always exists. Let \( i \neq j \). Due to the separation condition (II), since \( \| P_n^{t_i} - P_j^{t_i} \| \rightarrow 0 \) and \( P_j^{t_i} \in \mathcal{K}^t \), we have

\[
\| R_n^{t_i} R_n^{t_j} \| = \| E_n^{t_i}(P_n^{t_i}) E_n^{t_j}(P_n^{t_i}) \| \leq M^{t_j} \| E_n^{t_j}(P_n^{t_i}) P_n^{t_j} \| \rightarrow 0. \tag{2.4}
\]

We now prove that there is a number \( N \in \mathbb{N} \), such that, for all \( j = 1, \ldots, m \), \( k = 1, \ldots, k_j \), \( n \geq N \), all scalars \( \alpha_{i,j} \), and all \( y \in \mathcal{Y}_n := \bigcap_{i=1}^{m} \ker R_n^{t_j} \)

\[
|\alpha_{j,k}| \leq \gamma \left\| \sum_{i=1}^{m} \sum_{l=1}^{k_i} \alpha_{i,l} x_i^{n_{t_j}} + y \right\|, \quad \text{where } \gamma = 2 \max_{i=1,\ldots,m} M^{t_j} \| P_j^{t_j} \|. \tag{2.5}
\]

In other words, all of these \( x_i^{n_{t_j}} \) are linearly independent in a “uniform” manner.

By (2.4), for every \( \epsilon \in (0, 1) \), there exists a number \( N \in \mathbb{N} \) such that

\[
\sum_{i=1,\ldots,m, j \neq i} \| R_n^{t_i} R_n^{t_j} \| \leq \epsilon \quad \text{and} \quad \| R_n^{t_i} \| \leq M^{t_i} \| P_n^{t_i} \| \leq (1 + \epsilon) M^{t_i} \| P_n^{t_i} \| \tag{2.6}
\]

for all \( i = 1, \ldots, m \) and \( n \geq N \). Now let \( \alpha_{i,j} \) be arbitrary but fixed scalars and choose \( i_0, l_0 \) such that \( |\alpha_{i_0,l_0}| = \max_{i,l} |\alpha_{i,l}| \). Thanks to (2.3) and (2.6) we find (since \( x_i^{n_{t_j}} = R_n^{t_j} x_i^{n_{t_j}} \))

\[
\| R_n^{t_i} \| \left( \sum_{i=1}^{m} \sum_{l=1}^{k_i} \alpha_{i,l} x_i^{n_{t_j}} + y \right) \geq \left\| \sum_{l=1}^{k_i} \alpha_{i_0,l} R_n^{t_i} x_i^{n_{t_j}} \right\| - \left\| \sum_{l=1, l \neq i_0}^{k_i} \alpha_{i,l} R_n^{t_i} x_i^{n_{t_j}} \right\| \geq \alpha_{i_0,l_0} - \| \alpha_{i_0,l_0} \| \sum_{l=1, l \neq i_0}^{k_i} k_i \| R_n^{t_i} R_n^{t_i} \| \geq (1 - \epsilon) |\alpha_{i_0,l_0}|.
\]
Thus, for all $n \geq N$, for all $j = 1, \ldots, m$, and all $k = 1, \ldots, k_j$,

$$|\alpha_{j,k}| \leq |\alpha_{n,0}| \leq \left\| \frac{R_n^0}{1 - \epsilon} \right\| \left( \sum_{i=1}^{m} \sum_{l=1}^{k_i} \alpha_i x_{i,l}^n + y \right)$$

$$\leq \frac{1 + \epsilon}{1 - \epsilon} \max_{i=1, \ldots, m} (M^n \|P^i\|) \left( \sum_{i=1}^{m} \sum_{l=1}^{k_i} \alpha_i x_{i,l}^n + y \right).$$

Choosing $\epsilon = 1/3$ gives assertion (2.5). Obviously, $\mathcal{E}_n$ decomposes into the direct sum

$$\mathcal{E}_n = \text{span}\{x_{1,1}^n\} \oplus \cdots \oplus \text{span}\{x_{m,k_m}^n\} \oplus Y_n$$

for $n \geq N$, and we can introduce functionals $f_{i,j}^n \in \mathcal{E}_n^*$ by the rule

$$f_{i,j}^n \left( \sum_{k=1}^{k_i} \sum_{l=1}^{k_i} \alpha_k x_{k,l}^n + y \right) = \alpha_{i,j}^n 1 \leq i \leq m, 1 \leq j \leq k_i.$$

Then we always have $\|f_{i,j}^n\| \leq \gamma$ and $f_{i,j}^n(Y_n) = \{0\}$. Now, we denote by $R_n \in \mathcal{L}(\mathcal{E}_n)$ the linear operators

$$R_n x := \sum_{i=1}^{m} \sum_{j=1}^{k_i} f_{i,j}^n(x)x_{i,j}^n.$$

The operators $R_n$ are projections of rank $\dim R_n = k := \sum_{i=1}^{m} k_i$ and they are uniformly bounded with respect to $n$. Moreover, for every $x \in \mathcal{E}_n$ we have

$$\|A_n R_n x\| = \left\| A_n \sum_{i=1}^{m} \sum_{j=1}^{k_i} f_{i,j}^n(x)x_{i,j}^n \right\| \leq \sum_{i=1}^{m} \sum_{j=1}^{k_i} \|f_{i,j}^n(x)\| \|A_n R_n x_{i,j}\|$$

$$\leq \sum_{i=1}^{m} \sum_{j=1}^{k_i} \gamma \|x\| \|A_n R_n^i\| \|x_{i,j}\| = \gamma \|x\| \sum_{i=1}^{m} k_i \|A_n R_n^i\|.$$

Since $\|A_n R_n^i\| \to 0$ for each $i$ (see Proposition 2.8) it follows that

$$s_k(A_n) = \inf \{\|A_n + F\| : F \in \mathcal{L}(\mathcal{E}_n), \dim F \geq k\}$$

$$\leq \|A_n - A_n(L_n - R_n)\| = \|A_n R_n\| \leq \gamma k \max_i \|A_n R_n^i\| \to 0.$$

Due to Proposition 1.32 we can also choose $\mathcal{P}^t_\omega$-compact projections $\tilde{P}_i$, with $\tilde{P}_i W^i(\mathcal{A}) = 0$ and proceed in the same way to find the analogues $\tilde{k}_i \leq \dim \text{coker} W^i(\mathcal{A})$, $\tilde{f}_{i,j}^n$, $\tilde{R}_i$ and $\tilde{y}_n$, and to construct a bounded sequence $(\tilde{R}_n)$ of projections $\tilde{R}_n$, being of rank $\tilde{k} = \sum_{i=1}^{m} \tilde{k}_i$. Then

$$\|\tilde{R}_n A_n x\| = \left\| \sum_{i=1}^{m} \sum_{j=1}^{\tilde{k}_i} \tilde{f}_{i,j}^n(A_n x)\tilde{x}_{i,j}^n \right\| \leq \sum_{i=1}^{m} \sum_{j=1}^{\tilde{k}_i} \|\tilde{f}_{i,j}^n(\tilde{R}_i A_n x) + \tilde{f}_{i,j}^n((I - \tilde{R}_n) A_n x)\| \|\tilde{x}_{i,j}^n\|$$

$$\leq \sum_{i=1}^{m} \sum_{j=1}^{\tilde{k}_i} \left( \|\tilde{f}_{i,j}^n\| \|\tilde{R}_i A_n\| + \|\tilde{f}_{i,j}^n((I - \tilde{R}_n) A_n)\| \|x\| \right).$$
2.2. \( J^T \)-Fredholm sequences

where \( \| \hat{f}_{i,j}^n(I - \tilde{R}_n^i) \| \to 0 \) as \( n \to \infty \) due to the following observations: for each \( y \in \tilde{Y}_n \) we have

\[
\hat{f}_{i,j}^n((I - \tilde{R}_n^i)y) = \hat{f}_{i,j}^n(y) = 0,
\]

that is \( \hat{f}_{i,j}^n(I - \tilde{R}_n^i)(I - \tilde{R}_n) = 0 \), and further

\[
\| \hat{f}_{i,j}^n(I - \tilde{R}_n^i)\tilde{R}_n \| \leq \sum_{s=1}^{m} \sum_{t=1}^{k_s} |\hat{f}_{i,j}^n| \| \hat{f}_{i,j}^n(I - \tilde{R}_n^i)x_{s,t}^n \| \to 0
\]

since \( (I - \tilde{R}_n^i)x_{s,t}^n = 0 \) and \( \| \hat{f}_{i,j}^n(I - \tilde{R}_n^i)x_{s,t}^n \| = \| \hat{f}_{i,j}^n \tilde{R}_n x_{s,t}^n \| \leq \| \hat{f}_{i,j}^n \| \| \tilde{R}_n x_{s,t}^n \| \to 0 \) for \( s \neq i \) as \( n \to \infty \). Thus, we again obtain

\[
s'_k(A_n) \leq \| \tilde{R}_n A_n \| \to 0 \quad \text{as} \quad n \to \infty.
\]

Now, suppose that one operator \( W^t(A) \) is properly \( \mathcal{P}_t \)-deficient from the right (or left). Then for each \( k \in \mathbb{N} \) and each \( \epsilon > 0 \) there is a projection \( Q \in \mathcal{K}^t \), rank \( Q = k \) such that \( \| W^t(A)Q \| < \epsilon \) (or \( \| QW^t(A) \| < \epsilon \), respectively). Choosing a system of projections w.r.t. \( Q \), we deduce again from Proposition 2.8 that \( \limsup_n s'_k(A_n) = 0 \) (or \( \limsup_n s'_k(\tilde{A}_n) = 0 \)) since \( \epsilon \) can be chosen arbitrarily small.

Finally notice that \( W^t_i(A) \) being \( \mathcal{P}_t \)-Fredholm but not normally solvable implies \( \mathcal{P}_t \)-deficiency from both sides by Proposition 1.32.

2.2.3 Regular \( J^T \)-Fredholm sequences

**Definition 2.10.** We introduce the nullity \( \alpha(A) \) and deficiency \( \beta(A) \) of a sequence \( A \in \mathcal{F}_t \) by

\[
\alpha(A) := \sum_{t \in T} \dim \ker W^t(A) \quad \text{and} \quad \beta(A) := \sum_{t \in T} \dim \coker W^t(A).
\]

A \( J^T \)-Fredholm sequence \( A \in \mathcal{F}_t \) is said to be regular, if all operators \( W^t(A) \) are Fredholm (in the usual sense). Of course, due to Theorem 2.6, \( A \) is regular if and only if \( \alpha(A) \) and \( \beta(A) \) are finite, hence we are in a position to introduce the index of a regular sequence \( A \) by

\[
\text{ind}(A) := \alpha(A) - \beta(A).
\]

Applying Proposition 2.5 and the well known properties of Fredholm operators as stated in Theorem 1.2 it is not hard to prove the following result.

**Proposition 2.11.** Let \( A \in \mathcal{F}_t \) be a regular \( J^T \)-Fredholm sequence and \( B \in \mathcal{F}_t \).

- If \( \|B\| \) is sufficiently small then \( \alpha(A + B) \leq \alpha(A) \), \( \beta(A + B) \leq \beta(A) \) and \( \text{ind}(A + B) = \text{ind}(A) \).
- If \( B \in \mathcal{J}_t \) has only compact snapshots then \( A + B \) is regular and \( \text{ind}(A + B) = \text{ind}(A) \).
- If \( B \in \mathcal{G} \) then \( \alpha(A + B) = \alpha(A) \) and \( \beta(A + B) = \beta(A) \).
- If \( B \in \mathcal{F}_t \) is also a regular \( J^T \)-Fredholm sequence then \( \text{ind}(A + B) = \text{ind}(A) + \text{ind}(B) \).

Observe that, if all approximate identities \( P^t \) consist of finite rank operators only, then \( \mathcal{K}(E^t, \mathcal{P}_t) \) is a subset of \( \mathcal{K}(E^t) \), and hence every \( J^T \)-Fredholm sequence is regular.

\[\text{This already appeared in [30] and [50].}\]
2.2.4 The splitting property and the index formula

Proposition 2.12. Let \( \mathcal{A} = \{A_n\} \in \mathcal{F}^T \) be a \( \mathcal{F}^T \)-Fredholm sequence and suppose that all of its snapshots \( W^i(\mathcal{A}) \) are properly \( \mathcal{P}^i \)-Fredholm operators, respectively. If \( \alpha(\mathcal{A}) \) is finite then
\[
\liminf_n s_{\alpha(\mathcal{A})+1}(A_n) > 0,
\]
and if \( \beta(\mathcal{A}) \) is finite then
\[
\liminf_n s_{\beta(\mathcal{A})+1}(A_n) > 0.
\]

Proof. Assume that \( \alpha(\mathcal{A}) \) is finite. Due to Equation (2.1)
\[
\|\mathcal{A}\| = \| + \sum_{i=1}^m \{E_i(A^{(i)}_kB^i)\} + G
\]
from Proposition 2.5, all operators \( W^i(\mathcal{A}) \) with \( t \in T \setminus \{t_1, ..., t_m\} \) have kernels of dimension zero. Moreover, for every \( i = 1, ..., m \), Corollary 1.31 provides an operator \( B_i \in \mathcal{L}(E_i, \mathcal{P}_i) \) such that
\[
P^i := I^i - B^i W^i(\mathcal{A}) \in \mathcal{K}_i
\]
is a projection onto the kernel of \( W^i(\mathcal{A}) \). Define \( \hat{P}^i := I^i - P^i \) and furthermore, for every \( n \in \mathbb{N} \), (with \( \{B_n\} = \mathcal{B} \))
\[
D_n := B_n - \sum_{i=1}^m E_i(A^{(i)}_K^i B^i).
\]
It is obvious that \( \{D_n\} \in \mathcal{F}^T \) and due to the \( \mathcal{P}^i \)-strong convergence of \( A^{(i)}_K \) we see that
\[
\|E_i(A^{(i)}_K^i B^i)A_n - E_i(A^{(i)}_K^i \hat{P}^i)\| = \|E_i(A^{(i)}_K^i B^i)A_n - W^i(\mathcal{A}))\| \to 0
\]
for all \( i \), i.e. \( \sum_{i=1}^m E_i(A^{(i)}_K^i B^i)A_n - \sum_{i=1}^m E_i(A^{(i)}_K^i \hat{P}^i) \in G \). Thus
\[
D_n A_n = B_n A_n - \sum_{i=1}^m E_i(A^{(i)}_K^i B^i)A_n
\]
\[
= L_n + \sum_{i=1}^m E_i(A^{(i)}_K^i B^i) - \sum_{i=1}^m E_i(A^{(i)}_K^i \hat{P}^i) + H_n
\]
\[
= L_n + \sum_{i=1}^m E_i(A^{(i)}_K^i P^i) + H_n
\]
with \( \{H_n\} \in G \). Since \( \dim \ker W^i(\mathcal{A}) \) for all \( i \), we find
\[
\dim \ker \left( \sum_{i=1}^m E_i(A^{(i)}_K^i P^i) \right) \leq \alpha(\mathcal{A}).
\]
For sufficiently large \( n \) we have \( \|H_n\| < 1/2 \) and, applying Corollary 1.37, it follows
\[
\frac{1}{2} \leq \left( \|L_n + H_n\|^{-1} \right)^{-1} = s_{\alpha(\mathcal{A})}^i (L_n + H_n)
\]
\[
= \inf \{\|L_n + H_n - F\| : \dim \ker F \geq 1\}
\]
\[
\leq \inf \left\{ \left\| L_n + H_n - F + \sum_{i=1}^m E_i(A^{(i)}_K^i P^i) \right\| : \dim \ker F \geq \alpha(\mathcal{A}) + 1 \right\}
\]
\[
= \inf \{\|D_n A_n - F\| : \dim \ker F \geq \alpha(\mathcal{A}) + 1\}
\]
\[
\leq \inf \{\|D_n A_n - D_n F\| : \dim \ker F \geq \alpha(\mathcal{A}) + 1\}
\]
\[
\leq \|D_n\| \inf \{\|A_n - F\| : \dim \ker F \geq \alpha(\mathcal{A}) + 1\} \leq \|D_n\| s_{\alpha(\mathcal{A})+1}(A_n).
\]
If we start with Equation (2.2) we can proceed analogously to obtain
\[
\frac{1}{2} \leq \left( \|L_n + H_n\|^{-1} \right)^{-1} = s_{\alpha(\mathcal{A})}^i (L_n + H_n)
\]
\[
= \inf \{\|L_n + H_n - F\| : \dim \ker F \geq 1\}
\]
\[
\leq \inf \left\{ \left\| L_n + H_n - F + \sum_{i=1}^m E_i(A^{(i)}_K^i P^i) \right\| : \dim \ker F \geq \alpha(\mathcal{A}) + 1 \right\}
\]
\[
= \inf \{\|D_n A_n - F\| : \dim \ker F \geq \alpha(\mathcal{A}) + 1\}
\]
\[
\leq \inf \{\|D_n A_n - D_n F\| : \dim \ker F \geq \alpha(\mathcal{A}) + 1\}
\]
\[
\leq \|D_n\| \inf \{\|A_n - F\| : \dim \ker F \geq \alpha(\mathcal{A}) + 1\} \leq \|D_n\| s_{\alpha(\mathcal{A})+1}(A_n).
\]
We have that all operators $W^t(\mathcal{A})$ of a regular $\mathcal{J}^T$-Fredholm sequence $\mathcal{A}$ are Fredholm and $\mathcal{P}^t$-Fredholm, hence properly $\mathcal{P}^t$-Fredholm by Proposition 1.26. In view of Theorem 2.9 this yields the following result.

**Theorem 2.13.** Let $\mathcal{A} = \{A_n\} \in \mathcal{F}^T$ be a regular $\mathcal{J}^T$-Fredholm sequence. Then the approximation numbers from the right have the $\alpha(\mathcal{A})$-splitting property, that is

$$\lim_{n \to \infty} s^r_{\alpha(\mathcal{A})}(A_n) = 0 \quad \text{and} \quad \liminf_{n \to \infty} s^r_{\alpha(\mathcal{A})+1}(A_n) > 0.$$  

Analogously, the approximation numbers from the left have the $\beta(\mathcal{A})$-splitting property.

**Theorem 2.14.** Let $\mathcal{A} = \{A_n\} \in \mathcal{F}^T$ be a regular $\mathcal{J}^T$-Fredholm sequence. Then, for sufficiently large $n$, the operators $A_n$ are Fredholm and their indices coincide with $\text{ind} \mathcal{A}$, in other words

$$\lim_{n \to \infty} \text{ind} A_n = \sum_{t \in T} \text{ind} W^t(\mathcal{A}).$$

**Proof.** Theorem 2.13 shows that $s^r_{\alpha(\mathcal{A})+1}(A_n) > 0$ and $s^l_{\beta(\mathcal{A})+1}(A_n) > 0$ for sufficiently large $n$, hence $A_n$ is Fredholm by Corollary 1.38. Moreover, for large $n$ there is always an operator $F_n$ with $\text{dim ker} F_n \geq \beta(\mathcal{A})$ such that $\|A_n - F_n\| < s^l_{\beta(\mathcal{A})}(A_n) + 1/n$. We can choose a projection $S_n$ of rank $\beta(\mathcal{A})$ with norm less than $\beta(\mathcal{A}) + 2$ such that $\|S_n F_n\| < 1/n$ (see Proposition 1.6) and we deduce from Theorem 2.13 that

$$\frac{\|S_n A_n\|}{\beta(\mathcal{A}) + 2} < \frac{\|S_n A_n\| + \|S_n F_n\|}{\|S_n\|} \leq s^l_{\beta(\mathcal{A})}(A_n) + \frac{2}{n} \to n \to \infty 0.$$  

We set $S_n := 0$ for the remaining smaller $n$ and conclude that $\{S_n A_n\} \in \mathcal{G}$, that is the sequence $\{\mathring{A}_n\} := \{(L_n - S_n)A_n\} \in \mathcal{F}^T$ is $\mathcal{J}^T$-Fredholm with $\alpha(\{\mathring{A}_n\}) = \alpha(\mathcal{A})$, $\beta(\{\mathring{A}_n\}) = \beta(\mathcal{A})$ and $\\text{ind} \{\mathring{A}_n\} = \text{ind} \mathcal{A}$ (cf. Proposition 2.11). Analogously, we choose a sequence $\{R_n\} \in \mathcal{F}$ of projections $R_n$ of rank $\alpha(\mathcal{A})$ and $\|R_n\| \leq \alpha(\mathcal{A}) + 2$ for large $n$, such that $\{\mathring{A}_n R_n\} \in \mathcal{G}$.

We now consider the sequence $\mathcal{C} = \{C_n\} := \{(L_n - S_n)A_n (L_n - R_n)\}$ and we find that it is a regular $\mathcal{J}^T$-Fredholm sequence with $\alpha(\mathcal{C}) = \alpha(\mathcal{A})$, $\beta(\mathcal{C}) = \beta(\mathcal{A})$ and $\\text{ind} \mathcal{C} = \text{ind} \mathcal{A}$. More precisely, there is an $N \in \mathbb{N}$ and a constant $C > 0$ such that for all $n \geq N$

$$s^r_{\alpha(\mathcal{A})}(C_n) = s^l_{\beta(\mathcal{A})}(C_n) = 0 \quad \text{and} \quad s^r_{\alpha(\mathcal{A})+1}(C_n), s^l_{\beta(\mathcal{A})+1}(C_n) > C,$$  

which proves that the $C_n$ are Fredholm of the index $\alpha(\mathcal{A}) - \beta(\mathcal{A}) = \text{ind} \mathcal{A}$. Since $L_n - R_n$ and $L_n - S_n$ are Fredholm of index zero, we find that $A_n$ is Fredholm of index $\text{ind} \mathcal{A}$. 

**Remark 2.15.** Note that there is a bounded sequence $\{D_n\}$ such that for sufficiently large $n$ the operator $D_n$ is a generalized inverse for the $C_n$ from the previous proof. Moreover, $\mathcal{A} + \mathcal{C}$ belongs to $\mathcal{G}$, hence we find that $\mathcal{A} + \mathcal{C}$ is generalized invertible in $\mathcal{F}/\mathcal{G}$, whenever $\mathcal{A}$ is a regular $\mathcal{J}^T$-Fredholm sequence.

Furthermore, this shows that the nullity, deficiency and the index of a structured operator sequence $\mathcal{A}$ being $\mathcal{J}^T$-Fredholm as introduced in Definition 2.10 are universal characteristics of this sequence in the following sense: If these numbers exist for the $\mathcal{J}^T$-Fredholm sequence $\mathcal{A}$ in one setting $\mathcal{F}^T$ then they are the same in every setting $\mathcal{F}^T$ in which $\mathcal{A}$ is $\mathcal{J}^T$-Fredholm. In Section 2.3 we will recover this observation from a more general point of view.
2.2.5 Stability of sequences $\mathcal{A} \in \mathcal{F}^T$

**Definition 2.16.** A sequence $\mathcal{A} = \{A_n\} \in \mathcal{F}$ is called stable, if there is an index $n_0$ such that all operators $A_n$, $n \geq n_0$, are invertible and $\sup\{\|A_n^{-1}\| : n \geq n_0\} < \infty$.

It is a well known result of Kozak [39] that a sequence $\mathcal{A} \in \mathcal{F}$ is stable if and only if the coset $\mathcal{A} + \mathcal{G}$ is invertible in $\mathcal{F}/\mathcal{G}$. Utilizing the higher structure of the given setting, namely the existence of $\mathcal{P}^t$-strong limits $W^t(\mathcal{A})$, we can prove a stronger result. This observation has a long history and is known as the *Lifting Theorem* in the literature, see for example the books [12], Section 7.8ff, [29], Section 1.6 or [30], Section 5.3. It has its origin in [83].

**Theorem 2.17.** A sequence $\mathcal{A} \in \mathcal{F}^T$ is stable if and only if $\mathcal{A}$ is $\mathcal{J}^T$-Fredholm and all $W^t(\mathcal{A})$, $t \in T$, are invertible. In particular, $\mathcal{F}^T/\mathcal{G}$ is inverse closed in $\mathcal{F}/\mathcal{G}$.

**Proof.** Let $\mathcal{A} = \{A_n\}$ be $\mathcal{J}^T$-Fredholm and all $W^t(\mathcal{A})$ be invertible. Then $\mathcal{A}$ is regular and Theorem 2.13 yields an $N \in \mathbb{N}$ and a constant $C > 0$ such that $s_1^t(A_n) \geq C$ and $s_2^t(A_n) \geq C$ for all $n > N$. With Corollary 1.37 we deduce that $\mathcal{A}$ is stable. Conversely, let $\mathcal{A} = \{A_n\} \in \mathcal{F}^T$ be stable and define a sequence $\mathcal{B} = \{B_n\} \in \mathcal{F}$ by $B_n := A_n^{-1}$ if $A_n$ is invertible and $B_n := L_n$ otherwise. Then $\mathcal{B}A - I, \mathcal{A}B - I \in \mathcal{G}$. Notice that for every $t \in T$ the sequence

$$(A_t^n := (E_n^{-1}(A_n)L_n^1 + (I^t - L_n^0)) \in \mathcal{F}(\mathcal{E}^t, \mathcal{P}^t)$$

is also stable, tends to $W^t(\mathcal{A})$ and the operators $B_t^n := E_n^{-1}(B_n)L_n^1 + (I^t - L_n^0)$ are the inverses of $A_t^n$ for sufficiently large $n$. Proposition 1.29 shows that $W^t(\mathcal{A})$ is invertible and $(B_t^n)$ converges $\mathcal{P}^t$-strongly to $(W^t(\mathcal{A}))^{-1}$. Thus, $\mathcal{B} \in \mathcal{F}^T$. \hfill \square

### 2.3 A general Fredholm property

The notion of $\mathcal{J}^T$-Fredholmness for structured operator sequences $\mathcal{A} \in \mathcal{F}^T$ as it was introduced in the preceding sections depends on the underlying setting determined by $\mathcal{F}^T$ and $\mathcal{J}^T$. On the one hand, it is convenient to work in this context because there we can describe the Fredholm properties of a sequence $\mathcal{A}$ quite well in terms of Fredholm properties of its snapshots $W^t(\mathcal{A})$. On the other hand, the main disadvantage lies in the fact that a sequence which is Fredholm in one setting does not need to be Fredholm in another setting.\(^5\) Thus, in what follows, we introduce a “universal” Fredholm property for the larger framework of all bounded operator sequences in $\mathcal{F}$. This approach has been extensively studied in the case of $C^*$-algebras of operator sequences on finite dimensional spaces in [30], Chapter 6.3. and was derived by Roch [70]. In particular, we will recover the mentioned universal characteristics $\alpha(\mathcal{A})$, $\beta(\mathcal{A})$ and $\text{ind}(\mathcal{A})$ in this larger framework again.

To avoid degenerated cases we assume that $\lim \sup_n \dim E_n = \infty$.

**Definition 2.18.** A sequence $\{K_n\} \in \mathcal{F}$ is said to be of almost uniformly bounded rank if

$$\lim \sup_{n \to \infty} \text{rank} K_n < \infty.$$ 

Let $\mathcal{I}$ denote the closure of the set containing all sequences of almost uniformly bounded rank. The elements of $\mathcal{I}$ are referred to as compact sequences.

One easily checks that $\mathcal{I}$ forms a proper closed ideal in $\mathcal{F}$ which contains $\mathcal{G}$.

\(^5\)Clearly, $\mathcal{A} = I + \{K_t^t(L_n^t,K_t^t)\}$ with $K_t^t \in \mathcal{K}(\mathcal{P}^t, \mathcal{P}^t)$ is $\mathcal{J}^T$-Fredholm, but if we drop $t$ from the index set $T$ and consider $\mathcal{A}$ as a sequence in $\mathcal{F}^T$ with $\bar{T} = T \setminus \{t\}$ then we loose this property.
2.3. A GENERAL FREDHOLM PROPERTY

**Definition 2.19.** Now we are in a position to introduce a class of Fredholm sequences in $F$ by calling $A = \{A_n\} \in F$ Fredholm if $A + I$ is invertible in $F/I$.

Evidently, we have the typical statements as known for the classical Fredholmness

- Stable sequences are Fredholm and never compact.
- Products of Fredholm sequences are Fredholm.
- The sum of a Fredholm sequence and a compact sequence is Fredholm.
- The set of all Fredholm sequences is open in $F$.
- If $\{A_n\}$ is Fredholm, then $\{A_n^*\}$ is of Fredholm type.

For an equivalent characterization of Fredholm sequences we need the following definition.

**Definition 2.20.** Let $A = \{A_n\} \in F$. If there is a finite number $\alpha \in \mathbb{Z}^+$ with

$$
\lim \inf_{n \to \infty} s_{\alpha}^r(A_n) = 0 \quad \text{and} \quad \lim \inf_{n \to \infty} s_{\alpha+1}^r(A_n) > 0,
$$

then this number is called the $\alpha$-number of $A$ and it is denoted by $\alpha(A)$. Analogously, we introduce $\beta(A)$, the $\beta$-number of $A$, as the $\beta \in \mathbb{Z}^+$ with

$$
\lim \inf_{n \to \infty} s_{\beta}^l(A_n) = 0 \quad \text{and} \quad \lim \inf_{n \to \infty} s_{\beta+1}^l(A_n) > 0.
$$

Besides the well known result of Kozak [39], by applying Corollary 1.37, we immediately get the following characterization of stability in the large algebra $F$.

**Theorem 2.21.** For a sequence $A \in F$ the following are equivalent.

- $A$ is stable.
- $A + G$ is invertible in $F/G$.
- $\alpha(A) = \beta(A) = 0$.

**Theorem 2.22.** For a sequence $A \in F$ the following are equivalent.

- $A$ is Fredholm.
- There are sequences $B_1, B_2 \in F$ such that $B_1A - I$ and $AB_2 - I$ are of almost uniformly bounded rank.
- $A$ has an $\alpha$-number and a $\beta$-number.

**Proof.** Let $A = \{A_n\}$ be Fredholm. Then there are sequences $D, H_1, G_i \in F$ ($i = 1, 2$) such that $DA = I + H_1 + G_1$ and $AD = I + H_2 + G_2$, where $H_i$ are of almost uniformly bounded rank and $\|G_i\| < 1/2$. Since $I + G_i$ are invertible, we can define $B_1 := (I + G_1)^{-1}D$, $K_1 := (I + G_1)^{-1}H_1$ as well as $B_2 := D(I + G_2)^{-1}$, $K_2 := H_2(I + G_2)^{-1}$. This implies the second assertion.

Now let $\{K_n\} = \{B_n\}\{A_n\} - I$ and $n_0 \in \mathbb{N}$ with $k := \sup_{n \geq n_0} \text{rank} \ K_n < \infty$. For each $n \geq n_0$ we can introduce a projection $R_n$ with kernel of dimension $k$ and norm $\|R_n\| \leq k + 1$ such that $R_nK_n = 0$, due to Proposition 1.5. Then $R_nB_nA_n = R_n$. Moreover, for each $n \geq n_0$ we observe that $s_{k+1}^l(R_n) \geq 1$, because otherwise there would exist an operator $F$ with $\dim \ker F \geq k + 1$.
such that \( \| R_n - F \| < 1 \). Hence \( L_n - R_n + F \) would be invertible, but since \( \text{rank}(L_n - R_n) = k \) this yields a contradiction. Thus

\[
1 \leq s^k_{k+1}(R_n) = \inf \{ \| R_n - F \| : \dim \ker F \geq k + 1 \} \\
= \inf \{ \| R_n B_n A_n - F \| : \dim \ker F \geq k + 1 \} \\
\leq \inf \{ \| R_n B_n A_n - R_n B_n F \| : \dim \ker F \geq k + 1 \} \\
\leq \| R_n \| \| B_n \| s^k_{k+1}(A_n),
\]

hence \( \alpha(A) \leq k + 1 \) exists. Analogously we find a \( \beta \)-number for \( A \), that is, the third assertion holds.

Finally, let \( A \) have an \( \alpha \)-number and a \( \beta \)-number and let \( N \in \mathbb{N} \) be such that

\[
\inf \{ s^\alpha_{\alpha+1}(A_n), s^\beta_{\beta+1}(A_n) : n \geq N \} > 0.
\]

Then \( A_n, n \geq N, \) are Fredholm operators by Corollary 1.38. In view of Equation (1.11) there are a constant \( C > 0 \) and a sequence \( \{ R_n \} \in \mathcal{F} \) of projections \( R_n \) with kernels of dimension \( \alpha(A) \) such that

\[
\inf \{ \| A_n x \| : x \in \text{im} R_n, \| x \| = 1 \} \geq C \text{ for all } n \geq N.
\]

We consider the restrictions \( A_n|_{\text{im} R_n} \) of \( A_n \) to \( \text{im} R_n \), which are injective. The spaces \( \text{im}(A_n|_{\text{im} R_n}) \) are of the codimension not greater than \( \alpha(A) + \beta(A) \), hence they are closed and we can choose projections \( S_n \) onto \( \text{im}(A_n|_{\text{im} R_n}) \), which are uniformly bounded with respect to \( n \geq N \). Therefore, the operators \( A_n|_{\text{im} R_n} : \text{im} R_n \to \text{im} S_n \) are invertible and their inverses \( A_n^{(-1)} \) are (uniformly) bounded by \( C \).

For the operators \( B_n := R_n A_n^{(-1)} S_n \) we conclude that

\[
A_n B_n = A_n R_n A_n^{(-1)} S_n = A_n A_n^{(-1)} S_n = S_n, \\
B_n A_n = B_n A_n R_n + B_n A_n (L_n - R_n) = R_n A_n^{(-1)} S_n A_n R_n + B_n A_n (L_n - R_n) \\
in (2.7)
\]

Since the latter term is of uniformly bounded rank, this proves the Fredholmness of \( A \). \( \square \)

**Corollary 2.23.** Let \( \mathcal{A} = \{ A_n \} \in \mathcal{F} \) be a Fredholm sequence and let \( T, \), as well as \( E^t, \mathcal{P}^t, (L^t_n) \) and \( (E^t_n) \) be given as in Section 2.1 such that the Conditions (I), (II) are in force. Suppose that, for one \( t \in T \), a subsequence of \( (E^t_n(A_n) L^t_n) \) converges \( \mathcal{P}^t \)-strongly to an operator \( A^t \in \mathcal{L}(E^t, \mathcal{P}^t) \). Then \( A^t \) is Fredholm. If for all \( t \) in a certain subset \( T' \subset T \) and with respect to one common subsequence of \( \mathcal{A} \) such operators \( A^t \) exist then

\[
\sum_{t \in T'} \dim \ker A^t \leq \alpha(A), \quad \sum_{t \in T'} \dim \text{coker} A^t \leq \beta(A).
\]

**Proof.** The assertion immediately results from Theorem 2.9. \( \square \)

For a setting \( \mathcal{F}^T \) and for sequences \( \mathcal{A} \in \mathcal{F}^T \) we get from Theorem 2.13

**Corollary 2.24.** If \( \mathcal{A} \in \mathcal{F}^T \) is a regular \( \mathcal{J}^T \)-Fredholm sequence then \( \mathcal{A} \) is a Fredholm sequence in \( \mathcal{F} \), that is \( \mathcal{A} \) is invertible modulo \( \mathcal{T} \). Moreover, the numbers \( \alpha(A) \) and \( \beta(A) \) in the Definitions 2.10 and 2.20 are consistent.
2.4 Rich sequences

Recall our motivating questions on band-dominated operators from Section 1.5 and let us consider two examples on $l^p(\mathbb{Z}, \mathbb{C})$.

- Firstly, set $A := I + K$ with a $\mathcal{P}$-compact operator $K$. Then $(A_n)$ converges to $A$ whereas the two shifted copies $(A^1_n)$ and $(A^2_n)$ converge $\mathcal{P}$-strongly to the identity. This can be easily seen, e.g., for $(A^1_n)$ one has

$$A^1_n = V_n(P_n(I + K)P_n + Q_n)V_{-n} = V_n(I + P_nKP_n)V_{-n} = I + V_nP_nKP_nV_{-n},$$

and $\|P_nV_nP_nK\|, \|KP_nV_{-n}P_n\| \to 0$ as $n \to \infty$. We already promised (without having proved it yet!) that these snapshots are also sufficient to characterize the stability, that is we even claim that $(A_n)$ is stable if and only if $A$ is invertible.

- As a second example let $J$ be given by the rule $(x_i) \mapsto ((-1)^i x_i)$, that is its matrix representation equals diag(\ldots, -1, 1, -1, 1, -1, \ldots), and set $A := J + K$ with a $\mathcal{P}$-compact $K$. The central snapshot is again $A$, but the $\mathcal{P}$-strong limits of $(A^1_n)$ and $(A^2_n)$ do not exist anymore. To achieve convergence, we have to pass to subsequences, and this leads to two snapshots for $(A^1_n)$, namely $PJP + Q$ and $-PJP + Q$, where $P$ and $Q$ are the complementary projections $\chi_{\mathbb{Z}} J$ and $\chi_{\mathbb{Z}} \backslash J$, respectively. Similarly, $(A^2_n)$ provides two further snapshots.

The pending proof will be given in Section 2.5.1, but for the moment we learned something very important: Instead of considering the whole sequences $\mathcal{A}$ in $\mathcal{F}$ one might pass to subsequences to attain a simpler structure. More precisely, one could fix a strictly increasing sequence $g = (g_n)$ of positive integers and pass to the algebra $\mathcal{F}_g$ of all subsequences $\mathcal{A}_g = \{A_{g_n}\}$. Obviously, the theory for elements in the respective algebras of subsequences $\mathcal{G}_g$, $\mathcal{F}_g^T$, and $\mathcal{G}_g^T$ is analogous to what we have already done for the basic case $g_n = n$.

We call a sequence of operators $T$-structured, if for every $t \in T$ the respective snapshot $W^T(\cdot)$ exists. Our second example teaches that the finite section sequences of band-dominated operators are not $T$-structured in general, but one could pass to subsequences which are $T$-structured, and hence satisfy the requirements for our Fredholm theory.

**Definition 2.25.** We say that a sequence $\mathcal{A} \in \mathcal{F}$ is uniformly rich with respect to $T$ (or $T$-rich) if each strictly increasing sequence $g$ of positive integers has a subsequence $h \subset g$ such that $\mathcal{A}_h \in \mathcal{F}_h^T$.

To repeat this definition in other words, one may say that $\mathcal{A} \in \mathcal{F}$ is $T$-rich if and only if every subsequence of $\mathcal{A}$ has a $T$-structured subsequence.

Of course, one cannot expect the same behavior (like the existence of splitting numbers) under these weaker assumptions, but the following theorem shows that the stability, the general Fredholm property of Section 2.3 as well as the $\alpha$- and $\beta$-numbers can still be well described in terms of snapshots.

**Theorem 2.26.**

- The set $\mathcal{R}_T$ of all $T$-rich sequences is a Banach algebra $\mathcal{F}_T \subset \mathcal{R}_T \subset \mathcal{F}$. Moreover, the algebras $\mathcal{R}_T$ and $\mathcal{R}_T / \mathcal{G}$ are inverse closed in $\mathcal{F}$ and $\mathcal{F} / \mathcal{G}$, respectively.

- If every $T$-structured subsequence of $\mathcal{A} \in \mathcal{R}_T$ has a regularly $\mathcal{F}_h^T$-Fredholm subsequence then $\mathcal{A}$ is Fredholm and

$$\alpha(\mathcal{A}) = \max \sum_{t \in T} \dim \ker W^T(\mathcal{A}_g), \quad \beta(\mathcal{A}) = \max \sum_{t \in T} \dim \coker W^T(\mathcal{A}_g),$$

(2.8)
where the maximum is taken over all \((T\text{-structured})\) subsequences \(h_{ja} \in F^T_g\) of \(A\). Moreover, in this case, for every \((T\text{-structured})\) subsequence \(h_{ja} \in F^T_g\) of \(A\) the numbers \(\alpha(h_{ja})\) and \(\beta(h_{ja})\) are splitting numbers and

\[
\lim_{n \to \infty} \ind A_{gn} = \sum_{t \in T} \ind W^t(h_{ja}).
\]

- For \(A \in R^T\) the following are equivalent:
  1. \(A\) is stable.
  2. Every subsequence of \(A\) is stable.
  3. Every subsequence of \(A\) has a stable subsequence.
  4. Every \(T\text{-structured}\) subsequence of \(A\) is stable.
  5. Every \(T\text{-structured}\) subsequence of \(A\) has a stable subsequence.
  6. Every \(T\text{-structured}\) subsequence of \(A\) is \(J^T\)-Fredholm and all snapshots are invertible.
  7. Every \(T\text{-structured}\) subsequence of \(A\) has a \(J^T\)-Fredholm subsequence and all snapshots are invertible.

**Proof.** The proof of \(R^T\) being a closed algebra \(F^T \subset R^T \subset F\) is straightforward. For the second assertion assume that \(A = \{A_n\}\) does not have an \(\alpha\)-number. Then for each \(n \in \mathbb{N}\) there is a number \(j_n\) such that \(j_n > j_{n-1}\) and \(s_{\alpha}(A_{jn}) < 1/n\). Choose a \(J^T\)-Fredholm subsequence \(h_{ja} \in F^T_g\) of \(A\) and find a splitting number for \((s_{\alpha}(A_{jn}))\) due to Theorem 2.13, a contradiction. Thus, \(A\) has an \(\alpha\)-number and, analogously, a \(\beta\)-number, and Theorem 2.22 yields the Fredholm property of \(A\). From Theorem 2.9 or Corollary 2.23 we deduce the relations “\(\geq\)” in (2.8). Furthermore, there is a sequence \(j\) tending to infinity such that \(\lim_n s^j_{\alpha}(A_{jn}) = 0\). Choose a subsequence \(h_{ja} \in F^T_g\) of \(A\) and pass to the \(J^T\)-Fredholm subsequence \(h_{j}\). Notice that \(h_{ja}\) and \(h_{ja}\) have the same snapshots. Then Theorem 2.13 applies to \(h_{ja}\) and yields the equality for \(\alpha(A)\) in (2.8). Analogously we proceed for \(\beta(A)\).

Now, let \(h_{ja}\) be a \(T\text{-structured}\) subsequence of the Fredholm sequence \(A\). Then \(\alpha(A_{ja}) < \infty\). Assume that \(\alpha(A_{ja}) \neq \alpha(A_{ja})\) is not a splitting number. Then there is a subsequence \(h_{ja} \in F^T_g\) such that \(\lim_n s^j_{\alpha(A_{ja})}(A_{ja})\) exists and is strictly positive, hence \(\alpha(A_{ja}) < \alpha(A_{ja})\). Pass to a \(J^T\)-Fredholm subsequence \(h_{ja}\) of \(A\), which certainly has the splitting property with \(\alpha(A_{ja}) = \alpha(A_{ja})\) being the sum of the kernel dimensions of the snapshots, a contradiction. For the \(\beta(A)\) we proceed in an analogous manner.

We now turn our attention to the third assertion. If a sequence is stable then every subsequence is so, and it is obvious that \(1. \Rightarrow 2. \Rightarrow 3. \Rightarrow 5. \) and \(1. \Rightarrow 2. \Rightarrow 4. \Rightarrow 5. \). Furthermore, \(4. \Rightarrow 6. \) and \(5. \Rightarrow 7. \) hold (see Theorem 2.17). Statement 7. implies that \(A\) is Fredholm with \(\alpha(A) = \beta(A) = 0\) by the second assertion. Thus, Theorem 2.21 yields the stability of \(A\), i.e. \(7. \Rightarrow 1.\)

Finally, let \(B \in G\)-regularizer for \(A\) and \(g\) be a sequence tending to infinity. We pass to a subsequence \(h \subset g\) with \(h_{ja} \in F^T_g\). Then \(h_g + G_{h_g}\) is invertible in \(F^T_g/G_{h_g}\) by Theorem 2.17 and the (unique) inverse is given by \(B_{h_g} + G_{h_g}\), hence \(B_{h_g} \in F^T_g\). Since \(g\) was chosen arbitrarily, this shows that \(B \in R^T\) and finishes the proof of the first assertion. \(\square\)

**Remark 2.27.** We note that a \(T\text{-structured}\) subsequence \(h_{ja}\) of a Fredholm sequence \(A\) is not necessarily \(J^T_g\)-Fredholm. To see this simply take a setting where \(T\) is not a singleton, fix \(t \in T\) and a \(T^1\)-compact finite rank projection \(K\), set \(\tilde{T} := T \setminus \{t\}\) and consider the sequence \(A = \{A_n\} := I - \{E_t(L_t,K_L_t)\}\). Clearly, \(A\) is Fredholm and belongs to \(F^T\), but there is no subsequence \(h_{ja}\) being \(J^T_g\)-Fredholm. Moreover, the sequence \(B = \{B_n\}\), given by \(B_n = A_n\) if \(n\) is even and \(B_n = L_n\) otherwise, is Fredholm and \(\tilde{T}\)-structured, but has no splitting numbers.
Further, we introduce the closed ideals $\mathcal{J}_{\pm}$ of bounded linear operators $A$. The finite section algebra $\mathcal{F}_{\pm}(L_{\pm}^{n})$ is a closed algebra of $L_{\pm}^{n}$. Moreover, for $\mathcal{J}_{\pm}$, we can choose a subsequence $h \subseteq g$ which also provides $\mathcal{A}_{h}$, besides $A_{h} = A_{g}$. Thus $\mathcal{J}_{\pm} \subseteq \mathcal{R}_{\pm}$, and since $\mathcal{R}_{\pm}$ is a closed algebra of $\mathcal{F}$ we even find that $\mathcal{F}_{\pm} \subset \mathcal{R}_{\pm}$.

The following proposition is the key for the application of the Fredholm theory, which was introduced in the preceding sections, to the algebra $\mathcal{A}_{\pm}$.

2.5 Example: finite sections of band-dominated operators

Let us take another look at the band-dominated operators in $\mathcal{A}_{\pm}$. We now give a precise definition of the sequence-algebraic framework for the finite section sequences of band-dominated operators.

2.5.1 Standard finite sections

Let $\mathcal{H}_{\pm}$ stand for the set of all strictly increasing sequences of positive integers and, analogously, $\mathcal{H}_{\mp}$ for the set of all strictly decreasing sequences of negative integers. For a given sequence $h = (h_{n}) \in \mathcal{H}_{\pm}$ we set $L_{h_{n}} := P_{h_{n}}$, $E_{h_{n}} := \text{im} L_{h_{n}}$, $T := \{-1, 0, +1\}$, $I^{0} := I$, $I^{\pm} := \chi_{\mathbb{Z}_{\pm}} I$ and

$$
E^{0} := L^{0}(\mathbb{Z}, X), \quad L^{0}_{h_{n}} := L_{h_{n}}
$$

$$
E^{\pm} : L(\text{im} L^{0}_{h_{n}}) \to L(E_{h_{n}}), B \mapsto B
$$

for every $n$. By $\mathcal{P}^{t} := (L_{h_{n}}^{t})$ uniform approximate identities on $E^{t}$ are given such that $E^{t}$ have the $\mathcal{P}^{t}$-dichotomy, and the sequences $(L_{h_{n}}^{t})$ converge $\mathcal{P}^{t}$-strongly to the identities $I^{t}$ on $E^{t}$. As usual, we let $\mathcal{F}_{h}^{t}$ denote the Banach algebra of all bounded linear operators $A_{h_{n}} \in L(E_{h_{n}})$ for which there exist operators $W^{t}\{A_{h_{n}}\} \in L(E^{t}, \mathcal{P}^{t})$ for each $t \in T$ such that for $n \to \infty$

$$
E_{h_{n}}^{t-}(A_{h_{n}}) L_{h_{n}}^{t-} \to W^{t}\{A_{h_{n}}\} \quad \mathcal{P}^{t-}\text{-strongly.}
$$

Further, we introduce the closed ideals $\mathcal{G}_{h}$ and $\mathcal{J}_{h}$ in $\mathcal{F}_{h}^{T}$ by

$$
\mathcal{G}_{h} := \{G_{h_{n}} : \|G_{h_{n}}\| \to 0\},
$$

$$
\mathcal{J}_{h}^{T} := \text{span}\{E_{h_{n}}^{T}(L_{h_{n}}^{T} K L_{h_{n}}^{T}) \} : t \in T, K \in \mathcal{K}^{t}, \{G_{h_{n}}\} \in \mathcal{G}_{h}\}
$$

A sequence $\{A_{h_{n}}\} \in \mathcal{F}_{h}^{T}$ is said to be $\mathcal{J}_{h}^{T}$-Fredholm, if $\{A_{h_{n}}\} + \mathcal{J}_{h}^{T}$ is invertible in $\mathcal{F}_{h}^{T}/\mathcal{J}_{h}^{T}$.

The finite section algebra $\mathcal{F}_{\mathcal{A}_{\pm}}$. Let $\mathcal{F}$ stand for the algebra for all bounded sequences $\{A_{n}\}$ of bounded linear operators $A_{n} \in L(E_{n})$ and let $\mathcal{F}_{\mathcal{A}_{\pm}}$ denote the smallest closed subalgebra of $\mathcal{F}$ containing all sequences $\{L_{n} A L_{n}\}$ with some rich $A \in \mathcal{A}_{\pm}$.

For $\mathcal{A} = \{A_{n}\} \in \mathcal{F}_{\mathcal{A}_{\pm}}$ and a sequence $h = (h_{n})_{n \in \mathbb{N}} \in \mathcal{H}_{\pm}$ let $\mathcal{A}_{h}$ denote the subsequence $\{A_{h_{n}}\}$. It is obvious, that for all $\mathcal{A} = \{A_{n}\} \in \mathcal{F}_{\mathcal{A}_{\pm}}$ and each $h \in \mathcal{H}_{\pm}$ the central snapshot exists independently from the choice of $h$:

$$
W(\mathcal{A}) := \mathcal{P}\lim_{n \to \infty} A_{n} L_{n} = W^{0}(\mathcal{A}_{h}) = \mathcal{P}\lim_{n \to \infty} A_{h_{n}} L_{h_{n}}.
$$

Moreover, for $\mathcal{A} = \{L_{n} A L_{n}\}$, $A \in \mathcal{A}_{\pm}$ being rich, there is a sequence $g \in \mathcal{H}_{\pm}$ such that $A_{g}$ exists, and further we can choose a subsequence $h \subseteq g$ which also provides $\mathcal{A}_{h_{g}}$, besides $A_{h} = A_{g}$. Thus $\mathcal{A} \in \mathcal{R}^{T}$, and since $\mathcal{R}^{T}$ is a closed algebra of $\mathcal{F}$ we even find that $\mathcal{F}_{\mathcal{A}_{\pm}} \subset \mathcal{R}^{T}$.

The following proposition is the key for the application of the Fredholm theory, which was introduced in the preceding sections, to the algebra $\mathcal{A}_{\pm}$.
Proposition 2.28. Let $\mathcal{A} \in \mathcal{F}_{\mathcal{A}_P}$ be such that $W(\mathcal{A})$ is $\mathcal{P}$-Fredholm and let $g \in \mathcal{H}_+$. Then there is a $T$-structured subsequence $\mathcal{A}_h$ of $\mathcal{A}_g$ being $\mathcal{J}_T^T$-Fredholm.

Proof. We initially check that, for every $j \in \mathcal{H}_+$ and for each pair of operators $B, C \in \mathcal{A}_P$ such that $B_{\pm j}$ and $C_{\pm j}$ exist, we have

$$\mathcal{D} := \{L_{jn}BCL_{jn}\} - \{L_{jn}BL_{jn}\}\{L_{jn}CL_{jn}\} \in \mathcal{J}_j^T.$$  \hfill (2.9)

For this, put $Q_j^m := \{L_{jn} - m\}$ and $P_j^m := I_j - Q_j^m$ and check that $P_j^m$ belongs to $\mathcal{J}_j^T$. Since the limit operators $B_{\pm j}, C_{\pm j}$ exist, we have $\mathcal{D} \in \mathcal{F}_j^T$. For every $m$ the sequence $DQ_j^m$ is contained in $\mathcal{J}_j^T$ and

$$\|DQ_j^m\| \leq \|B\| \sup_{n \in \mathbb{N}} \| (I - L_{jn})CL_{jn} - m\|$$

which can be chosen arbitrarily small by Theorem 1.55. Thus, the closedness of $\mathcal{J}_j^T$ implies (2.9). We now show that

$$\mathcal{A}_h - \{L_{hn}W(\mathcal{A})L_{hn}\} \in \mathcal{J}_h^T$$  \hfill (2.10)

for a certain subsequence $h \subset g$. Note that there is a sequence (of sequences) $(\mathcal{A}^{(m)}) \subset \mathcal{F}_{\mathcal{A}_P}$ such that $\mathcal{A}^{(m)} \to \mathcal{A}$ in the norm of $\mathcal{F}_{\mathcal{A}_P}$ as $n \to \infty$, and such that all $\mathcal{A}^{(m)}$ are of the form

$$\mathcal{A}^{(m)} = \sum_{i=1}^{k_n} \prod_{j=1}^{l_m} \{L_nA_{ij}^{(m)}\}L_n,$$

where $A_{ij}^{(m)} \in \mathcal{A}_P$. We choose a subsequence $g^1$ of $g$ (applying (2.9)) such that all limit operators of all $A_{ij}^{(1)}$ exist and such that (2.10) holds for $\mathcal{A}^{(1)}_{g^2}$. Repeating this argument we can obtain a subsequence $g^2$ of $g^1$ such that all limits of all $A_{ij}^{(2)}$ exist and such that (2.10) holds for $\mathcal{A}^{(2)}_{g^2}$ as well, and so on. Hence, there are sequences $g \supseteq g^1 \supseteq g^2 \supseteq \ldots \supseteq g^m \supseteq \ldots$ such that $\mathcal{A}^{(m)}_{g^m} \in \mathcal{F}_T^T$ and (2.10) holds for $\mathcal{A}^{(m)}_{g^m}$ for every $m$ and $k \leq m$. Now let the sequence $h = (h_n)$ be defined by $h_n := g^m_n$. Since $\mathcal{A}_h$ is the norm limit of the sequence $(\mathcal{A}^{(m)}_h)_{m \in \mathbb{N}}$, we easily obtain (2.10) for $\mathcal{A}_h$.

Finally, let $B$ be a $\mathcal{P}$-regularizer for $A := W(\mathcal{A})$ and note that $B \in \mathcal{A}_P$ by Theorem 1.55. The construction of $h$ provides the limit operators $A_{\pm h}$, and the proof of Proposition 1.59 shows that $B_{\pm h}$ exist as well, thus we can apply (2.9) to the operators $A$ and $B$. This yields the $\mathcal{J}_h^T$-Fredholmness of $\{L_{hn}W(\mathcal{A})L_{hn}\}$ and hence also the $\mathcal{J}_h^T$-Fredholmness of $\mathcal{A}_h$, by (2.10). \hfill $\Box$

Definition 2.29. Let $\mathcal{H}_h \subset \mathcal{H}_+$ stand for the set of all sequences $h$ such that $\mathcal{A}_h \in \mathcal{F}_T^T$.

Corollary 2.30. Let $\mathcal{A} \in \mathcal{F}_{\mathcal{A}_P}$ and $W(\mathcal{A})$ be $\mathcal{P}$-Fredholm. Then $W^{\pm 1}(\mathcal{A}_g)$ are $\mathcal{P}^{\pm 1}$-Fredholm for every $g \in \mathcal{H}_h$.

Proof. Choose a subsequence $h \subset g$ such that $\mathcal{A}_h$ is $\mathcal{J}_h^T$-Fredholm by the previous proposition. Its snapshots $W^{\mp 1}(\mathcal{A}_h)$ equal $W^{\mp 1}(\mathcal{A}_g)$ and are $\mathcal{P}^{\mp}$-Fredholm, respectively (see Theorem 2.6). \hfill $\Box$

The main results Now we are in a position to adapt the general results of Part 2 to this specific class of approximation sequences of band-dominated operators and we are going describe the Fredholm properties of both, the $T$-structured subsequences of $\mathcal{A} \in \mathcal{F}_{\mathcal{A}_P}$ as well as the whole sequence $\mathcal{A}$. 
Theorem 2.31. Let $\mathcal{A} = \{A_n\} \in \mathcal{F}_{\mathcal{A}_p}$ and $g \in \mathcal{H}_h$.

- If $W(\mathcal{A}), W^{\pm 1}(\mathcal{A}_g)$ are Fredholm, then
  \[
  \lim_{n \to \infty} \text{ind } A_{g_n} = \text{ind } W(\mathcal{A}) + \text{ind } W^{+1}(\mathcal{A}_g) + \text{ind } W^{-1}(\mathcal{A}_g)
  \]
  and the approximation numbers from the right/left of the operators $A_{g_n}$ of $\mathcal{A}_g$ have the $\alpha$-/$\beta$-splitting property with
  \[
  \alpha = \dim \ker W(\mathcal{A}) + \dim \ker W^{+1}(\mathcal{A}_g) + \dim \ker W^{-1}(\mathcal{A}_g),
  \]
  \[
  \beta = \dim \coker W(\mathcal{A}) + \dim \coker W^{+1}(\mathcal{A}_g) + \dim \coker W^{-1}(\mathcal{A}_g).
  \]

- If one of the operators $W(\mathcal{A}), W^{\pm 1}(\mathcal{A}_g)$ is not Fredholm then, for each $k \in \mathbb{N}$, we have
  \[
  \lim s_k^n(A_{g_n}) = 0 \quad \text{or} \quad \lim s_k^n(A_{g_n}) = 0.
  \]

- $\mathcal{A}_g$ is stable if and only if $W(\mathcal{A})$ and $W^{\pm 1}(\mathcal{A}_g)$ are invertible.

Proof. Let all snapshots of $\mathcal{A}_g$ be Fredholm, pass to a subsequence $\mathcal{A}_h$ of $\mathcal{A}_g$ which is regularly $\mathcal{F}_{\mathcal{A}_p}$-Fredholm, by Proposition 2.28, and apply Theorem 2.26 to prove the first assertion. The second one easily follows from Theorem 2.9, and the last one again from Theorem 2.26 and Proposition 2.28.

For the whole sequence $\mathcal{A} \in \mathcal{F}_{\mathcal{A}_p}$ we further conclude from Theorem 2.26, Proposition 2.28 and Corollary 2.23.

Theorem 2.32. Let $\mathcal{A} = \{A_n\} \in \mathcal{F}_{\mathcal{A}_p}$. Then, $\mathcal{A}$ is a Fredholm sequence if and only if $W(\mathcal{A})$ and all operators $W^{\pm 1}(\mathcal{A}_h)$ with $h \in \mathcal{H}_h$ are Fredholm. In this case the $\alpha$- and $\beta$-number of $\mathcal{A}$ equal

\[
\alpha(\mathcal{A}) = \dim \ker W(\mathcal{A}) + \max_{h \in \mathcal{H}_h} \left[ \dim \ker W^{+1}(\mathcal{A}_h) + \dim \ker W^{-1}(\mathcal{A}_h) \right],
\]

\[
\beta(\mathcal{A}) = \dim \coker W(\mathcal{A}) + \max_{h \in \mathcal{H}_h} \left[ \dim \coker W^{+1}(\mathcal{A}_h) + \dim \coker W^{-1}(\mathcal{A}_h) \right].
\]

A sequence $\mathcal{A} \in \mathcal{F}_{\mathcal{A}_p}$ is stable if and only if all snapshots $W(\mathcal{A})$ and $W^{\pm 1}(\mathcal{A}_h)$ with $h \in \mathcal{H}_h$ are invertible.

Notice that every $\mathcal{P}$-Fredholm operator is Fredholm if $\dim X < \infty$. Thus, the last theorem, together with Corollary 2.30 reveal that the Fredholm property of $W(\mathcal{A})$ is already sufficient for the Fredholm property of $\mathcal{A}$ in this case.

The stability criterion of Theorem 2.32 often appears in a slightly different form in the literature and we want to close this paragraph with an outline of this alternative notation. For $\mathcal{A} = \{A_n\}$ in $\mathcal{F}_{\mathcal{A}_p}$ denote the set of all operators $B_h$ which are $\mathcal{P}$-strong limits of one of the sequences

\[
(V_{-n}(I - L_{h_n}) + L_{h_n}A_{h_n}L_{h_n})V_{h_n} \quad \text{with} \quad h \in \mathcal{H}_+(\text{or } \mathcal{H}_-)
\]

by $\sigma_{\text{stab}}^+(\mathcal{A})$ (or $\sigma_{\text{stab}}^-(\mathcal{A})$, respectively). Of course, if $B_h$ is in $\sigma_{\text{stab}}^{\pm}(\mathcal{A})$ then we can pass to a subsequence $g \in \mathcal{H}_h$ of $h$ and identify $B_h$ with the respective operator $W^{\pm 1}(\mathcal{A}_g)$.

Theorem 2.33. A sequence $\mathcal{A} \in \mathcal{F}_{\mathcal{A}_p}$ is stable if and only if all operators in

\[
\sigma_{\text{stab}}(\mathcal{A}) := \{W(\mathcal{A})\} \cup \sigma_{\text{stab}}^{+}(\mathcal{A}) \cup \sigma_{\text{stab}}^{-}(\mathcal{A})
\]

are invertible.
Further let the operators $A$ be such that $A = \sum_{i=1}^{m} \prod_{j=1}^{k} A_{ij}$ into band-dominated operators of simple structure (e.g., banded, triangular, or Toeplitz operators). This structure, which could permit the application of fast algorithms, gets lost if one applies the usual finite section method, based on the projections $P_n$, but it can be preserved, if one uses the composition of the single finite sections $L_n A_{ij} L_n$ instead. The results above provide the desired information on the stability and convergence also for such compositions.

### 2.5.2 Adapted finite sections

In the preceding section the aim was to check if for arbitrarily given band-dominated operators $A$ the “standard” finite section method, based on the projections $P_n$, applies. Providing that $A$ is rich, the answer is that one has to check the Fredholm properties and the invertibility of a possibly infinite set $\sigma_{\text{stab}}(A)$ of operators. Of course, we have seen that we can pass to subsequences to reduce the number of limit operators, but since the projections $L_n$ are always chosen symmetric w.r.t. $Z$, the limiting processes towards $\infty$ and $-\infty$ are somehow coupled, which seems to be artificial.

Now let $A \in A_{lp}$ be fixed and we ask if there is a more adapted sequence of projections which provides a specific “finite section like” method for this single operator, such that we only have to check one snapshot at $\infty$ and one snapshot at $-\infty$ and such that both can be chosen independently from each other. The idea is very natural and simple: Suppose that for $A \in A_{lp}$ (not necessarily rich) there is a sequence $l \in \mathcal{H}_-$, such that $A_l$ exists and, independently of $l$, let $u \in \mathcal{H}_+$ be another sequence such that $A_u$ exists. We show that the properties of the finite sections $\{L_n^{l,u} A L_n^{l,u}\}$, with

\[
L_n^{l,u} = \chi(l(n), \ldots, u(n)) I,
\]

are determined by the three operators $A$, $A_l$ and $A_u$.

**Theorem 2.35.** Let $A \in A_{lp}$, and let $l \in \mathcal{H}_-, u \in \mathcal{H}_+$ be such that $A_l, A_u \in \sigma_{\text{op}}(A)$ exist. Further let $\mathcal{A} := \{A_l^{l,u}\}$ denote the sequence of the operators $A_l^{l,u} := L_n^{l,u} A L_n^{l,u} \in \mathcal{L}(\text{im} L_n^{l,u})$. Then

- The operators $A^+ := \chi_{Z_+} A_l \chi_{Z_+} I + (1 - \chi_{Z_+}) I$ and $A^- := \chi_{Z_-} A_u \chi_{Z_-} I + (1 - \chi_{Z_-}) I$ are $\mathcal{P}$-Fredholm, whenever $A$ is $\mathcal{P}$-Fredholm.
- If $A$ and $A^\pm$ are Fredholm then $\lim_n \text{ind} A_l^{l,u}$ exists, equals $\text{ind} A + \text{ind} A^+ + \text{ind} A^-$ and the approximation numbers from the right/left of $A_l^{l,u}$ have the $\alpha/\beta$-splitting property with $\alpha = \dim \text{ker} A + \dim \text{ker} A^+ + \dim \text{ker} A^-$, and $\beta$ w.r.t. cokernels instead of kernels.
- If one of the operators $A_l, A_u$ is not Fredholm then, for each $k \in \mathbb{N}$, $\lim_n s_k^l(A_l^{l,u}) = 0$ or $\lim_n s_k^u(A_l^{l,u}) = 0$.
- $\mathcal{A}$ is stable if and only if $A$ and $A^\pm$ are invertible.
2.5. EXAMPLE: FINITE SECTIONS OF BAND-DOMINATED OPERATORS

Proof. Notice that \((L_{l,u}^{i})\) converges \(P\)-strongly to the identity and introduce \(T := \{-1, 0, +1\}\), homomorphisms \(E_{t}^{i}: \mathcal{L}(\text{im} L_{l,u}^{i}) \to \mathcal{L}(E_{l,u}^{i})\) and sequence algebras \(F^{l,u,T}, J^{l,u,T}\) in the same way as in Section 2.5.1. Then the finite section sequence of each \(P\)-regularizer of \(A\) is again in \(F^{l,u,T}\), is a \(J^{l,u,T}\)-regularizer for \(A\) (cf. (2.9)) and Theorems 2.6, 2.13, 2.14, 2.9, and 2.17 give the claim.

Of course, these two slightly different approaches can be combined to get more appropriate methods for classes of band-dominated operators having a certain common structure: First, one chooses the sequence \((L_{l,u}^{i})\) of projections which align with the common structure in a sense and one then proceeds as in Section 2.5.1 to prove Theorems 2.31 and 2.32 also in this setting. Furthermore, [79] and [80] present an idea how one can pass to modified finite sections which need not to be stable, but which are still generalized invertible (e.g. Moore-Penrose invertible). Also note that the equations \(Ax = y\) and \(V_{\kappa}Ax = V_{\kappa}y\) are equivalent for every \(\kappa \in \mathbb{Z}\) since \(V_{\kappa}\) is invertible, whereas at most one of them leads to a stable finite section sequence and therefore can be solved by the finite section method. The simplest example of an operator for which such a preconditioning procedure is indicated to get a stable finite section sequence is the operator \(A = V_{1}\) itself. This method is known as index cancellation and was already studied in [24]. A comprehensive discussion can also be found in [36].

A final remark The core of the simplifications which appear in this more specific situation of finite sections of band-dominated operators is given by Proposition 2.28: The Fredholm properties of a sequence \(A\) are already determined by the Fredholm properties of the central snapshot \(W(A)\), and hence the “\(J_{T}\)-Fredholmness” can be extinguished from all results. This may look nice and simple for the moment, but it is still unsatisfactory in a sense, because it can hardly be translated to other classes of operators or lifted to a more abstract level. Therefore, the next part is devoted to this gap, and the main tools there are localization techniques. This deeper study will provide further amazing relations between the sequences and their snapshots, for instance on the asymptotics of condition numbers and pseudospectra.
Part 3

Banach algebras and local principles

In the previous part we have seen how the stability and invertibility problem in certain sequence algebras can be replaced by the invertibility problem for a family of operators (the snapshots) and for a coset in a smaller quotient algebra (namely $\mathcal{F}^T / \mathcal{J}^T$). Of course, the latter condition is still unpleasant and can hardly be checked directly. Therefore it would be desirable to get a family of snapshots which is sufficient in the sense that the Fredholm property or invertibility of the snapshots can already provide the invertibility of the coset in question. It is not clear if or how this could be achieved in the very general context $\mathcal{F}^T$.

So, here we draw on well known local principles to attack this question for classes of sequences which will be referred to as localizable sequences in what follows. For this we start with a short introduction and description of the required tools in Section 3.1 and proceed with their application to sequence algebras in 3.2. We achieve our goal, the characterization of the Fredholm property of localizable sequences solely in terms of its snapshots, with Theorem 3.9. Furthermore, we consider the asymptotic behavior of norms, condition numbers and pseudospectra. This approach was already evolved and published for the finite sections of band-dominated operators in [82], but is new in this abstract framework which permits to consider further applications in Part 4.

The third section is devoted to the pending proofs in the theory on approximate projections. Here we look at this concept from a new, more general and purely Banach algebraic point of view.
3.1 Central subalgebras, localization and the KMS property

**Classical Gelfand theory**\(^1\) A mapping \(^*: A \mapsto A^*\) on a Banach algebra \(A\) is referred to as an involution if for all \(A, B \in A\) and all scalars \(\alpha, \beta \in \mathbb{C}\)

\[(A^*)^* = A, \quad (\alpha A + \beta B)^* = \alpha A^* + \beta B^* \quad \text{and} \quad (AB)^* = B^*A^*.\]

We say that the unital Banach algebra \(A\) with the involution \(^*\) is a \(C^*\)-algebra if for every \(A \in A\) the equality \(\|A^*A\| = \|A\|^2\) holds true. Note that involutions on \(C^*\)-algebras \(A\) are isometries, that is \(\|A^*\| = \|A\|\) for every \(A \in A\).

Further, a Banach algebra \(C\) is said to be commutative, if \(AB = BA\) holds for all \(A, B \in C\).

So, in what follows let \(C\) be a commutative \(C^*\)-algebra. A character of \(C\) is a non-zero homomorphism from \(C\) into the algebra \(\mathbb{C}\). We denote the set of all characters by \(\mathcal{M}_C\) and mention that every character is unital with norm 1. Every element \(A \in C\) defines a complex-valued function \(\hat{A}\) on \(\mathcal{M}_C\), its so-called Gelfand transform, by the rule \(\hat{A}(h) := h(A)\). The coarsest topology on \(\mathcal{M}_C\) which makes every \(\hat{A}\) continuous is referred to as the Gelfand topology.

An ideal \(I \subset C\) is said to be maximal, if the only ideal of \(C\) which is a proper superset of \(I\) is \(C\) itself.

**Theorem 3.1.** (Gelfand-Naimark Theorem)

Let \(C\) be a commutative \(C^*\)-algebra. Then the mapping \(h \mapsto \ker h\) is a bijection from the set \(\mathcal{M}_C\) of the characters \(h\) of \(C\) onto the set of the maximal ideals of \(C\). Provided with the Gelfand topology, \(\mathcal{M}_C\) becomes a compact Hausdorff space and the Gelfand transformation \(A \mapsto \hat{A}\) is an isometric isomorphism from \(C\) onto \(C(\mathcal{M}_C)\), the algebra of all continuous functions on \(\mathcal{M}_C\).

Thus, one can think of the elements \(h\) of \(\mathcal{M}_C\) either as characters on \(C\) or as maximal ideals of \(C\). In the literature both characterizations are often used synonymously, and one usually refers to \(\mathcal{M}_C\) as the maximal ideal space of \(C\).

Due to this theorem, \(C\) can be isometrically identified with the algebra of the continuous functions on the maximal ideal space \(\mathcal{M}_C\). Clearly, an element \(A \in C\) is invertible if and only if its Gelfand transform \(\hat{A}\) does not have any zeros on \(\mathcal{M}_C\) (one may say that it is locally invertible in every point). Furthermore the norm of an element \(A\) equals the norm of its Gelfand transform \(\hat{A}\).

Since the latter is a continuous function on the compact set \(\mathcal{M}_C\), we have

\[\|A\| = \max_{h \in \mathcal{M}_C} |\hat{A}(h)|. \quad (3.1)\]

Therefore, in all what follows, the elements of \(\mathcal{M}_C\) are denoted by \(x\) and are called points, the elements of \(\mathcal{C}\) are denoted by \(\varphi\) and are called functions, as a rule, and we will be a bit loosely and use the notation \(\varphi(x)\) for \(\varphi(x)\).

For the identification of the maximal ideal space \(\mathcal{M}_C\) one often benefits from the Banach-Stone theorem, whose proof can be found in [21], V.8.8.

**Theorem 3.2.** Let \(H, K\) be compact Hausdorff spaces, and let \(T\) be an isometric isomorphism between \(C(H)\) and \(C(K)\). Then \(H\) and \(K\) are homeomorphic, that is there is a continuous bijection \(\tau : H \rightarrow K\) whose inverse is continuous as well.

In the case of non-commutative Banach algebras the situation is much more involved and we here mainly build upon the paper [10] by Böttcher, Krupnik and Silbermann.

\(^1\)These results and their proofs can be found in every textbook on \(C^*\)-algebras (e.g. [30], Section 4.1).
3.1. CENTRAL SUBALGEBRAS, LOCALIZATION AND THE KMS PROPERTY

Local principles for non-commutative Banach algebras

Let $A$ be a unital Banach algebra. The center $\text{cen}A$ of $A$ is the set of all elements $C \in A$ with the property that $CA = AC$ for all $A \in A$. Let $C$ be a $C^*$-subalgebra of $\text{cen}A$ containing the identity.

For each point $x$ in the maximal ideal space $\mathcal{M}_C$ we introduce $\mathcal{J}_x$, the smallest closed ideal in $A$ containing $x$, and let $\phi_x$ denote the canonical mapping from $A$ to $A/\mathcal{J}_x$. In case $\mathcal{J}_x = A$ the algebra $A/\mathcal{J}_x$ consists of only one element $\Theta$ which is zero and the identity at the same time, and therefore we may say that it is invertible and has norm 0.

**Theorem 3.3.** (Allan [1], Douglas [19], see also [12], Theorem 1.35 or [63], Theorem 2.3.16)\footnote{In recent literature (e.g. [72]) such pairs $(A, C)$ are also called faithful localizing pairs.}

The element $A \in A$ is invertible if and only if $\phi_x(A)$ is invertible in $A/\mathcal{J}_x$ for every $x \in \mathcal{M}_C$.

The mapping $\mathcal{M}_C \to \mathbb{R}_+, x \mapsto \|\phi_x(A)\|$ is upper semi-continuous. If $\phi_x(A)$ is invertible then there is a neighborhood $U$ of $x$ in $\mathcal{M}_C$ such that $\phi_y(A)$ is invertible in $A/\mathcal{J}_y$ for every $y \in U$.

This again provides a “local characterization” of invertibility in $A$. We are also interested in a formula which represents the norm of an element $A$ in terms of local norms as in (3.1). Notice that the set of all finite sums $\sum_i \varphi_i A_i$ with some $\varphi_i \in C$, $\varphi_i(x) = 0$ and $A_i \in A$ forms a dense subset of $\mathcal{J}_x$, and, by Proposition 5.1 of [10],

$$\|\phi_x(A)\| = \inf \{\|\varphi A\| : \varphi \in C, 0 \leq \varphi \leq 1, \varphi \equiv 1 \text{ in a neighborhood of } x\}. \quad (3.2)$$

**Definition 3.4.** Let $A$ be a unital Banach algebra and $C$ a central $C^*$-subalgebra which contains the identity. The algebra $A$ is a KMS-algebra with respect to $C$ if for every $A \in A$ and $\varphi, \psi \in C$ with disjoint supports (i.e. $\varphi(x)\psi(x) = 0$ for every $x$)

$$\|(\varphi + \psi)A\| \leq \max\{\|\varphi A\|, \|\psi A\|\}. \quad (3.3)$$

**Theorem 3.5.** (Böttcher, Krupnik, Silbermann, [10], Theorem 5.3)\footnote{More precisely, $\mathcal{J}_x$ is the smallest closed ideal in $A$ containing the kernel of the character $x : C \to \mathbb{C}$.}

The algebra $A$ is a KMS-algebra with respect to $C$ if and only if for every $A \in A$

$$\|A\| = \max_{x \in \mathcal{M}_C} \|\phi_x(A)\|. \quad (3.4)$$

A look into the proof of this Theorem in [10] reveals that one only has to check for every single $A$ (separately) that (3.3) holds for all $\varphi, \psi \in C$ with disjoint supports if and only if (3.4) is true. Thus, we can flexibilize the notion of KMS-algebras a little bit without any change in Theorem 3.5:

**Definition 3.6.** Let $B$ be a unital Banach algebra and $C$ a central $C^*$-subalgebra containing the identity. A subalgebra $A \subset B$ is said to be a KMS-algebra with respect to $C$ if (3.3) holds for every $A \in A$ and $\varphi, \psi \in C$ with disjoint supports.

The aim of the following considerations is to use this concept of localization to find practicable criteria for the $\mathcal{J}^T$-Fredholm property of a structured operator sequence in $\mathcal{F}^T$ as introduced in Part 2. To be a bit more precise, we want to characterize a class of settings $\mathcal{F}^T$ where the $\mathcal{P}^T$-Fredholm property of the snapshots implies “local $\mathcal{J}^T$-Fredholmness” of the sequence and hence its $\mathcal{J}^T$-Fredholmness in the following sense: The local principle of Allan and Douglas suggests to look for a suitable $C^*$-subalgebra $C$ and to replace the $\mathcal{J}^T$-Fredholmness by the invertibility of all respective local cosets. As a second step one then might try to identify these local cosets with the snapshots to get the desired result. Both the existence of $C$ and the identification are not achievable in the very general context of sequence algebras $\mathcal{F}^T$, but under additional conditions.

In what follows we try to give a framework which is sufficiently restrictive to manage these two problems, but on the other hand, also sufficiently general to cover the algebras of more concrete operator sequences which we have in mind and which we take into the focus in the 4th part of this thesis.
3.2 Localization in sequence algebras

Let $\mathcal{F}^T$ be a sequence algebra as in Section 2.1 et seq.

3.2.1 Localizable sequences

A localizing setting Let $\mathcal{C} \subset \mathcal{F}^T$ be a closed algebra containing $\mathbb{I}$ such that every snapshot $W^T(\varphi)$ of every sequence $\varphi \in \mathcal{C}$ is just a multiple $\varphi(t)I^T$ of the respective identity, and such that for $t_1, t_2 \in T$, $t_1 \neq t_2$ there is always a $\varphi \in \mathcal{C}$ with $\varphi(t_1) \neq \varphi(t_2)$.

Further suppose that $\mathcal{C}/\mathcal{G} := \{\varphi + \mathcal{G} : \varphi \in \mathcal{C}\}$ can be provided with an involution which turns $\mathcal{C}/\mathcal{G}$ into a commutative $\mathbb{C}$-algebra. From Theorem 3.1 we know that $\mathcal{C}/\mathcal{G}$ can be identified with the set of all continuous functions on its maximal ideal space $\mathcal{M}_{\mathcal{C}/\mathcal{G}}$. Notice that, for every $t \in T$, the mapping $\mathcal{C}/\mathcal{G} \to \mathcal{C}$ which sends $\varphi + \mathcal{G}$ to $\varphi(t)$ is a character, that means $T$ can be understood as a subset of $\mathcal{M}_{\mathcal{C}/\mathcal{G}}$.

Let $M$ be a set $T \subset M \subset \mathcal{M}_{\mathcal{C}/\mathcal{G}}$ such that the closure of $\mathcal{M}_{\mathcal{C}/\mathcal{G}} \setminus M$ is either empty or intersects $M$. For every $x \in M \setminus T$ we also fix one “direction” $t = t(x) \in T$ and say that it is associated to $x$. In this way the objects $\mathcal{E}^x := \mathcal{E}^{t(x)}$, and analogously $\mathcal{P}^x$, $E_n^x$, $L_n^x$, etc., and consequently the snapshots $W^x$ are defined at every point $x \in M$.

Finally, for each $x \in M$, fix one function $\tilde{\mathcal{E}}^x \in C(\mathcal{M}_{\mathcal{C}/\mathcal{G}})$ (that is $\tilde{\mathcal{E}}^x \in \mathcal{C}/\mathcal{G}$) which takes values in $[0, 1]$ such that $\tilde{\mathcal{E}}^x \equiv 1$ in a neighborhood of $x$, choose one sequence $\tilde{\mathcal{E}}^x = (F_n^x) \in \tilde{\mathcal{E}}^x$, and set $(S_n^x) := (E_n^x(F_n^x)L_n^x)$. Now, we say that $\mathcal{C}$ defines a localizing setting. The functions $\tilde{\mathcal{E}}^x$ may be regarded as “local filter” in $x$, respectively.

Having such a localizing setting $\mathcal{C}$ we are going to single out a family of sequences $\tilde{A} \in \mathcal{F}^T$ which can be studied using localization with respect to $\mathcal{C}$, and whose local representatives are already characterized by the snapshots $W^T(\tilde{A})$.

Localizable sequences Suppose that $\mathcal{C}$ defines a localizing setting. A sequence $\tilde{A} \in \mathcal{F}^T$ is said to be C-localizable if, for every $x \in M$,

- $W^x(\tilde{A})$ is liftable, that is $\tilde{A}^x := \{E_n^x(S_n^x W^x(\tilde{A})S_n^x)\} = \mathcal{F}^x\{E_n^x(L_n^x W^x(\tilde{A}))\} \mathcal{F}^x \subset \mathcal{F}^T$,

- $W^x(\tilde{A})$ is almost commuting with $\mathcal{F}^x$, that is $||W^x(\tilde{A}), S_n^x|| \to 0$ as $n \to \infty$,

- $\tilde{A} + \mathcal{G}$ and $\tilde{A}^x + \mathcal{G}$ belong to the commutant of $\mathcal{C}/\mathcal{G}$ in $\mathcal{F}/\mathcal{G}$, that means they commute with every coset $\varphi + \mathcal{G}$, $\varphi \in \mathcal{C}$,

- $\tilde{A} + \mathcal{G}$ and $\tilde{A}^x + \mathcal{G}$ are locally equivalent in $x$, that is for every $\epsilon > 0$ there is a function $\varphi + \mathcal{G} \in \mathcal{C}/\mathcal{G}$ being equal to 1 in a neighborhood of $x$ such that $||\varphi(\tilde{A} - \tilde{A}^x) + \mathcal{G}|| < \epsilon$.

The set of all $\mathcal{C}$-localizable sequences shall be denoted by $\mathcal{L}^T = \mathcal{L}^T(\mathcal{C})$.

Finally, suppose that there is a sequence $\mathcal{T} = \{T_n\} \in \mathcal{L}^T$ of projections such that $\mathcal{T} + \mathcal{G}$ is locally equivalent to zero in every $x \in \mathcal{M}_{\mathcal{C}/\mathcal{G}} \setminus M$, and let $\mathcal{T}^T$ denote the set of all “truncated” sequences of the form $T \mathcal{T} + \lambda(\mathbb{I} - T) + \mathcal{J}$ with $\lambda \in \mathcal{L}^T$, a complex number $\lambda$ and a sequence $\mathcal{J} \in \mathcal{J}^T$.

Remark 3.7. Of course, it is possible (and even the rather common case) to choose $M = \mathcal{M}_{\mathcal{C}/\mathcal{G}}$ and $\mathcal{T} = \mathbb{I}$. Then also $\mathcal{T}^T = \mathcal{L}^T$ holds, as we obtain from the next proposition. In this case we say that $\mathcal{C}$ defines a globally localizing setting.

Otherwise $\mathcal{C}$ is said to define a partially localizing setting. Notice that this model will enable us to consider sequence algebras which contain the constant sequences $\tilde{A} = \{A\}$ of the operators of interest as well as the sequence $\mathcal{T}$ of projections which constitutes the “finite section method”, and to study the finite section sequences of the form $T \mathcal{T} + (1 - \mathcal{T})$. For an application of this kind see Section 4.3.
Proposition 3.8. The sets $T^T$ and $L^T$ are Banach algebras, fulfill $T^T \subset L^T$ and contain $G$ as well as $J^T$ as closed ideals.

The set $C/J^T := \{ \varphi + J^T : \varphi \in C \}$ forms a commutative $C^*$-subalgebra of $F^T/J^T$ being $*$-isomorphic to $C/G$ and $C(M_C/G)$.

Proof. Obviously, $L^T$ is a closed linear subspace of $F^T$. For $A, B \in L^T$ and $x \in M$ we have that $W^x(A\mathbb{B})$ is almost commuting with $F^x$, $A^* + G$ belongs to the commutant of $C/G$, $A^*B^x \in F^T$ and

\[
F^x(A\mathbb{B})^xF^x - A^*B^x = \{ E_n^x(S_n^x S_n^x W^x(A\mathbb{B}) S_n^x S_n^x - S_n^x W^x(A) S_n^x S_n^x W^x(\mathbb{B}) S_n^x) \} \in G
\]

since $W^x(A)$ and $W^x(\mathbb{B})$ are almost commuting with $F^x$, hence $F^x(A\mathbb{B})^xF^x \in F^T$, too. For every $t \in T$ with $W^x(F^T) = 0$ we have $W^x((A\mathbb{B})^x) = 0$, and for all other $t$ we know that the snapshot $W^t(F^x)$ is a non-zero multiple of the identity, hence $(E_n^x((A\mathbb{B})^x) L_n^t)$ converges $P^t$-strongly if and only if $(E_n^x(F_n^x(A\mathbb{B})^x F_n^x) L_n^t)$ converges $P^t$-strongly. The only if part is trivial, and for the if part let $K$ be $P^t$-compact. Then it holds that

\[
d^2 KE_n^{-1}((A\mathbb{B})^x L_n^t) = G KE_n^{-1}((F^x)^x (A\mathbb{B})^x L_n^t) = G KE_n^{-1}((F^x)^x (A\mathbb{B})^x F_n^x) L_n^t.
\]

as well as the dual equation with $K$ multiplied from the right. This shows that $(A\mathbb{B})^x \in F^T$, i.e. $W^x(A\mathbb{B})$ is liftable. Moreover, $(A\mathbb{B})^x + G$ belongs to the commutant of $C/G$ since

\[
\varphi(A\mathbb{B})^x = \varphi \{ E_n^x(S_n^x W^x(A\mathbb{B}) S_n^x) \} = G \{ E_n^x(S_n^x W^x(A) S_n^x W^x(\mathbb{B}) L_n^t) \}
\]

\[
= G \{ E_n^x(S_n^x W^x(A) S_n^x \varphi(z) W^x(\mathbb{B}) L_n^t) \} = G \{ E_n^x(S_n^x W^x(A) \varphi(z) W^x(\mathbb{B}) S_n^x) \}
\]

\[
= G \{ E_n^x(L_n^x W^x(A) \varphi(z) S_n^x W^x(\mathbb{B}) S_n^x) \} = G \{ E_n^x(L_n^x W^x(A) S_n^x W^x(\mathbb{B}) S_n^x \varphi(z)) \}
\]

\[
= G \{ E_n^x(S_n^x W^x(A\mathbb{B}) S_n^x) \} \varphi = (A\mathbb{B})^x \varphi
\]

for every $\varphi \in C$. These equalities also give that $A^*B^x + G$ and $(A\mathbb{B})^x + G$ are locally equivalent in $x$. The proof of $L^T$ being a Banach algebra can now easily be completed.

Furthermore, $T^T$ clearly forms an algebra since $T^2 = T$ and $J^T$ is a closed ideal in $F^T$, and a simple approximation argument yields its closedness. The set $G$ is a closed ideal in both algebras $T^T$ and $J^T$, as well as $J^T$ in $T^T$. To prove this for $J^T$ in $L^T$, it suffices to consider the generating sequences $A = (E_n^x(L_n^x K))$ with some $\tau \in T$ and $K \in K(E^T, P^T)$. Its snapshots are $P^T$-compact, respectively, and hence are liftable and almost commuting with $F^x$. The membership of $A$ to the commutant of $C/G$ follows from the fact that the snapshot $W^x(\cdot)$ of every $C$-sequence is a multiple of the identity. To prove the local equivalence of $A$ to the respective lifting notice that $A$ and $A^x$ are locally equivalent (they even coincide) in every $x \in M$ with $t(x) = \tau$, since $W^x(A) = K$ in these cases. Let $x \in M$ be such that $t(x) \neq \tau$. Then $W^x(A) = 0$. Choose a function $\varphi \in \mathcal{C}$ which is equal to 1 in a neighborhood of $\tau$ but vanishes in a neighborhood of $x$. Then $\varphi A - A \in G$ and $\varphi A$ is locally equivalent to zero in $x$. Hence $A$ is locally equivalent to zero in $x$ as well. Now the assertion easily follows for all $J^T$-sequences. Furthermore, the inclusions $T^T \subset L^T \subset F^T$ are proved.

Clearly, $C/J^T$ is a commutative subalgebra of $F^T/J^T$ and it inherits its involution from $C/G$. It remains to show that $\|\varphi + G\| = \|\varphi + J^T\|$ always holds. The estimate “$\geq$” is obvious. Assume that it is even proper for one $\varphi$, which means that there is an $\epsilon > 0$ and a sequence $J \in J^T$ such that $\|\varphi + G\| > \|\varphi + J\| + \epsilon \geq \|\varphi + J + G\| + \epsilon$. Without loss of generality we can also assume that $\|\varphi + G\| = 1$. Fix $x_0 \in M_C/G$ such that $d := (\varphi + G)(x_0)$ fulfills $|d| \geq 1 - \epsilon$ and $x_0$ either belongs

\[\text{(3.5)}\]

\[\text{Write } a_n = b_n \text{ if } \|a_n - b_n\| \to 0 \text{ as } n \to \infty.\]

\[\text{simply choose } \varphi \text{ such that } \varphi(1 - F^x) \in G.\]
to $T$ or to the interior of $\mathcal{M}_c/\mathcal{G} \setminus T$. In the latter case we choose a function $\psi + \mathcal{G} \in \mathcal{C}/\mathcal{G}$ equal to 1 in a neighborhood of $x_0$, equal to zero on $T$ and of norm one. In case $x_0 \in T$ we set $\psi := 1$. Then $\mathcal{B} + \mathcal{G} := dl - \psi(\phi + \mathcal{J}) + \mathcal{G}$ belongs to $\mathcal{F}^T/\mathcal{G}$ and is invertible with its inverse given by a Neumann series. Thus, all of its snapshots are invertible. In case $x_0 \in T$ this contradicts the fact that $W^{x_0}(\mathcal{B})$ is $\mathcal{P}^{x_0}$-compact. If $x_0 \notin T$ then $\psi J \in \mathcal{G}$, hence $W^{x_0}(\mathcal{B})$ is zero, a contradiction as well.

\textbf{Theorem 3.9.} The algebras $\mathcal{L}^T$, $\mathcal{L}^T/\mathcal{G}$ and $\mathcal{L}^T/\mathcal{J}^T$ are inverse closed in $\mathcal{F}$, $\mathcal{F}/\mathcal{G}$ and $\mathcal{F}^T/\mathcal{J}^T$, respectively. The same holds true with $\mathcal{L}^T$ replaced by $\mathcal{T}^T$.

A sequence $\mathcal{A} \in \mathcal{T}^T$ is $\mathcal{T}^T$-Fredholm if and only if all of its snapshots are $\mathcal{P}^t$-Fredholm. It is Fredholm if and only if all of its snapshots are Fredholm. It is stable if and only if all of its snapshots are invertible.

\textbf{Proof.} Let $\text{Com}(\mathcal{C}/\mathcal{G}) \subset \mathcal{F}/\mathcal{G}$ denote the commutant of $\mathcal{C}/\mathcal{G}$, that is, the set of all elements in $\mathcal{F}/\mathcal{G}$ which commute with every element in $\mathcal{C}/\mathcal{G}$. It is well known that $\text{Com}(\mathcal{C}/\mathcal{G}) \subset \mathcal{F}/\mathcal{G}$ is a closed subalgebra of $\mathcal{F}/\mathcal{G}$, which contains $\mathcal{L}^T/\mathcal{G}$ by definition. Furthermore, recall the homomorphisms $\phi_x$ from Section 3.1 which act on $\text{Com}(\mathcal{C}/\mathcal{G})$ and notice that two cosets $\mathcal{A} + \mathcal{G}, \mathcal{B} + \mathcal{G} \in \text{Com}(\mathcal{C}/\mathcal{G})$ are locally equivalent in $x \in \mathcal{M}$ if and only if $\phi_x(\mathcal{A} + \mathcal{G}) = \phi_x(\mathcal{B} + \mathcal{G})$. Indeed, the only if follows from the fact that $\phi_x(\varphi + \mathcal{G}) = \phi_x(1 + \mathcal{G})$ for every $\varphi \in \mathcal{C}$ being equal to 1 in a neighborhood of $x$. For the if part let $\phi_x(\mathcal{A} - \mathcal{B} + \mathcal{G}) = 0$. Then $\mathcal{A} - \mathcal{B} + \mathcal{G}$ can be approximated by finite sums of the form $\sum_n \varphi_n \mathcal{C}_n + \mathcal{G}$ with some $\varphi_n \in \mathcal{C}$, $\varphi_n(x) = 0$ and $\mathcal{C}_n \in \mathcal{F}$ as mentioned after Theorem 3.3, and the assertion follows. The respective local homomorphisms on the commutant $\text{Com}(\mathcal{C}/\mathcal{J}) \subset \mathcal{F}^T/\mathcal{J}^T$ of $\mathcal{C}/\mathcal{J}$ shall be denoted by $\Phi_x$.

Let $\mathcal{A} \in \mathcal{L}^T$ be $\mathcal{T}^T$-Fredholm and $\mathcal{B} \in \mathcal{F}^T$ one of its regularizers. We want to show that $\mathcal{B} \in \mathcal{L}^T$. Let $x \in \mathcal{M}$ and consider the snapshot $\mathcal{B} := W^x(\mathcal{B})$ which is a $\mathcal{P}^t$-regularizer for $A := W^x(\mathcal{A})$ by Theorem 2.6. We first prove that $\lim_n \|B_n\| = 0$ and that $B$ is liftable. We again write $a_n =_\mathcal{G} b_n$ if $\lim_n \|a_n - b_n\| = 0$, and since $I - AB, I - BA$ are $\mathcal{P}^t$-compact we have

$$B(I - S^\infty_n) =_\mathcal{G} B(I - S^\infty_n)AB =_\mathcal{G} BA(I - S^\infty_n)B =_\mathcal{G} (I - S^\infty_n)B.$$ 

The snapshot $W^x(\mathcal{B}^x)$ of the sequence $\mathcal{B}^x := \{E_n(S^n B S_n^m)\}$ at the point $x$ obviously exists, and for those $t \in T$ with $W^t(\mathcal{F}^t) = 0$ the snapshots $W^t(\mathcal{F}^t)$ exist as well and equal zero. Let $t \in T \setminus \{t(x)\}$ be such that $W^t(\mathcal{F}^t)$ (which is a multiple of $dt^t$ of the identity) does not vanish, and set $A^t := W^t(\mathcal{A}^t) = W^t(E_n(S^n B A(S^n B A))^t))$. For every $K \in \mathcal{K}(\mathcal{E}^t, \mathcal{P}^t)$,

$$\|E_n^{-t}(E_n(S^n B A(S^n B A))^t)K\| \geq \frac{\|E_n^{-t}(E_n(S^n B A(S^n B A))^t)K\|}{\|E_n^{-t}(E_n(S^n B A(S^n B A))^t)\|} \geq \frac{1}{M^t M^{t^x}} \sup_{n \in \mathbb{N}} \|S^n B A(S^n B A)^t\| \|E_n^{-t}(E_n(S^n B A(S^n B A))^t)K + G_n\|,$$

where $G_n := E_n^{-t}(E_n(S^n B A(S^n B A)^t)K$ tend to zero in the norm as $n \to \infty$. Thus, letting $n$ go to infinity, we find $\|A^t K\| \geq e\|K\|$ with a constant $c > 0$ independent of $K$, which shows that $A^t$ is injective, due to its $\mathcal{P}^t$-dichotomy. Analogously, one checks that $A^t$ is surjective, that is $(A^t)^{-1}$ exists, and it belongs to $\mathcal{L}(\mathcal{E}^t, \mathcal{P}^t)$ by Theorem 1.14. We even have that

$$H_n := d^t I - E_n^{-t}(E_n(S^n B A(S^n B A)))E_n^{-t}(E_n(S^n B A(S^n B A)))L_n^t$$

converges $\mathcal{P}^t$-strongly to zero, hence, for every $\mathcal{P}^t$-compact $K$,

$$\|E_n^{-t}(E_n(S^n B A(S^n B A)))L_n^t - d^t(A^t)^{-1}K\| \geq \|E_n^{-t}(E_n(S^n B A(S^n B A)))L_n^t[A^t - E_n^{-t}(E_n(S^n B A(S^n B A)))L_n^t](A^t)^{-1}K + H_n(A^t)^{-1}K\|$$
tends to zero as \( n \to \infty \). The dual relation with \( K \) multiplied from the left-hand side holds as well, thus the snapshot \( W^T(B^n) \) exists and equals \( d^I(A^n)^{-1} \). We conclude that \( B \) is liftable. In the same way as in the equations (3.5) and with the help of the fact \( B - BAB \in K(E^T, P^T) \) we find that \( B + G \) is in the commutant of \( C\mathcal{G} \). By \( B - BAB \in J^T \) and

\[
\varphi BAB = G, BAB = G, B\varphi AB = G, BAB\varphi
\]

we see that \( B + G \) is in the commutant of \( C\mathcal{G} \) as well. Hence it remains to check that \( B \) and \( B^x \) are locally equivalent. Since the coset

\[
\Phi_x(B - B^x + J^T) = \Phi_x(BA(B - B^x) + J^T) = \Phi_x(B(AB - AB^x) + J^T)
\]

equals zero, we have \( \phi_x(B - B^x + G) \in \phi_x(J^T) := \{ \phi_x(J + G) : J \in J^T \} \). If \( x \in M \setminus T \) then \( \phi_x(J) = \phi_x(G) \), hence \( \phi_x(B + G) = \phi_x(B^x + G) \). On the other hand, if \( x \in T \), then \( \phi_x(J^T) \) equals \( \phi_x(J^x) \), hence we find a \( P^\perp \)-compact \( K \) such that \( \phi_x(B - B^x + E_n(L_{K_n}^x) + G) = 0 \). Consequently, \( W^T(B - B^x + E_n(L_{K_n}^x)) = 0 \), and since \( W^T(B - B^x) = 0 \), it even holds \( K = 0 \). Therefore, we get again that \( \phi_x(B^x + G) = \phi_x(B^x + G) \). Thus, the local equivalence of \( B \) and \( B^x \) is proved for every \( x \in M \), \( B \in L^T \) follows, and yields the inverse closedness of \( \mathcal{T}^T/J^T \) in \( \mathcal{T}^T/J^T \). To show the respective assertions for \( L^T \) and \( L^T/G \) simply consult Theorem 2.17.

Now, let \( A \in \mathcal{T}^T \) be \( J^T \)-Fredholm and \( B \in L^T \) be a regularizer. Let us show that \( B \in \mathcal{T}^T \). If \( T = I \) then \( J^T = F^T \) and there is nothing to prove. Otherwise \( A = J^T TAT + \lambda(I - T) \) with \( \lambda \neq 0 \) and we can multiply the equations \( AB = A^T B \) and \( BA = J^T AB \) with \( (I - T) \) from the left (right, resp.) and we get \( \lambda(I - T)B = J^T AB(I - T) = J^T (I - T) \). Hence, \( B = J^T TB + \lambda^{-1}(I - T) \) and \( B = J^T TB + \lambda^{-1}(I - T) \). Putting the first of these equations into the second one we finally obtain \( B = J^T TB + \lambda^{-1}(I - T) \) that is \( B \in \mathcal{T}^T \). Therefore the assertions on the inverse closedness of \( \mathcal{T}^T, \mathcal{T}^T/G \) and \( \mathcal{T}^T/J^T \) easily follow as well.

Theorems 2.6, 2.17 and Corollary 2.23 show that all snapshots of a \((J^T\text{-Fredholm} / \text{Fredholm} / \text{stable})\) sequence \( A \in \mathcal{T}^T \) are \( P^T \)-Fredholm / Fredholm / invertible. For the converse let \( A \in \mathcal{T}^T \) be such that all snapshots are \( P^T \)-Fredholm. Due to the local principle of Allan and Douglas in Theorem 3.3 it suffices to check that \( \Phi_x(A + J^T) \) is invertible having an inverse \( \Phi_x(C^x + J^T) \) with \( C^x + J^T \in \mathcal{T}^T/J^T \) for every \( x \in M_{C\mathcal{G}} \), respectively. For \( x \in M \) we have that \( \Phi_x(A + J^T) \) equals \( \Phi_x(A^x + J^T) \), and since \( W^T(A) \) is \( P^T \)-Fredholm each \( P^T \)-regularizer \( B \) belongs to \( \mathcal{L}(E^T, P^T) \) by Theorem 1.14. Its lifting \( B^{x} := \{ E_n(S_n^x BS_n^x) \} \) belongs to \( J^T \) and \( B^{x} + \mathcal{G} \) is in the commutant of \( C\mathcal{G} \) as the above considerations show. Moreover,

\[
I - B^x A^x + J^T = \{ L_n - E_n^x(S_n^x BS_n^x)E_n^x(S_n^x AS_n^x) \} + J^T
\]

\[
= \{ L_n - E_n^x(S_n^x BA(S_n^x)^3) \} + J^T = \{ L_n - E_n^x((S_n^x)^4) \} + J^T,
\]

hence \( \Phi_x(I - B^x A^x + J^T) \) (as well as \( \Phi_x(I - A^x B^x + J^T) \)) equals zero. This gives the local invertibility in \( x \in M \). Now let \( x_0 \in M \) belong to the closure of \( M_{C\mathcal{G}} \setminus M \). From Theorem 3.3 we can derive that there is a point \( y_0 \in M_{C\mathcal{G}} \setminus M \) in a neighborhood of \( x_0 \) such that \( \Phi_{y_0}(A + J^T) \), which equals \( \Phi_{y_0}(\lambda I + J^T) \), is invertible as well, and consequently \( \lambda \neq 0 \). Since \( \lambda \) is independent of \( y \in M_{C\mathcal{G}} \setminus M \) we get the invertibility of all cosets \( \Phi_y(A + J^T) \) and hence the \( J^T \)-Fredholmness of \( A \).

Now, let all snapshots of \( A \in \mathcal{T}^T \) be Fredholm (or even invertible). Then each of them is \( P^T \)-Fredholm, respectively, and with the above we can conclude that \( A \) is regularly \( J^T \)-Fredholm. Corollary 2.24 gives the Fredholm property of \( A \), and in case of invertible snapshots we have splitting numbers \( \alpha(A) = \beta(A) = 0 \), hence stability from Theorem 2.21. \( \square \)
Remark 3.10. The advantages of the described setting are the following: Suppose that one is interested in a certain subalgebra $\mathcal{A} \subset \mathcal{F}^T$ and one wants to eliminate the unwieldy condition on the $\mathcal{J}^T$-Fredholm property in the results on stability, convergence of approximation numbers or the index formula for the sequences in $\mathcal{A}$. Then Proposition 3.8 shows that one only needs to find a suitable $C$ and to prove that the generators of $\mathcal{A}$ are localizable w.r.t. $C$, to guarantee that $\mathcal{A} \subset \mathcal{L}^T$. In many applications this simplifies matters essentially because the algebras under consideration are often spanned by only a few rather basic elements. We again refer to the application in Section 4.3 for a telling example.

Furthermore, Theorem 3.9 obviates the need for checking the $\mathcal{J}^T$-Fredholm property and, moreover, the stated inverse closedness provides control over the respective regularizers and inverses. The latter is a crucial aid in the next section where we deal with the convergence of condition numbers.

### 3.2.2 Convergence of norms and condition numbers

Within this section suppose that $\mathcal{C}$ defines a localizing setting in $\mathcal{F}^T$ such that, additionally,

- $\|\phi_x(\mathcal{A} + \mathcal{G})\| \leq \|W_x(\mathcal{A})\| \leq \|\mathcal{A}_g + \mathcal{G}_g\|$ holds for every $\mathcal{A} \in \mathcal{T}^T$, $g \in \mathcal{H}_+$ and $x \in M$;\(^{6}\)

- $\mathcal{T}^T/\mathcal{G}$ is a KMS-algebra with respect to $\mathcal{C}/\mathcal{G}$.

and say that $\mathcal{C}$ defines a faithful localizing setting.

Proposition 3.11. Let $\mathcal{A} = \{A_n\} \in \mathcal{T}^T$. The norms $\|A_n\|$ and $\|A_n^{-1}\|$ converge and

$$\lim_{n \to \infty} \|A_n\| = \max_{t \in T} \|W^t(\mathcal{A})\|, \quad \lim_{n \to \infty} \|A_n^{-1}\| = \max_{t \in T}(\|W^t(\mathcal{A})\|^{-1}).$$\(^{7}\)

Proof. At first, notice that $f : \mathcal{M}_{\mathcal{C}/\mathcal{G}} \to \mathbb{R}_+$, $x \mapsto \|\phi_x(\mathcal{A} + \mathcal{G})\|$ is upper semi-continuous on $\mathcal{M}_{\mathcal{C}/\mathcal{G}}$ by Theorem 3.3 and constant on $\mathcal{M}_{\mathcal{C}/\mathcal{G}} \setminus M$ since $\mathcal{A} \in \mathcal{T}^T$. If $\mathcal{M}_{\mathcal{C}/\mathcal{G}} \setminus M$ is not empty then choose $x_0 \in M$ which also belongs to the closure of $\mathcal{M}_{\mathcal{C}/\mathcal{G}} \setminus M$ and conclude that

$$\|\phi_x(\mathcal{A} + \mathcal{G})\| \leq \|\phi_{x_0}(\mathcal{A} + \mathcal{G})\| \quad \text{for all} \quad x \in \mathcal{M}_{\mathcal{C}/\mathcal{G}} \setminus M.$$

With the formula (3.4) from Theorem 3.5 this yields an $x_1 \in M$ such that for every $g \in \mathcal{H}_+$

$$\sup_{t \in T} \|W^t(\mathcal{A})\| \leq \|\mathcal{A}_g + \mathcal{G}_g\| = \limsup_{n \to \infty} \|A_{g_n}\| \leq \limsup_{n \to \infty} \|A_n\| = \|\mathcal{A} + \mathcal{G}\| = \max_{x \in \mathcal{M}_{\mathcal{C}/\mathcal{G}}} \|\phi_x(\mathcal{A} + \mathcal{G})\| = \|\phi_{x_1}(\mathcal{A} + \mathcal{G})\| \leq \|W^{x_1}(\mathcal{A})\|.$$

Since there is a $t_1 \in T$ with $W^{t_1}(\mathcal{A}) = W^{x_1}(\mathcal{A})$ we get that $\lim_n \|A_n\|$ exists and is determined as the maximum of the norms of all snapshots.

Now we consider the “inverses”. Suppose that one snapshot of $\mathcal{A}$ is not invertible. Then, by Theorem 2.17, the sequence $\mathcal{A}$ and each of its subsequences cannot be stable, that is there is no bounded subsequence of $\{A_n^{-1}\}$, hence $\lim_n \|A_n^{-1}\| = \infty$. If, conversely, all snapshots are invertible then Theorem 3.9 yields that $\mathcal{A}$ is stable, a sequence whose $n$th entry is $A_n^{-1}$ (if $A_n$ is invertible) is a $\mathcal{G}$-regularizer for $\mathcal{A}$ in $\mathcal{T}^T$ and its snapshots are $(W^t(\mathcal{A}))^{-1}$. Apply the above to this sequence to prove the second asserted equation. \(\square\)

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6 For the definition of the local homomorphisms $\phi_x$ on $\text{Com}(\mathcal{C}/\mathcal{G})$ see Section 3.1.

7 We again use the convention $\|B^{-1}\| = \infty$ if $B$ is not invertible.
Corollary 3.12. Let $A = \{A_n\} \in \mathcal{T}^T$ be stable. Then the condition numbers of the operators $A_n$ which are defined by $\text{cond}(A_n) := \|A_n\| \cdot \|A_n^{-1}\|$ converge and their limit is
\[ \lim_{n \to \infty} \text{cond}(A_n) = \max_{t \in T} \|W^t(A)\| \cdot \max_{t \in T} \|\!(W^t(A))^{-1}\!.\]

Remark 3.13. Let us give some more concrete criteria for a faithful localizing setting.

1. The first of the stated conditions in its definition is true if, for every $t \in T$,
\[ B_{t^*} = \lim_{n \to \infty} \|E_n^t\| = \lim_{n \to \infty} \|E_n^{-t}\| = \lim_{n \to \infty} \|\!S_n^t\!\| = 1. \]

2. For the second condition notice that for every $H$, hence, for every $\psi, \epsilon \in C$ with $\|\psi + G\| = \|\psi + G\| = 1$ and $\varphi \psi \in G$, the following holds
\[ \|\varphi_n A_n \varphi_n + \psi_n B_n \psi_n\| \leq \max\{\|A_n\|, \|B_n\|\} \quad \text{for every} \quad n \in \mathbb{N}. \]

The first part is straightforwardly proved. Indeed, for $A \in \mathcal{T}^T$ and $x \in M$
\[ \|\phi_x(A + G)\| = \|\phi_x(A^x + G)\| \leq \|A^x + G\| = \limsup_{n \to \infty} \|E_n^x(S_n^x W^x(A))S_n^x\| \leq \|W^x(A)\| \]

and Theorem 1.19 further yields for every $g \in \mathcal{H}^+$
\[ \|W^x(A)\| \leq \liminf_{n \to \infty} \|E_n^{-x}(A_n)S_n^x\| \leq \limsup_{n \to \infty} \|E_n^{-x}\| \|A_n\| \|S_n^x\| \leq \limsup_{n \to \infty} \|A_n\|. \]

For the second part notice that for every $\epsilon > 0$ there is an integer $N$ such that, for every $k \geq N$,
\[ \|\varphi_k A_k \varphi_k + \psi_k B_k \psi_k\| \leq \max\{\limsup_{n \to \infty} \|A_n\|, \limsup_{n \to \infty} \|B_n\|\} + \epsilon. \]

Hence, for every $A, B \in \mathcal{F}^T$ and $\varphi, \psi \in C$ with $\|\varphi + G\| = \|\psi + G\| = 1$ and $\varphi \psi \in G$,
\[ \|\varphi A \varphi + \psi B \psi + G\| \leq \max\{\|\varphi + G\|, \|\psi + G\|\}. \]

Now let $C \in \mathcal{T}^T$ and $f, g \in \mathcal{C}$ be such that the closures of the supports of $f + G$ and $g + G$ are disjoint. Choose $\varphi, \psi \in C$ such that the functions $\varphi + G, \psi + G$ are of norm one, have disjoint support and $f - \varphi f - g, g - \psi g \in G$. Then
\[ \|(f + g)C + G\| = \|\varphi^2 f C + \psi^2 g C + G\| = \|\varphi f C + \psi g C + G\| \leq \max\{\|f C + G\|, \|g C + G\|\}. \]

Since every pair of continuous functions on a compact set having disjoint support can be approximated by pairs of continuous functions having disjoint closed supports, we obtain the KMS property of $\mathcal{T}^T/G$ with respect to $C/G$.

---

8Fix $x \in \text{int}(M \setminus T)$ and a function $\varphi$ being equal to 1 in $x$ and identically zero on $T$. Then $\varphi + G$ and $\varphi^x + G$ are not locally equivalent in $x$.

9This idea already appeared in [10].
3.2.3 Convergence of pseudospectra

We turn our attention again to the asymptotic behavior of the pseudospectra $\text{sp}_{N,\epsilon}A_n$ which have been introduced and studied in Section 1.4 and we now suppose an additional condition which we adopt from [27] and [84].

**Definition 3.14.** A Banach space $X$ is called complex uniformly convex if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$x, y \in X, \quad \|y\| \geq \epsilon \quad \text{and} \quad \|x + \zeta y\| \leq 1 \quad \text{for all} \ \zeta \in \mathbb{C}, |\zeta| \leq 1$$

always implies $\|x\| \leq 1 - \delta$.

As already mentioned in Remark 1.51 this simplifies matters essentially. The precise result, taken from Shargorodsky [84], Theorem 2.6 and Corollary 2.7, reads as follows.

**Theorem 3.15.** Let $\Omega$ be a connected open subset of $\mathbb{C}$, let $X$ or $X^*$ be complex uniformly convex, and let $A: \Omega \to \mathcal{L}(X)$ be an analytic operator-valued function. Suppose that there exists a $\lambda_0 \in \Omega$ such that the derivative $A'(\lambda_0)$ of $A$ in $\lambda_0$ is invertible. If $\|A(\lambda)\| \leq M$ for all $\lambda \in \Omega$ then $\|A(\lambda)\| < M$ for all $\lambda \in \Omega$.

We note that every Hilbert space is complex uniformly convex (see [17]). For a further class of Banach spaces which are complex uniformly convex (and which are highly important for the present text), we borrow the following again from [84], preliminary to Theorem 2.5.

**Proposition 3.16.**

1. Let $(S, \Sigma, \mu)$ be an arbitrary measure space and let $X = L^p(S, \Sigma, \mu)$ with $1 \leq p < \infty$ denote the set of all (equivalence classes of) Lebesgue measurable functions $f$ on $(S, \Sigma, \mu)$ with the $p$-th power of $\|f\|$ being Lebesgue integrable. Then $X$ is complex uniformly convex. If $X = L^\infty(S, \Sigma, \mu)$ is the space of all Lebesgue measurable and essentially bounded functions then $X^*$ is complex uniformly convex.

2. If, additionally, $\mu(S) < \infty$ then $L^p(\mathbb{Z}, X)$ (or $(L^p(\mathbb{Z}, X))^*$, resp.) is complex uniformly convex. This particularly includes the cases $L^p(\mathbb{Z}, \mathbb{C}^N)$, $N \in \mathbb{N}$, where $\mathbb{C}^N$ is provided with the $p$-vector norm and the counting measure.

**Proof.** For the proof of $X$ (or $X^*$) being complex uniformly convex see the mentioned discussion in [84]. Let $\mu(S) < \infty$. Then $X := L^p(\mathbb{Z}, X)$ can be identified with $L^p(\hat{S}, \hat{\Sigma}, \hat{\mu})$, where the measure space $(\hat{S}, \hat{\Sigma}, \hat{\mu})$ is given by

$$\hat{S} := \mathbb{Z} \times S, \quad \hat{\Sigma} := \left\{ \bigcup_{k \in \mathbb{Z}} \{k\} \times A_k : A_k \in \Sigma \right\} \quad \text{and} \quad \hat{\mu} \left( \bigcup_{k \in \mathbb{Z}} \{k\} \times A_k \right) := \sum_{k \in \mathbb{Z}} \mu(A_k),$$

and the first part applies.

**Definition 3.17.** Let $M_1, M_2, \ldots$ be a sequence of non-empty subsets of $\mathbb{C}$. The uniform (partial) limiting set

$$u\text{-}\lim_{n \to \infty} M_n \quad \left( p\text{-}\lim_{n \to \infty} M_n \right)$$

of this sequence is the set of all $z \in \mathbb{C}$ that are (partial) limits of a sequence $(z_n)$ with $z_n \in M_n$.

These limiting sets are closed as shown with [30], Proposition 3.2.
3.2. LOCALIZATION IN SEQUENCE ALGEBRAS

Proposition 3.18. Let $\mathcal{C}$ define a faithful localizing setting and further let all spaces $E^T$ or their duals be complex uniformly convex. For $\mathcal{A} = \{A_n\} \in T^T$ and every $N \in \mathbb{Z}_+, \epsilon > 0$

$$w\lim_{n \to \infty} \text{sp}_{N,\epsilon} A_n = p\lim_{n \to \infty} \text{sp}_{N,\epsilon} A_n = \bigcup_{t \in T} \text{sp}_{N,\epsilon} W^t(\mathcal{A}).$$

Chapter 3 in the book [13] contains a concise discussion of the development and application of $\epsilon$-pseudospectra as well as the present proposition in this particular case $N = 0$. We particularly point out again that results similar to Theorem 3.15, which guarantee that the resolvent is not equal to $\epsilon$ exists and equals $\epsilon$. We further should mention [9] which has presented the convergence results locally constant, have been in that business from the very beginning, see Böttchers pioneering work [6], for example. We further should mention [9] which has presented the convergence results for $\epsilon$-pseudospectra of the finite sections of operators on $L^p$-spaces, $1 < p < \infty$, for the first time. The present proof of Proposition 3.18 is an adapted variant of [13], Theorem 3.17.

Proof. Suppose $z \in \text{sp} W^t(\mathcal{A})$, that is $W^t(\mathcal{A} - zI)\mathbb{C}^N$ and hence $W^t((\mathcal{A} - zI)^2N)$ are not invertible. Then $\mathcal{A} - zI\mathbb{C}^N$ is not stable and Proposition 3.11 yields that $\lim_n \|(A_n - zL_n)^{-2N}\| = \infty$ which implies $z \in \text{sp}_{N,\epsilon} A_n$ for sufficiently large $n$, and therefore $z \in w\lim_n \text{sp}_{N,\epsilon} A_n$.

So, now suppose that $\mathcal{A} - zI\mathbb{C}^N$ is stable, but $z \in \text{sp}_{N,\epsilon} W^t(\mathcal{A})$, i.e. $\|(W^t(\mathcal{A} - zI^t)^{-2N}\| \geq \epsilon^{-2N}$ for one $t \in T$. Let $U$ be an arbitrary open ball around $z$ such that $W^t(\mathcal{A}) - yI$ is invertible for all $y \in U$. If $\|(W^t(\mathcal{A} - yI^t)^{-2N}\|$ would be less than or equal to $\epsilon^{-2N}$ for every $y \in U$ then Theorem 3.15 would imply that $\|(W^t(\mathcal{A} - zI^t)^{-2N}\| < \epsilon^{-2N}$, a contradiction. For this notice that the function $y \mapsto (W^t(\mathcal{A} - yI^t)^{-2N}$ is analytic on $U$ and its derivative in $z$ is invertible. Hence there is a $y \in U$ such that $\|(W^t(\mathcal{A} - yI^t)^{-2N}\| > \epsilon^{-2N}$, that is we can find a $k_0$ such that

$$\|(W^t(\mathcal{A} - yI^t)^{-2N}\| > (\epsilon - \frac{1}{k})^{-2N} \quad \text{for all} \quad k \geq k_0.$$

Because $U$ was arbitrary we can choose a sequence $(z_m)_m$ s.t., for $m \geq 1/\epsilon$, $z_m$ belongs to the $(N, \epsilon - 1/m)$-pseudospectrum of $W^t(\mathcal{A})$ and $z_m \to z$ as $m \to \infty$. Since $\lim_n \|(A_n - z_mL_n)^{-2N}\|$ exists and equals $\max_{t \in T} \|(W^t(\mathcal{A} - z_mI^t)^{-2N}\|$, due to Proposition 3.11, it is greater than or equal to $(\epsilon - 1/m)^{-2N}$. Consequently, for sufficiently large $n$, $\|(A_n - z_mI)^{-2N}\| \geq \epsilon^{-2N}$ and thus $z_m \in \text{sp}_{N,\epsilon} A_n$. This shows that $z = \lim_m z_m$ belongs to the closed set $w\lim_n \text{sp}_{N,\epsilon} A_n$.

Finally consider the case that $\|(W^t(\mathcal{A} - zI^t)^{-2N}\| < \epsilon^{-2N}$ for all $t \in T$. Then

$$\lim_{n \to \infty} \|(A_n - zL_n)^{-2N}\| = \max_{t \in T} \|(W^t(\mathcal{A} - zI^t)^{-2N}\| < \epsilon^{-2N},$$

hence there are a $\delta > 0$ and an $n_0 \in \mathbb{N}$ such that $\|(A_n - zL_n)^{-2N}\| \leq \epsilon^{-2N} - \delta$ for all $n \geq n_0$. If $|y - z|$ is sufficiently small and $n \geq n_0$ we then further get

$$\|(A_n - yL_n)^{-2N}\| \leq \|(A_n - zL_n)(L_n + (z - y)(A_n - zL_n)^{-1})^{-2N}\|

= \|(A_n - zL_n)^{-2N}(L_n + (z - y)(A_n - zL_n)^{-1})^{-2N}\|

\leq \frac{\|(A_n - zL_n)^{-2N}\|}{(1 - |z - y|\|(A_n - zL_n)^{-1}\)2N} \leq \frac{\epsilon^{-2N} - \delta}{(1 - |z - y|\|(A_n - zL_n)^{-1}\))2N} < \epsilon^{-2N}.$$

Thus, $z$ does not belong to $p\lim_n \text{sp}_{N,\epsilon} A_n$. Since $w\lim_n \text{sp}_{N,\epsilon} A_n \subset p\lim_n \text{sp}_{N,\epsilon} A_n$ this completes the proof. \hfill \square
Proposition 3.6 in [30] states that for compact sets $M_n$ the limits $u\lim_n M_n$ and $p\lim_n M_n$ coincide if and only if $M_n$ converge w.r.t. the Hausdorff distance (to the same limiting set).\footnote{This observation is due to Hausdorff and can be found in his book [35] as well.} Thus, we can reformulate the preceding proposition as follows.

**Corollary 3.19.** Let $C$ define a faithful localizing setting and further let all spaces $E^t$ or their duals be complex uniformly convex. For a sequence $\mathcal{A} = \{A_n\} \in T^T$ the $(N,\epsilon)$-pseudospectra of the elements $A_n$ converge w.r.t. the Hausdorff distance to the union of the $(N,\epsilon)$-pseudospectra of all snapshots $W^t(\mathcal{A})$.

This now clarifies the observation that we made in the beginning of Section 1.4.4: In general, one cannot expect that for a given increasing sequence $(T_n)$ of projections the $(N,\epsilon)$-pseudospectra of the truncations $T_n A T_n$ of an operator $A$ converge to $sp_{N,\epsilon} A$. The reason is that $A$ captures only one facet of the asymptotic behavior of $(T_n A T_n)$, and (at least in many cases) further snapshots are needed to restore all relevant details of this sequence.

We finally pose the question what happens if the spaces $E^t$ are not known to be complex uniformly convex, and Theorem 3.15 is not available. A thorough look into the proof of Proposition 3.18 reveals that this additional condition is only needed to conclude $z \in u\lim_{n \to \infty} sp_{N,\epsilon} A_n$ for sufficiently large $n$ from $\| (W^t(\mathcal{A}) - z I^t)^{-2^k} \| \geq \epsilon^{-2^k}$ for one $t \in T$. Without it, we can arrive at the same conclusion if we suppose that $\| (W^t(\mathcal{A}) - z I^t)^{-2^k} \|$ is strictly greater than $\epsilon^{-2^k}$ for one $t$. Therefore, the picture for general Banach spaces is as follows: For every $\mathcal{A} \in T^T$ every $N \in \mathbb{Z}_+$ and $0 < \delta < \epsilon$

$$\bigcup_{t \in T} sp_{N,\beta} W^t(\mathcal{A}) \subset u\lim_{n \to \infty} sp_{N,\epsilon} A_n \subset p\lim_{n \to \infty} sp_{N,\epsilon} A_n \subset \bigcup_{t \in T} sp_{N,\epsilon} W^t(\mathcal{A}).$$

Thus we can use Theorem 1.44 to carry forward Theorem 1.48:

**Corollary 3.20.** Suppose that the index set $T$ of the algebraic framework $F^T$ is finite and $C$ defines a faithful localizing setting. Let $\mathcal{A} = \{A_n\} \in T^T$. For every $0 < \alpha < \epsilon < \beta$ there is an $N_0 \in \mathbb{Z}_+$ such that, for every $N \geq N_0$,

$$B_\alpha \left( \bigcup_{t \in T} sp W^t(\mathcal{A}) \right) \subset u\lim_{n \to \infty} sp_{N,\epsilon} A_n \subset p\lim_{n \to \infty} sp_{N,\epsilon} A_n \subset B_\beta \left( \bigcup_{t \in T} sp W^t(\mathcal{A}) \right).$$

**Remark 3.21.** Such results on the convergence of norms, condition numbers and pseudospectra have been proved for various classes of operators, such as Wiener-Hopf operators [4], Toeplitz operators [13], [8], band-dominated operators on $l^2$ [57], or convolution type operators on cones [50]. A systematic and abstract presentation for the $C^*$-algebra case is in [30]. The present text provides several extensions: It is in a more abstract and versatile axiomatic formulation, which covers more exotic situations such as $l^\infty$ spaces, in particular the class of band-dominated operators on all $l^p$-spaces as we will see in Section 3.2.5. Moreover, it treats $(N,\epsilon)$-pseudospectra, and it overcomes one of the most unpleasant hurdles in that business: the inverse closedness of the algebras under consideration is automatically given.

### 3.2.4 Rich sequences meet localization

Let us condense the previous results and apply them to rich sequences $\mathcal{A}$. For this, $H_\mathcal{A}$ again stands for the set of all strictly increasing sequences $g$ of positive integers such that $A_g$ is a $T$-structured subsequence of $\mathcal{A}$. Further, we denote by $\mathcal{TR}^T$ the set of all sequences $\mathcal{A} \in \mathcal{R}^T$...
which have the following property: Every $T$-structured subsequence $A_g$ has a subsequence $A_h$ and an algebra $C$ (that may depend on $A$ and $h$) which defines a faithful localizing setting (in $\mathcal{F}_h^T$), such that $A_h \in \mathcal{T}_h^T$.

**Theorem 3.22.** A sequence $A \in \mathcal{T}^T$ is Fredholm, if and only if all its snapshots are Fredholm. In this case

$$\alpha(A) = \max_{g \in \mathcal{H}_A} \left( \sum_{t \in T} \dim \ker W^t(A_g) \right), \quad \beta(A) = \max_{g \in \mathcal{H}_A} \left( \sum_{t \in T} \dim \coker W^t(A_g) \right).$$

Furthermore, for every $A = \{ A_n \} \in \mathcal{T}^T$,

$$\limsup_{n \to \infty} \| A_n \| = \max_{g \in \mathcal{H}_A} \left( \sum_{t \in T} \| W^t(A_g) \| \right), \quad \limsup_{n \to \infty} \| A_n^{-1} \| = \max_{g \in \mathcal{H}_A} \left( \sum_{t \in T} \| (W^t(A_g))^{-1} \| \right),$$

$$\liminf_{n \to \infty} \| A_n \| = \min_{g \in \mathcal{H}_A} \left( \sum_{t \in T} \| W^t(A_g) \| \right), \quad \liminf_{n \to \infty} \| A_n^{-1} \| = \min_{g \in \mathcal{H}_A} \left( \sum_{t \in T} \| (W^t(A_g))^{-1} \| \right),$$

and if, additionally, all spaces $E^t$ or their duals are complex uniformly convex then

$$p\text{-lim}_{n \to \infty} \text{sp}_{N, \epsilon} A_n = \bigcup_{g \in \mathcal{H}_A, t \in T} \text{sp}_{N, \epsilon} W^t(A_g).$$

A sequence $A \in \mathcal{T}^T$ is stable if and only if all its snapshots are invertible. In this case

$$\lim_{n \to \infty} \text{cond}(A_n) = \max_{g \in \mathcal{H}_A} \left( \sum_{t \in T} \| W^t(A_g) \| \cdot \max_{t \in T} \| (W^t(A_g))^{-1} \| \right),$$

$$\liminf_{n \to \infty} \text{cond}(A_n) = \min_{g \in \mathcal{H}_A} \left( \sum_{t \in T} \| W^t(A_g) \| \cdot \max_{t \in T} \| (W^t(A_g))^{-1} \| \right).$$

**Proof.** Let all snapshots of $A \in \mathcal{T}^T$ be Fredholm. Every $T$-structured subsequence $A_g$ of $A$ has a subsequence $A_h$ which belongs to $\mathcal{T}_h^T$, thus Theorem 2.9 yields that $A_h$ is Fredholm and regularly $\mathcal{F}_h^T$-Fredholm. Now, we derive the Fredholm property of $A$ and formulas for $\alpha(A)$ and $\beta(A)$ from Theorem 2.2. For the converse Corollary 2.23 can be employed. The stability can be tackled in the same way.

Recall from Proposition 3.11 that, for every $g \in \mathcal{H}_A$ and the respective localizable subsequence $A_h$,

$$\max_{t \in T} \| W^t(A_g) \| = \max_{t \in T} \| W^t(A_h) \| = \lim_{n \to \infty} \| A_{h_n} \| \leq \limsup_{n \to \infty} \| A_n \|.$$

Now, choose a sequence $j \in \mathcal{H}_A$ such that $\lim n \| A_{h_n} \| = \limsup_{n \to \infty} \| A_n \|$ and pass to a subsequence $g \in \mathcal{H}_A$ of $j$ to see $\max_{t \in T} \| W^t(A_g) \| = \lim_{n \to \infty} \| A_{g_n} \| = \limsup_{n \to \infty} \| A_n \|$. For the other equations we proceed analogously, with the aid of Propositions 3.11, 3.18 and Corollary 3.12.

### 3.2.5 Example: finite sections of band-dominated operators

Back in the $p$-setting, we want to apply the latest tools to the sequences in the algebra $\mathcal{F}_{A, p}$ which is generated by the finite section sequences of all rich band-dominated operators. By doing this, we recover and extend the observations of [63], Section 6.3, as well as those of Section 2.5 in the present text.

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11See also [69] and the preprint [57].
Theorem 3.23. Let $\mathcal{A} = \{A_n\} \in \mathcal{F}_{\mathcal{A}}$ and $g \in \mathcal{H}_\mathcal{A}$. Then $\mathcal{A}_g$ is $\mathcal{T}_g$-Fredholm if and only if its snapshots $W^t(\mathcal{A}_g)$ are $\mathcal{P}^t$-Fredholm, respectively. We further have the limits

$$C_1 := \lim_{n \to \infty} \|A_{g_n}\| = \max\{\|W^1(\mathcal{A})\|, \|W^1(\mathcal{A}_g)\|\},$$

$$C_2 := \lim_{n \to \infty} \|A_{g_n}^{-1}\| = \max\{\|(W^1(\mathcal{A}))^{-1}\|, \|(W^1(\mathcal{A}_g))^{-1}\|, \|(W^{-1}(\mathcal{A}_g))^{-1}\|\},$$

hence $\lim_n \text{cond}(A_{g_n}) = C_1C_2$ in the case of a stable sequence.

If, additionally, $\mathcal{P}^t$ or $(\mathcal{P})^*$ is complex uniformly convex then the $(N, \epsilon)$-pseudospectra of the operators $A_{g_n}$ converge w.r.t. the Hausdorff distance to the union of the $(N, \epsilon)$-pseudospectra of the snapshots.\(^{12}\)

Proof. The outline of this proof is very simple and straightforward: We are going to introduce an algebra $\mathcal{C}$ which defines a faithful globally localizing setting in $\mathcal{F}_{\mathcal{A}}^T$ and turns $\mathcal{A}_g$ into a $\mathcal{C}$-localizable sequence, hence allows to apply the results on the convergence of norms, condition numbers and pseudospectra.

First of all, we set $i_1 := 1$ and choose a subsequence $i = (i_k)$ of $g$ such that

$$\sup \left\{ \| (A_{g_n}^{(t_k)} - W^t(\mathcal{A}_g)) L_k \|, \| L_k (A_{g_n}^{(t_k)} - W^t(\mathcal{A}_g)) \| : t \in T, g_n \geq i_k \right\} < \frac{1}{k} \quad (3.6)$$

for every $k \geq 2$. This is possible since $A_{g_n}^{(t_k)}$ tends $\mathcal{P}^t$-strongly to $W^t(\mathcal{A}_g)$. Now, for $k \in \mathbb{N}$, define $\gamma_n := k/2$ for all $n \in \{i_k, \ldots, i_{k+1} - 1\}$, and let $b_n \colon \mathbb{R} \to [-1, 1]$ denote the continuous piecewise linear splines given by $b_n(\pm n) = \pm 1$ and $b_n(\pm (n - \gamma_n)) = \pm 1/2$ which are constant outside the interval $[-n, n]$, respectively.

For every continuous function $\varphi \in C[-1, 1]$ we let $\varphi_n^\gamma$ stand for the restriction of the function $\varphi \circ b_n^\gamma$ to $\mathbb{Z}$. Thus, the sequence $\{\varphi_n^\gamma L_n\}$ consists of operators of multiplication by inflated copies of $\varphi$ with a certain distortion (see Figure 3.1). With $\|\{\varphi_n^\gamma L_n\}\| \leq \|\varphi\|_\infty$ it straightforwardly follows that the set

$$\mathcal{C}^\gamma := \{\{\varphi_n^\gamma L_n\} : \varphi \in C[-1, 1]\}$$

forms a commutative subalgebra of $\mathcal{F}_g$, where $W^t(\{\varphi_n^\gamma L_n\}) = \varphi(t)I^t$ for every $t$ and every $\varphi$. By $^* : \{\varphi_n^\gamma L_n\} \mapsto \{\varphi_n^\gamma L_n\}$ an involution is given and the mapping $\varphi \mapsto \{\varphi_n^\gamma L_n\} + \mathcal{G}$ is a $^*$-homomorphism from $C[-1, 1]$ onto $\mathcal{C}^\gamma / \mathcal{G}$. We show that this mapping is even isometric, and therefore $\mathcal{C}^\gamma / \mathcal{G}$ proves to be a unital commutative $\mathcal{C}^\gamma$-algebra and its maximal ideal space can be identified with the interval $[-1, 1]$, due to Theorem 3.2. For this, we note that

$$\|\varphi\|_\infty \geq \|\{\varphi_n^\gamma L_n\}\| = \|\{\varphi_n^\gamma L_n\} + \mathcal{G}\|_{\mathcal{F}_g}$$

holds and, on the other hand, $\|\varphi\|_\infty = \sup\{\|\varphi(x)\| : x \in [-1, 1]\} \leq \|\{\varphi_n^\gamma L_n\} + \mathcal{G}\|_{\mathcal{F}_g}$ follows.

\(^{12}\)Without the complex uniform convexity Corollary 3.20 is still applicable.
from \(^\text{13}\)
\[
|\varphi(x)| = |\varphi(x)I| = \|T_n^\lim V_{-(b_n)}^{-1}(x)\varphi_n^*L_nV_{[(b_n)_{-1}(x)]}\|
\leq \lim_n \inf \|V_{-(b_n)}^{-1}(x)\varphi_n^*L_nV_{[(b_n)_{-1}(x)]}\| \leq \lim_n \sup \|\varphi_n^*L_n\| = \|\varphi_n^*L_n\| + G\|f/g\|
\]

Set \(M := [-1, 1]\). To every \(x \in (-1, 1)\) we associate the direction \(t(x) := 0 \in T\), and for every \(x \in [-1, 1]\) we fix a function \(F^x \in C[-1, 1]\) which equals 1 in a neighborhood of \(x\), equals 0 in a neighborhood of every \(t \in T \setminus \{x\}\) and takes only real values between 0 and 1. Then \(C^\gamma\) defines a faithful globally localizing setting. For this apply Remark 3.13 and notice that
\[
\|(\varphi_n^*A_n\varphi_n^* + \psi_n^*B_n\psi_n^*)a\| = \max\{\|\varphi_n^*A_n\varphi_n^*a\|, \|\psi_n^*B_n\psi_n^*a\|\} = \max\{\|A_n\|, \|B_n\|\}\|a\|
\]
holds for all \(a \in E_n\), all \(A_n, B_n \in F\) and all \(\varphi, \psi \in C[-1, 1]\) of norm 1 with disjoint support in the case \(p = \infty\). For \(1 \leq p < \infty\) we replace (3.7) by
\[
\|(\varphi_n^*A_n\varphi_n^* + \psi_n^*B_n\psi_n^*)a\|^p = \|\varphi_n^*A_n\varphi_n^*a\|^p + \|\psi_n^*B_n\psi_n^*a\|^p
\leq \|\varphi_n^*A_n\|^{\gamma\|\varphi_n^*a\|^p} + \|\psi_n^*B_n\|^{\gamma\|\psi_n^*a\|^p} \leq \max\{\|A_n\|^p, \|B_n\|^p\}\|a\|^p.
\]
It remains to show that \(A_y \in T_g^T\). For this assume that we can show, for every \(B \in A_p\), that
\[
\lim_n \sup_{k \in Z} \|(V_kBV_k, \varphi_n^*)\|^p = 0 \quad \text{for every} \quad \{ \varphi_n^*L_n \} \in C^\gamma.
\]
Then \(F_{A_p}/G\) is obviously contained in the closed subalgebra \(\text{Com}(C^\gamma/G)\) of \(F/G\) which yields that \(A_y + G_y\) commutes with every element of \(C^\gamma/G\). Every snapshot \(W^x(A_y)\) is again band-dominated (or at least the compression of a band-dominated operator to the space \(E^x\)) and, due to the choice of the functions \(F^x\), it is always liftable. Furthermore, by (3.8), it almost commutes with \(F^x\) and \(A_y^* + G_y^*\) belongs to \(\text{Com}(C^\gamma/G)\) as well. Moreover, for every \(x \in (-1, 1)\), equation (3.8) provides that sequences of the form \((I - L_n)B(I - L_n)\) and \((F^x_B(I - L_n))\), with \(B\) being band-dominated, tend to zero in the norm as \(n \to \infty\), hence \(A_y + G_y\) and \(A_y^* + G_y^*\) easily prove to be locally equivalent in \(x\). In case \(x = 1\) (or in an analogous way also for \(x = -1\)) we choose \(\varphi \in C([-1, 1]\) equal to 1 in a neighborhood of \(x\) and identically zero on \([-1, 1/2]\) (or \([-1/2, 1]\), respectively). Then \(\{nB_n(I - L_n)\}((A_y - A_y^*) \in G_y\) by the construction of the \((\gamma_n)\) and (3.6), which yields the local equivalence in \(x\) again. Thus \(A_y \in T_g^T\), and Theorem 3.9, Proposition 3.11 and Corollaries 3.12, 3.19 apply and give the assertions of the theorem.
So, let us check the relation (3.8) for band-dominated operators \(B\). It is easily proved for every shift operator and every operator of multiplication, hence it is clear for band operators and follows for band-dominated ones by a simple approximation argument.

Clearly, for every sequence \(A \in F_{A_p}\), the proof of the previous theorem shows that \(A \in T^{R, T}\), hence Theorem 3.22 applies, recovers and completes Theorem 3.32. More precisely, for every \(A = \{A_n\} \in F_{A_p}\), we have formulas for the \(\limsup\) and the \(\liminf\) of both \(\|A_n\|\) and \(\|A_n^{-1}\|\), in terms of norms of the snapshots and their inverses. In case of a stable sequence, this is true for \(\text{cond}(A_n)\) as well.

Moreover, if the space \(l^p\) or its dual is complex uniformly convex then, also by Theorem 3.22,
\[
\text{p-limit } sp_{N, \varepsilon} A_n = \bigcup_{g \in H_n, t \in T} sp_{N, \varepsilon} W^t(A_g).
\]

\(^{13}\)With \((b_n)^{-1}\) being the inverse of the bijective function \(b_n : [-n, n] \to [-1, 1]\), and \([\cdot]\) the floor function.

\(^{14}\)For a related observation in case \(p = 2\) see [63], Theorem 6.3.15.
3.3 $\mathcal{P}$-algebras

This section is devoted to the theory of Banach algebras with approximate projections. In particular, we give the pending proofs for the results that we stated in Section 1.2.1.

3.3.1 Basic definitions

**Definition 3.24.** Let $\mathcal{A}$ be a unital Banach algebra and let $\mathcal{P} = (P_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ be a bounded sequence of elements with the following properties:

- $P_n \neq I$ for all $n \in \mathbb{N}$, and $P_n \neq 0$ for large $n$,
- For every $m \in \mathbb{N}$ there is an $N_m \in \mathbb{N}$ such that $P_n P_m = P_m P_n = P_m$ if $n \geq N_m$.

Then $\mathcal{P}$ is referred to as an approximate projection. In all what follows we set $Q_n := I - P_n$ and we further write $m \ll n$ if $P_k Q_l = Q_l P_k = 0$ for all $k \leq m$ and all $l \geq n$.

**$\mathcal{P}$-compactness** Let $\mathcal{P}$ be an approximate projection on $\mathcal{A}$. An element $K \in \mathcal{A}$ is called $\mathcal{P}$-compact if $\|KP_n - K\|$ and $\|P_n K - K\|$ tend to zero as $n \to \infty$. By $\mathcal{A}(\mathcal{P}, K)$ we denote the set of all $\mathcal{P}$-compact elements and by $\mathcal{A}(\mathcal{P})$ the set of all elements $A \in \mathcal{A}$ for which $AK$ and $KA$ are $\mathcal{P}$-compact whenever $K$ is $\mathcal{P}$-compact.

**Theorem 3.25.** Let $\mathcal{P}$ be an approximate projection. $\mathcal{A}(\mathcal{P})$ is a closed subalgebra of $\mathcal{A}$, it contains the identity, and $\mathcal{A}(\mathcal{P}, K)$ is a proper closed ideal of $\mathcal{A}(\mathcal{P})$. An element $A \in \mathcal{A}$ belongs to $\mathcal{A}(\mathcal{P})$ if and only if, for every $k \in \mathbb{N}$,

$$\|P_k A Q_n\| \to 0 \quad \text{and} \quad \|Q_n A P_k\| \to 0 \quad \text{as} \quad n \to \infty.$$  \hspace{1cm} (3.9)

**Proof.** The following proof is an adaption of [63], Proposition 1.1.8.

The conditions (3.9) are clearly necessary for $A \in \mathcal{A}(\mathcal{P})$. Let, conversely, $A$ satisfy (3.9) and let $K \in \mathcal{A}(\mathcal{P}, K)$. Given $\epsilon > 0$, choose $k$ such that $\|Q_k K\| < \epsilon$, and choose $N$ such that $\|Q_n A P_k\| < \epsilon$ for all $n \geq N$. Then

$$\|Q_n A K\| \leq \|Q_n A\| \|Q_k K\| + \|Q_n A P_k\| \|K\| < \epsilon \|Q_n A\| + \|K\|$$

for all $n \geq N$. The other conditions can be checked similarly, thus $A \in \mathcal{A}(\mathcal{P})$.

It is immediate from the definition that $\mathcal{A}(\mathcal{P}, K), \mathcal{A}(\mathcal{P})$ are subalgebras of $\mathcal{A}$ and that $\mathcal{A}(\mathcal{P}, K)$ forms a proper ideal in $\mathcal{A}(\mathcal{P})$. To check the closedness of $\mathcal{A}(\mathcal{P}, K)$ let $(K_n) \subset \mathcal{A}(\mathcal{P}, K)$ converge to $K \in \mathcal{A}$ in the norm, and consider

$$\|Q_n K\| \leq \|Q_n K_m\| + \|Q_n\| \|K - K_m\|.$$  

The latter term becomes as small as desired, if $m$ is large, and the first one tends to zero as $n \to \infty$ for every $m$. Analogously, we find that $\|K Q_n\| \to 0$ as $n \to \infty$, hence $K \in \mathcal{A}(\mathcal{P}, K)$.

If $A$ is the (norm) limit of $(A_n) \subset \mathcal{A}(\mathcal{P})$ then $AK, KA$ are the limits of $(A_n K), (K A_n) \subset \mathcal{A}(\mathcal{P}, K)$ for every $K \in \mathcal{A}(\mathcal{P}, K)$, hence are $\mathcal{P}$-compact as well. Thus, $A \in \mathcal{A}(\mathcal{P})$. \hfill $\square$

**$\mathcal{P}$-Fredholmness** Here are two possible ways to introduce “almost invertibility”, in analogy to the Fredholm property in operator algebras.

- For $A \in \mathcal{A}$ we may assume that there is a $\mathcal{P}$-regularizer $B \in \mathcal{A}$, that is $I - AB, I - BA$ are $\mathcal{P}$-compact.
- For $A \in \mathcal{A}(\mathcal{P})$ we may assume that the coset $A + \mathcal{A}(\mathcal{P}, K)$ is invertible in the quotient algebra $\mathcal{A}(\mathcal{P})/\mathcal{A}(\mathcal{P}, K)$.

The big question is, if these definitions coincide for $A \in \mathcal{A}(\mathcal{P})$. 


3.3.2 Uniform approximate projections and the $\mathcal{P}$-Fredholm property

**Definition 3.26.** Given an algebra $\mathcal{A}$ with an approximate projection $\mathcal{P} = (P_n)$, we set $S_1 := P_1$ and $S_n := P_n - P_{n-1}$ for $n \geq 1$. Furthermore, for every finite subset $U \subset \mathbb{N}$, we define $P_U := \sum_{k \in U} S_k$. $\mathcal{P}$ is called uniform if $C_P := \sup \|P_U\| < \infty$, the supremum over all finite $U \subset \mathbb{N}$.

Two approximate projections $\mathcal{P} = (P_n)$ and $\hat{\mathcal{P}} = (\hat{P}_n)$ on $\mathcal{A}$ are said to be equivalent if for every $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that

$$P_mP_n = P_nP_m = P_m \quad \text{and} \quad F_mP_n = P_nF_m = F_m.$$  

In case of equivalent approximate projections $\mathcal{P}$ and $\hat{\mathcal{P}}$, one easily checks that $\mathcal{A}(\mathcal{P}, \mathcal{K}) = \mathcal{A}(\hat{\mathcal{P}}, \mathcal{K})$, hence also $\mathcal{A}(\mathcal{P}) = \mathcal{A}(\hat{\mathcal{P}})$.

**Theorem 3.27.** Let $\mathcal{P}$ be a uniform approximate projection on $\mathcal{A}$ and $A \in \mathcal{A}(\mathcal{P})$. Then there is an equivalent uniform approximate projection $\mathcal{P} = (F_n)$ on $\mathcal{A}$ with $C_P \leq C_P$ such that

$$\|F_n, A\| = \|AF_n - F_nA\| \to 0 \quad \text{as} \quad n \to \infty.$$  

**Proof.** Successively choose integers $1 = j_1 \ll i_2 \ll i_3 \ll \ldots$ such that for every $l$

$$\|P_s AQ_{j_l}\| \leq (2^{l+1}l)^{-1} \forall s \leq i_l \quad \text{and} \quad \|Q_l AP_{j_l}\| \leq (2^{l+5}l)^{-1} \forall t \geq i_{l+1}.$$  

(3.10)

Then, for all $k, n \in \mathbb{N}$ set $U^n_k := \{i_2n+k-1 + 1, \ldots, i_2n+k\}$ and $V^n_k := \{j_2n+k-2 + 1, \ldots, j_2n+k\}$, as well as $U^n_0 := \{1, \ldots, i_2n\}$ and $V^n_0 := \{1, \ldots, j_2n\}$, and find

$$\|P^n_{U_k} AQ^n_{J^{2n+k-2}}\| \leq (2^{k+2}n)^{-1} \quad \text{and} \quad \|P^n_{U_k} AQ^n_{J^{2n+k}}\| \leq (2^{k+2}n)^{-1}.$$  

Thus

$$\|P^n_{U_k} AQ^n_{V^n_k}\| \leq (2^{k+1}n)^{-1}, \quad \text{yielding} \quad \sum_{k \in \mathbb{Z}_+} \|P^n_{U_k} AQ^n_{V^n_k}\| \leq \frac{1}{n}.$$  

(3.11)

For $n \in \mathbb{N}$ we set

$$F_n := \sum_{k=0}^{n-1} \left( 1 - \frac{k}{n} \right) P^n_{U_k}$$

and deduce that $F_nF_{n+1} = F_{n+1}F_n = F_n$ as well as

$$\|F_n\| = \left\| \sum_{k=1}^{n} \frac{k}{n} P^n_{U_{n-k}} \right\| = \frac{1}{n} \left\| \sum_{h=1}^{n} kP^n_{U_{n-k}} \right\| \leq \frac{1}{n} \left\| \sum_{j=1}^{n} \sum_{k=j}^{n} P^n_{U_{n-k}} \right\| \leq C_P,$$  

(3.12)

that is $\mathcal{P} = (F_n)$ is an approximate projection. Further, $\mathcal{P}$ and $\hat{\mathcal{P}}$ are equivalent, since $P^n_{J^{2n-1}}F_n = F_nP^n_{J^{2n-1}}$ and $F_nP^n_{J^{2n+n-1}} = F_nP^n_{J^{2n+n-1}}$.

For bounded subsets $U \subset \mathbb{R}$ we finally introduce the elements $F_U$ as in Definition 3.26 and easily check that they can be represented in the form

$$F_U = \sum_{k=0}^{N-1} \frac{k}{N} P^n_{U_k}.$$
with certain disjoint finite sets $W_k \subset \mathbb{N}$, $k = 0, \ldots, N - 1$. By an estimate similar to (3.12) we deduce that $\mathcal{P}$ is uniform with $C_P \leq C_P$. It remains to consider the commutators of $A$ and $F_n$. As a start, notice that

$$AF_n = \sum_{k=0}^{n} P_{U_k^n} AF_n + Q_{I_2^{n+1}} AF_n$$

$$= \sum_{k=0}^{n} P_{U_k^n} AP_{V_k^n} F_n + \sum_{k=0}^{n} P_{U_k^n} AQ_{V_k^n} F_n + Q_{I_2^{n+1}} AP_{J_2^{n+1}} F_n,$$

where the last term is less than $C_P n^{-1}$ in the norm and the middle term is not greater than $C_P n^{-1}$ as well, due to the above estimate (3.11). The first one equals

$$\sum_{k=0}^{n} P_{U_k^n} AP_{V_k^n} \left( F_n - \left(1 - \frac{k}{n}\right)I \right) + \sum_{k=0}^{n} \left(1 - \frac{k}{n}\right) P_{U_k^n} AP_{V_k^n}$$

$$= \sum_{k=0}^{n} P_{U_k^n} AP_{V_k^n} (P_{U_{k-1}}^n - P_{U_{k+1}}^n) - \sum_{k=0}^{n} \left(1 - \frac{k}{n}\right) P_{U_k^n} AQ_{V_k^n} + F_n A,$$

where we redefine $P_{U_0^n} = P_{U_{n+1}^0} := 0$. The second item is smaller than $n^{-1}$ in the norm, thus

$$\|AF_n - F_n A\| \leq \frac{1}{n} \left\| \sum_{k=0}^{n} P_{U_k^n} A (I - Q_{V_k^n}) (P_{U_{k-1}}^n - P_{U_{k+1}}^n) \right\| + \frac{2C_P + 1}{n}$$

$$\leq \frac{1}{n} \left\| \sum_{k=0}^{n} P_{U_k^n} A (P_{U_{k-1}}^n - P_{U_{k+1}}^n) \right\| + \frac{2C_P n^{-1} + 2C_P + 1}{n}.$$ 

Now we obviously have

$$\left\| \sum_{k=0}^{n} P_{U_k^n} AP_{U_{k-1}^n} \right\| \leq 2 \left\| \sum_{l=0}^{\infty} \sum_{k \equiv l \text{ mod } 3} P_{U_k^n} AP_{U_{k-1}^n} \right\|$$

and this can be further estimated by

$$\sum_{l=0}^{2} \left\| \sum_{k=0}^{n} P_{U_k^n} A \left( \sum_{j=0}^{n} P_{U_{j-1}^n} \right) \right\| + \sum_{l=0}^{n} \left\| P_{U_k^n} A \left( - \sum_{j=0, j \neq k}^{n} P_{U_{j-1}^n} \right) \right\|$$

$$\leq \sum_{l=0}^{2} \left\| \sum_{k=0}^{n} P_{U_k^n} A \left( \sum_{j=0}^{n} P_{U_{j-1}^n} \right) \right\| + \sum_{l=0}^{n} \left\| P_{U_k^n} AQ_{V_k^n} \right\| \left\| \sum_{j=0, j \neq k}^{n} P_{U_{j-1}^n} \right\|.$$ 

We conclude

$$\left\| \sum_{k=0}^{n} P_{U_k^n} AP_{U_{k-1}^n} \right\| \leq \sum_{l=0}^{2} [C_P \|A\|C_P + n^{-1} C_P] = 3C_P (\|A\|C_P + n^{-1}),$$
and with a similar estimate for \( \sum P_{U_k} A P_{U_{k+1}} \) we finally get the assertion by

\[
\|AF_n - F_n A\| \leq \frac{6C_p(\|A\|C_p + n^{-1}) + 2C_p n^{-1} + 2C_p + 1}{n}.
\]

\( \square \)

Now, we can answer the question on the correct definition of \( \mathcal{P} \)-Fredholmness affirmatively.

**Theorem 3.28.** Let \( \mathcal{P} = (P_n) \) be a uniform approximate projection on \( \mathcal{A} \) and \( A \in \mathcal{A}(\mathcal{P}) \). Then the coset \( A + \mathcal{A}(\mathcal{P}, K) \) is invertible in the quotient algebra \( \mathcal{A}(\mathcal{P}) / \mathcal{A}(\mathcal{P}, K) \) if and only if there is a \( \mathcal{P} \)-regularizer \( B \in \mathcal{A} \) for \( A \). In particular, every \( \mathcal{P} \)-regularizer for \( A \in \mathcal{A}(\mathcal{P}) \) (if it exists) belongs to \( \mathcal{A}(\mathcal{P}) \), and \( \mathcal{A}(\mathcal{P}) \) is inverse closed in \( \mathcal{A} \).

**Proof.** Obviously, the invertibility of the coset implies the existence of a \( \mathcal{P} \)-regularizer \( B \).

Conversely, let \( B \in \mathcal{A} \) be such that \( P := I - BA, P' := I - AB \in \mathcal{A}(\mathcal{P}, K) \). Theorem 3.27 yields an equivalent uniform approximate projection \( \mathcal{P} = (F_n) \) such that \( \| [A, F_n] \| \to 0 \) as \( n \to \infty \). Thus, \( G_n := I - F_n \), we also have \( \| [A, G_n] \| \to 0 \) as \( n \to \infty \). Notice that \( \| PG_n \| \) and \( \| G_n P' \| \) tend to zero as \( n \to \infty \), hence (writing \( A_n \sim B_n \) if \( \| A_n - B_n \| \to 0 \) as \( n \to \infty \))

\[
BG_n \sim_n B G_n AB \sim_n BAG_n B \sim_n G_n B.
\]

Therefore \( \| [B, G_n] \| \to 0 \) as \( n \to \infty \), and we conclude, for every \( k \), that \( \| F_k B G_n \| \) and \( \| G_n B F_k \| \) tend to zero as \( n \to \infty \). Thus \( B \in \mathcal{A}(\mathcal{P}) = \mathcal{A}(\mathcal{P}) \) due to Theorem 3.25. \( \square \)

These results are completely new and clear up the open problem which has been in the background of the limit operator method [63], [44] for several years. They were recently published in [81] for the case of operator algebras \( \mathcal{L}(X, \mathcal{P}) \). This consistent picture can now be condensed into the following definition.

**Definition 3.29.** We say that \( A \in \mathcal{A}(\mathcal{P}) \) is \( \mathcal{P} \)-Fredholm if there is a \( \mathcal{P} \)-regularizer \( B \in \mathcal{A} \) for \( A \).

### 3.3.3 \( \mathcal{P} \)-centralized elements

We shortly discuss a class of elements which may be regarded as analogues to band-dominated operators in operator algebras, in a sense.

**Definition 3.30.** Let \( \mathcal{P} = (P_n) \) be a uniform approximate projection on the Banach algebra \( \mathcal{A} \) such that \( P_{n+1} P_n = P_n P_{n+1} = P_n \) for every \( n \in \mathbb{N} \). Say that \( A \in \mathcal{A} \) is \( \mathcal{P} \)-centralized if

\[
\lim_{m \to \infty} \sup \{ \| P_k A Q_{k+m} \|, \| Q_{k+m} A P_k \| : k \in \mathbb{N} \} = 0.
\]

**Proposition 3.31.** The set \( \mathcal{A}(\mathcal{P}, \mathcal{C}) \) of all \( \mathcal{P} \)-centralized elements is a closed unital subalgebra of \( \mathcal{A}(\mathcal{P}) \) and contains \( \mathcal{A}(\mathcal{P}, K) \) as a closed ideal.

If \( A \in \mathcal{A}(\mathcal{P}, \mathcal{C}) \) is \( \mathcal{P} \)-Fredholm then each of its \( \mathcal{P} \)-regularizers is \( \mathcal{P} \)-centralized as well.

In particular, \( \mathcal{A}(\mathcal{P}, \mathcal{C}) \) is inverse closed in \( \mathcal{A} \) and \( \mathcal{A}(\mathcal{P}) \).

This observation will be picked up in the applications part 4.4.6 in order to extend the results on the Fredholm property and the stability of the finite sections of band-dominated operators to a larger class of operators which will be called weakly band-dominated operators.
Proof. Apply Theorem 3.25 to get $A(P, C) \subset A(P)$, and straightforwardly check that $A(P, C)$ is even a closed linear subspace of $A(P)$ containing $A(P, K)$ and the identity. The estimate

$$\|P_k ABQ k+2m\| \leq \|P_k A\|\|P_{k+m} BQ k+2m\| + \|P_k AQ k+m\| \|BQ k+2m\|$$

and its dual for $\|Q_k+2m ABP_k\|$ obviates that $A(P, C)$ is an algebra.

Let $B \in A$ be a $P$-regularizer for $A \in A(P, C)$. By Theorem 3.28, $B$ already belongs to $A(P)$, and in fact the same arguments lead to the proof of $B$ belonging to $A(P, C)$: Since $A$ is $P$-centralized, we can replace (3.10) in the proof of Theorem 3.27 by the stronger conditions

$$\sup_{r \in \mathbb{Z}_+} \|P_{k+r} AQ_{j+r}\| \leq (2^{l+2})^{-1} \forall s \leq i_t \quad \text{and} \quad \sup_{r \in \mathbb{Z}_+} \|Q_{k+r} AP_{j+r}\| \leq (2^{l+5})^{-1} \forall t \geq i_t+1$$

and, instead of constructing only one approximate projection $\hat{P} = (F_n)$, we now consider the family $\hat{P}^r = (F_{n}^{r})$, $r \in \mathbb{N}$, given by

$$F_{n}^{r} := P_r + \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) P_{U_k}^{r},$$

where $U + r := \{u + r : u \in U\}$. Then we again find that these $\hat{P}^r$ are uniform approximate projections with $C_{P^r} \leq C_P$ and $\sup_{r \in \mathbb{N}} \|F_{n}^{r} A\| \to 0$ as $n \to \infty$. Moreover, there is a sequence $(m_n)$ of integers $m_n \geq n$ such that

$$P_{m_n + r} F_{n}^{r} = P_{m_n + r} F_{m_n + r} \quad \text{and} \quad P_{m_n + 1 + r} F_{n}^{r} = P_{m_n + 1 + r} F_{n}^{r}$$

for all $n, r \in \mathbb{N}$. Now, set $G_{n}^{r} := I - F_{n}^{r}$ and check that $\sup_{r \in \mathbb{N}} \|G_{n}^{r} B\| \to 0$ if $n \to \infty$ as it was done in the proof of Theorem 3.28. Thus, we find that for every fixed $n$ and all $m \geq m_{n+1}$

$$\sup_{k \in \mathbb{N}} \|P_{k} BQ_{k+m}\| = \sup_{k \in \mathbb{N}} \|P_{k} B G_{n}^{k} Q_{k+m}\| \leq \sup_{k \in \mathbb{N}} \|P_{k} B G_{n}^{k}\| \|BQ_{k+m}\| + \sup_{k \in \mathbb{N}} \|P_{k}\| \|G_{n}^{k} B\| \|Q_{k+m}\|,$$

where the first summand vanishes since $P_{k} G_{n}^{k} = 0$ and the second one becomes arbitrarily small as $n$ becomes large. With a dual estimate for $\sup_{k} \|Q_{k+m} B P_{k}\|$ this provides $B \in A(P, C)$ and easily finishes the proof.\[\square\]

3.3.4 $P$-strong convergence

Let $P = (P_n)$ be an approximate projection in the unitial Banach algebra $A$. A sequence $(A_n) \subset A$ converges $P$-strongly to $A \in A$ if, for all $K \in A(P, K)$, both $\|(A_n - A) K\|$ and $\| K (A_n - A) \|$ tend to $0$ as $n \to \infty$. In this case we write $A_n \to A$ $P$-strongly or $A = P$-$\text{lim}_n A_n$.

As for Proposition 1.16 we check that a bounded sequence $(A_n) \subset A$ converges $P$-strongly to $A \in A$ if and only if $\|(A_n - A) P_m\| \to 0$ and $\| P_m (A_n - A) \|$ $\to 0$ for every fixed $P_m \in P$.

**Definition 3.32.** By $F(A, P)$ we denote the set of all bounded sequences $(A_n) \subset A$, which possess a $P$-strong limit in $A(P)$. Furthermore, say that $P$ is an approximate identity if there is a universal constant $D_P > 0$ such that $\|A\| \leq D_P \sup_n \|A_n\|$ for every sequence $(A_n) \in F(A, P)$ and the respective $P$-strong limit $A$.

**Remark 3.33.** Notice that $P$ is an approximate identity on $A$ if and only if there is a universal constant $D > 0$ such that $\|A\| \leq D \sup_m \|P_m A\|$ for every $A \in A(P)$. Indeed, the implication only if is clear with $A_n := P_n A$, and the if part follows with $(A_n) \in F(A, P)$ from

$$\|A\| \leq D \sup_{m \in \mathbb{N}} \|P_m A\| = D \sup_{m \in \mathbb{N}} \|P_m A_n\| \leq D \sup_{m \in \mathbb{N}} \|P_m A_n\| \leq D \sup_{m \in \mathbb{N}} \|P_m\| \sup_{n \in \mathbb{N}} \|A_n\|.$$

\[\text{That is, it is independent of the sequence} \ (A_n).\]
3.3. $\mathcal{P}$-ALGEBRAS

**Theorem 3.34.** Let $\mathcal{P}$ be an approximate identity on the Banach algebra $A$ with identity $I$.

- If $(A_n) \subset A(\mathcal{P})$ converges $\mathcal{P}$-strongly to $A \in A$ then $(A_n)$ is bounded and $A \in A(\mathcal{P})$, i.e. $(A_n) \in \mathcal{F}(A, \mathcal{P})$.

- The $\mathcal{P}$-strong limit of every $(A_n) \in \mathcal{F}(A, \mathcal{P})$ is uniquely determined.

- Provided with the operations $\alpha(A_n) + \beta(B_n) := (\alpha A_n + \beta B_n)$, $(A_n)(B_n) := (A_n B_n)$, and the norm $\|(A_n)\| := \sup_n \|A_n\|$, $\mathcal{F}(A, \mathcal{P})$ becomes a Banach algebra with identity $I := (I)$.

The mapping $\mathcal{F}(A, \mathcal{P}) \to A(\mathcal{P})$ which sends $(A_n)$ to its limit $A = \mathcal{P}$-$\lim_n A_n$ is a unital algebra homomorphism and

$$\|A\| \leq D_{\mathcal{P}} \liminf_{n \to \infty} \|A_n\|. \quad (3.13)$$

- Every approximate projection $\mathcal{P}$ which is equivalent to $\mathcal{P}$ forms an approximate identity and $\mathcal{F}(A, \mathcal{P}) = \mathcal{F}(A, \mathcal{P})$ holds true.

- Let $\mathcal{P}$ be uniform. If $(A_n) \in \mathcal{F}(A, \mathcal{P})$ has an invertible limit $A$ and $(B_n) \subset A$ is a bounded sequence such that $(A_n B_n), (B_n A_n)$ converge $\mathcal{P}$-strongly to the identity, then $(B_n)$ belongs to $\mathcal{F}(A, \mathcal{P})$ with the limit $A^{-1}$.

**Proof.** For the first assertion let $K \in \mathcal{A}(\mathcal{P}, K)$. Then we have $A_n K, K A_n \in \mathcal{A}(\mathcal{P}, K)$ and further $\|A_n K - KA\|, \|K A_n - KA\| \to 0$. Since $\mathcal{A}(\mathcal{P}, K)$ is closed, this implies that $A K$ and $K A$ belong to $\mathcal{A}(\mathcal{P}, K)$, i.e. $A$ is in $\mathcal{A}(\mathcal{P})$. Assume that $(A_n)$ is unbounded. Set $B_{\mathcal{P}} := \sup_n \|P_n\|$ and $m_0 := 1$, and successively choose $m_{i-1} < k_i < l_i < p_i < m_i$ for all $i \in \mathbb{N}$ as follows: Given $m_{i-1}$ fix $k_i, l_i, p_i, m_i$ such that $\|Q_{k_i} A_{l_i}\| \geq 2D_{\mathcal{P}} B_{\mathcal{P}}^3$. This is possible since $(A_n)$ is unbounded and $(P_{m_i} A_n)_{n}$ is bounded, hence $(Q_m A_n)_{n}$ is unbounded for every fixed $m$. Since $\mathcal{P}$ is an approximate identity there is an $n$ such that $\|Q_{k_i} A_{l_i}\| \leq 2D_{\mathcal{P}} \|P_n Q_{k_i} A_{l_i}\|$ by Remark 3.33. Now choose $p_i \gg n, p_i \gg l_i$ and $m_i \gg p_i$ such that

$$\|Q_{k_i} A_{l_i}\| \leq 2D_{\mathcal{P}} \|P_{p_i} P_{m_i} A_{l_i}\| \leq 2D_{\mathcal{P}} B_{\mathcal{P}} \|P_{p_i} Q_{k_i} A_{l_i}\|,$$

hence $\|P_{p_i} Q_{k_i} A_{l_i}\| \geq i^3$. The series $\sum_{i=1}^{\infty} n^{-i} P_{m_{i+1}}$ converges absolutely, hence it defines an element $P$ in the Banach algebra $A$. This element easily proves to be $\mathcal{P}$-compact. Thus, $(P A_n)_{n}$ converges to $PA$ in the norm. Therefore $(P A_n)_{n}$ is bounded, which contradicts

$$\|P A_n\| \geq \frac{\|Q_{k_i} P_{p_i} P A_{l_i}\|}{\|Q_{k_i} P_{p_i}\|} \geq \frac{n^{-i} \|Q_{k_i} P_{p_i} P_{m_i} A_{l_i}\|}{(B_{\mathcal{P}} + 1)B_{\mathcal{P}}} = \frac{\|Q_{k_i} P_{p_i} A_{l_i}\|}{n^i (B_{\mathcal{P}} + 1)B_{\mathcal{P}}} \geq \frac{n}{(B_{\mathcal{P}} + 1)B_{\mathcal{P}}}.$$

The second assertion is quite obvious: If $\|P_m (A_n - A)\|$ and $\|P_m (A_n - B)\|$ tend to zero as $n$ goes to infinity for every fixed $m$, then $P_m (A - B) = 0$ for every $m$, hence $A - B = 0$ by Remark 3.33. The proof of $\mathcal{F}(A, \mathcal{P})$ being a normed algebra is straightforward, and we only note that if $(A_n), (B_n)$ are bounded and converge $\mathcal{P}$-strongly to $A, B \in A(\mathcal{P})$, respectively, then

$$\|K(A_n B_n - AB)\| \leq \|K(A_n - A)B_n\| + \|KA(B_n - B)\| \to 0$$

for every $K \in \mathcal{A}(\mathcal{P}, K)$, as $n \to \infty$, since $A \in \mathcal{A}(\mathcal{P})$ implies $KA \in \mathcal{A}(\mathcal{P}, K)$. The estimate (3.13) follows from the definition, since we can replace $(A_n)$ by one of its subsequences which realize the limit inf and then cancel arbitrarily but finitely many elements at the beginning of this subsequence. Finally, let $((C_n^m))_{m}$ be a Cauchy sequence of sequences $(C_n^m) \in \mathcal{F}(A, \mathcal{P})$, where $C_n^m$ shall denote the $\mathcal{P}$-strong limit of $(C_n^m)$, respectively. For every $n$, $(C_n^m)_{m}$ converges in $A$ to an element $C_n$, and the sequence $(C_n)$ is uniformly bounded. Furthermore, (3.13) yields that $(C_n^m)$ is a Cauchy
sequence with a limit $C \in \mathcal{A}(\mathcal{P})$. Now one easily checks that $(C_n)$ converges $\mathcal{P}$-strongly to $C$, thus $\mathcal{F}(\mathcal{A}, \mathcal{P})$ is complete.

For the fourth assertion apply the fact that $\mathcal{A}(\mathcal{P}, K) = \mathcal{A}(\hat{\mathcal{P}}, K)$, as well as Remark 3.33 with $\|A P_m\| \leq \|A F_k\|\|P_m\|$ and $\|A F_n\| \leq \|A P_k\|\|F_n\|$ for fixed $m$ and sufficiently large $k$.

Given the sequence $(B_n)$ in the last assertion, we set $B := A^{-1}$, find that $B \in \mathcal{A}(\mathcal{P})$ by Theorem 3.28, and consequently $K(B_n - B) = KB(A - A_n)B_n + KB(A_nB_n - I)$ tend to zero in the norm as $n \to \infty$ for every $K \in \mathcal{A}(\mathcal{P}, K)$. The same holds true for $(B_n - B)K$ and thus $(B_n) \in \mathcal{F}(\mathcal{A}, \mathcal{P})$ with limit $B$. \hfill \square

### 3.3.5 A look into the crystal ball

Notice that it would have been sufficient for this thesis to prove these theorems for the algebras $\mathcal{L}(\mathbf{X}, \mathcal{P})$ of operators in Part 1. Nevertheless, one argument for the present Section 3.3 is given by its simplicity and beauty. Working on this abstract level reveals that these results are purely Banach-algebraic. There is no need for additional tools or notions coming from function analysis or operator theory.

Another reason is that there is a natural connection between the local principle described in Section 3.1 and the present $\mathcal{P}$-algebraic framework. This will not be elaborated here in more detail, as it is not important for the classes of operators we have in mind here. We only mention that there are further families of operators and respective discretization methods in the literature - such as finite sections of Toeplitz operators having piecewise continuous symbol (see [29], [74]), or Cauchy singular integral operators and the collocation method (see [37], [38]) - for which the situation is much more involved, since there one needs to consider the algebras “modulo” $\mathcal{G}$ instead of $\mathcal{G}$ to achieve a sufficiently large center for the localization. Maybe the following observation can give a new perspective onto these topics and can contribute some better understanding of inverse closedness:

Let $\mathcal{A}$ be a unital Banach algebra and $\mathcal{C}$ be a commutative $C^*$-subalgebra, not necessarily in the center. We again think of the elements $\varphi \in \mathcal{C}$ as continuous functions on $\mathcal{M}_\mathcal{C}$. Now, for a fixed $x \in \mathcal{M}_\mathcal{C}$, we suppose that there is a sequence $(\varphi_n^x) \subset \mathcal{C}$ of functions of norm one, being equal to $1$ in a neighborhood of $x$ and having nested and contracting supports in the sense that $\varphi_{n+1}^x = \varphi_n^x\varphi_n^{x+1}$ and, for every $\psi \in \mathcal{C}$ which is equal to one in a neighborhood of $x$, there is an $n$ such that $\varphi_n^x \psi = \varphi_n^x$. Then $\mathcal{P}^x := (L_n^x)$ with $L_n^x := 1 - \varphi_n^x$ is a uniform approximate projection on $\mathcal{A}$ and an element $A \in \mathcal{A}$ is $\mathcal{P}^x$-compact iff it is locally equivalent to zero in $x$ in the sense that the norms $\|\psi A\|$ and $\|A \psi\|$ are small for functions $\psi$ with $\|\psi\| = \psi(x) = 1$ and small support. Denote the set of such elements by $\mathcal{J}_x$.

$\mathcal{A}(\mathcal{P}^x)$ is the subset of $\mathcal{A}$ for which $\mathcal{J}_x$ gives an ideal.

Furthermore, $A \in \mathcal{A}(\mathcal{P}^x)$ is $\mathcal{P}^x$-Fredholm iff there is a $B \in \mathcal{A}$ such that $I$, $AB$ and $BA$ are locally equivalent in $x$. In this case $B$ automatically belongs to $\mathcal{A}(\mathcal{P}^x)$, too. Moreover, for given $A \in \mathcal{A}(\mathcal{P}^x)$ we can replace $\mathcal{P}^x$ by an approximate projection, consisting of functions in $\mathcal{C}$ again, which asymptotically commutes with $A$.

Probably, these observations could lead to another abstract local principle with lower requirements on commutativity, in a sense. At the moment this is just a conjecture, but it is to be hoped that some future work will clear it up.
Part 4

Applications

Here we reap the fruit of our labor. We particularly want to demonstrate that the algebraic framework of the previous parts unifies several approaches which have been considered in the past, recovers lots of former results for special classes of operators and permits to extend them. Basically, these extensions are made into three directions:

1. Include more exotic cases such as $p = 1, \infty$ which stayed out of reach in most applications until now, and obliterate differences to the Hilbert space case and the case of reflexive Banach spaces with *-strong convergence.
2. Cover larger classes of operators, mainly by the notion of rich sequences.
3. State some new results, particularly on spectral approximation.

Firstly, we pick up again the operators of Section 1.4.4, being quasi-diagonal with respect to a certain sequence $(L_n)$ of projections, and we have a look at their finite sections with respect to $(L_n)$.

Section 4.2 is devoted to band-dominated operators on $L^p$-spaces and the whole bundle of results which we already know for their discrete analogues on $l^p$. We particularly rediscover the index formula for locally compact operators of Rabinovich, Roch and Roe [59], [58] and extend it into several directions.

The multi-dimensional convolution operators, which are subject of Section 4.3, have been intensively studied by Mascarenhas and Silbermann [48], [49], [50]. The present approach now permits to modify their proofs and to obtain the results for larger classes of such operators on a larger scale of spaces.

In the final Section 4.4 we briefly address some more concrete subclasses of $A_{lp}$ and, moreover, we want to catch a glimpse of further relatives beyond the world of band-dominated operators.
4.1 Quasi-diagonal operators

**Definition 4.1.** Let $X$ be a Banach space. Let further $A \in \mathcal{L}(X)$ and assume that there is a bounded sequence $(L_n)$ of projections on $X$, such that

$$\|[A, L_n]\| := \|AL_n - L_n A\| \to 0 \quad \text{as} \quad n \to \infty. \quad (4.1)$$

Then $A$ is said to be quasi-diagonal with respect to $(L_n)\,^1$.

**Proposition 4.2.** Let $A \in \mathcal{L}(X)$ be a quasi-diagonal operator w.r.t. $(L_n)$.

1. If $A$ is invertible and $\|[A, L_n]\| < (\|L_n\|^2\|A^{-1}\|)^{-1}$ then $L_n AL_n$ is invertible in $\mathcal{L}(\text{im} L_n)$.

2. If $A$ is invertible, $\|[A, L_n]\| < (\|L_n\|^2\|A^{-1}\|)^{-1}$, $y \in X$ is given, $x \in X$ is the unique solution to $Ax = y$, and $x_n \in \text{im} L_n$ is the unique solution to $L_n AL_n x_n = L_n y$, then

$$\|x - x_n\| \leq \|A^{-1}\| (\|[A, L_n]\|\|x_n\| + \|L_n y - y\|).$$

**Proof.** This proof is a slight modification of the one given with [14], Proposition 4.2. The estimate

$$\|L_n AL_n A^{-1}L_n - L_n\| = \|L_n(\|L_n A - L_n A\|A^{-1}L_n)\| \leq \|L_n\|^2\|A^{-1}\|\|[A, L_n]\| < 1 \quad (4.2)$$

yields the invertibility of $L_n AL_n A^{-1}L_n$ by a Neumann argument, hence the invertibility of $L_n AL_n$ from the right. With a similar estimate the invertibility from the left can be shown as well. The second assertion is also straightforward with $x_n = L_n x_n$ and:

$$\|x_n - x\| \leq \|A^{-1}\|\|A(x_n - x)\| = \|A^{-1}\|\|AL_n x_n - L_n AL_n x_n + L_n y - y\|$$

$$\leq \|A^{-1}\| (\|A L_n - L_n A\|\|x_n\| + \|L_n y - y\|). \quad \square$$

**Corollary 4.3.** Let $A \in \mathcal{L}(X)$ be a quasi-diagonal operator w.r.t. $(L_n)$ and let $y \in X$. If $A$ is invertible and $\|(I - L_n)y\| \to 0$ as $n \to \infty$ then $(L_n AL_n)$ is stable and the solutions $x_n$ of the equations $L_n AL_n x_n = L_n y$ converge in the norm to the solution $x$ of $Ax = y$.

**Proof.** It only remains to mention that for all $n$ such that $\|[A, L_n]\| < (2\|L_n\|^2\|A^{-1}\|)^{-1}$ we even get the estimate 1/2 at the right hand side of (4.2), and easily find that $\|(L_n AL_n A^{-1}L_n)^{-1}\| \leq 2$. Therefore the right inverses of $L_n AL_n$ (and also the left inverses) are uniformly bounded. \square

**Remark 4.4.** The notion of quasi-diagonality is due to Halmos [28] and was initially studied for bounded linear operators $A$ on a separable Hilbert space $X$. There, an operator $A \in \mathcal{L}(X)$ was said to be quasi-diagonal, if there was a sequence $(L_n)$ such that $A$ satisfied (4.1), and the projections $(L_n)$ were additionally supposed to be compact and orthogonal, and to converge strongly to the identity.

In such a setting, Halmos observed that $A$ is quasi-diagonal (in this stricter sense) if and only if $A = B + K$ where $B$ is a block diagonal operator with respect to some orthonormal basis of $X$ and $K$ is compact. This means that there exist compact and orthogonal projections $\tilde{L}_n$ tending strongly to $I$ which asymptotically commute with $A$ and even fulfill $L_{n+1}L_n = L_n L_{n+1} = L_n$ for every $n$.

Berg [3] pointed out that every normal operator is quasi-diagonal in this stricter sense, hence also every self-adjoint operator. So, this class contains a lot of interesting examples like almost Mathieu operators, discretized Hamiltonians, difference operators originated from PDE’s with constant coefficients, certain weighted shifts, etc. We will not go into great detail here, but we

\[1\text{This definition is slightly more general than Definition 1.52.}\]
4.1. QUASI-DIAGONAL OPERATORS

refer to the paper of Brown [14] which extensively treats the Hilbert space case, including the discussion of convergence rates for certain classes of quasi-diagonal operators and the definition of some explicit algorithms for two special cases. That paper is heavily based on the results of the Standard Algebra approach in [30], which provides the machinery to describe stability and several asymptotic properties of operator sequences in a Hilbert space setting. Of course, the approach of the present text suggests the expansion to the Banach space case, and this is what we do in the following.

The preceding observations reveal that a sequence of projections $(L_n)$ which meets the requirements given in the definition of a quasi-diagonal operator $A$ are fairly good candidates for the constitution of appropriate finite sections $L_nAL_n$. So, in what follows, we additionally assume that $(L_n)$ is a sequence of $\mathcal{P}$-compact projections tending $\mathcal{P}$-strongly to the identity, where $\mathcal{P} = (P_n)$ is a uniform approximate identity on $X$, which equips $X$ with the $\mathcal{P}$-dichotomy. Our aim is to apply the results of the general theory of Parts 2 and 3 and to improve Corollary 4.3.

**Proposition 4.5.** The set $QD_{\mathcal{P},(L_n)}$ of all operators which are quasi-diagonal w.r.t. $(L_n)$ is a Banach subalgebra of $\mathcal{L}(X,\mathcal{P})$ and contains the ideal $K(X,\mathcal{P})$. If $A \in QD_{\mathcal{P},(L_n)}$ is $\mathcal{P}$-Fredholm and $B$ is a $\mathcal{P}$-regularizer for $A$ then $B \in QD_{\mathcal{P},(L_n)}$. In particular, $QD_{\mathcal{P},(L_n)}$ is inverse closed in $\mathcal{L}(X,\mathcal{P})$.

**Proof.** Notice that for every fixed $m$ and arbitrary $k$

$$
\|(I - P_n)AP_m\| \leq \|(I - P_n)L_kAP_m\| + \|(I - P_n)(I - L_k)AP_m\|
$$

$$
\leq \|(I - P_n)L_kAP_m\| + \|(I - P_n)A(I - L_k)P_m\| + \|(I - P_n)[A, I - L_k]P_m\|
$$

$$
\leq \|(I - P_n)L_k\|[A]||C_P + (C_P + 1)||A\||(I - L_k)P_m||
$$

$$
+ (C_P + 1)C_P||[A, L_k]||,
$$

where the second and third term of the last estimate can be made arbitrarily small by choosing $k$ large, and the first term tends to zero as $n \to \infty$ for every fixed $k$, hence $\|(I - P_n)AP_m\| \to 0$ as $n \to \infty$ (and $\|P_nA(I - P_n)\| \to 0$ by an analogous estimate, too). Thus $QD_{\mathcal{P},(L_n)}$ is a subset of $\mathcal{L}(X,\mathcal{P})$. One easily checks that $QD_{\mathcal{P},(L_n)}$ is a closed subalgebra. For $K \in K(X,\mathcal{P})$ we have $\|K - P_lKP_l\| \to 0$ as $l \to \infty$, and for all $l$

$$
\|P_lKP_lL_n\| = \|(I - L_n)P_lKP_l - P_lKP_l(I - L_n)\| \leq \|P_l\|\|K\|\|(I - L_n)P_l\| + \|P_l(I - L_n)\|
$$

vanishes for $n \to \infty$. Thus $K \in QD_{\mathcal{P},(L_n)}$.

Let $A$ have a $\mathcal{P}$-regularizer $B$. Then $B \in \mathcal{L}(X,\mathcal{P})$ by Theorem 1.14 and further $I - AB, I - BA \in K(X,\mathcal{P})$, hence

$$
BL_n - L_nB = B(L_nA - AL_n)B + BL_n(I - AB) - (I - BA)L_nB
$$

$$
= B(L_nA - AL_n)B + B(L_n - I)(I - AB) - (I - BA)(L_n - I)B,
$$

which also tends to zero in the norm as $n \to \infty$. This finishes the proof.

**Remark 4.6.** At this juncture we want to point out again that the underlying general theory of the Parts 2 and 3 and all proofs also work in the classical setting (without $\mathcal{P}$). More precisely, the subsequent theorems remain true, if $(L_n)$ is a sequence of compact projections which converge $*$-strongly to the identity. In this case there is no need for an approximate projection $\mathcal{P}$ and we can replace $\mathcal{L}(X,\mathcal{P})$ by $\mathcal{L}(X)$, $K(X,\mathcal{P})$ by $K(X)$, $\mathcal{P}$-compact by compact, $\mathcal{P}$-Fredholm by Fredholm, $\mathcal{P}$-strong convergence by $*$-strong convergence and set $B_P := 1$.

The finite section sequences $(L_nAL_n)$ of quasi-diagonal operators $A$ obviously converge $\mathcal{P}$-strongly to $A$. So, let $\mathcal{F}_{QD}$ be the smallest closed subalgebra of $\mathcal{F}$ containing all sequences $(L_nAL_n)$ with $A \in QD_{\mathcal{P},(L_n)}$, where $\mathcal{F}$ is the Banach algebra of all bounded sequences $\{A_n\}$ with $A_n \in \mathcal{L}(im L_n)$. Notice that $\mathcal{W}(\mathcal{A}) := \mathcal{P}$-lim$_n A_nL_n$ exists for every sequence $\mathcal{A} = \{A_n\} \in \mathcal{F}_{QD}$. 


Theorem 4.7. A sequence $\mathcal{A} = \{A_n\} \in \mathcal{F}_{QD}$ is Fredholm if and only if $W(\mathcal{A})$ is Fredholm. In this case $\alpha(\mathcal{A}) = \dim \ker W(\mathcal{A})$ and $\beta(\mathcal{A}) = \dim \coker W(\mathcal{A})$ are splitting numbers and $\lim_n \text{ind} A_n = \text{ind} W(\mathcal{A})$.

If $W(\mathcal{A})$ is not Fredholm then infinitely many approximation numbers $s_k^{\mathcal{A}}(A_n)$ tend to zero as $n$ goes to infinity. A sequence $\mathcal{A} \in \mathcal{F}_{QD}$ is stable if and only if $W(\mathcal{A})$ is invertible.

Proof. Let $T := \{0\}$ and let $\mathcal{F}^T \subset \mathcal{F}$ denote the Banach algebra of all sequences $\mathcal{A} = \{A_n\}$ such that $(A_nL_n)$ converges $\mathcal{P}$-strongly. Denote the limit again by $W(\mathcal{A})$. Further, introduce the ideal $\mathcal{J}^T := \{(L_nKL_n) + \{G_n\} : K \in \mathcal{K}(X, \mathcal{P}), \|G_n\| \to 0\}$. Then $\mathcal{F}_{QD} \subset \mathcal{F}^T$.

The set $\mathcal{C} := \{\alpha I : \alpha \in \mathbb{C}\}$ is a commutative closed subalgebra of $\mathcal{F}^T$, the respective snapshots are multiples of the identity, $\mathcal{C}/\mathcal{G}$ turns into a $C^*$-algebra (provided with $(\alpha I + \mathcal{G})^* := \overline{\alpha I} + \mathcal{G}$) with $T$ as maximal ideal space. With $M = T$ and $\mathbb{F}^0 := 1$ it is clear that $\mathcal{C}$ defines a globally localizing setting.

For $\mathcal{A} := \{L_nAL_n\}$ with $A \in QD_{\mathcal{P},(L_n)}$ we have that $W(\mathcal{A}) = A$, hence this snapshot is liftable and almost commuting with $\mathbb{F}^0$. The coset $\mathcal{A} + \mathcal{G}$ as well as the lifting $\mathcal{A}^0 + \mathcal{G}$ belong to the commutant of $\mathcal{C}/\mathcal{G}$, and it is also obvious that $\mathcal{A} = \mathcal{A}^0$, i.e. they are locally equivalent in 0. Consequently, the whole algebra $\mathcal{F}_{QD}$ is contained in $\mathcal{C}^T = \mathcal{T}^T$ by Proposition 3.8.

Now apply Theorem 3.9 to prove that $\mathcal{A}$ is $\mathcal{J}^T$-Fredholm if and only if $W(\mathcal{A})$ is $\mathcal{P}$-Fredholm, and to derive the asserted criteria for the Fredholm property and the stability. Theorems 2.13, 2.14 and 2.9 provide the rest. \hfill $\square$

Theorem 4.8. Let $\mathcal{A} = \{A_n\} \in \mathcal{F}_{QD}$ and assume that $B_\mathcal{P} = \lim_n \|L_n\| = 1$. Then

$$\lim_{n \to \infty} \|A_n\| = \|W(\mathcal{A})\|, \quad \lim_{n \to \infty} \|A_n^{-1}\| = \|(W(\mathcal{A}))^{-1}\|, \text{ hence } \lim_{n \to \infty} \text{cond}(A_n) = \text{cond}(W(\mathcal{A})),$$

the latter only for stable $\mathcal{A}$. For every $0 < \alpha < \epsilon < \beta$ there is an $N_0 \in \mathbb{Z}_+$ such that

$$B_\alpha (\text{sp} W(\mathcal{A})) \subset \omega\text{-lim} \ sp_{N,\epsilon} A_n \subset p\text{-lim} \ sp_{N,\epsilon} A_n \subset B_\beta (\text{sp} W(\mathcal{A}))$$

for every $N \geq N_0$. If, additionally, $X$ or its dual is complex uniformly convex then, for every $\epsilon > 0$ and every $N \in \mathbb{Z}_+$, the $(N,\epsilon)$-pseudospectra of the operators $A_n$ converge w.r.t. the Hausdorff distance to the $(N,\epsilon)$-pseudospectrum of $W(\mathcal{A})$.

Proof. Clearly, this follows from Proposition 3.11 and Corollaries 3.12, 3.19 and 3.20 if $\mathcal{C}$ defines a faithful localizing setting, which is almost trivial if we take Remark 3.13 into account. We only note that two functions on $T$ have disjoint support iff at least one of them equals zero. Thus $\mathcal{T}^T/\mathcal{G}$ is a KMS-algebra w.r.t. $\mathcal{C}/\mathcal{G}$. \hfill $\square$
4.2 Band-dominated operators on $L^p(\mathbb{R}, Y)$

Let $Y$ be a Banach space. Here we consider the spaces $L^p = L^p(\mathbb{R}, Y)$, with $1 \leq p < \infty$, of all (equivalence classes of) Lebesgue measurable functions $f : \mathbb{R} \to Y$ for which the $p$-th power of $\|f\|_Y$ is Lebesgue integrable. By $L^\infty = L^\infty(\mathbb{R}, Y)$ we further denote the space of all (equivalence classes of) Lebesgue measurable and essentially bounded functions $f : \mathbb{R} \to Y$. Note that the Banach space structure is given by pointwise addition and scalar multiplication as well as the usual Lebesgue integral norm (or the essential supremum norm in the case $p = \infty$). In the same vein, we also define the spaces $L^p(\mathbb{R}^+, Y)$ and $L^p(\mathbb{R}^-, Y)$ of functions over the half axes.

Let $1 \leq p \leq \infty$ and let $P_n$ stand for the operator of multiplication by the characteristic function of the interval $[-n, n]$ acting on $L^p$. Then $\mathcal{P} := (P_n)$ forms a uniform approximate identity and $L^p$ has the $\mathcal{P}$-dichotomy. To prove the latter, we note that in case $1 \leq p < \infty$ the dual space $(L^p(\mathbb{R}, Y))^*$ can be identified with $L^q(\mathbb{R}, Y^*)$ where $1/p + 1/q = 1$, i.e. $\mathcal{P}^*$ is an approximate identity on $(L^p)^*$, and the space $L^\infty$ is complete w.r.t. $\mathcal{P}$-strong convergence. Now Theorem 1.27 applies.

For $\alpha \in \mathbb{R}$ we consider the operator

$$U_\alpha : L^p \to L^p, \quad (U_\alpha f)(t) := f(t - \alpha)$$

of shift by $\alpha$. Let $A \in \mathcal{L}(L^p, \mathcal{P})$ and $h = (h_n) \subset \mathbb{R}$ be a sequence with $|h_n| \to \infty$ as $n \to \infty$. The operator $A_h$ is called limit operator of $A$ with respect to $h$ if

$$U_{-h_n} A U_{h_n} \to A_h \quad \mathcal{P}\text{-strongly.}$$

The set $\sigma_{op}(A)$ of all limit operators of $A$ is again called the operator spectrum of $A$. Finally, an operator $A$ is called rich if every sequence $h \subset \mathbb{Z}$ whose absolute values tend to infinity has an infinite subsequence $g$ such that $A_g$ exists.

One may ask why the definition of rich operators is exclusively based on sequences of integers. The reason is that it would be a much (unnecessarily) stronger condition if all sequences of real numbers shall have a subsequence which leads to a limit operator, and the “set of rich operators” would dramatically shrink. Lindner gave a comprehensive explanation of this phenomenon in [44], Section 3.4.13. Nevertheless, one can replace $\mathbb{Z}$ in this definition by any other “two-sided infinite” set of isolated points (cf. Section 4.2.2).

**Definition 4.9.** An operator $A \in \mathcal{L}(L^p, \mathcal{P})$ is said to be band-dominated if, for every $\varphi \in \text{BUC}$,

$$\lim_{t \to 0} \sup_{r \in \mathbb{R}} \| [A, \varphi_{t, r}] I \| = 0.$$

Let $\mathcal{A}_{L^p}$ denote the set of all such operators.

Notice that this definition is in line with the class of (discrete) band-dominated operators as the third item of Theorem 1.55 shows, and also the treatment of the finite sections is very similar. Nevertheless we want to present a standalone consideration here and avoid to play on the results in the discrete setting which are scattered all over the whole text. In the very end of this section we finish with a short discussion of the connections between the discrete and the continuous case.

### 4.2.1 Standard finite sections

Set $L_n := P_n$, further define $E_n := \text{im } L_n$ and let $\mathcal{F}_{A_{L^p}} \subset \mathcal{F}$ denote the Banach algebra which is generated by all sequences $\{L_n A L_n\}$ with rich $A \in \mathcal{A}_{L^p}$. Clearly, for every sequence $\mathbb{A} = \{A_n\}$ in $\mathcal{F}_{A_{L^p}}$, the limit

$$W(\mathbb{A}) := \mathcal{P}\text{-lim } n \to \infty A_n L_n$$

is again a rich operator in $\mathcal{A}_{L^p}$.
exists. In this way, a first natural snapshot is defined and, as in the discrete case, there are two further points of interest, namely where the operators $L_n$ perform the truncation. More precisely, for a given sequence $h = (h_n)$ of positive integers tending increasingly to infinity we set $E_{h_n} := \text{im} L_{h_n}$, $T := \{-1, 0, +1\}$, $I^0 := I$, $I^\pm := \chi_{\mathbb{R}_\pm} I$ and

$$
E^0 := L^p(\mathbb{R}, Y) \quad \quad L^0_{h_n} := L_{h_n} \\
E^h_{h_n} := \text{im} L^h_{h_n} \quad \quad E^0_{h_n} : \text{L}(\text{im} L^0_{h_n}) \to \text{L}(E_{h_n}), B \mapsto B \\
E^h_{h_n} := U_{\pm h_n} L_{h_n} U_{\pm h_n} \quad \quad E^\pm_{h_n} : \text{L}(\text{im} L^{\pm}_{h_n}) \to \text{L}(E_{h_n}), B \mapsto U_{\pm h_n} B U_{\pm h_n}
$$

for every $n$. By $\mathcal{P}^i := (L^i_{h_n})$ uniform approximate identities on $E^i$ are given such that the $E^i$ have the $\mathcal{P}^i$-dichotomy and the sequences $(L^i_{h_n})$ converge $\mathcal{P}^i$-strongly to the identities $I^i$ on $E^i$. As usual, we let $\mathcal{F}^T_\mathcal{H}$ denote the Banach algebra of all bounded sequences $\{A_n\}$ of bounded linear operators $A_n \in \text{L}(E_{h_n})$ for which there exist operators $W^T \{A_n\} \in \text{L}(E^T_\mathcal{H})$ for each $t \in T$ such that

$$
E^h_{h_n}(A_n) L^h_{h_n} \to W^T \{A_n\} \quad \mathcal{P}^i\text{-strongly as } n \to \infty.
$$

Furthermore, introduce $\mathcal{H}_\mathcal{A}$ as the set of all strictly increasing sequences $h$ of positive integers such that $W^+(\mathcal{A}_h)$ and $W^-(\mathcal{A}_h)$ exist. Of course, for $\mathcal{A} = \{I_n, AL_n\}$ with rich $A \in A_{LP}$ and a strictly increasing sequence $h$ there is always a subsequence $g \in \mathcal{H}_\mathcal{A}$ of $h$. Since the set $\mathcal{R}^T$ of all rich sequences in $\mathcal{F}$ is a closed algebra (see Theorem 2.26), we see that every $\mathcal{A} \in \mathcal{F}_A$ is rich.

Proposition 4.10. Let $\mathcal{A} \in \mathcal{F}_A$ and $g \in \mathcal{H}_\mathcal{A}$. Then there is a subalgebra $C \subset \mathcal{F}^T_\mathcal{H}$ which defines a faithful globally localizing setting in $\mathcal{F}^T_\mathcal{H}$ and which turns $\mathcal{A}_g$ into a $C$-localizable sequence in $\mathcal{T}^T_\mathcal{A}$. In other words: $\mathcal{F}_A \subset \mathcal{T}^T_\mathcal{A}$.

Proof. ² Firstly, let us prove that $\sigma_{\text{op}}(B) \subset A_{LP}$ for every $B \in A_{LP}$. Let $C \in \sigma_{\text{op}}(B)$. From Theorem 1.19 we infer that $C \in \text{L}(L^p, \mathcal{P})$, but we assume that there is a function $\psi \in C[-1, 1]$, a sequence $(t_n) \in \mathbb{R}$, $t_n > 0$ and tending to zero, such that

$$
C := \lim_{n \to \infty} \sup_{r \in \mathbb{R}} \| [C, \psi_{t_n,r}] \| = \lim_{n \to \infty} \sup_{r \in \mathbb{R}} \| [U_rCU_{-r}, \psi_{t_n,r}] \| > 0.
$$

Clearly, we can choose a sequence $(r_n)$ such that $C = \lim_{n} \| [U_nCU_{-n}, \psi_{t_n,n}] \|$, and for all sufficiently large $n$ we fix an integer $m_n$ with $C/2 < \| [U_nCU_{-n}, \psi_{t_n,r}] P_{m_n}] \|$. Since $C$ is a limit operator of $B$, we can fix real numbers $s_n$ such that $C/4 < \| [U_nU_{-s_n}BU_{u_n}U_{-r}, \psi_{t_n,r}] P_{m_n}] \|$ for large $n$, hence $C/4 < \| [U_n CU_{-n}, \psi_{t_n,n}] \|$ for all large $n$, contradicting $B \in A_{LP}$.

Let $g \in \mathcal{H}_\mathcal{A}$, set $i_1 := 1$ and construct a subsequence $i := (i_k)$ of $g$ such that

$$
sup\{\| (A_{g_n}^{(t)} - W^t(\mathcal{A}_g))L_k \|, \| L_k(A_{g_n}^{(t)} - W^t(\mathcal{A}_g)) \| : t \in T, g_n \geq i_k \} < \frac{1}{k}
$$

for every $k \geq 2$. Now, for $k \in \mathbb{N}$, define $\gamma_n := k/2$ for all $n \in \{i_k, \ldots, i_{k+1} - 1\}$, and let $b_\gamma : \mathbb{R} \to [-1, 1]$ denote the continuous piecewise linear splines given by $b_\gamma(\pm n) = \pm 1$ and $b_\gamma(\pm(n - \gamma_n)) = \pm 1/2$ which are constant outside the interval $[-n, n]$, respectively. For every continuous function $\varphi \in C[-1, 1]$ we define $\varphi_\gamma := \varphi \circ b_\gamma$, a continuous function over $\mathbb{R}$ (see also Figure 3.1 on page 74). Thus, the sequence $\{\varphi_\gamma L_n\}$ consists of multiplication operators by inflated copies of $\varphi$ with a certain distortion. Since $\| \{\varphi_\gamma L_n\} \| = \| \varphi \|_\infty$ and with the involution $^* : \{\varphi_\gamma L_n\} \mapsto \{\varphi_\gamma L_n\}$ the set

$$
C^\gamma := \{\varphi_\gamma L_n : \varphi \in C[-1, 1]\}
$$

²Here, we proceed in the same manner as in the discrete case (Theorem 3.23).
becomes a commutative $C^*$-subalgebra of $\mathcal{F}^T$ being isometrically isomorphic to $C([-1,1])$, and $W^t(\{\varphi_n^m L_m\}) = \varphi(t)I^t$ for every $\varphi$ and $t \in T$. Moreover, the mapping $\varphi \mapsto \{\varphi_n^m L_n\} + \mathcal{G}$ is an isometric $*$-isomorphism and therefore $C^\gamma/\mathcal{G}$ also proves to be a unital commutative $C^*$-algebra whose maximal ideal space can be identified with the interval $[-1,1]$. For this, we note that for every $m$
\[
\|\varphi\|_\infty = \|\varphi_n^m L_m\| = \limsup_{n \to \infty} \|\varphi_n^m L_n\| = \|\{\varphi_n^m L_n\} + \mathcal{G}\|_{\mathcal{F}/\mathcal{G}}.
\]

Set $M := [-1,1]$. To every $x \in (-1,1)$ we associate the direction $t(x) := 0 \in T$, and for every $x \in [-1,1]$ we fix the function $F^x \in C([-1,1])$ which equals 1 in a neighborhood of $x$, equals 0 in a neighborhood of every $t \in T \setminus \{x\}$ and takes only real values between 0 and 1. Then $C^\gamma$ defines a faithful globally localizing setting. This is immediate from the definition and Remark 3.13 since in the case $p = \infty$,

\[
\|\varphi_n^m A_n \varphi_n^m + \psi_n^m B_n \psi_n^m\| = \max\{\|\varphi_n^m A_n \varphi_n^m\|, \|\psi_n^m B_n \psi_n^m\|\} \leq \max\{\|A_n\|, \|B_n\|\}\|f\| \quad (4.4)
\]

holds for all $f \in E_n$, all $\{A_n\}, \{B_n\} \in \mathcal{F}$ and all $\varphi, \psi \in C([-1,1])$ of norm 1 with disjoint support. For $1 \leq p < \infty$ we replace (4.4) by

\[
\|\varphi_n^m A_n \varphi_n^m + \psi_n^m B_n \psi_n^m\| = \max\{\|\varphi_n^m A_n \varphi_n^m\|, \|\psi_n^m B_n \psi_n^m\|\} \leq \max\{\|A_n\|, \|B_n\|\}\|f\| \quad (4.4)
\]

It remains to show that $\mathcal{A}_g$ belongs to $\mathcal{T}_g^T = \mathcal{L}_g^T$. For this assume that we can show, for every $B \in \mathcal{A}_L^T$, that

\[
\lim_{n \to \infty} \sup_{k \in \mathbb{Z}} \|\{[U^{-k}BUk, \varphi_n]\}\| = 0 \quad \text{for every} \quad \{\varphi_n^m L_n\} \in C^\gamma. \quad (4.5)
\]

Then $\mathcal{F}_L^T/\mathcal{G}$ is obviously contained in the closed subalgebra $\text{Com}(C^\gamma/\mathcal{G})$ of $\mathcal{F}/\mathcal{G}$ which yields that $\mathcal{A}_g + \mathcal{G}_g$ commutes with every element of $C^\gamma/\mathcal{G}_g$. Every snapshot $W^x(\mathcal{A}_g)$ is again band-dominated (or the compression of a band-dominated operator to the space $E^x$) and, due to the choice of the functions $E^x_n$, it is always liftable. Furthermore, by $(4.5)$, it “almost commutes with $\mathbb{R}^x_\sigma^\gamma$ and $\mathcal{A}_g^\gamma + \mathcal{G}_g$ belongs to $\text{Com}(C^\gamma/\mathcal{G}_g)$ as well. Moreover, for every $x \in (-1,1)$, the assumption $(4.5)$ provides that sequences of the form $(I - L_n)B\mathbb{R}_x^\gamma$ and $(F^x_n B(I - L_n))$, with $B$ being band-dominated, tend to zero in the norm as $n \to \infty$, hence $\mathcal{A}_g^\gamma + \mathcal{G}_g \text{ and } \mathcal{A}_g^\gamma + \mathcal{G}_g$ easily prove to be locally equivalent in $x$. If $x = 1$ (or in an analogous way also for $x = -1$) we choose $\varphi \in C([-1,1])$ equal to 1 in a neighborhood of $x$ and identically zero on $[-1,1/2]$ (or $[-1/2,1/2]$, respectively). Then $\{\varphi_n^m L_n\}(\mathcal{A}_g - \mathcal{A}_g^\gamma) \in \mathcal{G}$ by the relation (4.3) which yields the local equivalence in $x$ again and finishes the proof.

So, let us prove the relation (4.5) for a fixed $B \in \mathcal{A}_L^T$. Unfortunately, the $\varphi_n^m$ do not emerge from a linear inflation (as the functions in the Definition 4.9 of band-dominated operators do), since we used the piecewise linear splines $b_n^\gamma$. To tackle this, we simply decompose $\varphi$ into three continuous functions, each of them being non-constant on at most one of the intervals $[-1, -1/2]$, $[-1/2, 1/2]$, $[1/2, 1]$. Then the $b_n^\gamma$ inflate each of these functions linearly, with some additional shift and with disparate speed, and the definition of band-dominated operators easily gives the claim. Indeed, if $\varphi(-1/2) = 0$ then we can split $\varphi$ into the sum two functions $\varphi_1, \hat{\varphi}$ being supported only below or above $-1/2$, respectively. Otherwise, if $\varphi(-1/2) \neq 0$, then we can write the function as the product of two functions $\varphi_1, \tilde{\varphi}$, where $\varphi_1$ is constant above $-1/2$ and $\tilde{\varphi}$ is constant below this critical point. Now do the same with the function $\hat{\varphi}$ to split it into $\varphi_2, \varphi_3$ at the critical point $1/2$. 

\[\square\]
Theorem 4.11. Let $\mathcal{A} = \{A_n\} \in \mathcal{F}_{\mathcal{A}_{L^p}}$ and $g \in \mathcal{H}_A$.

- $\mathcal{A}_g$ is a Fredholm sequence if and only if $W(\mathcal{A}), W^{\pm 1}(\mathcal{A}_g)$ are Fredholm. In this case
  \[
  \lim_{n \to \infty} \text{ind} A_{g_n} = \text{ind} W(\mathcal{A}) + \text{ind} W^{+1}(\mathcal{A}_g) + \text{ind} W^{-1}(\mathcal{A}_g)
  \]  
  (4.6) and the approximation numbers from the right/left of the entries of $\mathcal{A}_g$ have the $\alpha$-$\beta$-splitting property with
  \[
  \alpha = \dim \ker W(\mathcal{A}) + \dim \ker W^{+1}(\mathcal{A}_g) + \dim \ker W^{-1}(\mathcal{A}_g),
  \]
  \[
  \beta = \dim \text{coker} W(\mathcal{A}) + \dim \text{coker} W^{+1}(\mathcal{A}_g) + \dim \text{coker} W^{-1}(\mathcal{A}_g).
  \]
- If one of the snapshots $W(\mathcal{A})$ and $W^{\pm 1}(\mathcal{A}_g)$ is not Fredholm then $\lim_n s^k_n(A_{g_n}) = 0$ or
  $\lim_n s^{\perp n}_k(A_{g_n}) = 0$ for each $k \in \mathbb{N}$.
- $\mathcal{A}_g$ is stable if and only if $W(\mathcal{A})$ and $W^{\pm 1}(\mathcal{A}_g)$ are invertible.

With the notation $\|B^{-1}\| := \infty$ if $B$ is a non-invertible operator, we further have
\[
C_1 := \lim_{n \to \infty} \|A_{g_n}\| = \max \{\|W(\mathcal{A})\|, \|W^{+1}(\mathcal{A}_g)\|, \|W^{-1}(\mathcal{A}_g)\|\},
\]
\[
C_2 := \lim_{n \to \infty} \|A_{g_n}^{-1}\| = \max \{\|W(\mathcal{A})\|^{-1}, \|W^{+1}(\mathcal{A}_g)\|^{-1}, \|W^{-1}(\mathcal{A}_g)\|^{-1}\},
\]
hence, for stable sequences $\mathcal{A}_g$, even $\lim_n \text{cond}(A_{g_n}) = C_1 C_2$ holds true.

In the case $L^p = L^p(\mathbb{R}, \mathbb{C})$ the $(N, e)$-pseudospectra of the operators $A_{g_n}$ converge w.r.t. the Hausdorff distance to the union of the $(N, e)$-pseudospectra of the snapshots.

Proof. Since $\mathcal{A}_g \in \mathcal{T}_g^f$, Theorem 3.9 tells us that $\mathcal{A}_g$ is $\mathcal{T}_g^f$-Fredholm if and only if its snapshots are $\mathcal{P}^f$-Fredholm, it is Fredholm if and only if all snapshots are Fredholm, and it is stable if and only if all snapshots are invertible. Theorems 2.13, 2.14, 2.9, Proposition 3.11 and Corollaries 3.12, 3.19 give the assertions. \hfill $\Box$

For the whole sequences $\mathcal{A} \in \mathcal{F}_{\mathcal{A}_{L^p}}$ we apply Theorem 3.22 and Proposition 3.16 to get

Theorem 4.12. A sequence $\mathcal{A} \in \mathcal{F}_{\mathcal{A}_{L^p}}$ is Fredholm, if and only if all of its snapshots are
Fredholm. In this case
\[
\alpha(\mathcal{A}) = \max_{g \in \mathcal{H}_A} \left( \sum_{t \in T} \text{dim} \ker W^t(\mathcal{A}_g) \right), \quad \beta(\mathcal{A}) = \max_{g \in \mathcal{H}_A} \left( \sum_{t \in T} \text{dim} \text{coker} W^t(\mathcal{A}_g) \right).
\]

A sequence $\mathcal{A} \in \mathcal{F}_{\mathcal{A}_{L^p}}$ is stable if and only if all its snapshots are invertible. Furthermore, for every $\mathcal{A} = \{A_n\} \in \mathcal{F}_{\mathcal{A}_{L^p}}$,
\[
\lim sup_{n \to \infty} \|A_n\| = \max_{g \in \mathcal{H}_A} \max_{t \in T} \|W^t(\mathcal{A}_g)\|, \quad \lim sup_{n \to \infty} \|A_n^{-1}\| = \max_{g \in \mathcal{H}_A} \max_{t \in T} \|(W^t(\mathcal{A}_g))^{-1}\|,
\]
\[
\lim inf_{n \to \infty} \|A_n\| = \min_{g \in \mathcal{H}_A} \min_{t \in T} \|W^t(\mathcal{A}_g)\|, \quad \lim inf_{n \to \infty} \|A_n^{-1}\| = \min_{g \in \mathcal{H}_A} \min_{t \in T} \|(W^t(\mathcal{A}_g))^{-1}\|,
\]
and, for stable $\mathcal{A}$, even
\[
\lim sup_{n \to \infty} \text{cond}(A_n) = \max_{g \in \mathcal{H}_A} \left( \max_{t \in T} \|W^t(\mathcal{A}_g)\| \cdot \max_{t \in T} \|(W^t(\mathcal{A}_g))^{-1}\| \right),
\]
\[
\lim inf_{n \to \infty} \text{cond}(A_n) = \min_{g \in \mathcal{H}_A} \left( \max_{t \in T} \|W^t(\mathcal{A}_g)\| \cdot \max_{t \in T} \|(W^t(\mathcal{A}_g))^{-1}\| \right).
\]

If $L^p = L^p(\mathbb{R}, \mathbb{C})$ and $\mathcal{A} \in \mathcal{F}_{\mathcal{A}_{L^p}}$ then
\[
p-\lim_{n \to \infty} \text{sp}_{N, \epsilon} A_n = \bigcup_{g \in \mathcal{H}_A, t \in T} \text{sp}_{N, \epsilon} W^t(\mathcal{A}_g).
\]
4.2. BAND-DOMINATED OPERATORS ON $L^p(\mathbb{R}, Y)$

4.2.2 Adapted finite sections

Until now, we have ingeniously used $L_n = P_n$ to get the simplest possible finite section method, but we had to pay the price that all operators under consideration needed to be rich. If, on the other hand, $A$ is a (not necessarily rich) band-dominated operator and $l, u$ are unbounded strictly decreasing or increasing sequences of negative/positive real numbers, respectively, such that $A_l, A_u \in \sigma_{sp}(A)$ exist, then one can also set $L_n^{l,u} = \chi_{[l(n), u(n)]} I$ and consider the adapted finite section sequence $\mathcal{A} = \{L_n^{l,u} AL_n^{l,u}\}$. Then naturally, $\mathcal{A}$ is $T$-structured in the respective modified setting $\mathcal{F}^T$, and its snapshots are

$$A, \quad \chi_- A|_{L^p(\mathbb{R}_-)}, \quad \text{and} \quad \chi_+ A|_{L^p(\mathbb{R}_+)},$$

with $\chi_-, \chi_+$ standing for the characteristic functions of the half axes $\mathbb{R}_-$ and $\mathbb{R}_+$, respectively. We again find

- $\mathcal{A}$ is Fredholm if and only if its three snapshots are Fredholm. In this case its $\alpha$- and $\beta$-numbers are determined as the sum of the kernel- (resp. cokernel-) dimensions of the snapshots. Furthermore

$$\lim_{n \to \infty} \text{ind} L_n^{l,u} AL_n^{l,u} = \text{ind} A + \text{ind} \chi_- A|_{L^p(\mathbb{R}_-)} + \text{ind} \chi_+ A|_{L^p(\mathbb{R}_+)}. \quad (4.7)$$

- If one snapshot is not Fredholm then all approximation numbers of the finite sections from at least one side tend to zero.

- $\mathcal{A}$ is stable if and only if its three snapshots are invertible.

- It holds that

$$\lim_{n \to \infty} \|L_n^{l,u} AL_n^{l,u}\| = \max\{\|A\|, \|\chi_- A|_{L^p(\mathbb{R}_-)}\|, \|\chi_+ A|_{L^p(\mathbb{R}_+)}\|\},$$

as well as an analogous equation for the inverses. This also provides a formula for the limit of the condition numbers in case of a stable sequence.

- If $L^p = L^p(\mathbb{R}, \mathbb{C})$ then the $(N, \epsilon)$-pseudospectra of the finite sections converge to the union of the $(N, \epsilon)$-pseudospectra of the snapshots with respect to the Hausdorff metric. For general $L^p(\mathbb{R}, Y)$ we set $S := \text{sp } A \cup \text{sp}(\chi_- A|_{L^p(\mathbb{R}_-)}) \cup \text{sp}(\chi_+ A|_{L^p(\mathbb{R}_+)})$. Then Corollary 3.20 ensures at least, that for every $0 < \alpha < \epsilon < \beta$ there is an $N_0 \in \mathbb{Z}_+$ such that $B_\alpha(S) \subset \lim_{n \to \infty} \text{sp}_{N, \epsilon}(L_n^{l,u} AL_n^{l,u}) \subset \lim_{n \to \infty} \text{sp}_{N, \epsilon}(L_n^{l,u} AL_n^{l,u}) \subset B_\beta(S)$ for every $N \geq N_0$.

In what follows we want to turn our attention to more concrete classes of band-dominated operators on $L^p$.

4.2.3 Locally compact operators

In [58] Rabinovich and Roch studied band-dominated operators of the form $A = I + K$ where $K$ is locally compact. At this, a band-dominated operator $K$ is said to be locally compact if $\varphi A$ and $A \varphi I$ are compact operators for each function $\varphi \in \text{BUC}$ with bounded support.

The operators $\chi_+ K \chi_- I$ and $\chi_- K \chi_+ I$ are compact for each locally compact $K$ (see [58]). Thus, the operators $\chi_+ A \in \mathcal{L}(L^p(\mathbb{R}_+))$ and $\chi_- A \in \mathcal{L}(L^p(\mathbb{R}_-))$ are Fredholm operators, whenever $A = I + K$ is Fredholm. We call

$$\text{ind}_+ A := \text{ind}(\chi_+ A) \quad \text{and} \quad \text{ind}_- A := \text{ind}(\chi_- A)$$
the plus- and the minus-index of $A$ and find that $\text{ind } A = \text{ind}_+ A + \text{ind}_- A$. Further notice that the limit operators of a locally compact operator are locally compact again.

Then the main result of [58] for operators on the spaces $L^p$, $1 < p < \infty$ reads as follows:

**Theorem 4.13.** Let $A = I + K$ with $K$ being a rich locally compact operator.

1. The operator $A$ is Fredholm iff all limit operators of $A$ are invertible and their inverses are uniformly bounded.

2. If $A$ is Fredholm then, for arbitrary limit operators $B, C \in \sigma(A)$ with respect to sequences $l, u \subset \mathbb{Z}$ tending to $-\infty$, $(+\infty)$, respectively,

$$\text{ind}_+ A = \text{ind}_+ B, \quad \text{ind}_- A = \text{ind}_- C, \quad \text{hence} \quad \text{ind } A = \text{ind}_+ B + \text{ind}_- C.$$ 

This can also be obtained and even generalized applying the results of the present text. It is easy to check that for all band-dominated operators $A = C + K$ with $C$ invertible and $K$ locally compact, and for all $p \in [1, \infty]$ the $P$-Fredholmness of $A$ already implies its Fredholmness (see e.g. [44], Proposition 2.15). Furthermore, the $P$-dichotomy of $A$ also yields the converse implication. Consequently, for every $p \in [1, \infty]$ and every rich band-dominated operator of the form $A = C + K$ we conclude from Theorem 1.58 that $A$ is Fredholm iff its limit operators are invertible and their inverses are uniformly bounded.

If $A = I + K$ is Fredholm then $\text{ind}_+ B$ and $\text{ind}_- B$ are finite numbers for every limit operator $B$ of $A$ and their sum equals zero since $B$ is invertible. Moreover, the finite sections of locally compact operators are compact, hence the finite sections of $A = I + K$ are Fredholm of index 0. Thus, Theorem 4.11 covers the second part of Theorem 4.13 and extends it to spaces $L^p$ with $p \in [1, \infty]$. Moreover, formula (4.6) gives an extension to the finite section sequences of general rich band-dominated operators and even to sequences in the algebra $\mathcal{F}_{A_{I,B}}$. By the analogon (4.7), also non-rich operators can be considered. A first version of such an index formula for band-dominated operators in terms of limit operators was derived by Rabinovich, Roch and Roe in [59].

### 4.2.4 Convolution type operators

We turn to another more concrete subclass of operators which were already studied by several authors. Thus, we omit repeating some details and refer the reader to e.g. [44], Section 4.2.

With every function $k \in L^1 = L^1(\mathbb{R}, \mathbb{C})$, we associate the operator that maps a given function $f \in L^p = L^p(\mathbb{R}, \mathbb{C})$ to the so-called convolution $k \ast f$ which is given by

$$(k \ast f)(x) = \int_{\mathbb{R}} k(x - y)f(y)dy, \quad x \in \mathbb{R}.$$ 

This operator is band-dominated and bounded by $\|k\|_1$, and it is usually denoted by $C(a)$ where $a$, the symbol of $C(a)$, is the Fourier transform of $k$. Notice further that $C(a)$ is always shift invariant, hence rich.

Let $\mathcal{B}_p$ denote the Banach subalgebra which is generated by all such convolution operators and all rich operators $bl$ of multiplication by a function $b \in L^\infty$. Moreover, introduce $\mathcal{B}^0_p$, the Banach algebra generated by all operators of the form $b_1 C(a) b_2 I$ where, again, $a$ is the Fourier transform of an $L^1$-function and $b_1 I, b_2 I$ are rich multiplication operators with functions $b_1, b_2 \in L^\infty$.

**Proposition 4.14.** (cf. [44], Lemma 4.10, Proposition 4.11)

- All operators in $\mathcal{B}^0_p$ are locally compact.
- The decomposition $\mathcal{B}_p = \{ bl : b \in L^\infty, bI \text{ rich} \} \oplus \mathcal{B}^0_p$ holds.
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Hence every operator $A \in \mathcal{B}_p$ is of the form $A = bI + B$ with $B \in \mathcal{B}_p^0$ and if $bI$ is invertible then the assumptions of Theorem 4.13 and its generalizations which were mentioned above hold.

In particular, the Fredholm property and the Fredholm index of $A$ are determined by its limit operators. For this notice that $A$ can be written in the form $A = bI(I + \tilde{B})$ with $\tilde{B} \in \mathcal{B}_p^0$, and the snapshots of $bI$ are invertible.

Of course, also Theorems 4.11 and 4.12 apply, telling us everything about stability, Fredholmness and further asymptotic properties.

The paper [4] already deals with such convolution operators and contains results on convergence of norms and pseudospectra. Several applications for boundary integral equations can be found in [16].

4.2.5 A remark on the connection to the discrete case

In the above discussion we avoided to exploit the analogies with the $l^p$-case, just to demonstrate seamlessly how the theory works in practice. However, we do not want to suppress this fact, but we shortly demonstrate the idea of discretization which transforms both cases into each other.

It is well known and has been frequently applied in the literature ([61], [63], [15], [44], [15]). Also in [81] this path was taken to derive the above results on the Fredholmness and stability. The convergence of the norms, condition numbers and pseudospectra for sequences in $\mathcal{F}_{A_{L^p}}$ appears here for the first time.

As a start, we mention that the $l^p$-theory which has been developed step by step in the Sections 2.5 and 3.2.5 can be also built upon the projections
\[ \tilde{L}_n : (x_i) \mapsto (\ldots, 0, x_{n-1}, x_n, \ldots) \]
without any change or loss in the outcome.

Now, let $\chi_0$ denote the characteristic function of the interval $I_0 := [0, 1)$ and set $X := L^p(I_0, Y)$.

The mapping $G$ which sends the function $f \in L^p$ to the sequence
\[ Gf = ((Gf)_k)_{k \in \mathbb{Z}}, \text{ where } (Gf)_k := \chi_0 U_{-k} f \]
is an isometric isomorphism from $L^p$ onto $l^p(\mathbb{Z}, X)$. Thus, the mapping
\[ \Gamma : \mathcal{L}(L^p) \to \mathcal{L}(l^p(\mathbb{Z}, X)), \quad A \mapsto GAG^{-1} \]
is an isometric algebra isomorphism. Moreover, we have $\Gamma(L_m) = \tilde{L}_m$ and the sets of compact ($\mathcal{P}$-compact) operators translate under the discretization operator $\Gamma$. Hence, $A \in \mathcal{L}(L^p, \mathcal{P})$ is Fredholm ($\mathcal{P}$-Fredholm, properly $\mathcal{P}$-Fredholm, or properly $\mathcal{P}$-deficient) if and only if $\Gamma(A)$ is so.

The results of Section 3.1 from [63] together with Theorem 1.55 tell us that

**Proposition 4.15.** Let $A \in \mathcal{L}(L^p, \mathcal{P})$ and $h = (h_n) \in \mathcal{H}_+$. The limit operator $A_h$ exists if and only if the limit operator $(\Gamma(A))_h$ of $\Gamma(A)$ w.r.t. $h$ exists. Then $(\Gamma(A))_h = \Gamma(A_h)$. In particular, $A$ is rich if and only if $\Gamma(A)$ is rich.

Moreover, $\Gamma(A_{L^p}) = \mathcal{A}_{l^p}$ and there is a natural identification of sequences $h = \{A_n\} \in \mathcal{F}_{A_{L^p}}$ with sequences in $\mathcal{F}_{A_{l^p}}$ via $\Gamma$. 

4.3 Multi-dimensional convolution type operators

4.3.1 The operators and their finite sections

Functions, spaces and operators Here we consider the spaces $L^p(K)$, $p \in [1, \infty]$ of the respective ($p$-integrable or essentially bounded) functions defined on a cone $K \subset \mathbb{R}^2$. By a cone $K$ (with vertex at the origin) we mean an angular sector in $\mathbb{R}^2$, i.e. a set $K$ which includes $\{\gamma y : \gamma \geq 0\}$ for each $y \in K$ and such that $\{x \in K : \|x\| = 1\}$ forms a connected closed infinite subset of the unit circle. Clearly, $\mathbb{R}^2$ itself is a cone.

The papers [49], [50] of Mascarenhas and Silbermann deal with the finite section method for convolution type operators which are introduced as follows: Let $u \in L^1(\mathbb{R}^2)$ and $\lambda \in \mathbb{C}$. We denote by $C(\lambda)$ the convolution operator on $X := L^p(\mathbb{R}^2)$ defined by

$$C(\lambda) : X \to X, \quad g \mapsto \lambda g(\cdot) + \int_{\mathbb{R}^2} u(\cdot - s)g(s)ds,$$

where $a$ is given by $a(x) = \lambda + (F(u))(x)$, with $F$ being the Fourier transform on $\mathbb{R}^2$. The function $a$, usually called the symbol of the operator $C(\lambda)$, is continuous in $\mathbb{R}^2$ and tends to $\lambda$ at infinity. It is well known that $C(\lambda)$ is a linear bounded operator on $X$, its invertibility is equivalent to the invertibility of the symbol $a$, and $C(\lambda_1)C(\lambda_2) = C(\lambda_1\lambda_2)$ holds for all symbols $\lambda_1, \lambda_2$. The set of all such symbols

$$W(\mathbb{R}^2) := \{a = \lambda + F(u) : u \in L^1(\mathbb{R}^2), \lambda \in \mathbb{C}\}$$

is a subalgebra of $L^\infty(\mathbb{R}^2)$, closed in the norm $\|a\|_W := |\lambda| + \|u\|_{L^1}$, and is called the Wiener algebra. Let $U_1(0)$ be the open unit disc in $\mathbb{R}^2$ and $\xi : x \mapsto \frac{x}{1-|x|}$ be the homeomorphism which maps $U_1(0)$ onto $\mathbb{R}^2$. We denote by $C(\mathbb{R}^2)$ the set of all continuous functions $f$ on $\mathbb{R}^2$ for which $f \circ \xi$ admits a continuous extension $\tilde{f} \circ \xi$ onto the closed unit disc $B_1(0)$.

Now, the convolution type operators on a cone $K$ under consideration in [49] (the case $p = 2$) and [50] (the cases $1 < p < \infty$) have been of the form

$$\chi_K A \chi_K I + (1 - \chi_K)I$$

with $A$ belonging to the algebra

$$\mathcal{A} := \text{alg}\{C(\lambda), f : a \in W(\mathbb{R}^2) \text{ and } f \in C(\mathbb{R}^2)\} \subset \mathcal{L}(X).$$

The present approach permits to recover the results of these papers on the finite sections and to extend them to the cases $p \in [1, \infty]$ as well as to some larger classes of operators. Notice that the pioneering stability results are due to Kozak [39] and that the Fredholmness and invertibility of convolution operators have already been studied in several further publications, e.g. [85], [23], [20], [47], [7]. See also [44], Section 4.2, [72], Section 5.8 and the notes in [13], Chapter 3.

Finite section domains Let $\Omega \subset \mathbb{R}^2$ be a closed bounded set containing $0$ as interior point and further assume that for each $t \in \partial\Omega$ there is a cone $K_t$ with vertex in $t$, neighborhoods $U_t$ and $V_t$ of $t$ and a $C^1$-diffeomorphism $\rho_t : \mathbb{R}^2 \to \mathbb{R}^2$, such that $\rho_t^{-1}$ are bounded on $\mathbb{R}^2$ and $\rho_t(t) = t$, $\rho_t'(t) = I$, $\rho_t(U_t \cap \partial\Omega) = V_t \cap K_t$.

Here we say that $K_t$ is a cone with vertex $x \in \mathbb{R}^2$, if $K_t = x + K_0^\Omega$, where $K_0^\Omega$ is a cone with vertex at 0, respectively. Figuratively speaking, the domain $\Omega$ looks (locally) like a cone in each of its

\footnote{See [44], Section 4.2, for example.}
boundary points. This property will make a major contribution to the construction of snapshots which shall capture the asymptotics at the boundary points. But firstly, let us precisely define the finite section method: For \( n \in \mathbb{N} \) we let \( n\Omega \) denote the inflated copy \( \{nx : x \in \Omega \} \) of \( \Omega \) and introduce the finite section projection \( T_n := \chi_{n\Omega} I \). Then the finite section sequence of a bounded linear operator \( A \in \mathcal{L}(X) \) is given by \( \{T_n AT_n + (I - T_n)\} \).

**A sequence algebraic framework** From the general theory and the above examples we already got an impression, that we will require several snapshots, and it is standing to reason that a “central” snapshot at the origin and further ones at every boundary point of \( \Omega \) are the natural candidates.

We introduce the operators \( P_n = \chi_{B_n(0)} I \) of multiplication by the characteristic function \( \chi_{B_n(0)} \) of the closed disc with radius \( 1.27 \). For \( n \Omega \) mate identity which provides \( X := L^p(\mathbb{R}^2) \) with the \( \mathcal{P} \)-dichotomy (the latter follows from Theorem \ref{thm:1.27}). For \( \alpha \in \mathbb{R}^2 \) we introduce the operator \( V_\alpha \) of shift by \( \alpha \) on \( X \)

\[
V_\alpha : X \to X, \quad (V_\alpha f)(t) := f(t - \alpha).
\]

Now, for given \( \Omega, t \in \delta \Omega \) and the respective diffeomorphism \( \rho_t \), we define operators \( R_n^t : X \to X \) and their inverses by the rules

\[
(R_n^t g)(y) := g\left(n\rho_t \left(\frac{y}{n}\right)\right), \quad ((R_n^t)^{-1} g)(y) = g\left(n\rho_t^{-1} \left(\frac{y}{n}\right)\right)
\]

and we complete this family with \( R_0^0 := I \).

We set \( T := \delta \Omega \cup \{0\} \) and, for every \( n \in \mathbb{N} \) and every \( t \in T \), let \( E_n := X, E_n^t := X \), and \( E_n^t(\cdot) := (R_n^t)^{-1}V_{nt} \cdot V_{-nt}R_n^t \). Further, set \( L_n^0 := I \) for every \( n \) and \( t \), and introduce the algebra \( \mathcal{F}^T \) of all bounded sequences \( \{A_n\} \) of operators \( A_n \in \mathcal{L}(X) \) for which the snapshots

\[
W^t(A) := \lim_{n \to \infty} E_n^{-1}(A_n) \in \mathcal{L}(X, \mathcal{P}), \quad t \in T
\]

exist. Also the ideals \( \mathcal{G} \) and \( \mathcal{J}^T \) are defined as usual (see Section \ref{sec:2.1} et seq.). Notice that the mappings \( E_n^t \) are algebra isomorphisms and all required assumptions, such as the uniformity and separation conditions \((I)\) and \((\Pi)\), are fulfilled.

So, until now, we constructed a setting which allows to study finite section sequences \( \{T_n AT_n + (I - T_n)\} \) (which arise from a given finite section domain \( \Omega \)) by the Fredholm theory of Part 2. As a next step we seek for a localizing setting which should help to obviate the \( \mathcal{J}^T \)-Fredholm property in the main results.

**A localizing setting** Throughout the previous examples we were interested in very large and abstract classes of (band-dominated) operators and the arising finite section sequences. It was necessary to consider the properties of subsequences, and we had to construct the suitable localizing setting for every sequence individually. Here we tread a simpler path, define a localizing setting in the very beginning and commit ourselves to use it for all occurring (sub)sequences. Of course, the price that we have to pay for that is the limitation to sequences which are compatible with this predefined setting. Nevertheless, it will work sufficiently well for all we are interested in.

For each function \( \varphi \in C(\overline{\mathbb{R}^2}) \), and for each \( t > 0 \), we again define inflated copies by \( \varphi_t(x) := \varphi\left(\frac{x}{t}\right) \) and we consider the set \( C := \{\varphi_n I : \varphi \in C(\overline{\mathbb{R}^2})\} \) which belongs to \( \mathcal{F}^T \). The central snapshot \( W_0^t(\varphi_n I) \) of \( \{\varphi_n I\} \) in \( C \) equals \( \varphi(0) I \) and, for \( t \in \delta \Omega \), we have \( W^t(\varphi_n I) = \varphi(t) I \), since

\[
\|(V_{-nt}R_n^t\varphi_n(R_n^t)^{-1}V_{nt} - \varphi(t) I)P_m\| = \sup_{s \in B_n(0)} \left| \varphi\left(s - \frac{\rho_t \left(\frac{y}{n}\right)}{n}\right) - \varphi(t) \right| \to 0
\]

\(^{4}\)To check the uniform boundedness of the \( R_n^t \) apply the estimate \((4.13)\) with \( \psi \equiv 1 \).
as \( n \to \infty \) for every fixed \( m \), and the dual estimate with \( P_m \) multiplied from the left holds as well.

Provided with the involution \( \cdot^* : \{ \varphi_n I + G \} \to \{ \varphi_n I + \overline{G} \} \), the set \( C \) becomes a commutative \( \mathrm{C}^* \)-algebra which is isometrically isomorphic to \( C(R^2) \) and \( C(B_1(0)) \) since \( \| \varphi_n I \|_\infty = \| \varphi \|_\infty = \| \varphi \circ \xi \|_\infty \) for every \( n \) and \( \varphi \), hence \( \| \varphi \|_\infty = \limsup_n \| \varphi_n I \| = \| \{ \varphi_n I \} + G \|_{\mathcal{F}/G} \). Their maximal ideal spaces can be identified (see Theorem 3.2) and for our purposes it is convenient to think of this space as the compactification \( \overline{R^2} \) with a sphere of points at infinity and with the Gelfand topology.

More precisely, a character on \( C(R^2) \) is either of the form \( x : f \mapsto f(x) \) with \( x \in R^2 \), or of the form \( \theta_\infty : f \mapsto \int \circ \xi (\theta) \) with \( \theta \in \delta B_1(0) \).

Here we fix \( \Omega := \Omega \subset \overline{R^2} \) and to every \( x \in \Omega \setminus T \) we associate \( t(x) = 0 \). Now, we have to choose the functions \( F^x \in C(R^2) \) which take their values in the interval \([0,1]\) and are identically 1 in a neighborhood of \( x \in \Omega \), respectively. We do this in such a way that in case \( x \in \delta \Omega \) the support of \( F^x \) is contained in \((U_x \cap V_x) \setminus \{0\} \), and in case \( x \in \text{int } \Omega \) the function \( F^x \) shall vanish on \( \delta \Omega \).

Finally notice that the sequence \( \mathbb{T} = \{ T_n \} \) belongs to \( \mathcal{L}^T \): Its central snapshot \( W^0(\mathbb{T}) \) is the identity, and for \( t \in \delta \Omega \) the snapshot \( W^t(\mathbb{T}) \) is the operator of multiplication by the characteristic function \( \chi_{K^0} \) of the cone \( K^0 \). The localizability of \( \mathbb{T} \) is immediate from the definition.

Consequently, \( C \) defines a partially localizing setting and the results of Section 3.2.1 apply. Also notice that

\[
\{ E_n^{\pm 1}(\varphi_n I) \} \in C \quad \text{for every } \{ \varphi_n I \} \in C \text{ and every } t \in T. \tag{4.9}
\]

### 4.3.2 The main result for localizable sequences

**Theorem 4.16.** The set \( \mathcal{L}^T \) of all \( C \)-localizable sequences is a Banach algebra. For every sequence \( \mathcal{A} = \{ A_n \} \) of the form \( \mathbb{T} \mathcal{A} \mathbb{T} + (1 - \mathbb{T}) \) with \( \mathcal{A} \in \mathcal{L}^T \) the following hold

- \( \mathcal{A} \) is a Fredholm sequence if and only if all of its snapshots are Fredholm. In this case

\[
\lim_{n \to \infty} \dim \ker A_n = \sum_{t \in T} \dim W^t(\mathcal{A}) \tag{4.10}
\]

and the approximation numbers from the right/left of the \( A_n \) have the \( \alpha-/\beta \)-splitting property with the finite splitting numbers

\[
\alpha = \sum_{t \in T} \dim \ker W^t(\mathcal{A}) \quad \text{and} \quad \beta = \sum_{t \in T} \dim \text{coker } W^t(\mathcal{A}).
\]

- If one of the snapshots \( W^t(\mathcal{A}) \) is not Fredholm then, for each \( k \in \mathbb{N} \),

\[
\lim_{n \to \infty} s_k^-(A_n) = 0 \quad \text{or} \quad \lim_{n \to \infty} s_k^+(A_n) = 0.
\]

- The norms \( \| A_n \| \) and \( \| A_n^{-1} \| \) converge and \( ^5 \)

\[
\lim_{n \to \infty} \| A_n \| = \max_{t \in T} \| W^t(\mathcal{A}) \|, \quad \lim_{n \to \infty} \| A_n^{-1} \| = \max_{t \in T} \| (W^t(\mathcal{A}))^{-1} \|.
\]

- \( \mathcal{A} \) is stable if and only if all snapshots are invertible. In this case

\[
\lim_{n \to \infty} \text{cond}(A_n) = \max_{t \in T} \| W^t(\mathcal{A}) \| \cdot \max_{t \in T} \| (W^t(\mathcal{A}))^{-1} \|.
\]

- The \( (N,\epsilon) \)-pseudospectra of the operators \( A_n \) converge w.r.t. the Hausdorff distance to the union of the \( (N,\epsilon) \)-pseudospectra of the snapshots.

\(^5\)We again use the convention \( \| B^{-1} \| = \infty \) if \( B \) is not invertible.
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Proof. Recall from Section 3.2.1 the definition of $T^T$, the set of all sequences being of the form $\mathcal{T}^\Lambda + \lambda(1 - \mathcal{T}) + j$ with $\Lambda \in \mathcal{L}^\Lambda$, a complex number $\lambda$ and a sequence $j \in \mathcal{F}^T$. Proposition 3.8 provides that $\mathcal{L}^\Lambda$ and $\mathcal{T}^T$ are Banach algebras and Theorem 3.9 yields the criteria for the stability and the Fredholm property. Then we derive the index formula from Theorem 2.14, the splitting property from Theorem 2.13, and the convergence of the approximation numbers in the case of a non-Fredholm snapshot from Theorem 2.9.

We check that $C$ even defines a faithful localizing setting. In the case $p \leq \infty$ we estimate

$$\|(\phi_n A_n \varphi_n + \psi_n B_n \psi_n)\| = \max\{\|\phi_n A_n \varphi_n\|, \|\psi_n B_n \psi_n\|\} \leq \max\{\|A_n\|, \|B_n\|\} \|f\|$$

(4.11)

holds for all $f \in \mathcal{X}$, all $\{A_n\}$, $\{B_n\} \in \mathcal{F}^T$, all $n \in \mathbb{N}$ and all $\varphi, \psi \in C(\mathcal{R}^2)$ of norm 1 with disjoint support. For $1 \leq p < \infty$ we replace (4.11) by

$$\|(\phi_n A_n \varphi_n + \psi_n B_n \psi_n)\|^p = \|\phi_n A_n \varphi_n\|^p + \|\psi_n B_n \psi_n\|^p \leq \|\phi_n A_n\|^p \|\varphi_n\|^p + \|\psi_n B_n\|^p \|\psi_n\|^p \leq \max\{\|A_n\|^p, \|B_n\|^p\} \|f\|^p,$$

and we conclude from Remark 3.13 that $T^T / G$ is a KMS-algebra with respect to $C / G$ in all cases $p \in [1, \infty]$. We now show that for every $\Lambda \in \mathcal{T}^T$, every $g \in \mathcal{H}_+$ and every $x \in \Omega$

$$\|(\phi_x (\Lambda + G))\| \leq \|W^X(\Lambda)\| \leq \|A_g + G\|.$$  

(4.12)

We start with the following observation: Let $x > 0$ and $t \in \Omega$ be given. Then there is a function $\psi \in C(\mathcal{R}^2)$ such that $\|\psi\|_\infty = 1$, $\psi(t) = 1$, $E^t\psi = \psi$ and $\|\psi_n (R_n^T)^{-1}\| \|R_n^T \psi_n I\| \leq 1 + \epsilon$ for all $n$. In the case $p = \infty$ this follows from the fact $(R_n^T)^{-1} = (R_n^T) = 1$. If $1 \leq p < \infty$ then

$$\|(R_n^T \psi_n)f\|^p = \int_{\mathbb{R}^2} |(R_n^T \psi_n f)(s)|^p ds = \int_{\mathbb{R}^2} \psi_n (nt \left( \frac{s}{n} \right)) \left| f \left( n pt \left( \frac{s}{n} \right) \right) \right|^p ds = \int_{\mathbb{R}^2} |\psi_n(u)f(u)|^p \left| \det(\rho_n^{-1})'(\left( \frac{u}{n} \right)) \right| du \leq \sup_{u \in \text{supp } \psi_n} \left| \det(\rho_n^{-1})'(\left( \frac{u}{n} \right)) \right| \|f\|^p$$

(4.13)

together with $(\rho_n^{-1})'(t) = I$, the continuity of $(\rho_n^{-1})'$ and an analogous argument for $(R_n^T)^{-1}$ give the claim for sufficiently small support of $\psi$. We conclude for every $x \in \Omega$

$$\|(\phi_x (\Lambda + G))\| = \|(\phi_x (\Lambda^x + G))\| = \|(\psi_n (\Lambda^x \psi_n I + G))\|$$

$$\leq \|(\psi_n I) \Lambda^x \psi_n I + G\| = \limsup_{n \to \infty} \|\psi_n E_n^z(W^X(\Lambda)) \psi_n I\|$$

$$\leq \limsup_{n \to \infty} \|\psi_n (R_n^T)^{-1}\| \|V_n(t, x)\| \|W^X(\Lambda)\| \|V_n(t, x)\| \|R_n^T \psi_n I\| \leq (1 + \epsilon) \||W^X(\Lambda)\||.$$

Theorem 1.19 further yields

$$\||W^X(\Lambda)\|| \leq \liminf_{n \to \infty} \|E_{g_n}^z(\psi_{g_n} A_{g_n} \psi_{g_n})\|$$

$$\leq \limsup_{n \to \infty} \|V_{g_n}(t, x)\| \|R_{g_n}^T \psi_{g_n} I\| \|A_{g_n}\| \|\psi_n (R_n^T)^{-1}\| \|V_{g_n}(t, x)\| \leq (1 + \epsilon) \limsup_{n \to \infty} \|A_{g_n}\|.$$

Since $\epsilon$ was arbitrary, we arrive at (4.12) and thus, in view of Remark 3.13, we have proved that $C$ defines a faithful localizing setting. Now, Proposition 3.11 provides the convergence of $\|A_n\|$ and $\|A_n^{-1}\|$ and Corollary 3.12 the convergence of the approximation numbers in case $\Lambda$ is a stable sequence. From Proposition 3.16 we conclude that all spaces $\mathcal{E}^t$ or their duals are complex uniformly convex, hence Corollary 3.19 yields the convergence of the pseudospectra. \(\square\)

We note that the snapshots $A^t = W^t(\mathcal{T}^\Lambda + (1 - \mathcal{T}))$ are operators acting on the respective “local” cones $K^t_\Lambda$ since they are of the form $A^t = \chi_{K^t_\Lambda} A^t \chi_{K^t_\Lambda} I + (1 - \chi_{K^t_\Lambda}) I$.  


4.3.3 Some members of $\mathcal{L}^T$

Since $\mathcal{L}^T$ is a Banach algebra, we get a whole closed subalgebra of $\mathcal{L}^T$ and the above results for each of its elements for free, if we only have the generators of this algebra. So, let us start finding interesting individuals in $\mathcal{L}^T$.

**Some basic ingredients** We have already seen that $\mathbb{T}$ is $\mathcal{C}$-localizable, and that its snapshots $W^t(\mathbb{T})$, $t \in \delta \Omega$ are the projections $\chi_{K_n^t}I$, respectively. Also the identity $I$ belongs to $\mathcal{L}^T$.

Furthermore, all sequences $\mathcal{A} \in \mathcal{J}^T$ are $\mathcal{C}$-localizable and have $\mathcal{P}$-compact snapshots. This particularly includes all sequences $\{J\}$ with a $\mathcal{P}$-compact operator $J$, and hence all $\{gI\}$ arising from operators of multiplication by functions $g \in L^\infty(\mathbb{R}^2)$ with $\|(I - T_n)g\|_\infty \to 0$ as $n \to \infty$. The respective snapshots are $W^0\{J\} = J$, $W^0\{gI\} = gI$, and all other snapshots vanish.

**Convolution operators** Of course, the most challenging operators here are the convolutions $C(a)$ with $a \in W(\mathbb{R}^2)$, and we are going to show that $\{C(a)\}$ is in $\mathcal{L}^T$. Actually, we do not need to consider all symbols $a = \lambda + F(u)$ with $\lambda \in \mathbb{C}$ and $u \in L^1(\mathbb{R}^2)$ since $\mathcal{L}^T$ is a Banach algebra containing the identity $I$, but we can assume that $\lambda = 0$ and $u$ is continuous with compact support.

For the latter notice that every $u \in L^1(\mathbb{R}^2)$ can be approximated by continuous functions having compact support and that $\|C(F(u))\| \leq \|u\|_{L^1}$.

**Proposition 4.17.** Let $u$ be continuous with compact support.

1. The commutator $\{C(F(u)), \{\varphi_n I\}\}$ belongs to $\mathcal{G}$ for every $\varphi \in C(\mathbb{R}^2)$.

2. For every $\varepsilon > 0$ and every $t \in T$ there is a function $\psi \in C(\mathbb{R}^2)$ of norm one, identically 1 in a neighborhood of $t$ and with small support such that $\|\psi_n(E^x_n(C(F(u)) - C(F(u)))\| < \varepsilon$ for sufficiently large $n$.

3. For every compact set $M$, both $\chi_M C(F(u))$ and $C(F(u)) \chi_M I$ are compact operators.

**Proof.** Let the support of $u$ be contained in the ball $B_R(0)$. From $\|C(F(u))\| \leq \|u\|_{L^1}$ and

$$(\varphi_n C(F(u)) - C(F(u)) \varphi_n I)(x) = \int_{B_R(x)} u(x - y) \left(\varphi \left(\frac{x}{n}\right) - \varphi \left(\frac{y}{n}\right)\right) f(y) dy,$$

using the Hölder inequality, we conclude that $\|\varphi_n C(F(u)) - C(F(u)) \varphi_n I\| \leq c_n \|u\|_{L^1}$, where

$$c_n := \sup_{x \in \mathbb{R}^2} \sup_{y \in B_R(x)} \left|\varphi \left(\frac{x}{n}\right) - \varphi \left(\frac{y}{n}\right)\right| = \sup_{r \in \mathbb{R}^2} \sup_{s \in B_{\frac{r}{2}}(r)} \left|\varphi \left(r\right) - \varphi \left(s\right)\right|.$$ 

Since $|\xi^{-1}(r) - \xi^{-1}(s)| \leq |r - s|$ we find

$$c_n = \sup_{r \in \mathbb{R}^2} \sup_{s \in B_{\frac{r}{2}}(r)} \left|\psi \circ \xi \left(\xi^{-1}(r)\right) - \psi \circ \xi \left(\xi^{-1}(s)\right)\right| \leq \sup_{a \in U_\gamma(0)} \sup_{b \in U_\gamma(a) \cap U_\gamma(0)} \left|\psi \circ \xi \left(a\right) - \psi \circ \xi \left(b\right)\right|$$

and the uniform continuity of $\varphi \circ \xi$ yields $c_n \to 0$ as $n \to \infty$, the first assertion.

For the second assertion let $t \in \delta \Omega$ and notice that, due to the shift invariance of $C(F(u))$ and the first part, it suffices to show that $\|\chi_n W(R_n^{-1}C(F(u)) R_n - C(F(u))) \psi_n I\| < \varepsilon/2$ for every $n$, where $W$ denotes the support of $\psi$. Set $\gamma := \varepsilon/(4\|\chi_{B_{2R}(0)}\|_{L^1})$. 

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<sup>6</sup>See [44], Example 1.45, for instance.

<sup>7</sup>The proofs are modifications of those in [49] and [72].
For \( f \in X \) and \( x \in nW \) we find that \( \chi_{nW}(R_n^{-t}C(F(u))R_n^t - C(F(u)))\psi_n f(x) \) equals
\[
\chi_{nW}(x) \left[ \int_{\mathbb{R}^2} u \left( n\rho_t^{-1} \left( \frac{x}{n} \right) - s \right) \psi \left( n\rho_t \left( \frac{x}{n} \right) \right) f \left( n\rho_t \left( \frac{x}{n} \right) \right) ds - \int_{\mathbb{R}^2} u(x-s)\psi_n(s)f(s)ds \right] 
\]
\[
= \int_{nW} \chi_{nW}(x) \left[ u \left( n\rho_t^{-1} \left( \frac{x}{n} \right) - n\rho_t^{-1} \left( \frac{y}{n} \right) \right) \left| \det(\rho_t^{-1})' \left( \frac{y}{n} \right) \right| - u(x-y) \right] \psi_n(y)f(y)dy.
\]
By Taylor’s Theorem we have
\[
\Delta := n \left| \rho_t^{-1}(v) - \rho_t^{-1}(w) \right| - (v-w) \leq n \left| \rho_t^{-1}(v) - (\rho_t^{-1})'(v) (v-w) + h(v-w)(v-w) - (v-w) \right| \leq \|h(v-w)\| \to 0 \text{ as } |v-w| \to 0.
\]
Fix \( \delta \in (0,R) \). Due to the properties of the diffeomorphism \( \rho_t \) we can choose \( W \) sufficiently small such that \( \|\rho_t^{-1}'(v) - I + h(v-w)\| \leq \delta(2R)^{-1} \) for all \( v, w \in W \). Then, for every \( n \) and \( v, w \in W \), \( \Delta \leq \delta \) holds in case \( n|v-w| \leq 2R \), and \( |n|v-w| - \Delta| \geq R \) if \( n|v-w| \geq 2R \). Moreover, \( |(\rho_t^{-1})'(v)| \) is continuous and equals 1 in \( t \), that is, by an appropriate choice of \( W \), we can guarantee that \( |1 - |(\rho_t^{-1})'(v)|| \leq \gamma/\|u\|_\infty \) on \( W \). Notice that the continuous function \( u \) with a compact support is uniformly continuous. Hence the \( \delta \in (0,R) \) can be fixed such that
\[
\sup_{x,y \in nW} \sup_{|\alpha| \leq \delta} |u(x-y + \alpha) - u(x-y)| \leq \gamma.
\]
Puzzling all these parts together we obtain for the expressions
\[
E_n(x,y) := \chi_{nW}(x) \left[ u \left( n\rho_t^{-1} \left( \frac{x}{n} \right) - n\rho_t^{-1} \left( \frac{y}{n} \right) \right) \left| \det(\rho_t^{-1})' \left( \frac{y}{n} \right) \right| - u(x-y) \right] \psi_n(y)
\]
that \( E_n \) is supported in \( \{(x,y) : x,y \in nW, |x-y| \leq 2R\} \), and \( \sup_{x,y \in \mathbb{R}^2} |E_n(x,y)| \leq 2\gamma \) for every \( n \). Now, in the cases \( 1 \leq p < \infty \),
\[
\|\chi_{nW}(R_n^{-t}C(F(u))R_n^t - C(F(u)))\psi_n f\|^p = \int_{nW} \left[ \int_{\mathbb{R}^2} E_n(x,y) f(y)dy \right]^p dx 
\]
\[
\leq \int_{nW} \left( \int_{B_{2R}(x)} |E_n(x,y) f(y)|dy \right)^p dx \leq (2\gamma)^p \int_{nW} \left( \int_{B_{2R}(x)} |f(y)|dy \right)^p dx 
\]
\[
= (2\gamma)^p \|C(\chi_{B_{2R}(0)}) f\|_p \leq (2\gamma)^p \|\chi_{B_{2R}(0)}\|_p \|f\|_p = \left( \frac{\epsilon}{2} \right)^p \|f\|_p,
\]
and if \( p = \infty \), then we estimate \( \|\chi_{nW}(R_n^{-t}C(F(u))R_n^t - C(F(u)))\psi_n f\| \) by
\[
\operatorname{ess sup}_{x \in nW} \int_{nW} E_n(x,y) f(y)dy \leq \operatorname{ess sup}_{x \in nW} \int_{B_{2R}(x)} 2\gamma |f(y)|dy = 2\gamma \|C(\chi_{B_{2R}(0)}) f\| \leq \frac{\epsilon}{2} \|f\|.
\]
The assertion is trivial for \( t = 0 \). For the third part let \( B_r(0) \) be a ball containing \( M \), fix a continuous function \( \zeta \) of norm one, equal to 1 on the set \( B_{r+R}(0) \) and supported in the set \( B_{2r+R}(0) \), and set \( D := \zeta C(F(u))\zeta I \). Then \( \chi_M C(F(u)) = \chi_M D \) and \( C(F(u))\chi_M I = D\chi_M I \), whereas \( D \) is an integral operator with a continuous kernel function which is compactly supported. It is well known \(^8\) that such integral operators are compact.

\(^8\)For a detailed proof see [63], Lemma 3.2.4, which works for all \( p \in [1,\infty] \).
Together with (4.9) this is everything that we need to directly check the $\mathcal{C}$-localizability of $A$: All snapshots of $A$ equal $C(F(u))$, hence are liftable and their liftings also belong to $F^T$. We only note that, for all $t, \tau \in T$,

$$E_{n}^{-\tau}[E_{n}(C(u))] - C(u) = E_{n}^{-\tau}[E_{n}(C(u) - C(u))] + E_{n}^{-\tau}[C(u) - E_{n}(C(u))].$$

The commutation properties and the local equivalence are immediate from the previous proposition and (4.9) as well.

### Multiplications by continuous functions

For a given function $f \in C(\mathbb{R}^2)$ and the sequence $A := \{f\}$ it obviously holds that $W^0(A) = f I$. If $t$ is in $\delta \Omega$ then we conclude from the definitions that the respective snapshot $W^t(A)$ equals $(f \circ \xi)(t/|t|)I$. Now it is easy to check that $A \in \mathcal{L}^T$.

**Corollary 4.18.** The algebra $\mathcal{A}$ is embedded in $\mathcal{L}^T$ by taking $A \in \mathcal{A}$ as the sequence $\{A\}$.

### Cones

Let a finite section domain $\Omega$ be given and let $K$ be a cone with vertex at 0 such that $A := \{\chi_K I\} \in \mathcal{F}^T$. We prove $A \in \mathcal{L}^T$.

Clearly, $W^0(A) = \chi_K I$ and, for every $x \in \text{int} \Omega$, the conditions in the definition of localizable sequences are fulfilled, that is $W^r(A)$ are “liftable”, the commutation properties hold and $A + \mathcal{G}$, $A^{\tau} + \mathcal{G}$ are locally equivalent. The cases $t \in \delta \Omega \cap \text{int} K$ with $W^t(A) = I$, and $t \in \delta \Omega \setminus K$ with $W^t(A) = 0$ are obvious as well. So, let $t \in \delta \Omega \cap \delta K$. For the description of the snapshot $W^t(A)$ we note that there is a uniquely determined half space $H_t$ such that $K \cap B_r(t) = H_t \cap B_r(t)$ for sufficiently small $r > 0$, and we claim $W^t(A) = \chi_H I$.

The operators $E_{n}^t(\chi_H I)$ are operators of multiplication by the characteristic function of a certain set, and they converge $\mathcal{P}$-strongly, since $A \in \mathcal{F}^T$. Let $A'$ denote the limit. Further, all $P_n(E_{n}^t(\chi_H I) - E_{n}^t(\chi_H I))$ are operators of multiplication by a characteristic function. For all sufficiently large $k, m$, say $k, m \geq N$, their norms are less than one, due to the convergence to $A'$, hence they are zero. Consequently, $P_n(E_{k}^t(\chi_H I) - E_{n}^t(\chi_H I)) = 0$ for all $n$ and all $k, m \geq nN$, which implies $P_n(E_{k}^t(\chi_H I) - A') = 0$ all $n$ and all $k \geq nN$, therefore $A'$ must be an operator of multiplication by a characteristic function as well. One now easily checks that $A' = \chi_H I$, and $A$ proves to be $\mathcal{C}$-localizable.

The situation is analogous for cones $K$ with vertex at 0 which comply with the weaker condition $A := \{\chi_K T_0\} \in \mathcal{F}^T$. In this case $W^t(A) = \chi_K T_0 \chi_H I$ for $t \in \delta \Omega \cap \delta K$.

**Example 4.19.** We want to point out at this juncture that sequences arising from the operators in $A$ belong to $\mathcal{L}^T$ for every finite section domain $\Omega$, whereas the operators of multiplication by the characteristic function of a cone require a suitable choice of the finite section domain and the rectifying diffeomorphisms $\rho_t$, at least for the points in $\delta \Omega \cap \delta K$. Here are two examples:

1. Of course, it is rather obvious that $\Omega = B_1(0)$ can be provided with diffeomorphisms $\rho_t$ for every $t \in \delta \Omega$ which act as the identity on the line through $t$ and the origin, and which make $\Omega$ appear locally as a half space in $t$. Then every cone $K$ with vertex at the origin defines a $\mathcal{C}$-localizable sequence $\{\chi_K I\}$. In particular, for every operator $A$ with $\{A\} \in \mathcal{L}^T$ and its compression $A_K := \chi_K A \chi_K I + (1 - \chi_K) I$ to the cone $K$ we get $\{A_K\} \in \mathcal{L}^T$ as well, hence we can apply the finite section method for such operators acting on cones: $A := T[A_K] + (1 - T)$. The snapshots at all $t$ “outside” the cone are the identity, for $t$ from the interior of $K$ we obtain $W^t(A) = \chi_K T W^t(A) \chi_K I + (1 - \chi_K) I$, that are operators acting on the respective cones $K^0$ which are half spaces in this particular case $\Omega = B_1(0)$. Solely the snapshots at the points $t \in \delta \Omega \cap \delta K$ are operators acting on rectangular cones, namely $K^0 \cap H_t$. To avoid this phenomenon, and to ensure that all snapshots at points $t \in \delta \Omega$ become operators on half spaces, one could make another construction as follows:
2. For a given cone $K$ we choose $\Omega$ such that its boundary $\delta \Omega$ is a smooth curve in every point $t \in \delta \Omega \cap \text{int} \, K$, merges into the boundary $\delta K$ of $K$, and is a circular arc outside $K$ (see Figure 4.1). The diffeomorphisms can be chosen in such a way that $\{ \chi_K T_n \} \in \mathcal{F}$. Now, for an operator $A$ with $\{ A \} \in \mathcal{F}$ the compression $A_K := \chi_K A \chi_K^* I + (1 - \chi_K) I$ has a $\mathcal{C}$-localizable finite section sequence $\mathcal{T}(A_K)\mathcal{T} - (1 - \mathcal{T})$ whose snapshots are either trivial or operators on half spaces for every $t \in \delta \Omega$. We particularly note again that $W^1 \{ \chi_K T_n \} = \chi_K^* \chi_K I$ for $t \in \delta \Omega \cap \delta K$.

The advantage of the latter example lies in the fact that for convolutions on half spaces the invertibility can effectively be checked.

**Proposition 4.20.** Let $a$ be in the Wiener algebra $W(\mathbb{R}^2)$, $H, H'$ be half spaces and $K = H \cap H'$ be a cone. Set $A := C(a)$ as well as $A_H := \chi_H A \chi_H I + (1 - \chi_H) I$ and $A_K := \chi_K A \chi_K I + (1 - \chi_K) I$.

Then the following are equivalent: $a$ is invertible; $A$ is Fredholm; $A$ is $\mathcal{P}$-Fredholm; $A_H$ is invertible; $A_H$ is Fredholm; $A_H$ is $\mathcal{P}$-Fredholm; $A_K$ is Fredholm; $A_K$ is $\mathcal{P}$-Fredholm.

**Proof.** It is a well known fact that the Fredholmness of $A_K$ and $A$ are both equivalent to the invertibility of the symbol $a$. Its proof goes back to Simonenko [85] and can also be found in [12], 9.57. (for $1 \leq p < \infty$, and for $p = \infty$ by duality and the help of Proposition 1.34).

It is also clear that the Fredholmness of $A_K$ yields that it is properly $\mathcal{P}$-Fredholm, hence it implies its $\mathcal{P}$-Fredholmness. So, let $A_K$ be $\mathcal{P}$-Fredholm, choose a non-zero $\alpha \in \delta K$ and consider $\mathcal{P}$-$\text{lim}_n V_{-\alpha} A_K V_{\alpha}$. Due to the shift invariance of $A$ this limit exists and equals $A_H$. Further notice that this limit is invertible by the discussion after Theorem 1.58.

The implications $(A_H$ invertible $\Rightarrow A_H$ Fredholm $\Rightarrow A_H$ $\mathcal{P}$-Fredholm$)$ are obvious, and by passing to the $\mathcal{P}$-strong limit $\mathcal{P}$-$\text{lim}_n V_{-\beta} A_H V_{\beta}$ with $\beta \in \text{int} \, H$ we find in the same way that $(A_H$ $\mathcal{P}$-Fredholm $\Rightarrow A$ invertible $\Rightarrow A$ Fredholm $\Rightarrow A$ $\mathcal{P}$-Fredholm$)$ holds.

Employing this argument a third time we finally get that $A$ $\mathcal{P}$-Fredholm provides itself as an invertible limit, thus $(A$ $\mathcal{P}$-Fredholm $\Rightarrow A$ invertible $\Rightarrow a$ invertible $\Rightarrow A_K$ Fredholm$)$.

**On some simplifications and side effects of Theorem 4.16** Let $B_0$ be the (non-closed) algebra of all finite sums of products of convolution operators $C(a)$ with $a \in W(\mathbb{R}^2)$, multiplications $f I$ by $f \in C(\mathbb{R}^2)$, compact operators $J \in \mathcal{L}(X, \mathcal{P})$ and multiplications $g I$ by functions $g \in L^\infty(\mathbb{R}^2)$ for which $\| (I - T_n) g \| \rightarrow_{n \rightarrow \infty} 0$ holds. Further let $\mathcal{B}$ denote the closure of $B_0$ in $\mathcal{L}(X)$, which obviously includes $\mathcal{A}$. The following observation has also been made in [50].
Corollary 4.21. Let $A \in B_0$, $A_K$ its compression to a cone $K$ with vertex at 0, and $\Omega$ be chosen in such a way that all snapshots $W^t(\chi_K T_n)$, $t \in \delta \Omega$, are either trivial or operators of multiplication by the characteristic function of a half space. Then, for $K := T \{A_K \} T + (I - T)$,

- $A$ is Fredholm iff $A_K$ is Fredholm.
- $A$ is stable iff $A_K$ is invertible.

Proof. Let $A_K$ be Fredholm. We show that all snapshots $A^t := W^t(\chi_K T_n)$, $t \in \delta \Omega$, are invertible. Then, since $A \in L^T$, Theorem 4.16 provides the asserted criteria for the Fredholm property and stability, as well as formulas on the $\alpha$- and $\beta$-number and the index.

The snapshots $B^t := W^t(A)$ with $t \in \delta \Omega$ are invertible by analogous arguments as in the discussion after Theorem 1.58. Further it is easy to see that, for $t \in \delta \Omega \setminus K$, we have $A^t = I$.

If $t \in \delta \Omega \cap K$, then the $B^t$ are of the form $C(a_t)$ with some $a_t \in W(\mathbb{R}^3)$, thereby $A^t$ (which is the compression of $B^t$ to the respective half space $K^t_0$) is invertible as well by the previous proposition.

Moreover, Theorem 4.16 tells us that, for large $n$, $A_n$ is a Fredholm operator. Note that $A_n$ consists of compact operators, operators of multiplication by certain functions (which commute with $T_n$) and convolutions $C(a_t)$, $a = F(u)$, $u \in L^1$. From Proposition 4.17 we conclude that $T_n C(a_t)$ and $C(a_t) T_n$ are compact, hence $A_n = f_n I + C_n$ with an operator $f_n I$ of multiplication and a compact operator $C_n$. Consequently, $\operatorname{ind} A_n = 0$, which specifies the index formula (4.10).

Corollary 4.22. Let $A \in B$ and $A_K$ be its compression to a cone $K$ with vertex at 0. If $A_K$ is Fredholm then its index is zero.

Proof. By Theorem 1.2 we can choose an operator $B \in B_0$ in a neighborhood of $A$ such that its compression $B_K$ is Fredholm of the same index as $A_K$. Furthermore, we can choose a finite section domain $\Omega$ with the respective diffeomorphisms as in the second part of Example 4.19. Then we are in a position to apply the previous corollary to find $\operatorname{ind} B_K = 0$.

Finally, as another example, consider a finite section domain $\Omega$ being a polygon and all diffeomorphisms $\rho_t$ being the identical mappings. Define the algebra $E^T \subset L^T$ by the generators $\{C(a)\}$ with $a \in W(\mathbb{R}^2), \{f\}$ with $f \in C(\mathbb{R}^2)$ and $f \circ \rho$ being constant on the unit sphere $\delta B(0)$, $\{J\}$ with compact $J \in L(X,P)$, $\{gI\}$ where $g \in L^\infty(\mathbb{R}^2)$ with $\|I - T_n\| g \rightarrow \infty$, $\{\chi_{K} I\}$ with cones $K$ which intersect $\delta \Omega$ solely in its vertices, and by the sequence $T$. Then, for a sequence $A = T \alpha T + (I - T) \in E^T$, the snapshots $W^t(A)$ for all $t$ in the relative interior of one edge of $\delta \Omega$ coincide. Moreover, if the snapshot $W^v(\chi)$ at a vertex $v$ of $\Omega$ is Fredholm then the snapshots $W^t(\chi)$ for all $t$ in the interior of the flanking edges can be obtained as $P$-strong limits of $W^v(\chi)$, hence are invertible. Further, by Theorem 1.19, they fulfill

$$\|W^t(\chi)\| \leq \|W^v(\chi)\| \quad \text{and} \quad \|W^t(\chi)^{-t}\| \leq \|W^v(\chi)^{-t}\|. $$

For the latter we again refer to the proof of Theorem 1.58. Thus, in this particular setting, we can replace the index set $T$ in Theorem 4.16 by the finite set which consists of 0 and the vertices of $\Omega$. This was also discussed in [50] and picks up results of Maximenko [51].

4.3.4 Rich sequences

The consideration of rich sequences in the present text permits to include some more classes of multiplication operators which require the passage to subsequences, but then provide convenient snapshots again. Here we consider the set $R E^T \supset L^T$ of all sequences $\alpha \in F$ with the property
that every subsequence of $A$ has a $C_0$-localizable subsequence $A_g$, that is $A_g \in \mathcal{L}_g^T$. Moreover, we let $\mathcal{R}T^T$ denote the set of all sequences $TAT + \lambda(\mathbb{Z} - T) + J$ with $A \in \mathcal{R}L^T$, $\lambda \in \mathbb{C}$ and $J \in \mathcal{F}T$.

Note that, in contrast to the set $\mathcal{T}R^T$ as introduced in Section 3.2.4, we now have one common localizing setting for all subsequences under consideration. Of course, these definitions are much more restrictive, but their greatest advantage is that $\mathcal{R}L^T$ and $\mathcal{R}T^T$ can straightforwardly be proved to be Banach algebras. Therefore we again only need to investigate some generators. Nevertheless, $\mathcal{R}T^T$ is included in the set $\mathcal{T}R^T$, hence the main results of Theorem 3.22 are applicable to all sequences in $\mathcal{R}T^T$.

Thus, for the Fredholm and stability criteria, the $\alpha$- and $\beta$-numbers and for the formulas which describe the asymptotics of the norms, condition numbers and pseudospectra of each sequence $A \in \mathcal{R}T^T$ we refer to Theorem 3.22 without stating the results again. Here we only get to know some elements in $\mathcal{R}L^T$.

**Very slowly oscillating functions** We say that a bounded and continuous function $f$ is very slowly oscillating, if for each $x \neq 0$ and each $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(rs) - f(rx)| < \epsilon \quad \text{for all} \ s \in U_\delta(x) \text{ and all} \ r > 0.$$  

The set VSO of all very slowly oscillating functions is a Banach algebra and includes $C(\mathbb{R}^2)$. A basic example for a function $f \in \text{VSO} \setminus C(\mathbb{R}^2)$ is given by $f(x) = \sin \ln |x|, |x| \geq 1$.

**Proposition 4.23.** For every $f \in \text{VSO}$ the sequence $\mathcal{A} = \{fI\}$ belongs to $\mathcal{R}L^T$ with $W^0(\mathcal{A}) = fI$ and the other snapshots $W^t(\mathcal{A}_g)$ at the boundary points $t \in \delta \Omega$ are multiples of the identity.

**Proof.** Clearly, if we have a $T$-structured subsequence $A_g$ of $A$ then it follows immediately from the definition that $A_g$ belongs to $L_g^T$ with the snapshots as asserted. Therefore, it suffices to show that $A$ is rich. Let $h \in \mathcal{H}_+$, and choose a dense and countable subset $\{t_i\}_{i \in \mathbb{N}}$ of $T$. Then there is a subsequence $g^i \subset h$ such that $W^t(\mathcal{A}_g^i)$ exists. To see this, notice that the sequence $(f(h_nt_i))_{n \in \mathbb{N}}$ has a convergent subsequence $(g^i_n t_i)_{n \in \mathbb{N}}$ with limit $f_{t_i}$ due to the Bolzano-Weierstrass Theorem. Now one easily derives from the definition of VSO that $W^t(\mathcal{A}_g) = f_{t_i} I$. By this argument we successively find subsequences $g^{i+1}_n \subset g^i_n$ such that $W^{t_{i+1}}(\mathcal{A}_g^{i+1})$ exists. Define $g_n := g^i_n$ for every $n$ and find that $W^s(\mathcal{A}_g)$ exists for every $s$ in this dense subset $\{t_i\}_{i \in \mathbb{N}}$ of $T$. For an arbitrary $t \in T$ there is a sequence $(s_m)_m \subset \{t_i\}_{i \in \mathbb{N}}$ which converges to $t$. One now easily checks that $(W^{s_m}(\mathcal{A}_g)_m$ converges to an operator $B$ and that $B$ provides the $\mathcal{P}$-strong limit of $(E^{-s_i}(fI))$. Thus, $\mathcal{A}_g \in \mathcal{F}_g^T$. \hfill \Box

We note that the snapshots of VSO-sequences are shift invariant, hence the above Corollaries 4.21 and 4.22 stay valid if we enrich the algebras $\mathcal{B}_0$ and $\mathcal{B}$ by very slowly oscillating functions. Also the algebra $\mathcal{E}_T^T$ permits an extension in this vein. Moreover, we want to point out that the sequences arising from VSO are treatable by arbitrary finite section domains $\Omega$. Finally, we consider a further class of multiplications which depends on multiplicities which shape of $\Omega$.

**Periodic functions** Let the finite section domain $\Omega$ be given. A function $f \in C(\mathbb{R}^2)$ is said to be $\Omega$-periodic (or periodic with respect to $\Omega$) with period $P$ if, for every $t \in \delta \Omega$, the functions $f_t : [1, \infty) \to \mathbb{C}, \alpha \mapsto f(\alpha t)$ are periodic with period $P$, that is $f_t(\alpha) = f_t(\alpha + P)$ for every $\alpha$.

A simple example for such a function is shown in Figure 4.2, where $\Omega = B_1(0)$ and $P$ equals $\frac{2\pi}{T}$.

Let $f$ be $\Omega$-periodic, $h$ be a strictly increasing sequence of positive integers, and decompose each $h_n = k_nP + r_n + 1$ with $k_n \in \mathbb{Z}$ and $r_n \in [0, P)$. Due to the Bolzano-Weierstrass theorem there is a
Figure 4.2: The function \( f(z) := \sin(7|z|) \sin(5 \arg(z)) \), \( z \neq 0 \) is \( \Omega \)-periodic for \( \Omega = B_1(0) \).

convergent subsequence of \((r_n)\) with limit \( r \), and we denote by \( g \) the respective subsequence of \( h \). Then, for every \( t \in \delta \Omega \), the operator \( W^t(A_g) \) exists and equals to the operator of multiplication by the continuous function which takes the value \( f((1 + r + s)t) \) on the set \( \delta K_t^0 + nP + s \) for every \( n \in \mathbb{Z} \) and \( s \in [0, P] \).

If, in particular, for one \( t \in \delta \Omega \), the boundary \( \delta K_t \) of the local cone \( K_t \) contains the origin then \( f \) proves to be constant along the ray \( \alpha t, \alpha \in [1, \infty) \), hence \( W^t(A_g) = f(t)I \).

Now, one easily verifies

**Proposition 4.24.** For every \( \Omega \)-periodic function \( f \) the sequence \( A = \{fI\} \) belongs to \( \mathcal{RL}^T \).

**Remark 4.25.** We finally want to mention, without giving details, that bounded and continuous functions \( f \) which are constant along each curve \( \delta(\alpha \Omega) \), \( \alpha > 0 \), and which are slowly oscillating also define operators \( A = fI \) of multiplication that can be tackled by the tools of the present text. Here and in the literature a continuous function is called slowly oscillating if, for every \( r > 0 \),

\[
\sup\{|f(x) - f(y)| : y \in U_r(x)\} \to 0 \quad \text{as} \quad |x| \to \infty.
\]

The idea is to check that the arising sequences \( A \in \mathcal{F} \) are rich, to construct an adapted “blowing-up-procedure” and to define a localizing setting as in Proposition 4.10 such that one finally gets \( A \in \mathcal{LR}^T \).

The present ideas and proofs for convolution type operators on \( L^p(\mathbb{R}^2) \) also work for analogous operators on \( L^p(\mathbb{R}^N) \) and even on \( L^p(\mathbb{R}^N, \mathbb{C}^m) \), the product of \( m \) copies of the space \( L^p(\mathbb{R}^N) \). The discrete multidimensional setting \( L^p(\mathbb{Z}^N, X) \) is more involved, since it is not possible to rectify the boundary \( \delta \Omega \) by diffeomorphisms as in the continuous case. At least for finite sections with respect to polyhedra \( \Omega \) which have their vertices in \( \mathbb{Z}^N \) there are comprehensive results [62], [60], Section 6.2 in [63], [69] for the case \( p = 2 \), and Section 4.1 in [44]. For deeper results on the Fredholm theory for such finite section sequences and its consequences we refer also to [69], which considers the case \( p = 2 \) and already provides results on the asymptotics of norms, condition numbers and pseudospectra. An extension of the Fredholm theory to the cases \( p \in [1, \infty) \) with finite dimensional \( X \) can be found in [80]. By the tools of the present text this now can easily be done for arbitrary Banach spaces \( X \), and can be completed by the statements on the norms, condition numbers and pseudospectra, by the index formula and by certain further classes of rich sequences. Notice that Lindner [43] also considered the stability of finite section sequences with respect to more general starlike domains \( \Omega \).
4.4 Further relatives of band-dominated operators

At the end of the 4th part we return to our prototypic example, the class of band-dominated operators on $l^p(Z, X)$, which has been a guide through the whole text, and we touch upon some related families of operators.

At that we have two different aims in mind. On the one hand we want to translate what we obtained in the already studied case to these similar settings. We weaken the notion of band-dominated operators into several directions without loss of the essential results on the applicability of the finite section method. Just to give some keywords, we deal with weakly band-dominated, triangular, quasi-triangular and triangle-dominated operators.

On the other hand, for more specific families of operators, such as some comfortable classes of Toeplitz operators or slowly oscillating operators, also the conditions in the theorems and the respective conclusions turn into a simpler and more concrete form which we want to expose.

Since this is just to give some links to well known forerunners and to sound out some possible future work, we keep the presentation as brief as possible.

4.4.1 Band-dominated operators on $l^p(N, X)$

Recall the approximate projection $P$ and the sequence $(L_n)$ in $L(l^p(Z, X))$. Let $P \in L(l^p(Z, X))$ denote the canonical projection onto the subspace of all sequences $(x_i)$ whose entries $x_i$ vanish for $i \leq 0$. This subspace can be (isometrically) identified with the space $l^p(N, X)$ of all one-sided infinite and $p$-summable (for $1 \leq p < \infty$) or bounded (for $p = \infty$, resp.) sequences. For every bounded linear operator $A$ on $l^p(Z, X)$ the compression $PAP$ yields a bounded linear operator on $l^p(N, X)$, and vice versa for every $B \in L(l^p(N, X))$ we have $PBP + Q \in L(l^p(Z, X))$, where $Q := I - P$.

We introduce the algebra $A_{l^p(N)}$ of band-dominated operators on $l^p(N, X)$ as the set of all compressions of band-dominated operators on $l^p(Z, X)$, and note that we would obtain exactly the same if we would build upon shifts and operators of multiplication by bounded sequences as generators for the algebra $A_{l^p(N)}$. Analogously, we take the compressions $PL_nP$ as the finite section projections, denote them again by $L_n$, and let $\mathcal{F}_{A_{l^p(N)}}$ stand for the Banach algebra which is generated by all finite section sequences of rich band-dominated operators. Then we can completely translate Theorems 2.31, 2.32, 2.33, 2.35 and 3.23 in the sense of this mentioned embedding. Of course, for every $\mathcal{A} \in \mathcal{F}_{A_{l^p(N)}}$, we have the basic snapshot $W^0(\mathcal{A})$ which appears in the form $P W^0(\mathcal{A})P + \lambda Q$, and only one further “direction” which provides relevant snapshots $W^1(\mathcal{A}_g)$ since the snapshots $W^{-1}(\mathcal{A}_g)$ are always trivial. One usually applies the flip operator $J$, given by $(x_i) \mapsto (x_{i-1})$, to achieve that these $W^1(\mathcal{A}_g)$ can be regarded as operators $JW^1(\mathcal{A}_g)J$ acting on $l^p(N, X)$ as well.

Instead of using such flipped versions $JE_n^{-1}(A_n)L_nJ = JV_nA_nL_nV_{-n}J$ of $E_n^{-1}(A_n)$ for the construction of the snapshots at $t = 1$, one can also employ the operators

$$W_n : l^p(N, X) \to l^p(N, X), \quad (x_1, x_2, x_3, \ldots) \mapsto (x_n, x_{n-1}, \ldots, x_1, 0, 0, \ldots)$$

and set $E_n^{-1}(A_n) := W_nA_nW_n$ to obtain exactly the same outcome. Actually, the second approach has been the initial one [83] and constitutes a cornerstone in the history of the Banach algebra techniques which make up the largest part of this thesis. Before we turn to the class of Toeplitz operators that has been the engine in the development throughout the last decades we shortly examine a class of band-dominated operators on $l^p(N, C)$ whose finite sections possess surprisingly simple Fredholm and stability criteria.
4.4.2 Slowly oscillating operators

We call a function $a \in L^\infty(\mathbb{N}, \mathbb{C})$ slowly oscillating if, for every $r > 0$,

$$\sup \{|a(x) - a(y)| : y \in \mathbb{N} \text{ such that } |x - y| \leq r\} \to 0 \quad \text{as} \quad |x| \to \infty.$$ 

A band-dominated operator $A$ is said to be a slowly oscillating operator if its coefficients are slowly oscillating, that is $A$ is the norm limit of a sequence of band operators having only slowly oscillating coefficients.

**Theorem 4.26.** Let $A$ be a slowly oscillating operator and $\mathcal{A} := \{L_n AL_n\}$ be the sequence of its finite sections. Then $\mathcal{A}$ is Fredholm if and only if $A$ is a Fredholm operator. In this case $\alpha(\mathcal{A}) = \beta(\mathcal{A}) = \max\{\dim \ker A, \dim \text{coker} A\}$ are splitting numbers.

In particular, the invertibility of $A$ is necessary and sufficient for the stability of $\mathcal{A}$.

This result on the stability has been established by Roch, Lindner and Rabinovich in [46] ($p = 2$, banded operators) and [68] ($1 < p < \infty$) for slowly oscillating operators, whereas it is well known for a much longer time for large classes of Toeplitz operators (see the pioneering work in [24], [83] or the comprehensive monographs [12], [13]). For the remaining cases $p = 1, \infty$ Theorem 4.26 is new. Actually, the additional snapshots of the sequences in this Theorem are Toeplitz, and the proof immediately follows if we invest some facts on such operators. Therefore we defer the proof to the end of the next section.

4.4.3 Toeplitz operators with continuous symbol

A one-sided infinite matrix $T(a) := (a_{j-k})_{j,k \in \mathbb{N}}$ with complex entries $a_i$ is called Toeplitz matrix. Suppose that $T(a)$ induces a bounded linear operator on $l^p(\mathbb{N}, \mathbb{C})$ via $(x_i) \mapsto (\sum_{k \in \mathbb{N}} a_{i-k} x_k)$. Then there is a function $a \in L^\infty(\mathbb{T})$ on the unit circle $\mathbb{T}$ whose Fourier coefficients form the sequence $(a_i)_{i \in \mathbb{Z}}$; we further denote this operator again by $T(a)$ and call it the Toeplitz operator with the generating function (or symbol function) $a$.

The set of all such symbols $a$ which define a Toeplitz operator $T(a)$ shall be denoted by $M^p$ in what follows, and it is well known that, provided with pointwise operations and the norm $\|a\|_{M^p} := \|T(a)\|_{l^p(\mathbb{N}, \mathbb{C})}$, $M^p$ forms a Banach algebra, the so-called multiplier algebra.

Recall the shift operators $V_k$ and the finite section projections $L_n$ on $l^p(\mathbb{Z}, \mathbb{C})$, as well as the canonical projection $P$ which maps $l^p(\mathbb{Z}, \mathbb{C})$ onto $l^p(\mathbb{N}, \mathbb{C})$ and compresses the projections $PV_kP$ and $PL_nP$ again by $V_k$ and $L_n$. Clearly, the shift $V_k$ on $l^p(\mathbb{N}, \mathbb{C})$ is a Toeplitz operator having the trigonometric polynomial $t \mapsto t^k$ as generating function. So, we see that for every trigonometric polynomial $a$ the corresponding Toeplitz operator $T(a)$ is banded.

Let $C^p$ denote the closure of the set of all trigonometric polynomials in $M^p$ and refer to its elements as continuous symbols. This definition immediately gives that every Toeplitz operator with continuous symbol is band-dominated.

It is well known that $M^2 = L^\infty(\mathbb{T})$ and $C^2 = C(\mathbb{T})$, but for $p \neq 2$ the equalities must be replaced by proper inclusions “$\subset$”. Moreover, $M^1 = C^1 = M^\infty = C^\infty = W$ is the so-called Wiener algebra $W$ of all functions $a \in L^\infty(\mathbb{T})$ whose Fourier coefficients $(a_i)$ fulfill $\sum_{i \in \mathbb{Z}} |a_i| < \infty$. All these definitions and results are taken from [12], Chapter 2 for the cases $1 \leq p < \infty$. The case $p = \infty$ is tackled by its duality to $l^1$ and the observation $T(a)^* = T(\overline{a})$ in [8], for example.

**Theorem 4.27.** (cf. [12], 2.47. and Coburn’s theorem [12], 2.38.)

Let $a \in C^p$. Then the Toeplitz operator $T(a)$ is Fredholm if and only if $a$ does not have any zeros on $\mathbb{T}$. In this case, $T(a)$ is one-sided invertible and its index equals $-\text{wind}(a,0)$, that is minus the winding number of the curve $a$ around the origin. Moreover, $a^{-1} \in C^p$ and $T(a^{-1})$ is a regularizer for $T(a)$. 

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Theorem 4.28. Let $\mathcal{A} = \{A_n\} := \{L_n T(a) L_n\}$ with $a \in C^p$. Then

- $\mathcal{A}$ is Fredholm if and only if $a$ does not vanish on $\mathbb{T}$. In this case the approximation numbers of the $A_n$ have the splitting property with
  \[ \alpha(h) = \beta(h) = |\text{ind} T(a)| = \max(\dim \ker T(a), \dim \coker T(a)) = |\text{wind}(a, 0)|. \]

Otherwise, all approximation numbers tend to zero as $n \to \infty$.

- The norms $\|A_n\|$ and $\|A_n^{-1}\|$ converge to the limits (with $\tilde{a} := a(z^{-1})$)
  \[ \max\{\|T(a)\|, \|T(\tilde{a})\|\} \text{ and } \max\{\|T(a)^{-1}\|, \|T(\tilde{a})^{-1}\|\}, \]

- For every $\epsilon > 0$ and every $N \in \mathbb{Z}_+$ the $(N, \epsilon)$-pseudospectra $\text{sp}_{N, \epsilon} A_n$ of the finite sections converge with respect to the Hausdorff distance to the union $\text{sp}_{N, \epsilon} T(a) \cup \text{sp}_{N, \epsilon} T(\tilde{a})$ as $n$ goes to infinity.

- $\mathcal{A}$ is stable if and only if $a$ does not vanish on $\mathbb{T}$ and its winding number is zero. In this case the condition numbers $\text{cond}(A_n)$ converge to
  \[ \max\{\|T(a)\|, \|T(\tilde{a})\|\} \cdot \max\{\|T(a)^{-1}\|, \|T(\tilde{a})^{-1}\|\}. \]

This theorem has grown to several expansion stages since the middle of the 20th century, and it holds even for much larger classes of Toeplitz operators, such as for piecewise continuous symbols. A beautiful treatise on this topic with a clear and standalone presentation of the theory around Toeplitz operators and its history can be found in the already mentioned monograph [13] by Böttcher and Silbermann. The particular case of banded Toeplitz matrices is subject of the book [8] by Böttcher and Grudsky. For the most complete resource on this topic and its development we refer to the comprehensive monograph [12] by Böttcher and Silbermann. In the present work, Theorem 4.28 is just a special case of what we know for band-dominated operators on $l^p(\mathbb{N}, \mathbb{C})$ from Section 4.4.1 and Theorem 4.26, taking Theorem 4.27 into account. We also want to record that the smallest closed subalgebra of $L(l^p(\mathbb{N}, \mathbb{C}))$ containing all Toeplitz operators with continuous symbol is of the form
\[ \{T(a) + K : a \in C^p, K \mathcal{P}\text{-compact}\}, \]
and the smallest closed subalgebra of $\mathcal{F}$ containing all respective finite section sequences equals
\[ \{\{L_n T(a) L_n + L_n K L_n + W_n L W_n + G_n\} : a \in C^p, K, L \mathcal{P}\text{-compact}, \{G_n\} \in \mathcal{G}\}. \]

To check this take [12], 2.14, 7.7, and Proposition 1.20 into account.

We close this section with the pending

Proof of Theorem 4.26. Let $A$ be a slowly oscillating Fredholm operator and $\mathcal{A} = \{L_n A L_n\}$ be its finite section sequence. As in Section 4.4.1 we see that Corollary 2.30 can be translated to the $l^p(\mathbb{N}, \mathbb{C})$-case, hence provides that every snapshot of $A$ is Fredholm. Notice that $A$ is rich, due to Corollary 1.60. Let $g \in \mathcal{H}_+$ be such that $W^1(a_g)$ exists, and choose a sequence of banded slowly oscillating operators $A^n$ which converge to $A$ in the norm as $n$ goes to infinity. Now, pass to a subsequence $h^1$ of $g$ such that $W^1\{L_{h^1} A^1 L_{h^1}\}$ exists, then to a subsequence $h^2$ of $h^1$ such that $W^1\{L_{h^2} A^2 L_{h^2}\}$ exists, and so on. Then define $h = (h_n)$ by $h_n := h_n^n$. It is obvious from

\[\text{The additional operators } T(\tilde{a}) \text{ cannot be omitted in general, see [13], Example 7.14 for instance.} \]
the definition that the snapshots \( W^1 \{ L_{h_n} A^k L_{h_n} \} \) are banded Toeplitz operators for every \( k \) and converge to \( W^1(A) \), which equals \( W^1(\mathbb{A}) \), as \( k \) goes to infinity hence \( W^1(\mathbb{A}) \) is a Fredholm Toeplitz operator with a continuous generating function. Moreover, from Theorem 2.31 we get that \( 0 = \lim_n \text{ind} L_{h_n} AL_{h_n} = \text{ind} A + \text{ind} W^1(\mathbb{A}) \). Therefore

\[
\dim \ker A + \dim \ker W^1(\mathbb{A}) = \dim \text{coker} A + \dim \text{coker} W^1(\mathbb{A})
\]

where, by Theorem 4.27, at least one of these four non-negative integers is equal to zero. Employing Theorem 2.31 again, we find the asserted splitting numbers at least for all subsequences \( \mathbb{A} \) which have a snapshot \( W^1(\mathbb{A}) \), and Theorem 2.32 provides the Fredholm property of \( \mathbb{A} \) and the \( \alpha \)- and \( \beta \)-number of the whole sequence \( \mathbb{A} \). Assume that \( \alpha(\mathbb{A}) \) is not a splitting number for \( \mathbb{A} \). Then there is a subsequence \( \mathbb{A}_j \) of \( \mathbb{A} \) such that \( \lim_n s_{\alpha(\mathbb{A})}(A_{j_n}) \) exists and is larger than zero. Now pass to a subsequence \( \mathbb{A}_j \) of \( \mathbb{A}_j \) which has a snapshot \( W^1(\mathbb{A}_j) \) and gives a contradiction. The same argument works for \( \beta(\mathbb{A}) \). The stability criterion follows from Theorem 2.21. \( \square \)

### 4.4.4 Cross-dominated operators and the flip

Here, we consider the closed algebra \( \mathcal{X}_0 \) of bounded linear operators on \( l^p(\mathbb{Z}, X) \) which is generated by the shift operators \( V \), \( \alpha \in \mathbb{Z} \), by the operators of multiplication \( aI \) by sequences \( a \in l^\infty(\mathbb{Z}, \mathcal{L}(X)) \) and, in contrast to the algebra \( \mathcal{A}_{lp} \) of band-dominated operators, additionally by the flip \( J \) which maps the sequence \( (x_i) \) to \( (x_{-i}) \). We refer to such operators as cross-dominated operators, which is reasonable since the matrix representations of finite combinations of the generators look like a “cross” consisting of two orthogonal bands of finite width. For some similar material on an algebra of convolution type operators and a flip we refer to [71].

Clearly, the approach of Section 2.5 is not appropriate here, since the homomorphisms \( E^{\pm 1}_a \) as defined there do not provide a \( \mathcal{P}^{\pm 1} \)-strongly convergent copy \( (E^{\pm 1}_a(A))_{a \in \mathbb{A}} \) of the finite section sequence \( \{ A_n \} := \{ L_n JL_n \} \) associated to the flip operator, not even for any subsequence. Hence, one gets the impression that the present setting is more involved. Actually, this is false. To see this, we use a tricky transformation which shuffles the entries of the sequences under consideration:

\[
S : l^p(\mathbb{Z}, X) \to l^p(\mathbb{N}, X), \quad (x_i)_{i \in \mathbb{Z}} \mapsto (y_k)_{k \in \mathbb{N}} \quad \text{with} \quad y_k = \begin{cases} x_{\frac{k}{2}} & \text{if } k \text{ is even} \\ x_{\frac{k-1}{2}} & \text{if } k \text{ is odd}. \end{cases}
\]

Obviously, this is an isometric isomorphism, and it can now straightforwardly be checked, by considering the generators, that \( \Psi : A \mapsto SAS^{-1} \) provides an isometric algebra isomorphism which translates \( \mathcal{X}_0 \) into the algebra \( \mathcal{A}_{lp(\mathbb{N})} \) of band-dominated operators on \( l^p(\mathbb{N}, X) \). Also the finite section projections \( L_n \) are transformed into the respective finite section projections

\[
\hat{L}_n : l^p(\mathbb{N}, X) \to l^p(\mathbb{N}, X) : (y_i)_{i \in \mathbb{N}} \mapsto (y_1, y_2, \ldots, y_{2n+1}, 0, \ldots).
\]

Thus, all results on the Fredholm property, on stability, on the formulas for the \( \alpha \)- and \( \beta \)-numbers and the index, on the asymptotic behavior of norms, condition numbers and on the convergence of pseudospectra are available for cross-dominated operators as well. To be a bit more precise we note that the isometry \( \Psi \) translates the operator of multiplication by the sequence \( (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots) \in l^\infty(\mathbb{Z}, \mathcal{L}(X)) \) into the operator of multiplication by the bounded sequence \( (a_0, a_1, a_{-1}, a_2, a_{-2}, \ldots) \in l^\infty(\mathbb{N}, \mathcal{L}(X)) \), and vice versa. Furthermore, using \( d := (I, 0, I, 0, I, 0, \ldots) \), \( V_k := PV_kP \) and \( \hat{L}_1 := PL_1P \), we straightforwardly check that

\[
\Psi(V_1) = SV_1S^{-1} = \hat{d}\hat{V}_{-2} + (I - \hat{d})\hat{V}_2 + \hat{V}_1\hat{L}_1 \quad \text{and} \quad \Psi(J) = SJJS^{-1} = d\hat{V}_{-1} + \hat{V}_1(I - dI).
\]
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In matrix notation this is

\[ \Psi(V_1) = \begin{pmatrix} 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Psi(J) = \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ I & 0 \end{pmatrix}. \]

Conversely, \( \Psi^{-1}(\hat{V}_1) = V_1 J P + J (I - P) \) and \( \Psi^{-1}(\hat{V}_1) = J P \) if \( L_1 = \hat{V}_1 \), i.e.

\[ \begin{pmatrix} \vdots \\ 0 & I \\ I & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & I \end{pmatrix}. \]

4.4.5 Quasi-triangular operators

**Definition 4.29.** Let \( X \) be a Banach space. Let further \( A \in \mathcal{L}(X) \) and assume that there is a bounded sequence \((L_n) \subset \mathcal{L}(X)\) of projections, such that

\( \| (I - L_n) A L_n \| \to 0 \) as \( n \to \infty \).

Then \( A \) is said to be quasi-triangular with respect to \((L_n)\). If even \( (I - L_n) A L_n = 0 \) holds for every \( n \) then \( A \) is called triangular w.r.t. \((L_n)\).

For a collection of several basic properties and applications of quasi-triangular operators on Hilbert spaces see [2]. By some straightforward adoptions we find that analogues of Proposition 4.2 and Corollary 4.3 hold for an invertible quasi-triangular operator \( A \) whose inverse is quasi-triangular, too. So, we again see that a sequence of projections \((L_n)\) which makes an operator \( A \) quasi-triangular is quite well suited to define a finite section method. In what follows, we assume that \((L_n)\) itself converges in a convenient manner: Let \( \mathcal{P} = (P_n) \) be a uniform approximate identity on \( X \), which equips \( X \) with the \( \mathcal{P}\)-dichotomy, and suppose that all \( L_n \) are \( \mathcal{P}\)-compact projections which tend \( \mathcal{P}\)-strongly to the identity. (We do not grow tired of noting that the classical situation with compact \( L_n \) which converge \( \ast \)-strongly, and without a \( \mathcal{P}\), is covered by our proofs as well.)

**Proposition 4.30.** The set \( QT_{\mathcal{P}, (L_n)} \) of all operators in \( \mathcal{L}(X, \mathcal{P}) \) which are quasi-triangular w.r.t. \((L_n)\) is a Banach subalgebra of \( \mathcal{L}(X, \mathcal{P}) \) and contains \( QD_{\mathcal{P}, (L_n)} \) the set of all quasi-diagonal operators (w.r.t. \((L_n)\)) as well as the ideal \( K(X, \mathcal{P}) \).

The proof of this proposition is easy. Furthermore the finite section sequence \((L_n A L_n)\) of each operator \( A \in QT_{\mathcal{P}, (L_n)} \) converges \( \mathcal{P}\)-strongly to \( A \), since \( A \in \mathcal{L}(X, \mathcal{P}) \) and \( L_n \to I \) \( \mathcal{P}\)-strongly.

So, we introduce \( \mathcal{F}_{QT} \) as usual, namely as the closed algebra which is generated by all sequences \( \{L_n A L_n\} \) with \( A \in QT_{\mathcal{P}, (L_n)} \). Notice that \( W(\hat{A}) := \mathcal{P}\)-lim \( A_n L_n \) exists for every sequence \( \hat{A} = \{A_n\} \in \mathcal{F}_{QT} \) by Theorem 1.19, and that \( W(\hat{A}) \) is quasi-triangular.
Theorem 4.31. Let $\mathcal{A} = \{A_n\} \in \mathcal{F}_{QT}$ and suppose that $W(\mathcal{A})$ is Fredholm with a regularizer in $QT_{\mathcal{P}}(L_n)$. Then the sequence $\mathcal{A}$ is Fredholm, $\alpha(\mathcal{A}) = \dim \ker W(\mathcal{A})$ and $\beta(\mathcal{A}) = \dim \coker W(\mathcal{A})$ are splitting numbers and $\lim_{n \to \infty} A_n = \text{ind} W(\mathcal{A})$. If $W(\mathcal{A})$ is not Fredholm then infinitely many approximation numbers $s_k^0(A_n)$ or $s_k^1(A_n)$ tend to zero as $n \to \infty$.

The sequence $\mathcal{A}$ is stable if $W(\mathcal{A})$ is invertible and its inverse belongs to $QT_{\mathcal{P}}(L_n)$. It is not stable if $W(\mathcal{A})$ is not invertible.

Proof. We take up the sequence algebraic framework $\mathcal{F}^T$ as defined for quasi-diagonal operators in Section 4.1 and note that $\mathcal{A} \in \mathcal{F}^T$.

Firstly, we check that $\mathcal{A} - \{L_nW(\mathcal{A})L_n\} \in \mathcal{G}$: This obviously holds true if $\mathcal{A}$ is the finite section sequence $\{L_nAL_n\}$ of a single $A \in QT_{\mathcal{P}}(L_n)$, hence also for a finite product of the form $\Pi_i \{L_nA_iL_n\}$ with $A_i \in QT_{\mathcal{P}}(L_n)$ (where we use that $\|(I - L_n)A_iL_n\| \to 0$), and since $\mathcal{G}$ is a closed ideal the assertion follows for all $\mathcal{A} \in \mathcal{F}_{QT}$. Now, let $B = QT_{\mathcal{P}}(L_n)$ be a regularizer for $A := W(\mathcal{A})$. Then $B := \{L_nBL_n\}$ belongs to $\mathcal{F}^T$ as well and

$$A_nL_nBL_n = L_nABL_n - L_nA(I - L_n)BL_n + (A_n - L_nAL_n)L_nBL_n = L_n + L_n(AB - I)L_n - L_nA(I - L_n)BL_n + (A_n - L_nAL_n)L_nBL_n$$

where $\{L_nA(I - L_n)BL_n\}, \{(A_n - L_nAL_n)L_nBL_n\} \in \mathcal{G}$ and $AB - I$ is $\mathcal{P}$-compact. We can make an analogous observation for $L_nBL_nA_n$ using the quasi-triangularity of $A$ and we get that $\mathcal{A}$ is regularly $\mathcal{F}^T$-Fredholm, hence Corollary 2.24 together with Theorems 2.13, 2.14, 2.9 and 2.21 give the assertion. 

Notice that the additional condition - the regularizer belongs to $QT_{\mathcal{P}}(L_n)$ - is not redundant in general, as a simple example demonstrates: The shift operator $V_{-1}$ on $l^p(\mathbb{Z}, \mathbb{C})$ is triangular with respect to the standard sequence $(L_n)$, but its regularizer $V_1$ is neither triangular nor quasi-triangular w.r.t. $(L_n)$. Since all regularizers coincide modulo the ideal of compact operators, none of them is quasi-triangular.

Even for the case of an invertible operator $A$ in a setting with projections $L_n$ having infinite rank there are counterexamples: Set $X = l^p[0, 2]$ and consider the operators $C, D$ on $X$ given by

$$(Cf)(x) = \begin{cases} f(2x) & \text{if } x \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad (Df)(x) = \begin{cases} f(2x - 2) & \text{if } x \in [1, 2] \\ 0 & \text{otherwise} \end{cases}.$$ 

Clearly, both of them are invertible from the left with the left inverses $(C^{-1}(x))(x) = g(x/2)$ and $(D^{-1}(x))(x) = g(x/2 + 1)$, respectively. One easily checks that the following operator $B$ on $X^3$ is invertible

$$B = \begin{pmatrix} C & D & 0 \\ 0 & 0 & C^{-1} \\ 0 & 0 & D^{-1} \end{pmatrix} \quad \text{and} \quad B^{-1} = \begin{pmatrix} C^{-1} & 0 & 0 \\ D^{-1} & 0 & 0 \\ 0 & C & D \end{pmatrix}. $$

Now take $A \in \mathcal{L}(l^{p}(\mathbb{Z}, X))$ as the block diagonal operator having the blocks $A_n$ along its diagonal. Clearly, $A$ is invertible and triangular w.r.t. $(L_n)$, but its inverse isn’t.

Nevertheless, if all $L_n$ are finite rank operators, $A$ is quasi-triangular w.r.t. $(L_n)$ and invertible, then $A^{-1}$ is also quasi-triangular. To see this, note that

$$\|L_n - L_nA^{-1}L_nAL_n\| \leq \|L_n\|\|A^{-1}\|\|(I - L_n)AL_n\| \to 0 \quad \text{as } n \to \infty,$$

that is, for large $n$, we get the invertibility of $L_nAL_n$ from the left, and there is a uniformly bounded sequence of left inverses in $\mathcal{L}(\text{im} L_n)$. Since $\dim \text{im} L_n < \infty$ the $L_nAL_n$ are even
invertible with uniformly bounded inverses, hence
\[
\|L_n - L_n AL_n A^{-1} L_n\| = \|L_n AL_n (L_n - L_n A^{-1} L_n AL_n) (L_n AL_n)^{-1} L_n\| \\
\leq \|L_n AL_n\| \|L_n - L_n A^{-1} L_n AL_n\| \|(L_n AL_n)^{-1} L_n\| \to 0.
\]
Thus, \(A^{-1} \in QT_{\mathcal{P}, (L_n)}\) follows from
\[
\|(I - L_n) A^{-1} L_n\| \leq \|A^{-1}\| \|A(I - L_n) A^{-1} L_n\| = \|A^{-1}\| \|L_n - AL_n A^{-1} L_n\| \\
\leq \|A^{-1}\| \|L_n - L_n AL_n A^{-1} L_n\| + \|A^{-1}\| \|(I - L_n) AL_n\| A^{-1} L_n\| \to 0.
\]

### 4.4.6 Triangle-dominated and weakly band-dominated operators

Encouraged by the latest observations we ask if we can enlarge the class of band-dominated operators on \(l^p(\mathbb{Z}, X)\) in such a way, that also quasi-triangular operators are included, and (at least some of) the results on the stability or almost stability of the respective finite sections still hold. So, take the usual finite section projections \((L_n)\) of Section 2.5 for the following definition.

**Definition 4.32.** Let \(\mathcal{T}_p\) denote the subset of \(\mathcal{L}(l^p, \mathcal{P})\) whose elements \(A\) additionally fulfill
\[
\lim_{m \to \infty} \sup_{n > m} \|(I - L_n) A L_{n - m}\| = 0. \quad (4.14)
\]
The operators in \(\mathcal{T}_p\) are called triangle-dominated (with respect to \((L_n)\)).

Clearly, all band-dominated operators (by Theorem 1.55) as well as all operators which are quasi-triangular w.r.t. \((L_n)\) are contained in \(\mathcal{T}_p\), and \(\mathcal{T}_p\) easily proves to be a Banach algebra. We know that the band-dominated operators require at least three “directions” from which we should look at their finite section sequences to get enough information: Besides the central snapshot \(W_0(A)\) we need to observe the two critical points where the \(L_n\) perform the truncation. Therefore, we let \(\mathcal{F}_{\mathcal{T}_p}\) denote the smallest Banach algebra which contains all finite section sequences of rich triangle-dominated operators, and we use an algebraic framework \(\mathcal{F}^T\) as in Section 2.5.1 in order to study these sequences. Notice that \(W_0(A)\) is triangle-dominated for every \(A \in \mathcal{F}_{\mathcal{T}_p}\) and, proceeding exactly as in the proof of Proposition 2.28, we find

**Proposition 4.33.** Let \(A \in \mathcal{F}_{\mathcal{T}_p}\) be such that \(W_0(A)\) is \(\mathcal{P}\)-Fredholm and has a \(\mathcal{P}\)-regularizer in \(\mathcal{T}_p\), and let \(g \in \mathcal{H}_p\). Then there is a \(T\)-structured subsequence \(h_k\) of \(h_n\) which is \(J_k^T\)-Fredholm.

Now, also the assertions of Theorems 2.31, 2.32 and 2.33 hold true for all sequences \(A \in \mathcal{F}_{\mathcal{T}_p}\) having the additional property that one \(\mathcal{P}\)-regularizer of \(W_0(A)\) (if one exists) automatically belongs to \(\mathcal{T}_p\).\(^{10}\) That means, we have criteria for the stability and Fredholm property of such sequences, and we know that the asymptotic behavior of the approximation numbers and indices of their entries is closely connected with the kernel dimensions and indices of the snapshots.

Actually, the need for the additional condition on the regularizers \(B\) of \(W_0(A)\) stems from the proof of Proposition 2.28, where we use that for both, \(W_0(A)\) and \(B\), the relation (4.14) holds. This can be achieved if we assume that \(W_0(A)\) fulfills, besides (4.14), also the dual relation for \(\|L_{n-m} W_0(A)(I - L_n)\|\):

**Definition 4.34.** An operator \(A \in \mathcal{L}(l^p)\) is called weakly band-dominated (w.r.t. \((L_n)\)), if
\[
\lim_{m \to \infty} \sup \{\|(I - L_n) AL_{n-m}\|, \|L_{n-m} A(I - L_n)\| : n > m\} = 0
\]
holds. The set of all weakly band-dominated operators shall be denoted by \(\mathcal{W}_p\).

\(^{10}\)This even covers the above counterexamples for the “quasi-triangular setting”, but it remains an open question if, or under which conditions the regularizers of a triangle-dominated operator are of that type again.
Since this is exactly what we know as $\mathcal{P}$-centralized elements from Section 3.3.3, $W_{lp}$ proves to be a Banach subalgebra of $\mathcal{L}(l^p, \mathcal{P})$. For $\mathcal{F}_{W_{lp}}$, the smallest closed algebra which contains all finite section sequences of rich weakly band-dominated operators, we again adapt Proposition 2.28 and take account of Proposition 3.31 to prove the following

**Proposition 4.35.** Let $A \in \mathcal{F}_{W_{lp}}$ be such that $W^0(\mathbb{A})$ is $\mathcal{P}$-Fredholm and let $g \in \mathcal{H}_+$. Then there is a $T$-structured subsequence $\mathbb{A}_h$ of $\mathbb{A}_g$ which is $\mathcal{F}_h^T$-Fredholm.

This permits to replace $A_{lp}$ by $W_{lp}$ in the Theorems 2.31, 2.32 and 2.33 without any further restrictions. We note that clearly $A_{lp} \subset W_{lp}$ by Theorem 1.55 and the inclusion is proper in general as the next section reveals.

### 4.4.7 Toeplitz operators with symbol in $C_p + \overline{H_p}^\infty$

After this short course on generalizations of band-dominated operators on $l^p(\mathbb{Z}, X)$ we again look at Toeplitz operators on $l^p(\mathbb{N}, \mathbb{C})$, and we mention that the notions of triangle-dominated and weakly band-dominated operators translate to these spaces in a natural way, as in Section 4.4.1. Here, only the spaces with $1 < p < \infty$ are of interest, since $M^1 = C^1 = M^\infty = C^\infty = W$ are already completely resolved. Notice that the set of compact and the set of $\mathcal{P}$-compact operators coincide for every $1 < p < \infty$, respectively.

Let $\overline{H_p}^\infty$ stand for the set of all functions $a \in L^\infty(T)$ whose Fourier coefficients fulfill $a_i = 0$ for all $i > 0$, and set $\overline{H_p}^\infty := M^p \cap \overline{H_p}^\infty$. Furthermore, let $\chi_1$ denote the function $\chi_1(x) = x$. From [12], 2.51 we know “It is obvious that $\overline{H_p}^\infty$ is a closed subalgebra of $M^p$. Let $\text{alg}(C_p, \overline{H_p}^\infty)$ denote the smallest closed subalgebra of $M^p$ containing $C_p$ and $\overline{H_p}^\infty$. It is clear that $\text{alg}(C_p, \overline{H_p}^\infty)$ coincides with $\text{alg}(\chi_1, \overline{H_p}^\infty)$. The discontinuous function $(1 - \overline{\chi_1})^{1/2} (\beta \in \mathbb{R} \setminus \{0\})$ can be shown to belong to $\overline{H_p}^\infty$ for all $1 < p < \infty$. ([12], Theorem 6.42 et seq.) This shows that $\text{alg}(C_p, \overline{H_p}^\infty)$ is strictly larger than $C_p$.” Moreover, Sarason [77] proved that this algebra even coincides with

$$C_p + \overline{H_p}^\infty := \{ f + g : f \in C_p, g \in \overline{H_p}^\infty \},$$

and we use this notation in what follows. Böttcher and Silbermann observed that for a function $a \in C_p + \overline{H_p}^\infty$ the Toeplitz operator $T(a)$ is Fredholm on $l^p(\mathbb{N}, \mathbb{C})$ if and only if $a$ is invertible in $C_p + \overline{H_p}^\infty$. In this case $T(a^{-1})$ is a regularizer for $T(a)$. ([12], Theorems 2.53, 2.60). Also recall Coburn’s theorem ([12], Theorem 2.38) which yields that every Fredholm Toeplitz operator is one-sided invertible.

We condense the previous observations to the following generalization of both Theorem 4.28 and the stability criterion in [12], Theorem 7.20.

**Theorem 4.36.** Let $\mathbb{A} = \{ A_n \} := \{ L_n A L_n \}$ with $A = T(a) + K$, $a \in C_p + \overline{H_p}^\infty$ and $K$ compact. Then $\mathbb{A}$ is Fredholm if and only if $A$ is Fredholm. In this case the approximation numbers of the $A_n$ have the splitting numbers $\alpha(\mathbb{A}) = \beta(\mathbb{A}) = \max \{ \dim \ker A, \dim \text{coker } A \}$. Otherwise, all approximation numbers tend to zero as $n \to \infty$. $\mathbb{A}$ is stable if and only if $A$ is invertible.

**Proof.** Clearly, $A$ belongs to $T_{lp}(\mathbb{N})$, the set of all triangle-dominated operators on $l^p(\mathbb{N})$, and if $A$ is Fredholm then the regularizer $T(a^{-1})$ is in $T_{lp}(\mathbb{N})$, too. The snapshot $JW^1(\mathbb{A})J$, as discussed in Section 4.4.1, is the Toeplitz operator with the symbol $\tilde{a} \in C_p + \overline{H_p}^\infty$, $\tilde{a}(t) := a(t^{-1})$. Hence, the Fredholmness of $T(a)$ and Proposition 4.33 imply the Fredholmness of $W^1(\mathbb{A})$, and Coburn’s theorem applied to $JW^1(\mathbb{A})J$ even provides its one-sided invertibility. The rest is as in the proof of Theorem 4.26. \hfill $\Box$

---

11 Analogously, $H^\infty$ consists of all functions $a$ with vanishing Fourier coefficients $a_i$, $i < 0$. 
This particularly covers the Toeplitz operators with so-called quasi continuous symbols, that are functions in $QC^p := \{(C^p + H^\infty_p) \cap (C^p + \overline{H^\infty_p})\}$. Notice that such Toeplitz operators are weakly band-dominated and, by [12], 2.80, $C^2$ is a proper subset of $QC^2$. For some further reading we also refer to [76], or Appendix 4 in [52], and the literature cited there.

**Remark 4.37.** We also want to mention that there is an analogon to Sarasons observation (4.15) for band-dominated operators. We denote the set of all rich triangular operators in $\mathcal{L}(l^p(\mathbb{N}, \mathbb{C}), \mathcal{P})$ by $\Delta_{l^p(\mathbb{N})}$ and note that it forms a Banach algebra. Now, it also holds that

$$\mathcal{A}_{l^p(\mathbb{N})} + \Delta_{l^p(\mathbb{N})} = \text{alg}\{\mathcal{A}_{l^p(\mathbb{N})}, \Delta_{l^p(\mathbb{N})}\}, \tag{4.16}$$

hence this sum is a Banach algebra as well. To prove this we proceed in the same way as in [12] for the Toeplitz case. We recall the mappings $S_R$ which were defined by (1.18) and whose compressions $S_R : \mathcal{L}(l^p(\mathbb{N}), \mathcal{P}) \to \mathcal{L}(l^p(\mathbb{N}), \mathcal{P})$ are well defined, linear and bounded by 2 as well. Proposition 1.56 further yields that $S_R(A)$ are band operators and $\|S_R(A) - A\| \to 0$ as $R \to \infty$ for every band-dominated $A$. Moreover, $S_R(A)$ are triangular whenever $A$ is triangular, and if for $h = (h_n) \in \mathcal{H}_+$ the limit operator $A_h$ exists then there is a subsequence $g \subset h$ such that $(S_R(A))_g$ exists as well (Consider the decompositions $h_n = c_n R + r_n$ with $c_n \in \mathbb{N}$ and remainders $0 \leq r_n < R$, and choose the subsequence $g$ such that the respective remainders converge). If, in particular, $A \in \mathcal{L}(l^p(\mathbb{N}), \mathcal{P})$ is rich then all $S_R(A)$ are rich.

Having these properties of the mappings $S_R \in \mathcal{L}(\mathcal{L}(l^p(\mathbb{N}), \mathcal{P}))$ we can apply a lemma of Zalcman and Rudin (see [12], 2.52) to prove that $\mathcal{A}_{l^p(\mathbb{N})} + \Delta_{l^p(\mathbb{N})}$ is a closed linear space. If $A_i = B_i + T_i$, $i = 1, 2$ with $B_i \in \mathcal{A}_{l^p(\mathbb{N})}$ and $T_i \in \Delta_{l^p(\mathbb{N})}$ then the operators $A_i^R := S_R(B_i) + T_i$ converge to $A_i$ in the norm, respectively. Clearly, $A_1^R A_2^R$ belong to $\mathcal{A}_{l^p(\mathbb{N})} + \Delta_{l^p(\mathbb{N})}$ and converge to $A_1 A_2$ as $R \to \infty$, hence $A_1 A_2 \in \mathcal{A}_{l^p(\mathbb{N})} + \Delta_{l^p(\mathbb{N})}$ as well, that is (4.16) is proved.

### 4.4.8 A larger class of slowly oscillating operators

We finally enlarge the class of slowly oscillating operators by calling an operator $A \in \mathcal{L}_{l^p(\mathbb{N})}$ slowly oscillating if it is rich and all its limit operators are shift invariant, that is

$$V_k A V_{-k} = A_k \quad \text{for all } k \in \mathbb{Z} \text{ and all } A_k \in \sigma_{\text{op}}(A).$$

Since for every sequence $\mathcal{H} := \{L_n AL_n\}$ with a slowly oscillating operator $A$ and every $h \in \mathcal{H}_\mathcal{A}$ the snapshot $JW^A(k_h)J$ is a Toeplitz operator, we can plug Coburns theorem (which provides one-sided invertibility for every Fredholm Toeplitz operator) into the results for triangle-dominated operators. In analogy to Theorem 4.26 we then obtain

**Theorem 4.38.** Let $A$ be a slowly oscillating operator and $\mathcal{H} := \{L_n AL_n\}$ be the sequence of its finite sections. If $A$ is a Fredholm operator with a triangle-dominated regularizer then $\mathcal{H}$ is Fredholm. If $A$ is invertible in $\mathcal{L}_{l^p(\mathbb{N})}$ then $\mathcal{H}$ is stable.

Conversely if $\mathcal{H}$ is Fredholm then $A$ is Fredholm and $\alpha(\mathcal{H}) = \beta(\mathcal{H}) = \max\{\dim \ker A, \dim \text{coker } A\}$ are splitting numbers of $\mathcal{H}$.

It is not clear at the moment if the condition on the regularizer/the inverse of $A$ can be dropped in general. At least, there is a positive answer if $A = B + K$ where $K$ is $\mathcal{P}$-compact and $B$ decomposes into a finite product of operators which are one of the following

- a Fredholm weakly band-dominated operator,
- a Fredholm Toeplitz operator with symbol in $C^p + \overline{H^\infty_p}$,
- a sum $I + C$ with a triangle-dominated $C$ having norm less than 1,
- an invertible quasi-triangular operator.
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Theses

on the thesis

“On some Banach Algebra Tools in Operator Theory”
presented by Dipl.-Math. Markus Seidel

(1) This text is concerned with the investigation of operator sequences $A = \{A_n\}$ which typically arise from approximation methods for bounded linear operators. The heart of the present approach is to capture the asymptotic properties of such sequences in terms of so-called snapshots, a family of operators where each of them displays one basic facet of the asymptotics, but together they give a comprehensive picture of $A$. The snapshots are usually taken by a slight transformation of the sequence and a limiting process. One may interpret this as looking at $A$ from different angles, and one is interested in results as

- The stability of a sequence is equivalent to the invertibility of all snapshots.
- A sequence $\{A_n\}$ is almost stable (we say Fredholm) iff all snapshots are Fredholm. In this case the dimensions of the kernels and cokernels of the snapshots as well as their indices provide information on the splitting property and the indices of the $A_n$.
- The norms $\|A_n\|$ converge to the maximum of the norms of the respective snapshots. Analogous conclusions hold for the norms of the inverses and the condition numbers.
- The pseudospectra of the elements $A_n$ approximate the union of the pseudospectra of all snapshots.

(2) The stability plays a crucial role for the applicability of approximation methods which replace a linear equation $Ax = b$ by (hopefully simpler) equations $A_n x_n = b_n$ since, as a rule, it holds that the convergence of $b_n$ to $b$ and the convergence of $A_n$ to $A$ together with the stability of $\{A_n\}$ imply the existence of the solutions $x_n$ and their convergence to the solution $x$ of the initial equation. Different types of convergence can be studied.

(3) Besides the singular values which are well known and available for operators on Hilbert spaces, there are further geometric characteristics being their natural generalizations to the Banach space case, such as the so-called approximation numbers and the related Bernstein and Mityagin numbers. These characteristics provide the right language to describe the behavior of almost stable (i.e. Fredholm) sequences and constitute the connecting link between the Fredholm properties of the snapshots of an operator sequence $\{A_n\}$ and the Fredholm properties of its entries $A_n$. More precisely, the so-called splitting property determines how many approximation numbers (or singular values in case of Hilbert spaces) of the $A_n$ tend to zero as $n$ gets large. Moreover, the indices of the $A_n$ converge and the limit is nothing but the sum of the indices of all snapshots.
The above observations particularly apply to band-dominated operators on sequence spaces $l^p$ or function spaces $L^p$ with $1 \leq p \leq \infty$, to quasi-diagonal operators and to convolution type operators on cones. They recover lots of well known results for more concrete classes of operators, such as Toeplitz operators, but translate also to larger and more general classes: quasi-triangular, triangle-dominated, or weakly band-dominated operators.

In that business it turns out to be extremely fruitful to replace the classical triple (compactness, Fredholmness, strong convergence) by a similar concept which defines the substitutes $\mathcal{P}$-compactness, $\mathcal{P}$-Fredholmness and $\mathcal{P}$-strong convergence for operators on a Banach space $X$ with the help of so-called uniform approximate identities $\mathcal{P}$. One restricts considerations to the set $\mathcal{L}(X, \mathcal{P})$ of operators $A$ which are compatible with $\mathcal{P}$-compactness in the sense that the product of $A$ with any $\mathcal{P}$-compact operator is $\mathcal{P}$-compact again. In this set $\mathcal{P}$-Fredholmness means invertibility up to $\mathcal{P}$-compact operators, and a sequence converges $\mathcal{P}$-strongly iff, multiplied with a $\mathcal{P}$-compact operator, this always turns into norm convergence. Often, the approximate identities are intrinsically given by an approximation or discretization method, e.g. a sequence of nested projections, and one may say that the $\mathcal{P}$-notions are induced by the underlying method.

Actually, the set $\mathcal{L}(X, \mathcal{P})$ forms a beautiful and perfectly self-contained universe: It is a closed and inverse closed algebra, all $\mathcal{P}$-regularizers (the “almost inverses”) of $\mathcal{P}$-Fredholm operators are still included, and the $\mathcal{P}$-strong limits of sequences in $\mathcal{L}(X, \mathcal{P})$ belong to that set again. The notion of invertibility at infinity, an alternative way how to generalize the classical Fredholm property, proves to coincide with $\mathcal{P}$-Fredholmness. In fact, many concrete settings, such as $l^p$ or $L^p$ spaces, provide the mentioned approximate identities in a very natural way (e.g. sequences of canonical projections), where the classical Fredholm property can be completely embraced by the $\mathcal{P}$-notions, and large classes of operators (Toeplitz and Hankel operators, convolution type and even singular integral operators, band-dominated and quasi-diagonal operators) perfectly fit into this framework.

This alternative $\mathcal{P}$-concept covers and generalizes the classical one in large parts, and it mimics most of the important interactions between approximation methods and the three basic notions compactness, Fredholmness and strong convergence, which are usually exploited. Nevertheless, it displays its full power in settings where the classical approach fails, e.g. if the discretizations $A_n$ are of infinite rank, hence not compact in general, or in cases where the strong convergence is not available anymore (such as $l^\infty$).

The above mentioned Fredholm theory for sequences with a certain asymptotic structure (namely the existence of snapshots) can be embedded into a more general approach which does not require any structure and provides a transcendent notion of Fredholmness. Surprisingly, the approximation numbers still provide a characterizing description for this general Fredholm property via so-called $\alpha$- and $\beta$-numbers, an analogon to the splitting property for the structured case.

Between these two models another one can be established. The so-called rich sequences do not need to be structured in the previous sense, but shall have a rich amount of structured subsequences, and hence they strike a balance and benefit from both models. This dramatically enlarges the class of sequences whose asymptotics can be captured by snapshots (or snapshots of subsequences, in this case) but keeps large parts of the desired results.